Transport and Projective Reduction in a Source–Readout Derivation of Einstein Gravity

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Abstract

We construct a complete source–readout realization of the massless spin-two self-coupling derivation of nonlinear Einstein–matter gravity. An admitted source support, its literal preparation tuple and the physical raw sector are specified separately. Smooth multiplicative transport reconstructs an affine connection, and a path-space regularity theorem supplies its local velocity-linear first jet together with a finite-resolution bound. The gravitational parent is extended to an unrestricted affine connection. Its algebraic distortion Hessian has only the projective one-form kernel in dimension greater than two. A volume-restored transport observable removes that ambiguity and returns the Levi–Civita transport on the connection equations, so torsion-free kinematics need not be imposed in advance. On the quotient, an explicit off-shell stress formula and residual identity recover the entire cubic bootstrap and the Einstein–matter equations, including the required horizontal auxiliary-metric variation. We also prove a restricted minimal-scalar selection statement and exhibit inequivalent signatures, actions and matter couplings compatible with the target-neutral source architecture. Thus the construction reduces several separate realization hypotheses while exposing the remaining physical selection problem. Exact finite algebra, independent curvature and variational implementations, and controlled numerical reconstruction checks accompany the complete derivation.

1 Problem, contribution and logical dependencies

An architecture that represents an established physical theory faces two distinct mathematical tasks. It must recover the theory, including its operations and solution domain, and it must identify which features of the represented theory its own principles select. For gravity, an equality between two named actions addresses only part of the first task. A complete construction must recover the current that drives self-coupling, the allowed variations, the connection equation, matter response and the nonlinear field equations on one common realization.

Deser's first-order derivation supplies a particularly effective starting point [Deser, 1970, 2010]. A free massless spin-two field couples to its total symmetric stress. In contravariant metric-density and independent-affinity variables, the appropriate cubic interaction closes the bootstrap in one step. The derivative part of the stress is essential: it is generated by the nonlinear density–connection relation rather than by inserting the full current directly into the action. The endpoint is the nonlinear Einstein system, with universal metric coupling in the declared matter class.

We construct that entire chain within the source–readout architecture of Rodgers [2026b]. The architecture distinguishes an admitted source, a literal finite preparation support and a physical

raw-sector construction. Its lift is a common-witness homotopy fibre. This distinction permits an exact realized gravitational sector while keeping the selection of its physical law visible. Geometric descent and observable-quotient machinery provide useful related contracts [Rodgers, 2026a,c].

The principal results concern three dependencies. First, a precise smoothness hypothesis on multiplicative path transport supplies the local velocity-linear first-jet rule and a quantitative reconstruction estimate. The connection–transport correspondence is established mathematics [Schreiber and Waldorf, 2009]; we give the local functional-analytic argument and the bounds needed by this realization. Second, the affine connection can be unrestricted at the kinematic level. The classical Einstein–Palatini projective freedom [Dadhich and Pons, 2012] is handled by an explicit volume-restored source observable. The connection equations then recover Levi–Civita transport on shell. Third, the full stress identity and bootstrap extend to the projective quotient, where the connection Hessian has a controlled kernel. This yields an off-shell residual identity as well as the complete Einstein–matter endpoint.

We establish the precise scope of these reductions. Path-space smoothness is a realization condition. Projective reduction requires an observable contract: bare vector transport changes under the projective symmetry. Minimal scalar matter follows in a specified natural, metric-ultralocal quadratic class, while common-density coupling alone permits nonminimal curvature terms. The target-neutral source principles admit inequivalent raw-sector choices, so they do not by themselves determine dimension, signature, the Einstein action or its coupling. A dependency ledger in Section 7 records the actual warrants.

The construction is classical and local on a prepared smooth chart. It includes finite integration patches and relative action variations with compact support. The gravitational and matter fields, path operations and boundary restrictions share one source witness. The physical four-dimensional Lorentzian setting and the free spin-two constitutive law are stated inputs. Results about arbitrary source fibres, quantum measures or a full all-sector physical compatibility equalizer are separate questions.

1.1 Conventions

Physical equations use an oriented four-dimensional chart U, signature $(-,+,+,+)$, and natural units. We use $n > 2$ only when proving dimension-independent affine algebra. The first lower index of an unrestricted connection is the differentiation direction:

$$(A_\mu)^\alpha{}_\beta = \Gamma^\alpha{}_{\mu\beta}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \qquad R_{\mu\nu} = F^\alpha{}_{\mu\alpha\nu}. \tag{1}$$

No symmetry of the two lower connection indices is assumed in the extended parent. A symmetric density couples to $R_{(\mu\nu)}$. All contractions sum full index ranges; derivatives with respect to symmetric variables use the corresponding off-diagonal multiplicities.

Write $B^{\mu\nu} = v g^{\mu\nu}$, $v = \sqrt{|\det g|}$, and choose the normalized Cartesian reference density $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. Reference indices are raised and lowered with η and its inverse; in particular the reference volume density is one. Matrix estimates use a fixed induced operator norm in the prepared frames, so that each matrix entry is bounded by that norm. In four dimensions,

$$B = \eta + \lambda h, \qquad A = \lambda c, \qquad v = \sqrt{-\det B}, \qquad g^{\mu\nu} = B^{\mu\nu}/v. \tag{2}$$

Here $\lambda \neq 0$ restores the canonical coupling. The canonical affinity c has dimension length^{-2} and is unrelated to the symbol for a signal speed. The physical coupling will be $\kappa_E = \lambda^2/2$.

2 Admitted support and realized source variables

2.1 A native source family

The source signature of Rodgers [2026b, Chapter 2] contains a dependency preorder, deposition-potential tokens, a marked subsort, finite supports, lineage, invariants and intrinsic boundary data. It contains no metric, coordinate, field, action or desired outcome. Source admission is the least closure of a given finite witness class under strong generated maps, generator-relative attachments, suitable retracts, stabilizing directed colimits and isomorphisms. Strong maps retain the strict marked pairs and their lineages.

The walking marked dependency $X_\star$ is explicitly in the witness class: its two source vertices satisfy $0 \prec 1$, with one marked token supported on $\{0,1\}$. Its proper predecessor-closed source face is the root face $F_0 = \{0\}$. Faces may be attachment domains even when they have no marked token and are not admitted source candidates themselves.

Proposition 2.1 (Admitted walking-cell support)**.** *For every nonempty small index set I, the rooted fan X_I consisting of a root r, distinct leaves ℓ_i, dependencies $r \prec \ell_i$ and copies of the walking marked token with supports $\{r, \ell_i\}$ is admitted by the stated generated closure. All invariant and lineage data are inherited from the walking cells.*

Proof. Attach a fresh walking cell along its root face to the inherited root of the previous cell. For a finite number of attachments, the pushout has precisely the common root and the distinct leaf/cell copies: maps agreeing on the root extend uniquely. Transitivity adds no leaf–leaf dependency. Every marked support retains its two distinct elements, strict pair and inherited lineage. The root-face map is strong and mark-reflecting, and all old cell data remain unchanged. This verifies the source conditions for these particular generator-relative attachments and proves finite admission by induction.

Take the directed family over finite nonempty subsets of I. Once a finite cell is present, its support, strict pair, lineage and invariant data are unchanged by every later inclusion. The allowed stabilizing directed colimit is exactly X_I. Permuting I is a source isomorphism. The construction uses only existing walking cells, with no freely adjoined physical label. □

This proposition supplies an admission presentation for the chosen family. The branch labels carry no spatial or temporal calibration. The result has no assumption about the raw physical law that the support will realize.

2.2 Literal preparation and the raw sector

For a fixed marked token m_*, the monograph's literal preparation is

$$Y = J_{\mathrm{real}}(X_I, m_*) = (X_I, m_*, S_{m_*}, \preceq_{m_*}, \ell_{m_*}, \mathrm{Inv}_{m_*}, \mathrm{Bnd}_{m_*}). \tag{3}$$

Here S_{m_*} and its inherited data are the finite lineaged support; X_I records its ambient admitted source. This is the preparation tuple of equations (3.1)–(3.2) in the monograph. Physical fields belong instead to the gravitational raw-sector relation

$$\mathcal{K}_{\mathrm{grav}}(Y) \subseteq \mathsf{O}^{\mathrm{raw}}_{\mathrm{grav}} \times \mathsf{D}_{\mathrm{grav}}. \tag{4}$$

The discrepancy column retains both zero and nonzero equation, composition and interface residuals. Einstein solutions are not used to define source admission.

Choose a physical preparation consisting of U, its coordinate/tangent frames, a path family, reference density, coupling and boundary convention. A bare marked-occurrence assignment β covers the prepared piecewise smooth paths, their subarcs, reversals and compositions. It assigns whole marked occurrences to paths; it is not a map sending the shared abstract source root to every physical path endpoint. Endpoint coincidences and path composition are added raw-sector operations.

A raw lifted history has the form

$$z = (\beta, T, B, \chi, \text{preparation}, \text{residuals}), \qquad T_\gamma : T_xU \longrightarrow T_yU \quad (\gamma : x \to y). \tag{5}$$

B is a smooth section of $\mathrm{Sym}^2 TU \otimes \mathrm{Dens}(U)$ on a nondegenerate Lorentz branch; χ denotes the chosen metric-matter fields. The transport is invertible, multiplicative, reparameterization invariant and identity on constant paths. Section 3 states its regularity precisely. The raw sector contains off-shell histories and retains their residuals. Its physical equation selection is the variational law developed below.

This is an explicit realization rule over the literal support (3). The smooth manifold, transport class, density and constitutive action are physical/raw-sector inputs. Their placement in (4) preserves the source signature's target neutrality and makes the remaining selection freedom explicit.

2.3 Common witnesses and recovery

For this gravitational sector, $\mathsf{Comp}_{\mathrm{GR}}$ has metric, connection, stress and matter configurations as objects, with their required compatibility equations; arrows are the allowed gauge transformations. Let $\mathcal{E}_{\mathrm{GR}}$ consist of the admitted fan, its preparation and the histories (5), with arrows preserving these data and their readouts. The projection π reconstructs $A[T]$ and g_B, and evaluates the stress and matter maps defined below. A recovery witness for a physical configuration O is a history z together with an isomorphism $\varphi : \pi(z) \simeq O$. These witnesses form the gravitational homotopy fibre

$$\mathrm{Lift}(O) = \mathcal{E}^{\simeq}_{\mathrm{GR}} \underset{\mathsf{Comp}^{\simeq}_{\mathrm{GR}}}{\times^h} \{O\}. \tag{6}$$

This is the gravitational instance of the common-witness definition in Rodgers [2026b, Section 1.3.3]. In particular B, T, χ, the action, its allowed variations and the stress response belong to the same history. The explicit projection, inverse and variation calculations in Sections 3–6 establish its recovery properties.

The construction proves recovery on this specified branch of (6), including complete solution coverage in its declared domain. It makes no assertion about arbitrary other source antecedents of the same physical configuration.

3 Transport regularity, reconstruction and resolution

3.1 The first jet as a consequence of path-space regularity

Use the prepared trivialization to regard transports as matrices. Around each constant path, $C^1([0,1], U)$ is an open part of a Banach space. The differentiability below is with respect to its C^1 norm and includes constant paths. Smoothness only on immersed paths is a weaker condition.

Theorem 3.1 (Regular transport reconstruction)**.** *Suppose T is multiplicative on successive subarcs, identity on constants and invariant under orientation-preserving reparameterization. Suppose T is continuously Fréchet differentiable on open C^1 neighborhoods of the constant paths and the map $x \mapsto DT_{\mathrm{const}(x)}$ is smooth as a map to bounded linear operators. Assume the corresponding joint higher differentiability for parameterized variations and higher derivatives used below. Then there is a smooth matrix-valued one-form A such that*

$$\frac{\mathrm{d}}{\mathrm{d}s} T_{\gamma|[0,s]} = -A_{\gamma(s)}(\dot\gamma(s)) T_{\gamma|[0,s]}, \qquad T_{\gamma|[0,0]} = \mathrm{I}. \tag{7}$$

The first jet is linear in velocity and depends only on the current point and velocity. Conversely a smooth A defines such a smooth transport family. The two constructions are inverse on this regular class.

Proof. At the constant path at x, set

$$A_x(v) = -DT_{\mathrm{const}(x)}[t \mapsto tv]. \tag{8}$$

The Fréchet derivative is linear on its tangent space, so the right side is linear in v. For a C^2 curve, its actual short subarc satisfies

$$\gamma(s+\epsilon t) = \gamma(s) + \epsilon t \dot\gamma(s) + O(\epsilon^2) \quad \text{in } C^1([0,1]). \tag{9}$$

Differentiability implies $T_{\gamma|[s,s+\epsilon]} = \mathrm{I} - \epsilon A_{\gamma(s)}(\dot\gamma(s)) + o(\epsilon)$. The continuous derivative makes this remainder uniform on compact families of the short paths. Composition therefore makes $s \mapsto T_{\gamma|[0,s]}$ continuous and gives its right derivative in (7), with a continuous right-hand side. The integral equation follows, and hence the ordinary derivative and (7). Piecewise smooth paths follow by composition across the pieces.

Conversely solve the matrix integral equation associated with (7). Its Picard series converges uniformly on compact path images, with the jth iterated integral bounded by $M^j/j!$ for an appropriate integrated generator bound. Uniqueness gives composition, reversal and reparameterization invariance. On a compact controlled path neighborhood, the Nemytskii map $\gamma \mapsto A(\gamma)\dot\gamma$ has the regularity of A. Differentiating the integral equation, or its uniformly bounded Picard series, proves the corresponding path-space smoothness. The generator of this reconstructed family is (8). Uniqueness then proves the inverse identity for every family satisfying the hypotheses. □

This is a local sufficient realization condition for the established connection–transport correspondence [Schreiber and Waldorf, 2009]. It discharges a separate first-jet postulate within the stated smooth-functor model. The smoothness condition itself is physical/raw-sector data, rather than a consequence of the finite source signature.

Proposition 3.2 (A regularity obstruction)**.** *Composition, reversal and reparameterization invariance alone permit a nonlinear velocity law. In two coordinate directions let*

$$F(v,w) = \frac{v^3}{v^2+w^2} \quad ((v,w) \neq 0), \qquad F(0,0) = 0, \qquad T_\gamma = \exp\Big[-\int F(\dot\gamma^1, \dot\gamma^2)\,\mathrm{d}s\Big]\,\mathrm{I}. \tag{10}$$

This family has the three stated algebraic properties and is smooth on controlled immersed-path neighborhoods. It is not Fréchet differentiable at the constant paths in the sense of Theorem 3.1.

Proof. F is odd and positively homogeneous of degree one. These identities give reversal and reparameterization invariance; additivity of the integral gives composition. At a constant path, its derivative along the straight variation $t \mapsto t(v,w)$ would have to be $-F(v,w)\mathrm{I}$. But $F(1,1) = 1/2$, whereas $F(1,0) + F(0,1) = 1$. A Fréchet derivative cannot have these directional values. □

3.2 Admissible variations and curvature

For a compactly supported generator variation δA, differentiation of (7) gives

$$\delta T_\gamma(1) = -\int_0^1 T_\gamma(1,s)\,\delta A_\mu(\gamma(s))\dot\gamma^\mu(s)\,T_\gamma(s,0)\,\mathrm{d}s. \tag{11}$$

Its infinitesimal readout is precisely δA. Conversely, differentiate the reconstructed transport equation in a jointly smooth one-parameter family: its tangent obeys the inhomogeneous equation with zero initial value whose unique solution is (11). Thus every regular tangent transport variation arises from its generator variation, and the readout covers all the allowed field variations. Commutation of the parameter derivative with the short-path derivative uses the joint second differentiability stipulated here; it is not inferred from pointwise path differentiability alone.

Compose the four transports around a positively oriented small coordinate rectangle. Expanding the two-parameter integral equations gives

$$T_{\square_{\mu\nu}} = \mathrm{I} - F_{\mu\nu}\,\Delta x^\mu \Delta x^\nu + O\big((\Delta x^\mu)^2\Delta x^\nu + \Delta x^\mu(\Delta x^\nu)^2\big). \tag{12}$$

The commutator in (1) is generated by the ordered matrix products. For constant matrices the loop order is $e^{A_\nu t}e^{A_\mu s}e^{-A_\nu t}e^{-A_\mu s}$, whose mixed coefficient is $-[A_\mu, A_\nu]$. This fixes the sign convention. Endpoint derivatives and small-loop limits therefore recover the original differential and nonlinear connection operations.

3.3 A reconstruction bound and its limitation

Proposition 3.3 (Finite-resolution transport)**.** *Suppose the second path derivatives are bounded on the relevant neighborhood and γ is C^2. Then its subarc defect has a bound*

$$\left\| T_{\gamma|[s,s+h]} - \mathrm{I} + hA(\gamma(s))\dot\gamma(s) \right\| \le Ch^2. \tag{13}$$

The constant is controlled by the first two derivatives of T and the first two derivatives of γ. If the generator along a parameter interval of length L has norm at most M, the left-endpoint product on a partition of mesh $h_{\max}$ obeys

$$\left\| T_\gamma - \overleftarrow{\prod_j}[\mathrm{I} - h_j A(\gamma(s_j))\dot\gamma(s_j)] \right\| \le CLh_{\max}e^{ML}. \tag{14}$$

Proof. Taylor expansion of the Fréchet map, using (9), proves (13). Both exact subarc transports and Euler factors have norm bounded by e^{Mh_j}. Telescope the difference of the ordered products. Each local defect is multiplied by factors bounded in product by e^{ML}, and $\sum_j h_j^2 \le Lh_{\max}$. □

If a measured short transport has error at most ϵ and the forward first-jet remainder is at most Kh^2, the generator estimate has error

$$\left\| \widetilde{A}_h - A \right\| \le \epsilon/h + Kh. \tag{15}$$

For an admissible optimizing scale this is at most $2\sqrt{K\epsilon}$. With a controlled third derivative, centered differences give $\epsilon/h + K_3h^2$ and optimized order $\epsilon^{2/3}$. These statements price differentiation; neither assumes exact finite localization nor demonstrates an available physical instrument.

Curvature requires additional spatial derivative control. If $E = \widetilde{A} - A$, with $\max_\mu \|A_\mu\| \le M$, $\max_\mu \|E_\mu\| \le \delta_0$ and $\max_{\mu,\nu} \|\partial_\mu E_\nu\| \le \delta_1$, then

$$\left\| F_{\mu\nu}(\widetilde{A}) - F_{\mu\nu}(A) \right\| \le 2\delta_1 + 4M\delta_0 + 2\delta_0^2. \tag{16}$$

The proof expands the derivative and three commutator differences and uses submultiplicativity. Componentwise Ricci errors have at most the additional contraction factor n. This is a local residual estimate, rather than stability of the nonlinear Einstein initial-value problem.

The derivative requirement is substantive. In a flat chart choose only ${\Gamma^0}_{11} = \sin(kx^0)/k$ nonzero. This connection is torsion-free, its generator tends uniformly to zero and its transports tend to identity on bounded-length path families. Yet $R_{11}(0) = 1$ for every k. Along $x^0 = x^1 = s$, $0 \le s \le 1$, the transport difference has norm at most $2/k^2$, since the generator is a fixed nilpotent matrix times $\sin(ks)/k$. Small uniform source readout errors alone cannot certify small gravitational equation residuals.

4 Unrestricted affine connection and projective reduction

4.1 Curvature with a definite index convention

For an unrestricted ${\Gamma^\alpha}_{\mu\beta}$, equation (1) gives

$$R_{\mu\nu}(A) = \partial_\alpha {\Gamma^\alpha}_{\nu\mu} - \partial_\nu {\Gamma^\alpha}_{\alpha\mu} + {\Gamma^\alpha}_{\alpha\rho}{\Gamma^\rho}_{\nu\mu} - {\Gamma^\alpha}_{\nu\rho}{\Gamma^\rho}_{\alpha\mu}. \tag{17}$$

This order matters. The expression obtained by extending the familiar symmetric-affinity contraction ${\Gamma^\alpha}_{\mu\nu}{\Gamma^\beta}_{\alpha\beta} - {\Gamma^\alpha}_{\beta\mu}{\Gamma^\beta}_{\alpha\nu}$ without imposing its lower-index symmetry is a different polynomial. For ${\Gamma^\alpha}_{\mu\beta} = \delta^\alpha_\beta \omega_\mu$, that polynomial is $(n-1)\omega_\mu\omega_\nu$, while the quadratic term in (17) vanishes.

Define symmetric component operators

$$D_{\mu\nu}(c) = \mathrm{Sym}_{\mu\nu}(\partial_\alpha {c^\alpha}_{\nu\mu} - \partial_\nu {c^\alpha}_{\alpha\mu}), \tag{18}$$

$$Q_{\mu\nu}(c) = \mathrm{Sym}_{\mu\nu}({c^\alpha}_{\alpha\rho}{c^\rho}_{\nu\mu} - {c^\alpha}_{\nu\rho}{c^\rho}_{\alpha\mu}). \tag{19}$$

Then $R_{(\mu\nu)}(\lambda c) = \lambda D_{\mu\nu}(c) + \lambda^2 Q_{\mu\nu}(c)$. On symmetric lower indices these recover Deser's contractions.

4.2 The algebraic kernel

Consider the affine density action $\int B^{\mu\nu} R_{\mu\nu}(A)$, with a nondegenerate metric density and matter independent of A. Write

$$A = \mathrm{LC}(g) + C, \qquad C \in \Gamma(TU \otimes T^*U \otimes T^*U). \tag{20}$$

Since $\nabla^g B = 0$, the terms linear in $\nabla^g C$ integrate to the matched boundary current. The bulk distortion action is exactly

$$q_B(C) = B^{\mu\nu}({C^\alpha}_{\alpha\rho}{C^\rho}_{\nu\mu} - {C^\alpha}_{\nu\rho}{C^\rho}_{\alpha\mu}). \tag{21}$$

Thus every non-gauge distortion is an auxiliary variable, rather than a new connection wave.

Theorem 4.1 (General affine stationary connection)**.** *For a smooth nondegenerate metric density in dimension $n > 2$, the stationary points of (21) are exactly*

$${C^\alpha}_{\mu\beta} = \delta^\alpha_\beta \omega_\mu. \tag{22}$$

The Hessian has rank $n^3 - n$, independently of signature. Consequently the connection equation of the affine Einstein action, with metric matter, gives $A = \mathrm{LC}(g) + \mathrm{I}\omega$.

Proof. At a point divide by v and vary the independent component $C^{\ell}{}_{pq}$. The equation is

$$\delta^p_\ell g^{mn} C^q{}_{nm} + g^{pq} C^a{}_{a\ell} - g^{mp} C^q{}_{\ell m} - g^{qn} C^p{}_{n\ell} = 0. \tag{23}$$

Lower the first index of C, and put $U_a = g^{bc} C_{abc}$ and $V_b = g^{ac} C_{cab}$. Lowering p, q in (23) gives

$$C_{abc} + C_{cab} = g_{bc} U_a + g_{ca} V_b. \tag{24}$$

The three cyclic equations imply

$$2C_{abc} = g_{bc}(U_a - V_a) + g_{ca}(U_b + V_b) + g_{ab}(V_c - U_c). \tag{25}$$

Its b, c trace yields $(n-2)(U_a - V_a) = 0$. Hence $U = V$ and $C_{abc} = g_{ac} V_b$, which is (22). Conversely direct substitution satisfies every equation. The quadratic form's kernel is therefore precisely the n-dimensional projective space, proving the rank. No definiteness assumption entered the argument. □

The projective equivalence is a classical result [Dadhich and Pons, 2012]. The source construction uses its precise observable implications. The raw connection has torsion $\delta^\alpha_\nu \omega_\mu - \delta^\alpha_\mu \omega_\nu$ and need not be metric compatible. Calling it uniquely Levi–Civita before treating the projective freedom would be incorrect.

4.3 A source readout that fixes the observable ambiguity

Under a projective shift,

$$A_\mu \longmapsto A_\mu + \omega_\mu \mathrm{I}, \qquad F_{\mu\nu} \longmapsto F_{\mu\nu} + (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)\mathrm{I}. \tag{26}$$

Only an antisymmetric tensor is added to Ricci, so the symmetric-density action is invariant. The same shift changes the full source transport by

$$T_\gamma \longmapsto e^{-\int_\gamma \omega} T_\gamma. \tag{27}$$

Bare open-vector transport is consequently not an invariant of this symmetry. A projective quotient needs an explicit raw-observable contract.

For $n > 2$ set $v_B = |\det B|^{1/(n-2)}$ and, in the prepared endpoint frames, define

$$T^B_\gamma = \left[\frac{v_B(x)}{v_B(y) \det T_\gamma}\right]^{1/n} T_\gamma, \qquad \gamma : x \to y. \tag{28}$$

Transports continuously generated from identity have positive determinant, so the positive root is unambiguous. In physical four dimensions $v_B = \sqrt{-\det B}$.

Theorem 4.2 (Volume-restored transport)**.** *On connected coordinate patches the readout $(B, T) \mapsto (B, T^B)$ is projectively invariant, multiplicative and natural under diffeomorphic coordinate changes, including orientation reversal. Its infinitesimal connection is*

$$A^B_\mu = A_\mu - \frac{\operatorname{tr} A_\mu}{n}\mathrm{I} + \frac{\partial_\mu \log v_B}{n}\mathrm{I}. \tag{29}$$

It identifies the projective transport quotient with the regular transport families satisfying $\det T^B_\gamma = v_B(x)/v_B(y)$. On the connection equations of Theorem 4.1, $A^B = \mathrm{LC}(g_B)$ and T^B is exactly metric parallel transport.

Proof. The scalar factor in (27) cancels its nth power in the determinant of (28). For two successive paths, the intermediate volume and determinant factors cancel, proving composition. Under a pullback with endpoint Jacobians J_x, J_y, $T' = J_y^{-1} T J_x$ and $v'(x) = |\det J_x| v(\Phi x)$. The sign of $\det J$ is constant along the path, so $\det J_x / \det J_y = |\det J_x| / |\det J_y|$. Hence the scalar multiplier is invariant and T^B transforms as a tangent map.

Differentiate (28) on a short path, using $\mathrm{d} \log \det T / \mathrm{d}s = -\operatorname{tr}(A_\mu)\dot{\gamma}^\mu$. This gives (29). Its trace is $\partial_\mu \log v_B$. Every projective class has this unique representative: subtract the one-form $(\operatorname{tr} A - \mathrm{d} \log v_B)/n$. The inverse-transport theorem then proves the quotient/volume-family inverse identities for all regular histories. Finally Theorem 4.1 gives $A = \mathrm{LC}(g_B) + \mathrm{I}\omega$, and $\operatorname{tr} \mathrm{LC}(g_B) = \mathrm{d} \log v_B$, so (29) removes exactly ω. □

Equation (29) fixes trace, not torsion, off shell. The unrestricted quotient has 60 connection coordinates in four dimensions, while the torsion-free parent has 40. The additional twenty coordinates are removed by the algebraic stationary equations. The on-shell metric transport and Einstein equations agree; the two whole off-shell parents are different.

We use B, T^B and metric matter as the extended model's projectively invariant readouts. If bare transport or a projectively sensitive boundary norm is retained as an observable, this quotient is unavailable. Equivalently, a trace-fixed boundary collar may be used with only transformations preserving that collar. These are concrete restrictions on the realization and its observables, rather than consequences of source admission alone.

The normalization is itself an operation on source data: it uses endpoint densities and the determinant of the given transport. It does not insert the Levi–Civita formula into the source kinematics. That formula is recovered by the action equations. Variation may be performed before quotienting. If the volume section is used during variation, its moving trace contributes a projective tangent; its contraction with the connection Euler derivative vanishes by the trace identity proved below.

5 The complete source bootstrap on the quotient

5.1 An explicit source action and projection

Let $A[T]$ be the generator reconstructed in Theorem 3.1. The realized source action is built from its endpoint and short-path operations:

$$S_{\mathrm{src}}[T, B, \chi] = \lambda^{-2} \int_U \left[(B - \eta)^{\mu\nu} D_{\mu\nu}(A[T]) + B^{\mu\nu} Q_{\mu\nu}(A[T]) \right] \mathrm{d}^4 x + S_m[B, \chi]. \tag{30}$$

This is a chosen physical variational law over (4). Its free quadratic part is Deser's starting spin-two law. Its cubic part must be verified against the total-current requirement. Neither part is a condition for primitive source admission.

Projection by (2) gives the two source stages

$$S_2 = \int (h^{\mu\nu} D_{\mu\nu}(c) + \eta^{\mu\nu} Q_{\mu\nu}(c)), \qquad S_3 = \lambda \int h^{\mu\nu} Q_{\mu\nu}(c). \tag{31}$$

The inverse is $B = \eta + \lambda h$ and $T = \mathrm{Hol}(\lambda c)$, proved by the transport ODE. All generator and density variations in the declared domain lift through (11). Thus the action and its variational operations are being recovered, rather than only its pointwise values.

5.2 Explicit quotient operators

In four dimensions let $c_I = c^{\alpha}{}_{\mu\beta}$ range over all 64 independent components and define

$$q_z(c) = z^{\mu\nu} Q_{\mu\nu}(c), \qquad H_z = \frac{\partial^2 q_z}{\partial c_I \partial c_J}, \qquad J(z) = D^* z. \tag{32}$$

The formal adjoint is fixed by compact variations: $\int z^{\mu\nu} D_{\mu\nu}(u) = \int J(z)^I u_I$ after its matched boundary integration. Explicitly,

$$J(h)_\ell{}^{pq} = -\partial_\ell h^{qp} + \delta^p_\ell \partial_n h^{qn}. \tag{33}$$

There is no inverse differential operator in these definitions.

Let $P\omega$ have components $\delta^\alpha_\beta \omega_\mu$. In the fixed component basis $P^T P = 4\mathrm{I}$ and $\Pi = PP^T/4$ projects onto the kernel of $H = H_\eta$. Theorem 4.1 implies that $H + \Pi$ is invertible. Set

$$G = (H + \Pi)^{-1} - \Pi, \qquad HG = GH = \mathrm{I} - \Pi, \qquad G^T = G. \tag{34}$$

This is a coordinate formula for the inverse on the quotient. A different valid inverse changes its relevant outputs only by a projective vector, annihilated by D. No physical Euclidean metric is selected by this component-space construction.

For every symmetric z, $H_z P = 0$ and $P^T J(z) = 0$. The first identity follows from projective invariance of q_z; the second follows from $D(P\omega) = 0$. Define

$$L(h) = -GJ(h), \qquad K(h,c) = GH_h c, \qquad \tau_{\mu\nu}(h,c) = Q_{\mu\nu}(c) - D_{\mu\nu} K(h,c). \tag{35}$$

The free connection equation is $Hc + J(h) = 0$, so $c = L(h) + P\omega$. The symmetric Ricci readout is independent of ω. The linearized Levi–Civita connection of

$$\phi_{\mu\nu} = -h_{\mu\nu} + \tfrac{1}{2}\eta_{\mu\nu} \operatorname{tr}_\eta h \tag{36}$$

solves this full equation and differs from $L(h)$ only projectively. Therefore $r(h) = DL(h)$ is exactly Deser's metric spin-two operator:

$$2r_{\mu\nu}(h) = \Box h_{\mu\nu} - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} - \tfrac{1}{2}\eta_{\mu\nu}\Box \operatorname{tr}_\eta h. \tag{37}$$

Its trace-adjusted Einstein form has identically zero flat divergence. The linear gauge transformations follow by substitution: $\delta h^{\mu\nu} = \eta^{\mu\nu}\partial_\alpha \xi^\alpha - \partial^\mu \xi^\nu - \partial^\nu \xi^\mu$ and $\delta c^\alpha{}_{\mu\nu} = \partial_\mu \partial_\nu \xi^\alpha$, together with the projective freedom. Gauge identities are consequently properties of the selected quadratic law, not additional independent equations.

5.3 Horizontal auxiliary variation and the full stress

Introduce an auxiliary inverse metric density ψ near η. In this operation h is a weight-one tensor density and c is a general $(1,2)$ tensor. With the auxiliary background connection $C(\psi) = \mathrm{LC}(g(\psi))$, covariantize

$$S_2(\psi) = \int \big[h^{\mu\nu}(\nabla_\alpha c^\alpha{}_{\nu\mu} - \nabla_\nu c^\alpha{}_{\alpha\mu}) + \psi^{\mu\nu} Q_{\mu\nu}(c)\big]. \tag{38}$$

The torsion-free auxiliary background is a computational prescription; the dynamical c remains unrestricted. Its direct connection variation is

$$\delta_C S_{2,\mathrm{kin}} = \int h^{\mu\nu}\big[\delta C^\alpha{}_{\alpha\rho} c^\rho{}_{\nu\mu} - \delta C^\rho{}_{\alpha\nu} c^\alpha{}_{\rho\mu} - \delta C^\rho{}_{\alpha\mu} c^\alpha{}_{\nu\rho} + \delta C^\rho{}_{\nu\mu} c^\alpha{}_{\alpha\rho}\big]. \tag{39}$$

This is also $\int (H_h c)^T \delta C$. One may verify that equality directly, or expand $BR(A)$ about $(\psi, C(\psi))$ and take its quadratic coefficient.

Theorem 5.1 (Off-shell source stress)**.** *For the prescribed auxiliary tensor/density variation,*

$$\left.\frac{\delta S_2(\psi)}{\delta \psi^{\mu\nu}}\right|_{\psi=\eta} = Q_{\mu\nu}(c) - D_{\mu\nu}(GH_h c) = \tau_{\mu\nu}(h,c). \tag{40}$$

It holds off shell for all smooth h, c in the stated domain, including unrestricted torsion components.

Proof. The background variation δC is the linearized Levi–Civita connection of $\delta\psi$. It solves $H\delta C + J(\delta\psi) = 0$, so $\delta C = -GJ(\delta\psi) + P\zeta$ for some one-form ζ. Since $H_h P = 0$,

$$\int (H_h c)^T \delta C = -\int (GH_h c)^T J(\delta\psi) = -\int \delta\psi^{\mu\nu} D_{\mu\nu}(GH_h c). \tag{41}$$

The explicit ψQ term supplies Q. The last integration uses the same compact-variation boundary contract as (32). This proves every component of (40), including derivatives of both h and c. □

The source operation implementing this variation is

$$B(\psi) = \psi + \lambda h, \qquad T(\psi) = \mathrm{Hol}[C(\psi) + \lambda c], \qquad c_\psi = \frac{A[T] - C(\psi)}{\lambda}. \tag{42}$$

The auxiliary action (38) is evaluated on this relative tensor generator. Holding c_ψ fixed is different from holding the physical transport fixed. If $B(\psi)$ follows (42) but T is frozen, the off-shell chain-rule defect is

$$\tau_{\text{fixed T}} = \tau + \lambda^{-1} D(GE_c^{(2)}), \qquad E_c^{(2)} = J(h) + Hc. \tag{43}$$

Indeed $\delta c_\psi = -\delta C/\lambda$, and the same adjoint calculation proves (43). For $h = 0$ and only ${c^0}_{11} = x^0$ nonzero, the correct τ_{11} is zero but the defect is $1/\lambda$. Separately, omitting the kinetic covariant variation altogether keeps Q alone and loses $-DK$. These are distinct failures. Equation (42) repairs the intended operation in the source parent.

5.4 Cubic closure and a complete residual identity

Theorem 5.2 (Closed bootstrap and off-shell recovery)**.** *For $S_2 + S_3$, define the first-order source Euler residuals*

$$E_h = D(c) + \lambda Q(c), \qquad E_c = J(h) + Hc + \lambda H_h c. \tag{44}$$

Then the exact off-shell identity is

$$r(h) + \lambda\tau(h,c) = E_h - D(GE_c). \tag{45}$$

In particular all source solutions satisfy $r(h) = -\lambda\tau$. The cubic interaction produces no further auxiliary stress, so this self-coupling closes in one step.

Proof. Apply G to the second residual: $GE_c = -L(h) + (\mathrm{I} - \Pi)c + \lambda K(h,c)$. Since $D\Pi c = 0$, subtraction from the first residual gives (45). On a connection solution this is the exact relation $c = L(h) - \lambda K(h,c) + P\omega$, whose derivative supplies the derivative stress term in the h equation. The interaction λhQ has neither explicit auxiliary-background dependence nor explicit derivatives when the relative tensor c is held fixed under (42). Its auxiliary stress is zero. Thus the current already equals the total self-current in this prescription. □

Equation (45) is stronger than a check on exact solutions: it specifies the derivative price for transferring connection-equation errors to the second-order readout. It gives $\|r + \lambda\tau\| \le \|E_h\| + \|DGE_c\|$ in matching norms. It does not turn a small undifferentiated connection residual into a certified Einstein residual.

The action is now identified, after verifying its current:

$$\lambda^{-2} B^{\mu\nu} R_{\mu\nu}(A) = \lambda^{-1} \eta^{\mu\nu} D_{\mu\nu}(c) + h^{\mu\nu} D_{\mu\nu}(c) + \eta^{\mu\nu} Q_{\mu\nu}(c) + \lambda h^{\mu\nu} Q_{\mu\nu}(c). \tag{46}$$

The first term is the displayed constant-density divergence. No infinite series has been assumed to converge. The current construction and nonlinear closure recover Deser's gravitational equations on the entire declared domain, rather than reproducing only an isolated connection identity.

6 Einstein–matter recovery, symmetry and boundary data

6.1 The metric equations and normalization

Take a connected nondegenerate Lorentzian density domain containing η, and let the matter action depend on B and matter fields χ, but not on the independent affine connection. Up to the explicit divergence in (46), the completed action is

$$S = \lambda^{-2} \int B^{\mu\nu} R_{\mu\nu}(A)\, \mathrm{d}^4x + S_m[B, \chi]. \tag{47}$$

Its connection equation gives $A = \mathrm{LC}(g) + \mathrm{I}\omega$. Substitution is legitimate for the bulk equations: the eliminated connection Euler derivative vanishes, including its projective directions, so the chain-rule contribution to the remaining variation is zero. Moreover $B^{\mu\nu} R_{\mu\nu}(A) = vR(g)$ on this solution family. The reduced action is exactly the Einstein–Hilbert action with $1/(2\kappa_E) = \lambda^{-2}$.

Define the Hilbert tensor by $\delta S_m = -\frac{1}{2} \int v T_{\mu\nu} \delta g^{\mu\nu}$, with the matter variations displayed separately. Since $B^{\mu\nu} = v g^{\mu\nu}$,

$$\delta v = \tfrac{1}{2} g_{\rho\sigma} \delta B^{\rho\sigma}, \qquad \delta g^{\mu\nu} = v^{-1} \left(\delta B^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} g_{\rho\sigma} \delta B^{\rho\sigma} \right), \tag{48}$$

$$M_{\mu\nu} := \frac{\delta S_m}{\delta B^{\mu\nu}} = -\tfrac{1}{2} \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \right). \tag{49}$$

The density equation is therefore

$$R_{\mu\nu}(g) = \kappa_E \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \right), \qquad G_{\mu\nu}(g) = \kappa_E T_{\mu\nu}, \qquad \kappa_E = \lambda^2/2. \tag{50}$$

These are the full nonlinear equations. Their metric solutions lift by choosing any smooth projective one-form ω and taking $T = \mathrm{Hol}(\mathrm{LC}(g) + \mathrm{I}\omega)$. The invariant readout returns $T^B = \mathrm{Hol}(\mathrm{LC}(g))$. Conversely every source solution in the declared sector projects to (50) and the matter equations. This is solution coverage modulo the stated gauge symmetries, not a claim that all off-shell affine configurations already describe metric connections.

For the bootstrap readout, the matter contribution adds λM to both E_h and $r + \lambda\tau$. Consequently (45) remains true with these additions. Matter does not alter the connection equation under the stated independence assumption.

6.2 What the matter prescription supplies

Extend the reference density to an auxiliary smooth field for the response operation. The common-density response condition, for arbitrary compactly supported auxiliary and field variations, is

$$\frac{\delta S_m}{\delta \eta^{\mu\nu}} = \lambda^{-1} \frac{\delta S_m}{\delta h^{\mu\nu}}. \tag{51}$$

At fixed $B = \eta + \lambda h$ and fixed matter fields, an infinitesimal background change has $\delta h = -\delta\eta/\lambda$. Equation (51) makes its action derivative zero. Integration along a path in such a fixed-B fibre proves that the action factors through B on neighborhoods with path-connected fibres, modulo the specified boundary equivalence. This is a local statement, sufficient near the prepared background; connectedness of the total field domain alone would not imply connectedness of every fibre. The condition selects common metric-density dependence, not all matter interactions.

For a real scalar, choose

$$S_m = -\tfrac{1}{2} \int \left(B^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + v m^2 \varphi^2 \right) \mathrm{d}^4 x. \tag{52}$$

The matter equation and stress are

$$v^{-1} \partial_\mu (B^{\mu\nu} \partial_\nu \varphi) - m^2 \varphi = 0, \tag{53}$$

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \tfrac{1}{2} g_{\mu\nu} \left(g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + m^2 \varphi^2 \right). \tag{54}$$

Thus the complete nonlinear scalar system, including its trace and normalization, is present. The construction is not restricted to vacuum or the linearized matter source.

Proposition 6.1 (Restricted scalar selection)**.** *Suppose the scalar Lagrangian is a natural scalar density, polynomial of degree at most two in a real scalar and its first derivatives, with no derivatives of the metric and no additional fields or prescribed tensors. Matching its flat-space action fixes the minimal curved-space action* (52)*, apart from a vacuum constant and matched divergences.*

Proof. At a point, a linear frame change reduces the metric to its canonical inertia form. Its orthogonal stabilizer leaves no nonzero vector and leaves only multiples of the inverse metric among symmetric contravariant rank-two tensors. A quadratic first-derivative term is therefore a constant times $v g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$; a term linear in $\partial\varphi$ has no invariant coefficient vector. The remaining algebraic terms are $v(a\varphi^2 + b\varphi + c)$. There is no zeroth-jet scalar of the metric alone other than a constant on a fixed-signature orbit. Flat matching fixes the kinetic coefficient, a and b. The constant c is separately a vacuum term. These statements apply pointwise and hence to the natural density throughout the patch. □

The absence of metric derivatives is an additional realization hypothesis. Without it, $\xi v R(g) \varphi^2$ obeys common-density dependence and agrees with the minimal kinetic action in flat space, but changes the curved-space equations. The proposition does not remove this freedom by renaming it.

Connection-dependent matter is a separate extension. For example, the covariant term $\int j^\mu \operatorname{tr}[A_\mu - \mathrm{LC}(g)_\mu]$ uses a vector density j and the trace of a tensorial connection difference. A projective variation gives $n \int j^\mu \omega_\mu$. The gravitational connection Euler derivative has identically vanishing projective trace, so unrestricted stationarity requires $j^\mu = 0$ for this term. Choosing a representative of ω cannot remove that obstruction. The equivalence established here uses metric matter; additional hypermomentum or spinor couplings require their own consistent connection action.

6.3 Gauge covariance and constraints

For a diffeomorphism Φ with Jacobian J, define the pulled-back history by

$$B'(x) = |\det J_x| J_x^{-1} B(\Phi x) J_x^{-T}, \tag{55}$$

$$T'_\gamma = J_y^{-1} T_{\Phi\circ\gamma} J_x, \qquad \gamma : x \longrightarrow y. \tag{56}$$

The transport composition and inverse identities are preserved. Differentiating (56) gives the usual affine connection transformation, including its inhomogeneous Jacobian derivative. The determinant/volume factors in (28) transform together, so the restored transport is natural. The symmetric-curvature density and metric-matter action are invariant under these transformations. Projective shifts supply an additional affine redundancy with invariant (B, T^B) readout.

Noether's identity for the reduced diffeomorphism-invariant action gives $\nabla^\mu G_{\mu\nu} = 0$. For the scalar it gives

$$\nabla^\mu T_{\mu\nu} = (\Box_g \varphi - m^2\varphi)\partial_\nu \varphi. \tag{57}$$

Thus stress conservation follows on the matter equation. The normal Einstein equations on a spacelike hypersurface are the usual constraint equations because the full field equations and admissible metric variations have been recovered. Constraint propagation follows from the same Bianchi and matter identities, under the regularity and evolution assumptions of the chosen Einstein–matter initial-value formulation. No independent global existence theorem or new canonical constraint algebra is asserted here.

6.4 Boundary, scale and limiting contracts

For local Euler equations we take smooth compactly supported variations, or variations vanishing with the necessary derivatives in a fixed boundary collar. The explicit divergence relating the source and affine actions must be retained for other boundary problems. Fixing transport on paths lying entirely in the collar realizes this local restriction. Fixing every path with endpoints on the boundary would instead constrain interior transport and is not the variational domain used here. Projective transformations must preserve the chosen boundary readouts, as specified in Section 4.

In units with $\hbar = c = 1$, a metric is dimensionless, A has inverse-length dimension and λ has length dimension; $h = (B - \eta)/\lambda$ and $c = A/\lambda$ have dimensions L^{-1} and L^{-2}. Both hDc and $\eta Q(c)$ then have dimension L^{-4}. The matter normalization in (50) is consequently dimensionally consistent. The $\lambda \to 0$ free limit is a family with $B = \eta + \lambda h$ and $A = \lambda c$ for bounded h, c, not a limit holding an arbitrary finite curved metric fixed. Nondegeneracy and the connected-signature condition remain required throughout. Dimension two, degenerate densities and nonregular transport histories are excluded for explicit reasons, rather than silently included in the inversion.

7 Dependency reduction and the remaining selection problem

7.1 A dependency ledger

The logical chain is now explicit. Native admission supplies the fan support and compatible literal preparation. A declared raw relation supplies regular transport, a nondegenerate density and matter histories. Smooth multiplicative transport yields a connection and covers its variations. The selected free spin-two law and horizontal response determine the stress. Cubic total-current coupling closes that law. The general affine equation eliminates distortion modulo projective transport; the volume-restored observable returns metric transport. The density and matter variations then give (50) and (53).

Table 1 records the origin of each consequential premise. Here "source" means a consequence of the target-neutral admission/preparation architecture; "realization" means additional physical or regularity structure placed in its raw sector. This distinction separates admission of a representation from selection of a physical law.

Table 1: Premises, provenance and dependence.

Premise	Status	Consequence or remaining dependence
Source fan, common witness	Source-derived	Proposition 2.1; common-witness equations restrict a sector but do not choose its action.
Literal preparation tuple	Source definition	Equation (3); physical fields belong to the raw relation (4).
Smooth manifold, dimension four	Added realization	Enables the physical density inversion and Einstein–matter sector; admission does not select dimension.
Lorentzian background	External physical input	Signature is inherited by a connected nondegenerate density branch.
Nondegeneracy	Domain condition	A quantitative weak-field sufficient condition follows below; no general exclusion of degeneracy follows from admission.
Multiplicative regular transport	Added realization	Smoothness at constant paths implies the local first-jet law, Theorem 3.1.
Torsion-free dynamics	Derived on shell modulo projective symmetry	Theorem 4.1 eliminates nonprojective distortion; Theorem 4.2 returns the unique metric transport.
Projective-invariant observables	Added observable choice	Required to treat projective shifts as redundancy; bare vector transport does not descend.
Free massless spin-two action	External physical law	Deser's quadratic starting point; source admission permits alternatives.
Linear gauge identity	Consequence of chosen free law	Follows from (37) and its trace-adjusted form.
Horizontal auxiliary response	Specified operation	Equation (42) is necessary for the off-shell stress theorem.
Cubic current coupling	Bootstrap requirement	Theorem 5.2 proves closure, conditional on the free law and total-current requirement.
Coupling magnitude	Open physical selection	$\kappa_E = \lambda^2/2$ fixes conversion, not its measured value.
Metric matter	Added constitutive sector	Common-density response gives dependence on B; restricted minimal scalar selection also needs metric ultralocality.
Nonlinear gauge and constraints	Derived from completed law	Pullback invariance, Bianchi and matter Noether identities; well-posedness uses the chosen evolution domain.
Boundary and regularity classes	Added analytic contract	Matched collar/divergence data and differentiability control projection and variation.

The reductions are linked. Constant-path smoothness replaces a separate first-jet postulate within the regular transport realization; it is not a deduction from the combinatorial source alphabet alone. The general affine action replaces torsion-free kinematics by an algebraic field equation and a precise observable quotient. Gauge invariance need not be added again after selecting its quadratic and covariant actions. Minimal scalar coupling is selected only in the restricted natural class of Proposition 6.1. These are genuine conditional reductions with stated premises, not a proof that the target-neutral architecture uniquely selects general relativity.

7.2 Competing realizations and sharp obstructions

Proposition 7.1 (Target-neutral underdetermination)**.** *The native source-admission and literal-preparation principles used in Section 2, together with the generic raw-relation schema, do not determine a spacetime dimension, signature, gravitational action or matter curvature coupling.*

Proof. Keep the same admitted fan, marking, order, lineage and invariant tuple. In one raw sector choose regular transport and nondegenerate Lorentzian densities on a four-dimensional patch. In another choose Euclidean densities, or a patch of a different dimension greater than two. In each sector the same raw-relation schema can retain the corresponding fields and their nonzero residuals. Neither the support conditions nor the literal preparation tuple refers to these physical choices. Within a fixed four-dimensional Lorentzian raw sector, choose the affine Einstein action or its metric reduction augmented by $\alpha \int vR(g)^2$. Both define relations of the same type, but have inequivalent Euler equations. Similarly the minimal scalar law and its $\xi vR\varphi^2$ extension fit that schema. Thus there are explicit competing realizations satisfying the stated target-neutral premises. □

This is an obstruction to deriving these selections from the listed core premises. It is not a counterexample to every additional physical postulate in the full Shadow monograph. Any further selection claim must identify which additional postulate excludes the displayed competitor.

Several useful consequences do follow once a branch is chosen. If $\|\eta^{-1}\lambda h\| < 1$ in a submultiplicative matrix norm, the Neumann series makes B invertible. The path $\eta + s\lambda h$, $0 \le s \le 1$, stays nondegenerate, and continuity of the inertia of a real symmetric matrix fixes its signature to that of η. This derives preservation of the selected signature in a controlled neighborhood, not selection of the signature itself.

Within the Einstein–Hilbert family with a vacuum term $-2\Lambda v$, requiring the given flat metric and zero matter to solve the completed equations forces $\Lambda = 0$: its Einstein tensor vanishes while its metric is nondegenerate. The same condition does not exclude R^2 terms, which vanish with their first variation on flat space. Thus the flat-background condition supplies one parameter restriction, not uniqueness of the action.

The alternatives also change concrete responses. For the Palatini affine theory $f(R) = R + \alpha R^2$ with connection-independent matter, the trace equation in four dimensions gives $R = -\kappa_E T$. At an algebraic dust response point with $T = -\rho$ and orthonormal $T_{00} = \rho$,

$$R_{00} = \frac{\kappa_E \rho - f/2}{1 + 2\alpha\kappa_E \rho}, \qquad R_{00} - \frac{\kappa_E \rho}{2} = -\frac{3\alpha\kappa_E^2 \rho^2}{2(1 + 2\alpha\kappa_E \rho)}. \tag{58}$$

Here $R_{\mu\nu}$ is the symmetric Ricci tensor of the independent affine connection and $R = g^{\mu\nu}R_{\mu\nu}$. Its connection equation uses the density $f'(R)B$, not B. In a constant-trace neighborhood with $f'(R) \neq 0$ its metric connection agrees with $\mathrm{LC}(g)$, whereas for varying trace it generally does not. Equation (58) is a pointwise algebraic response comparison, not a constructed global dust solution. It distinguishes the law from (50) where the denominator is nonzero.

For the scalar alternative, take $g = e^{2t^2}\eta$ and constant massless $\varphi = 1$. The minimal scalar residual vanishes, while the nonminimal residual is $-\xi R$ in the convention $(\Box_g - \xi R)\varphi = 0$, with

$$R = 12(1 + 2t^2)e^{-2t^2}. \tag{59}$$

Flat matching and common-density dependence therefore cannot establish minimal coupling without the additional restriction used in Proposition 6.1.

7.3 What the architecture explains

The source architecture supplies an admitted support, a common preparation framework, explicit composition operations and a controlled route between transport and geometric dynamics. The regular realization adds a reconstruction theorem and an error transfer bound. The projective extension explains why torsion-free kinematics can be relaxed while preserving the full metric endpoint, and identifies exactly which transport information is discarded. The stress identity explains how the derivative self-current survives projection and why holding the wrong source variable fixed fails. These are structural explanations of the realization and bootstrap, rather than independent experimental evidence for a microscopic carrier.

The strongest remaining selection problem is the free raw law and observable class. A further source principle would need to distinguish the chosen quadratic spin-two/current structure from the explicit alternative action sectors while preserving the native admission rules. Merely requiring an Einstein readout, or defining the raw relation to have Einstein residuals, would insert the desired conclusion. The present derivation instead exposes its physical starting law and proves its complete source realization and nonlinear recovery.

8 Verification and discriminating controls

8.1 Proofs and executable evidence

The universal statements are the arguments in Sections 2–7. The accompanying finite checks test their implementations, conventions and failure modes. They comprise 43 reference checks, 53 extended affine/transport checks and three coordinate/density controls, for 99 checks in total: 91 exact finite-algebra checks and eight numerical estimates. Replaying a reference implementation tests reproducibility; independent evidence comes from the different algorithms and variational routes described below. These finite checks support the implementations and fixtures; the general smooth-field results are established by the proofs.

The reference suite checks the original torsion-free stress, nonlinear endpoint, transport reconstruction, source fan and gauge/boundary controls. The extended suite checks the unrestricted affine Hessian against a separately assembled Euler matrix, the covariant kinetic response against the quotient-adjoint formula, curvature from matrix commutators against the component-action equation, and the full off-shell residual against an explicit metric spin-two operator. Exact rational jets keep these algebra checks independent of floating-point cancellation. The stress and residual fixtures use different rational data, with reproducible seeds 271049 and 271050 respectively. The scripts retain the constructed coefficients and component results.

The unrestricted Hessian ranks are 24 in dimension three and 60 in dimension four, with both Lorentzian and Euclidean four-dimensional representatives. Dimension two has rank four among eight components, providing a control against extending the $n^3 - n$ statement through its excluded case. Restriction to symmetric lower indices recovers the original 40-component action and its

Suite	Exact	Numerical	Total
Reference source realization	40	3	43
General affine/projective algebra	20	0	20
General horizontal stress	9	0	9
Transport, matter and competitors	13	5	18
Independent off-shell residual	3	0	3
Regularity and scalar controls	3	0	3
Coordinate and density controls	3	0	3
Total	91	8	99

Table 2: Executable checks by scope. A check may compare several tensor components; check counts are not counts of independent theorems.

invertible four-dimensional connection Hessian. The pure-projective countercontrol detects the incorrect unrestricted extension of the symmetric-affinity contraction.

For the stress identity the direct covariant kinetic variation is evaluated componentwise before comparison with $-DGH_h c$. Twenty component/fixture comparisons, including derivatives of both fields, are retained inside the nine reported checks. The residual fixture has quadratic h and affine unrestricted c, with nonzero Euler residuals, and verifies all ten independent components of (45). These tests would not be satisfied merely by testing an exact vacuum solution.

8.2 Nonlinear metric and matter fixtures

One exact fixture is Kasner geometry

$$g = \mathrm{diag}(-1, t^{-2/3}, t^{4/3}, t^{4/3}), \qquad t > 0, \qquad A = \mathrm{LC}(g) + \mathrm{I}(t\,\mathrm{d}t + t\,\mathrm{d}x). \tag{60}$$

The raw connection has projective torsion and $R_{01} = 1$, $R_{10} = -1$, but vanishing symmetric Ricci. Along $t : 1 \to 2$ with fixed spatial coordinates,

$$T^{\mathrm{LC}} = \mathrm{diag}(1, 2^{1/3}, 2^{-2/3}, 2^{-2/3}), \qquad T = e^{-3/2} T^{\mathrm{LC}}, \qquad v(1)/v(2) = 1/2. \tag{61}$$

The source formula (28) restores T^{LC} exactly. This tests finite transport, not only infinitesimal projective cancellation.

A nonvacuum fixture uses null coordinates (u, v, y, z) and

$$\mathrm{d}s^2 = -2\mathrm{d}u\mathrm{d}v + H\mathrm{d}u^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \quad H = -\tfrac{1}{2}\kappa_E a^2 (y^2 + z^2), \quad \varphi = au. \tag{62}$$

Its volume density is one, $R_{uu} = \kappa_E a^2$, and the scalar gradient is null. The code adds a nonclosed projective one-form and independently builds curvature from connection matrices. All ten symmetric density equations and the scalar equation hold with $\kappa_E = \lambda^2/2$. The antisymmetric projective curvature is retained, so the test would detect accidentally imposing raw metric compatibility instead of recovering the invariant metric sector.

The competing-law tests retain the nonminimal scalar residual (59), the higher-curvature response (58), the flat-vacuum restriction on Λ, the projective trace Ward identity, and the degeneration of $B/\sqrt{-\det B}$. The constant-path regularity control checks the nonadditive directional derivative of (10). Independent Lorentz-stabilizer linear systems leave a one-dimensional symmetric rank-two invariant space and no invariant vector, supporting the restricted scalar classification. Three further exact controls check finite restored transport in both coordinate orientations, generator naturality under the nonlinear map $\Phi(t, x, y, z) = (t^2, x, y, z)$ for $t > 0$, and direct

determinant/metric/scalar-density derivatives in all ten symmetric directions at a nonunit-density Lorentzian point. The generator control includes the inhomogeneous Jacobian derivative.

8.3 Finite-resolution reconstruction

For a separate two-component transport problem, take

$$A(t) = \begin{pmatrix} t & 1 \\ 0 & 2t \end{pmatrix}, \quad 0 \le t \le 1. \tag{63}$$

The exact endpoint transport is

$$U(1) = \begin{pmatrix} e^{-1/2} & -e^{-1/2}\sqrt{\pi/2}\,\operatorname{erf}(1/\sqrt{2}) \\ 0 & e^{-1} \end{pmatrix}. \tag{64}$$

The numerical test uses 60-digit arithmetic, left-endpoint Euler products and the induced infinity norm. It compares the independent analytic solution with the reconstructed transport on meshes $N = 10, 20, 40, 80$. The bound $3e^{2.2}/N$ is a conservative instance of Proposition 3.3. Table 3 shows the saved results.

N	Transport error	Analytic upper bound
10	0.08414675745	2.70750405
20	0.04097061846	1.35375202
40	0.02021324661	0.67687601
80	0.01003910875	0.33843801

Table 3: Finite transport reconstruction. Successive error orders are 1.038318, 1.019289 and 1.009670.

The declared acceptance window is 0.85 to 1.15 for the final two orders and a finest-mesh error below 0.03. All bounds and criteria pass. The oscillatory connection in Section 3 separately verifies why transport convergence in a uniform norm cannot replace the derivative controls needed for curvature.

8.4 Reproduction contract

The public package includes the reference and extended implementations, settings, saved outputs, dependency versions, a SHA-256 manifest and a claim-to-evidence index. It requires Python with SymPy 1.14.0 and mpmath 1.3.0. After extraction, use a new empty output directory and run

```
python run_all.py --output ../new_replay
```

The runner checks the package hashes, executes each suite, validates its pass flags and counts, and compares complete saved scientific result objects. It also regenerates the verification tables and compares them with the saved tables used in the manuscript. Tables are generated as UTF-8 with LF line endings; their text comparison ignores only CRLF versus LF. Three separate portability regressions check line-ending equivalence, rejection of altered content and deterministic generation. They are additional to the 99 scientific checks. Exact symbolic strings must match; floating-point values use relative tolerance 10^{-10} and absolute tolerance 10^{-12}. Environment metadata is reported separately. Output logs preserve failures for diagnosis. Hash integrity establishes which files were used; it does not itself validate the physics. The manuscript source is separately compilable, so the mathematical arguments remain accessible independently of the scripts.

9 Discussion

The completed construction realizes the entire massless spin-two bootstrap in a native source–readout sector and recovers nonlinear Einstein–matter gravity. Its source support, literal preparation, raw physical histories, path operations, projection, inverse and admissible variations have distinct definitions. The off-shell stress and residual identities carry the derivative self-current through the source representation, while the cubic coupling supplies the full nonlinear endpoint. This preserves Deser's physical achievement and identifies the extra work needed to lift its operations rather than only its final equations.

Two structural improvements reduce separate kinematic hypotheses. First, multiplicative transport that is smoothly differentiable at constant paths has a velocity-linear local generator, with quantitative finite-resolution and noise bounds. The nonadditive immersed-path counterexample shows exactly why smoothness at constants matters. Second, the unrestricted affine parent removes an imposed torsion-free condition: its equations eliminate all nonprojective distortion, and a density-normalized transport readout removes the remaining projective ambiguity. This extends the off-shell parent while preserving the full on-shell metric transport and Einstein–matter system. The projective Palatini kernel is established background; its integration with the explicit source observable and quotient stress is the construction used here.

The dependency analysis also gives a restricted scalar selection theorem, derives the gauge identities from the chosen laws, and relates flat-background stationarity to the cosmological term. These deductions have different origins. Source admission establishes the supporting object and preparation compatibility. Regularity, naturality, the quadratic spin-two law, total-current coupling and the observable contract supply additional premises. Neither their conjunction nor the exact recovery map is evidence that the target-neutral core independently chooses four dimensions, Lorentzian signature, the gravitational coupling or a unique matter action. The displayed competing realizations make that remaining freedom mathematically concrete.

The scientific claim is therefore a complete, strengthened conditional source realization of Einstein–matter gravity, with explicit assumption reductions, an unrestricted projective connection extension and controlled reconstruction. It is not a microscopic selection of general relativity from admission alone. The most consequential next problem is to find and test a target-neutral principle that restricts the free raw action and observable class strongly enough to exclude the explicit competitors. Such a principle must have content before the desired Einstein endpoint is specified. The present results provide the exact return maps and discriminating alternatives against which that proposal can be judged.

A Boundary integrations and density inversion

The affine derivative variation with the index convention of (17) has derivative part

$$\delta R_{\mu\nu} = \partial_\alpha \delta A^\alpha{}_{\nu\mu} - \partial_\nu \delta A^\alpha{}_{\alpha\mu} + \delta\big(A^\alpha{}_{\alpha\rho} A^\rho{}_{\nu\mu} - A^\alpha{}_{\nu\rho} A^\rho{}_{\alpha\mu}\big). \tag{65}$$

Integration against a symmetric density produces the boundary current

$$\Theta^\rho = B^{\mu\nu} \delta A^\rho{}_{\nu\mu} - B^{\mu\rho} \delta A^\alpha{}_{\alpha\mu}. \tag{66}$$

Its bulk derivative coefficient is exactly $J(B)$ in (33). For the source action the constant-density divergence is

$$\lambda^{-1} \eta^{\mu\nu} D_{\mu\nu}(c) = \partial_\rho \left[\lambda^{-1} (\eta^{\mu\nu} c^\rho{}_{\nu\mu} - \eta^{\mu\rho} c^\alpha{}_{\alpha\mu})\right]. \tag{67}$$

The contraction $c^{\alpha}{}_{\alpha\mu}$ cannot be exchanged with a different trace without lower-index symmetry. Equations (66) and (67) specify the matched integrations used in the stress and recovery arguments. A different boundary functional changes the boundary variational problem and must be supplied explicitly.

In dimension $n > 2$, $B = vg^{-1}$ gives $|\det B| = v^{n-2}$ and hence

$$v = |\det B|^{1/(n-2)}, \qquad g^{-1} = B/v. \tag{68}$$

The determinant identity follows from $|\det g| = v^2$. Differentiating gives $\delta v = (n-2)^{-1} g_{\mu\nu}\delta B^{\mu\nu}$, from which (48) follows for $n = 4$. This also explains why dimension two requires a different treatment: the density determinant no longer recovers the conformal scale. Signature is fixed by the connected nondegenerate component, not by taking an absolute determinant.

B Variation domains and the fixed-transport obstruction

Work locally on a smooth coordinate patch, with a finite regularity class high enough for each displayed derivative, or in the smooth category. Path neighborhoods are taken in the C^1 Banach topology for the differentiability-at-constants theorem; controlled C^2 paths and bounded second derivatives are used for the reconstruction bound. Joint differentiability in base point and path perturbation is part of that realization. Ordinary finite jets do not replace these domain requirements.

The maps between A and regular transport are inverse by uniqueness of the Volterra equation. Their differentials are inverse by (11) and its short-path derivative. The density map is a smooth local diffeomorphism on its nondegenerate branch by the determinant inversion above. These facts prove coverage of compactly supported affine, density and matter variations. In the projective quotient the tangent directions removed are exactly $P\omega$; $P^T E_c = 0$ identically. Moving the volume section as B varies consequently contributes no additional connection Euler term. Eliminating the other distortion directions uses their stationary equations and is an on-shell operation.

For completeness, the fixed-transport error follows before imposing any field equation. Along the auxiliary metric-density variation,

$$\delta C = -GJ(\delta\psi) + P\zeta, \qquad \delta c_\psi|_T = -\lambda^{-1}\delta C. \tag{69}$$

Contracting the latter with $E_c^{(2)}$ and integrating the adjoint gives

$$\delta S_2|_{T,B=\psi+\lambda h} = \int \delta\psi^{\mu\nu}\left[\tau_{\mu\nu} + \lambda^{-1} D_{\mu\nu}(GE_c^{(2)})\right]. \tag{70}$$

The projective term vanishes by $P^T E_c^{(2)} = 0$. If B is also frozen instead of following $\psi + \lambda h$, the relative h varies by $-\delta\psi/\lambda$, producing the additional term $-E_h^{(2)}/\lambda$. Thus neither fixed-source operation equals the prescribed horizontal stress off shell. The distinction persists even when special solutions accidentally make the defect vanish.

C Analytic checks behind the reconstruction table

For the triangular matrix used in Section 8, the second diagonal entry solves $u_{22}' = -2tu_{22}$, while $u_{11}' = -tu_{11}$. Variation of constants gives

$$u_{12}(t) = -e^{-t^2/2}\int_0^t e^{-s^2/2}\,\mathrm{d}s, \tag{71}$$

which yields (64) without a numerical differential-equation solver. In the induced infinity norm, $\|A\| \le 2$ and $\|A'\| \le 2$ on the interval. Along a subinterval of length h, the exact fundamental solution has norm at most e^{2h}, and its second derivative is $(-A'+A^2)U$. Taylor's integral remainder therefore bounds the local Euler defect by $3h^2e^{2h}$. In the product telescoping identity, the other exact and Euler factors contribute at most $e^{2(1-h)}$. Summing N equal intervals yields a global error at most $3e^2/N$, already smaller than the declared $3e^{2.2}/N$. The saved comparison uses the larger bound consistently; no contractivity assumption is required.

For the oscillatory curvature control, write $A_1 = (\sin(kx^0)/k)E_{01}$ and all other $A_\mu = 0$. Since $E_{01}^2 = 0$, its path exponential is exactly $\mathrm{I} - E_{01}\int \sin(kx^0)\mathrm{d}x^1/k$. On bounded-length families the difference from identity is $O(k^{-1})$. The diagonal probe $x^0 = x^1 = s$ has integral $(1-\cos k)/k^2$, but $\partial_0(A_1)^0{}_1 = \cos(kx^0)$ gives $R_{11}(0) = 1$. This explicit limit distinguishes transport reconstruction from curvature reconstruction and explains the derivative norms in (16).

D Scope of the recovery statements

There are three different equivalences. First, regular source transport and affine generators are off-shell inverse representations, including their allowed variations. Second, projective classes of those generators and volume-restored transport families are off-shell inverse representations at fixed nondegenerate density. Third, the unrestricted affine action reduces to the metric Einstein–matter equations after eliminating nonprojective distortion by its algebraic field equation. The original torsion-free parent and the new unrestricted quotient agree at this third endpoint, not as entire off-shell field spaces. Their respective 40 and 60 connection coordinates make the distinction visible.

All statements are local in the declared smooth sector, or apply to global histories whose patching and boundary data satisfy these same contracts. A global nontrivial tangent bundle requires compatible local frames and transport descent. An arbitrary global spacetime topology, singular extension or complete all-sector Shadow equalizer has not been deduced from the finite fan construction. No such enlargement is required for the full local nonlinear Einstein–matter result proved here.

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