

# Quantum Source Completions and Memory–Environment Tradeoffs

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## Abstract

We quantify stationary quantum-source resources under explicit readout and retained-output contracts. Orthogonal-current-symbol completions of stationary triple data have exact minimum dimension equal to the sum of their complex positive-semidefinite slice ranks. For delayed phase-circle processes we give a dimension-four emitted-symbol realization and prove noiseless optimality. The delayed trine specialization has classical HMM minimum nine for every non-iid parameter, quantum minimum four at sufficiently low noise and three on a certified high-noise interval. A continuous qutrit family attains one-step recorded-channel Choi rank nine, while a four-dimensional latitude source attains three, giving an exact two-point Pareto frontier on that interval. At noise .99, a mixed-support qutrit source lowers complete-history Choi rank to 50, 152 and  $4563^{n-4}$  at horizons two, three and  $n \geq 4$ ; universal lower bounds are 48, 145 and  $4373^{n-4}$ . Full-word proofs, algebraic CP certificates and output-specific optima separate internal memory from retained-history dilation cost. The exact intermediate memory boundary and full-history optimum remain open. These are operational quantum-source results under standard probabilities, not ontology or total-hardware theorems.

## 1 Question, contributions and operational scope

This paper develops quantum-source constructions within Shadow Theory’s pre-existing source–readout programme [13]. The stationary classical completion used for comparison is established in Rodgers [12]. An observed stochastic law may admit several source realizations with different internal memories and different retained-output costs. We study these resources separately for stationary quantum instruments, emphasizing the delayed trine law. The results combine a general positive-semidefinite factorization completion theorem, exact memory regimes and a two-objective memory/environment frontier with a constructive mixed-support source. For  $0 \leq \lambda \leq 1$ , put  $T_{ii} = \lambda/3$  and  $T_{ij} = (3 - \lambda)/6$  for  $i \neq j$ . The target has uniform adjacent pairs and full word law  $P(y_1 \cdots y_n) = 9^{-1} \prod_{t=1}^{n-2} T_{y_t, y_{t+2}}$ . It is the interleaving of two independent stationary chains. A source consists of a stationary density matrix and time-homogeneous CP symbol maps whose sum is trace preserving. The emitted trit is retained in an append-only classical archive; archive feedback, correlated future tapes and uncounted returning controllers are excluded. All history-correlated systems accessible to future operations belong to the reusable causal memory. The current exact results are: minimum classical HMM size nine for every  $\lambda < 1$ ; minimum quantum dimension four on  $[0, 3 \cdot 10^{-8}]$  and three on  $[\lambda_+, 1)$ , where  $\lambda_+ = 1 - \sqrt{71 - 17\sqrt{17}}/4$ . The intermediate dimension is three or four; the iid endpoint needs dimension one. In the admitted high-noise interval the exact one-step Pareto points are  $(d, k) = (3, 9)$  and  $(4, 3)$ . Here  $k$  is the Choi rank of the channel retaining the classical symbol and final memory, equivalently its minimum initially pure Stinespring remainder dimension. At  $\lambda = .99$  a different qutrit source has one-step cost fifteen but lower complete-history cost, with certified bounds  $48 \leq R_2^{\min} \leq 50$ ,

$145 \leq R_3^{\min} \leq 152$  and  $437 3^{n-4} \leq R_n^{\min} \leq 456 3^{n-4}$  for  $n \geq 4$ . These costs concern distinct tasks: reusable memory, complete arbitrary-input channels retaining final memory, fixed-input marginal states, and archive-only channels. They are not interchangeable and are not hardware, energy or thermodynamic minima. The complete-channel statements retain arbitrary inert input references. Initial mixed preparation and any purifier, clocks, controls and independent fresh ports remain declared resources. Equality of passive output laws is not equality under arbitrary interventions.

**Conventions and reading guide.** For  $\Phi : M_d(\mathbb{C}) \rightarrow M_d(\mathbb{C})$  we use the unnormalized Choi matrix  $J(\Phi) = \sum_{p,q} |p\rangle\langle q| \otimes \Phi(|p\rangle\langle q|)$ . Its rank counts Kraus operators; the complex-linear rank of  $\Phi$  counts independent image operators. Neither is the internal Hilbert dimension  $d$ . The sector-restricted cost  $D$  includes the orthogonal current-symbol register, whereas  $d$  need not store that symbol internally. All trine-law and seed parameters are dimensionless; no physical time or energy scale is inferred from them.

Section 2 establishes the general completion and emitted-symbol interfaces. Sections 3 and 4 give the exact low- and high-noise memory regimes. The remaining sections compare one-step and complete-history costs, then demonstrate access-dependent distinctions between passive-law-equivalent sources. The technical companion retains alternative constructions, approximation statements and additional accounting tasks.

The quantum stochastic-realization problem and memory compression have established antecedents [7, 11, 15]; a systematic compression method for nondeterministic quantum implementations is given in Elliott [3]. These works address general realization, representation or memory/thermal questions; they do not by themselves give the particular delayed-trine Kraus and retained-history optima proved here. Normal measurement-model and multi-time resource accounting are established [1, 10]. Our contributions are the stated constructions, exact process-specific bounds and their operational separation. Targeted literature examination does not establish publication priority. Conventional Born probabilities are assumed; no new Born-selection, collapse, consciousness or unique-ontology claim is inferred.

## 2 Stationary quantum completions and the readout interface

A nonnegative stationary triple array  $p(i, y, j)$  has middle slices  $M_y(i, j) = p(i, y, j)$  and adjacent masses  $w(i, y) = \sum_j p(i, y, j) = \sum_k p(k, i, y)$ . The classical deterministic-readout minimum  $\sum_y \text{rank}_+(M_y)$  is an existing comparison theorem, not claimed as a new result here. The exact classical comparison is

$$N_{\min}(p) = \sum_y \text{rank}_+(M_y), \quad (1)$$

with its full stationary gluing proof established in Rodgers [12]. We give its quantum counterpart in a distinct orthogonal-current-symbol class, then distinguish that class from general emitted-symbol instruments.

### 2.1 The exact quantum completion theorem

Let  $p(i, y, j)$  be a normalized stationary triple array on a finite alphabet:

$$w(i, y) := \sum_j p(i, y, j) = \sum_k p(k, i, y). \quad (2)$$

Omit symbols with zero stationary probability. Write  $\text{rank}_{\text{psd}}^{\mathbb{C}}(M)$  for the least  $d$  admitting  $d$ -dimensional complex Hermitian PSD factors  $M_{ij} = \text{Tr}(A_i B_j)$ . Positive-semidefinite rank and its static correlation-generation interpretation are established [4, 6]; the additional requirement below is one stationary recurrent source, not a separate factorization for each observation.

The declared source class has a Hilbert space  $\mathcal{H} = \bigoplus_y \mathcal{H}_y$ , block-diagonal memory states, and current readout  $y$  given by its orthogonal sector. A time-homogeneous Markov step is specified by CP maps

$$\Phi_{j \leftarrow y} : \mathcal{B}(\mathcal{H}_y) \longrightarrow \mathcal{B}(\mathcal{H}_j), \quad \sum_j \Phi_{j \leftarrow y}^*(I) = I.$$

Reading the sector projectively does not change a block-diagonal state. The source cost is the total Hilbert dimension  $D = \sum_y \dim \mathcal{H}_y$ , including its visible current-symbol register. An open Markov channel, with fresh environment degrees of freedom at each step, is allowed explicitly. This parallels the fresh randomness allowed in a classical stochastic Markov source; it is not a count of all environmental or archival degrees of freedom in a closed universe.

**Theorem 2.1** (Exact quantum completion with orthogonal current readout). *In the declared source class the exact minimum for the triple array is*

$$D_{\min}(p) = \sum_y \text{rank}_{\text{psd}}^{\mathbb{C}}(M_y). \quad (3)$$

*An optimum can be chosen with measure-and-prepare block channels. It realizes the entire canonical stationary order-two completion of  $p$ . Consequently, for a specified full stationary order-two law, the same expression is its exact minimum in this class. No full-law claim for higher-order input processes is made.*

*Proof.* For the lower bound let  $\tau_y$  be the stationary positive block of an arbitrary admissible source, with  $\sum_y \text{Tr} \tau_y = 1$ . Put

$$A_i^y = \Phi_{y \leftarrow i}(\tau_i), \quad B_j^y = \Phi_{j \leftarrow y}^*(I).$$

Both are positive and  $p(i, y, j) = \text{Tr}(A_i^y B_j^y)$ . Hence each sector dimension is at least its slice's complex PSD rank; summation proves the bound.

For attainment choose minimum PSD factors separately in each sector. They may be normalized without increasing their dimensions. Specifically, set  $S_y = \sum_j B_j^y$ , restrict to its support, and replace

$$A_i^y \mapsto S_y^{1/2} A_i^y S_y^{1/2}, \quad B_j^y \mapsto S_y^{-1/2} B_j^y S_y^{-1/2}$$

on that support. Cyclicity of trace and support of every  $B_j^y$  preserve all pairings, while  $\sum_j B_j^y = I$ . Thus  $\text{Tr} A_i^y = w(i, y)$ . Define  $\tau_y = \sum_i A_i^y$  and restrict further to its support, compressing all factors there. Pairings and the effect sum are again preserved, and  $\tau_y$  is now faithful. Since

$$\text{Tr}(B_j^y \tau_y) = \sum_i p(i, y, j) = w(y, j),$$

a zero  $w(y, j)$  implies  $B_j^y = 0$  on this support. The destination matrix  $A_y^j$  also vanishes because its trace is  $w(y, j)$ .

For positive  $w(y, j)$  set

$$\sigma_y^j = \frac{A_y^j}{w(y, j)}, \quad \Phi_{j \leftarrow y}(X) = \text{Tr}(B_j^y X) \sigma_y^j; \quad (4)$$

set the map to zero otherwise. These are CP maps with effects  $B_j^y$ , so each input sector's summed map is trace preserving. The constructed state is stationary because

$$\sum_y \Phi_{j \leftarrow y}(\tau_y) = \sum_y A_y^j = \tau_j.$$

It is normalized since  $\sum_y \text{Tr} \tau_y = \sum_{i,y} w(i,y) = 1$ . After any positive-probability observed pair  $(i,y)$  the normalized memory state is exactly  $\sigma_i^y$ . Its next-output law and next state are

$$\Pr(j \mid i, y) = \text{Tr}(B_j^y \sigma_i^y) = \frac{p(i, y, j)}{w(i, y)}, \quad \sigma_y^j.$$

The channel therefore gives, for  $n \geq 2$ ,

$$\Pr(y_1 \cdots y_n) = w(y_1, y_2) \prod_{t=2}^{n-1} \frac{p(y_{t-1}, y_t, y_{t+1})}{w(y_{t-1}, y_t)}. \quad (5)$$

Forbidden transitions have probability zero; no division is used along them. This is the canonical order-two word law. Support reductions cannot strictly reduce the final dimension below a slice's minimum PSD rank, by the already proved lower bound. Equality follows.  $\square$

The arbitrary-past quantum conditional state in this theorem need not be perfectly distinguishable from the states induced by other pasts. That is the source of the compression. Neither exact recovery of the past symbol nor preservation of all inaccessible purifying records is asserted.

## 2.2 A full-support, strictly smaller quantum realization

For a finite circular phase representation, informed by the geometric phase and angular-register descriptions in [8, abstract and Sections 2–3] and [9, pp. 64–68], use the following quantum encoding and symbol maps.

Let  $m = 3$ ,  $\theta_i = 2\pi i/3$ ,  $\mathbf{n}_i = (\cos \theta_i, \sin \theta_i, 0)$ , and let  $\boldsymbol{\sigma}$  be the Pauli matrices. For a dimensionless noise parameter  $0 \leq \lambda < 1$ , define

$$\rho_i = \frac{I + \mathbf{n}_i \cdot \boldsymbol{\sigma}}{2}, \quad E_j^\lambda = \frac{I - (1 - \lambda)\mathbf{n}_j \cdot \boldsymbol{\sigma}}{3}. \quad (6)$$

The states are pure; the effects are positive and sum to identity. The probability matrix is

$$T_{ij}^\lambda = \text{Tr}(E_j^\lambda \rho_i) = \frac{1 - (1 - \lambda)\cos(\theta_i - \theta_j)}{3} = \begin{cases} \lambda/3 & i = j, \\ 1/2 - \lambda/6 & i \neq j. \end{cases} \quad (7)$$

It is symmetric doubly stochastic with eigenvalues  $1, -(1 - \lambda)/2, -(1 - \lambda)/2$ . Interleave two independent stationary  $T^\lambda$ -chains. Their full law is

$$\Pr(y_1 \cdots y_n) = \frac{1}{9} \prod_{t=1}^{n-2} T_{y_t, y_{t+2}}^\lambda, \quad n \geq 2. \quad (8)$$

Every finite word is positive when  $0 < \lambda < 1$ ; the noiseless model has forbidden events  $Y_{t+2} = Y_t$ . Adjacent pairs are uniform for all these models.

Each triple slice is  $M_y = T^\lambda/9$ . It has ordinary rank three, so its nonnegative rank is exactly three. The quantum factors  $A_i^y = \rho_i/9$ ,  $B_j^y = E_j^\lambda$  have size two, and PSD rank one would force ordinary rank one. Hence Theorem 2.1 and the classical baseline give the exact comparison

$$\boxed{D_{\text{sector}} = 6, \quad N_{\text{classical, det}} = 9.} \quad (9)$$

The explicit channel is  $\Phi_{j \leftarrow y}(X) = \text{Tr}(E_j^\lambda X) \rho_y$  with stationary blocks  $\tau_y = I/6$ . Thus the reduced memory is the physical state of one qubit inside each of three orthogonal current-output sectors; no hidden classical controller remembering the preceding label is permitted in the resource count.

For the *full* law, nine is also the exact minimum among all finite classical HMMs with memoryless stochastic emissions: their past-two/future-two joint matrix factors through the hidden state, whereas here

$$H_{(a,b),(c,d)} = \frac{T_{ac}^\lambda T_{bd}^\lambda}{9} = \frac{(T^\lambda \otimes T^\lambda)_{(a,b),(c,d)}}{9} \quad (10)$$

has rank nine. A nine-state causal-pair source attains the law. This is a use of Rodgers's existing Hankel-rank method, not a newly invented lower-bound method.

There is a finite-data margin. The smallest singular value of each  $M_y$  is  $(1 - \lambda)/18$ . If a normalized triple distribution  $p'$  satisfies  $\text{TV}(p', p) < (1 - \lambda)/36$ , then each slice perturbation has operator norm at most its entrywise  $\ell_1$  norm, at most  $2 \text{TV}(p', p)$ , and every perturbed slice retains rank three. Consequently every deterministic-instantaneous classical source reproducing  $p'$  still needs at least nine states. For arbitrary stochastic emissions, the four-time version follows from  $s_{\min}(H) = (1 - \lambda)^2/36$ : four-time TV error less than  $(1 - \lambda)^2/72$  retains the rank-nine lower bound. These are dimension certificates under their respective source interfaces, not device-independent claims that an unknown apparatus has quantum memory.

### 2.3 Circular grids of arbitrary resolution

For any  $m \geq 3$  use equally spaced phases  $\theta_i = 2\pi i/m$  and the same pure states  $\rho_i$ . Define

$$E_j = \frac{I - \mathbf{n}_j \cdot \boldsymbol{\sigma}}{m}, \quad T_{ij} = \frac{1 - \cos(\theta_i - \theta_j)}{m}. \quad (11)$$

The POVM is normalized because  $\sum_j \mathbf{n}_j = 0$ . The transition is symmetric doubly stochastic, has zero diagonal and strictly positive off-diagonal entries, and has rank three. To check its rank explicitly, write  $c_i = \cos \theta_i$ ,  $s_i = \sin \theta_i$ . Then

$$T = \frac{J - cc^T - ss^T}{m},$$

and the mutually orthogonal vectors  $\mathbf{1}, c, s$  have squared norms  $m, m/2, m/2$ . The three nonzero eigenvalues are  $1, -1/2, -1/2$ . The order-two law is  $m^{-2} \prod_t T_{y_t, y_{t+2}}$ . Every triple slice has PSD rank exactly two; its exact quantum sector cost is  $2m$ .

**Proposition 2.2** (Classical sector cost grows with angular resolution). *Let  $r_m$  be the smallest integer  $r$  with  $m \leq \binom{r}{\lfloor r/2 \rfloor}$ . Every classical deterministic-instantaneous source for the phase-circle triple law has at least  $mr_m$  states.*

*Proof.* In any nonnegative factorization  $T = AB$  with  $r$  factors let  $S_i = \{\ell : A_{i\ell} > 0\}$ . If  $S_i \subseteq S_j$ , the zero  $T_{jj}$  implies  $B_{\ell j} = 0$  for every  $\ell \in S_j$ , hence  $T_{ij} = 0$ . This contradicts  $T_{ij} > 0$  for  $i \neq j$ . Thus the  $m$  supports are an antichain. To recall the counting bound without relying on a citation, a uniformly random permutation of  $r$  elements has at most one initial segment belonging to an antichain. A set  $S$  is such an initial segment with probability  $\binom{r}{|S|}^{-1}$ . Taking expectations yields  $\sum_i \binom{r}{|S_i|}^{-1} \leq 1$ , hence  $m \leq \binom{r}{\lfloor r/2 \rfloor}$ , the classical Sperner bound[14]. Apply this to every slice of (1).  $\square$

| Phase count $m$ | $r_m$ | Quantum sector minimum | Classical sector lower bound |
|-----------------|-------|------------------------|------------------------------|
| 180             | 10    | 360                    | 1800                         |
| 360             | 11    | 720                    | 3960                         |
| 720             | 12    | 1440                   | 8640                         |

The grid sizes follow the finite angular registers in [9, pp. 64–68]; the probabilities are those of the instrument defined here, not inferred from geometric line density. The classical values are lower bounds,

not asserted exact nonnegative ranks. Known low-PSD-rank/high-nonnegative-rank distance geometry already supplies the static phenomenon; the added stationary channel realizes it at every time and proves the full observed law.

## 2.4 Removing the stored classical output: exactly two qubits

The sector restriction matters. A general iterated quantum instrument may emit the current output into an external archive without preserving it in an orthogonal sector of the internal memory. In this broader class a quantum source consists of a density matrix on  $\mathbb{C}^d$ , CP maps  $\Psi_j$  whose sum is trace preserving, and a stationary density matrix. Word probabilities are traces of successive maps. All outputs are retained; no null, failure or loss outcome is postselected away.

**Theorem 2.3** (Exact emitted-symbol instrument dimension). *For every noiseless phase-circle law (11),  $m \geq 3$ , the minimum internal Hilbert dimension among stationary iterated quantum instruments is exactly four.*

*Proof.* For attainment use two qubits and define

$$\Psi_j(X) = \text{Tr}_1[(E_j \otimes I)X] \otimes \rho_j. \quad (12)$$

The partial trace equals the result of sandwiching  $X$  by  $\sqrt{E_j} \otimes I$  before tracing the first qubit, so this is CP. Its effect is  $E_j \otimes I$ ; summing the effects gives identity. On product inputs,

$$\Psi_j(\rho_a \otimes \rho_b) = T_{aj}\rho_b \otimes \rho_j.$$

Starting from  $I_4/4$ , the first two outputs are independent uniform symbols; the conditional memory then equals  $\rho_{y_1} \otimes \rho_{y_2}$ . Every later step emits  $y_{t+2}$  with probability  $T_{y_t, y_{t+2}}$  and shifts the two states. This proves every finite word of the target law, with no accumulated approximation. Also

$$\sum_j \Psi_j(I_4/4) = \sum_j \frac{I_2}{2m} \otimes \rho_j = I_4/4,$$

which proves stationarity.

For the lower bound the four-time cut matrix is  $H_{ab,cd} = T_{ac}T_{bd}/m^2$ , of rank nine. Any  $d$ -dimensional instrument has PSD factors  $H_{ab,cd} = \text{Tr}(A_{ab}B_{cd})$ , where  $A_{ab}$  is the unnormalized memory state after emitting  $a, b$  and  $B_{cd} = \Psi_c^* \Psi_d^*(I)$  is the effect of the next two outputs. All these factors are nonzero, because every adjacent pair has probability  $1/m^2$ . Their real vector space has dimension at most  $d^2$ ; thus  $d \geq 3$ .

Suppose  $d = 3$ . The diagonal zeros of  $T$  imply zero pairing whenever  $a = c$  or  $b = d$ . Zero trace pairing of PSD matrices implies orthogonal ranges. Put

$$R_a = \text{span}_d \text{ran } B_{ad}, \quad C_b = \text{span}_c \text{ran } B_{cb}.$$

The nonzero range of  $A_{ab}$  is orthogonal to  $R_a + C_b$ , so  $\dim(R_a + C_b) \leq 2$  for every  $a, b$ . If some  $R_a$  has dimension two, every  $C_b \subseteq R_a$ , and all effects  $B_{cd}$  are supported on that same two-dimensional subspace. Compressing all factors to it gives  $\text{rank } H \leq 4$ , contradicting nine. The same reasoning excludes a two-dimensional  $C_b$ . Otherwise all these nonzero spaces are lines. Since every  $B_{ab} \neq 0$  has range contained in both  $R_a$  and  $C_b$ , all lines are equal; this would give rank at most one, another contradiction. Therefore  $d = 3$  is impossible, and the four-dimensional construction is optimal.  $\square$

The theorem gives a completed dimension lower bound beyond the generic  $\sqrt{\text{rank } H}$  estimate. It holds for arbitrary complex quantum instruments, not only for the particular shift update or for

orthogonal current-symbol sectors. Its diagonal-zero hypothesis is load bearing. For the positive trine family, the same construction with  $E_j^\lambda$  still gives  $d \leq 4$  and the rank argument gives  $d \geq 3$ . The following sections prove dimension four on a low-noise interval and dimension three on a high-noise interval; the intervening boundary remains unresolved.

For  $m = 3$  the unrestricted comparison is exactly four quantum dimensions versus exactly nine classical HMM states. For larger  $m$ , marginalizing  $H$  over  $b, d$  gives  $T/m$ . Nonnegative rank cannot increase under this operation, so every finite classical HMM needs at least  $\max\{9, r_m\}$  states, whereas the quantum minimum remains four. Thus a resolution-dependent classical lower bound survives the broader interface, although it is weaker than the sector bound. No exact classical optimum for  $m > 3$  is claimed.

### 3 A quantitative positive-noise interval requiring four quantum dimensions

The diagonal-zero argument proves exact dimension four at zero noise. Its zeros disappear under strictly positive trine noise, so that proof cannot simply be reused. The following quantitative bound establishes a nonempty positive-noise interval with the same minimum dimension. It is an explicit mathematical robustness result; the small numerical constant is not a practical apparatus noise tolerance.

For  $0 \leq \lambda < 1$ , put  $r = 1 - \lambda$  and

$$T_{ii}^\lambda = \lambda/3, \quad T_{ij}^\lambda = 1/2 - \lambda/6 \quad (i \neq j), \quad C_\lambda = T^\lambda \otimes T^\lambda.$$

All entries and parameters are dimensionless. Multiplication by the stationary factor  $1/9$  does not change PSD rank. Every three-dimensional quantum instrument reproducing the full delayed law would give a complex Hermitian PSD factorization of  $C_\lambda$  of size three through its two-past/two-future cut. The known two-qubit shift construction provides size four and reproduces the entire process, not only this cut.

**Proposition 3.1** (A positive-noise dimension-four interval). *For*

$$0 \leq \lambda \leq \lambda_0 := 3 \times 10^{-8},$$

*the complex PSD rank of  $C_\lambda$  is four. Consequently the delayed trine process has exact minimum internal quantum memory dimension four throughout this interval, among autonomous CP instruments with an emitted classical symbol and no orthogonal current-symbol storage requirement.*

*Proof.* The eigenvalues of  $T^\lambda$  are  $1, -r/2, -r/2$ . Hence  $C_\lambda$  has ordinary rank nine, and its second and fifth singular values are both  $r/2$ . A PSD factorization of size  $d$  has ordinary rank at most  $d^2$ , excluding  $d \leq 2$ .

Suppose a size-three factorization exists. Normalizing the sum of its row factors by congruence gives effects  $A_{ab} \geq 0$ ,  $\sum_{a,b} A_{ab} = I_3$ , and density matrices  $b_{cd} \geq 0$ ,  $\text{Tr } b_{cd} = 1$ , such that

$$\text{Tr}(A_{ab}b_{cd}) = T_{ac}^\lambda T_{bd}^\lambda.$$

The normalization is legitimate: a singular sum would restrict all row factors to dimension at most two, already contradicting ordinary rank nine. Column sums of  $C_\lambda$  are one, giving the density normalization. Define positive row and column sums of the preparation states,

$$R_a = \sum_d b_{ad}, \quad Q_b = \sum_c b_{cb}.$$

Each has trace three and operator norm at most three. Row sums of  $C_\lambda$  give  $1 = \text{Tr}(A_{ab} \sum_{cd} b_{cd}) \leq 9 \text{Tr} A_{ab}$ , so the largest eigenvalue of  $A_{ab}$  is at least  $1/27$ . Moreover

$$\text{Tr}[A_{ab}(R_a + Q_b)] = 2\lambda/3.$$

Evaluating on a largest-eigenvalue eigenvector of  $A_{ab}$  therefore yields

$$\lambda_{\min}(R_a + Q_b) \leq \epsilon := 18\lambda \quad \text{for every } a, b. \quad (13)$$

Set  $s = 1/2000$  and split into two cases.

*Case 1: some  $R_a$  or  $Q_b$  has second eigenvalue at least  $s$ .* By symmetry take  $R_a$ . Let  $P$  project onto its two largest eigenvectors, and let  $z$  be a unit vector in the complementary line. For each  $b$ , choose a unit  $n_b$  attaining the minimum in (13). Positivity gives  $\|Pn_b\|^2 \leq \epsilon/s$  and  $\langle n_b, Q_b n_b \rangle \leq \epsilon$ . After choosing the phase of  $n_b$ ,  $\|n_b - z\| \leq \sqrt{2\epsilon/s}$ . Thus

$$\langle z, Q_b z \rangle \leq \left( \sqrt{\epsilon} + \sqrt{3} \sqrt{2\epsilon/s} \right)^2 =: L = \epsilon(1 + \sqrt{6/s})^2.$$

Since  $b_{cd} \leq Q_d$ , every preparation has complementary mass  $\text{Tr}[(I - P)b_{cd}] \leq L$ . The projection/gentle bound gives  $\|b_{cd} - Pb_{cd}P\|_1 \leq 2\sqrt{L}$ . Replacing every preparation by its two-dimensional compression creates a real matrix  $C'$  of ordinary rank at most four. The complete POVM  $\{A_{ab}\}$  contracts the trace norm of each Hermitian preparation difference, so every column error has  $\ell^1$ , hence  $\ell^2$ , norm at most  $2\sqrt{L}$ . There are nine columns; therefore

$$\|C_\lambda - C'\|_{\text{op}} \leq \|C_\lambda - C'\|_{\text{F}} \leq 6\sqrt{L}.$$

But the best rank-four approximation has error at least the fifth singular value  $r/2$ . This contradicts the numerical bounds below.

*Case 2: every  $R_a, Q_b$  has second eigenvalue below  $s$ .* Their top-eigenvector projections, denoted  $P_a$  and  $Q'_b$ , leave residual trace below  $2s$ . Since  $b_{ab} \leq R_a, Q_b$ ,

$$D(b_{ab}, P_a) \leq \sqrt{2s}, \quad D(b_{ab}, Q'_b) \leq \sqrt{2s},$$

where  $D$  is half trace distance. For completeness, the pure-state upper bound  $D(\rho, |v\rangle\langle v|) \leq \sqrt{1 - \langle v, \rho v \rangle}$  follows by convexity and Jensen. Fix  $P_0$ . Comparing through  $b_{0b}$  gives  $D(P_0, Q'_b) \leq 2\sqrt{2s}$ , hence  $D(b_{ab}, P_0) \leq 3\sqrt{2s}$  for every pair. Replacing all preparations by  $P_0$  produces a rank-one matrix  $C''$ . Each error column is a difference of normalized probability vectors and has TV at most  $3\sqrt{2s}$ . For a zero-sum real vector  $v$ ,  $\|v\|_2^2 \leq 2(\|v\|_1/2)^2$ , by separately bounding its positive and negative parts. Thus each column has  $\ell^2$  norm at most  $6\sqrt{s}$ , so

$$\|C_\lambda - C''\|_{\text{op}} \leq 18\sqrt{s}.$$

This contradicts the necessary rank-one approximation error  $r/2$ .

Here are explicit conservative constants for both contradictions. Since  $\sqrt{12000} < 110$ , the Case-1 squared error is at most

$$6^2 18\lambda_0 111^2 = 0.23952024 < 0.49^2,$$

and the Case-2 squared error is  $18^2 s = 0.162 < 0.49^2$ . Meanwhile  $r/2 \geq (1 - \lambda_0)/2 > 0.49$ . Also  $\epsilon/s < 1$  throughout the interval, justifying the eigenvector estimate. Both cases are impossible. Thus PSD rank is at least four; the supplied two-qubit construction supplies the matching upper bound.  $\square$

**Scope and benefit.** This turns the isolated zero-noise dimension obstruction into an explicit positive-noise interval. The constants are deliberately loose; no optimal transition noise or exact middle interval is asserted. CP instruments permit fresh environment ports and retained classical output archives. Internal memory dimension does not count those resources or prove equality of complete microscopic environment laws. Sources with equal complete declared symbol laws remain operationally equivalent for that experiment family.

## 4 A continuous family of minimum-Kraus qutrit sources

A coordinate-plane Weyl encoding gives exact stationary qutrit sources with recorded one-step Choi rank nine throughout an explicit noise interval. Its two real seed weights and complex phase can be classified exactly within the minimum-Kraus subfamily. This class is not the class of all qutrit sources. In particular, its fixed full-history cost does not preclude the mixed-support improvement constructed below. The source laws and CP maps are defined explicitly below.

### 4.1 Process, resources and the exact update interface

For  $0 < r < 1$  define the trit transition matrix

$$T_{aj}(r) = \begin{cases} (1-r)/3 & a = j, \\ (1+r/2)/3 & a \neq j, \end{cases} \quad \lambda = 1-r. \quad (14)$$

The stationary delayed-trine law is

$$P_r(y_1 \cdots y_n) = \frac{1}{9} \prod_{v=1}^{n-2} T_{y_v, y_{v+2}}(r), \quad n \geq 2, \quad P_r(y_1) = 1/3. \quad (15)$$

It is the interleaving of two independent stationary chains. A source is a stationary density matrix on a  $d$ -dimensional memory with time-homogeneous completely positive maps  $\Phi_j$ , whose sum is trace preserving. The complete output at each step is the observed classical trit. The emitted-record channel is

$$\mathcal{A}(X) = \sum_{j=0}^2 |j\rangle\langle j| \otimes \Phi_j(X), \quad k = \text{rank } J(\mathcal{A}) = \sum_j \text{rank } J(\Phi_j).$$

Here  $k$  is the minimum pure-environment dimension for a Stinespring dilation of this *complete cq channel*. A coherent emitted record is a different channel. The retained archive is external to the reusable memory, independent fresh ports are admitted, and all reusable history-bearing controllers belong in the source cut. This is an information-theoretic resource comparison, not a claim about hardware dimensions, energy, preparation cost or mixed baths.

The four-symbol cut  $H_{ab,cd} = T_{ac}T_{bd}/9$  has rank nine, because  $T$  has eigenvalues  $1, -r/2, -r/2$ . A  $d$ -dimensional quantum instrument factors this cut through a space of dimension at most  $d^2$ ; a classical  $m$ -state HMM factors it through dimension  $m$ . Thus  $d \geq 3$  and  $m \geq 9$ . The classical pair source attains nine. These are standard representation-rank methods [7, 11]; our contribution is explicit completely positive attainment with controlled environmental resources, not the rank inequality itself.

### 4.2 A variable seed and its Choi blocks

We use the established finite Weyl framework [5] and the standard Choi criterion for complete positivity [2]. Let  $\omega = e^{2\pi i/3}$ ,  $Z = \text{diag}(1, \omega, \omega^2)$  and  $X|p\rangle = |p+1 \bmod 3\rangle$ . Choose  $0 < a < 1$ ,  $t > 0$ ,  $t^2 < a(1-a)$

and initially set  $\zeta = -t\omega^2 = te^{i\pi/3}$ . Define

$$\chi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & \zeta \\ 0 & \bar{\zeta} & 1-a \end{pmatrix}, \quad S_{uv} = Z^u X^v \chi X^{-v} Z^{-u}. \quad (16)$$

Here  $u, v$  are trit indices and  $a$  is the real seed weight. Write

$$q = a(1-a), \quad D = 1 - 3q > 0, \quad D_b = X^b \text{diag}(0, a, 1-a) X^{-b}, \quad F_b = |b+1\rangle\langle b+2|.$$

Then

$$S_{uv} = D_v + \zeta \omega^{-u} F_v + \bar{\zeta} \omega^u F_v^\dagger.$$

The three  $D_v$  span the diagonal matrices: their coordinate determinant is  $a^3 + (1-a)^3 = D$ . The nonconstant Fourier components in  $u$  span the six  $F_v, F_v^\dagger$ . Therefore the nine states form a Hermitian basis. Each has rank two, with support on the plane missing  $|v\rangle$ , and  $\sum_{uv} S_{uv} = 3I$ .

Define the unique linear maps on this basis by

$$\Phi_j(S_{uv}) = T_{uj}(r) S_{vj}. \quad (17)$$

Equivalently,

$$\Phi_j(D_v) = S_{vj}/3, \quad (18)$$

$$\Phi_j(F_v) = -\frac{r\omega^j}{6\zeta} S_{vj}, \quad \Phi_j(F_v^\dagger) = -\frac{r\omega^{-j}}{6\bar{\zeta}} S_{vj}. \quad (19)$$

These identities specify the map on every qutrit operator, including coherences. They are not merely an update on a finite list of states. Covariance gives  $\Phi_j = \text{Ad}_{X^j} \circ \Phi_0 \circ \text{Ad}_{Z^{-j}}$ .

Use  $J(\Phi) = \sum_{pq} |p\rangle\langle q| \otimes \Phi(|p\rangle\langle q|)$ . In the input Fourier basis  $f_k = 3^{-1/2} \sum_p \omega^{pk} |p\rangle$ , the nonzero part of  $J(\Phi_0)$  consists of three blocks on  $\text{span}\{f_k \otimes 1, f_{k+1} \otimes 2\}$ :

$$B_0 = \begin{pmatrix} ae_- & c_0 \\ \bar{c}_0 & (1-a)e_- \end{pmatrix}, \quad B_1 = \begin{pmatrix} ae_- & c_1 \\ \bar{c}_1 & (1-a)e_+ \end{pmatrix}, \quad B_2 = \begin{pmatrix} ae_+ & c_1 \\ \bar{c}_1 & (1-a)e_- \end{pmatrix}, \quad (20)$$

$$e_- = \frac{1}{3} - \frac{r}{6t}, \quad e_+ = \frac{1}{3} + \frac{r}{3t}, \quad L = (a-1) + (2a-1)\omega, \quad (21)$$

$$c_0 = -\frac{\omega^2(Dr + tL)}{3D}, \quad c_1 = -\frac{\omega^2(-Dr + 2tL)}{6D}. \quad (22)$$

All other output- $|0\rangle$  directions vanish. Since  $|L|^2 = D$  and  $\text{Re } L = -1/2$ ,

$$|c_0|^2 = \frac{Dr^2 - rt + t^2}{9D}, \quad |c_1|^2 = \frac{Dr^2 + 2rt + 4t^2}{36D}.$$

Consequently the two distinct determinants are

$$d_0 = \frac{q(2t-r)^2}{36t^2} - \frac{Dr^2 - rt + t^2}{9D}, \quad (23)$$

$$d_1 = d_2 = \frac{q(2t-r)(t+r)}{18t^2} - \frac{Dr^2 + 2rt + 4t^2}{36D}. \quad (24)$$

These are derived by exact change of basis from (19); the verification code independently reconstructs the entire Choi matrix.

### 4.3 All three blocks have rank one on a continuous curve

**Theorem 4.1** (A continuous minimum-Kraus construction). *For  $0 < h < 2$ , set*

$$q = \frac{h}{3h+2}, \quad a = \frac{1}{2} \left( 1 + \sqrt{\frac{2-h}{3h+2}} \right), \quad t^2 = \frac{h(2-h)}{2(h+1)(3h+2)}, \quad r = ht. \quad (25)$$

*The states and maps above are a stationary gutrit instrument generating every finite word of (15). Each branch has Choi rank three and the complete emitted-record channel has rank nine. The attained processes include every*

$$0 < r \leq r_{\max} := \frac{\sqrt{71-17\sqrt{17}}}{4} = 0.2381181907818797\dots, \quad (26)$$

$$\lambda_{\min} := 1 - r_{\max} = 0.7618818092181203\dots \leq \lambda < 1.$$

*The minimum internal memory dimension is exactly three throughout this interval. The iid endpoint  $\lambda = 1$  has minimum dimension one and is not included in that assertion.*

*Proof.* For  $0 < h < 2$ ,  $0 < q < 1/4$  and  $t^2/q = (2-h)/[2(h+1)] < 1$ . Thus all seed states are positive of rank two and their basis is nonsingular. Also  $D = 2/(3h+2)$ ,  $e_- = (2-h)/6 > 0$ ,  $e_+ = (1+h)/3 > 0$ . Multiplication of (24) by  $36Dt^2$  and substitution  $r = ht$  reduce the determinant equations to

$$qD(2-h)^2 = 4t^2(Dh^2 - h + 1),$$

$$2qD(2-h)(1+h) = t^2(Dh^2 + 2h + 4).$$

Both are identities under (25). Every block therefore has positive diagonal and zero determinant: it is positive of rank one. Covariance transfers this to every branch, proving complete positivity and the stated Choi ranks.

Trace preservation follows on the basis  $S_{uv}$  from  $\sum_j T_{uj} = 1$ . With initial state  $I/3$ , direct application gives

$$\Phi_j(I/3) = D_j/3, \quad \Phi_k(D_j) = S_{jk}/3, \quad \sum_j D_j = I.$$

The initial state is stationary, and the first two symbols have probability  $1/9$  and normalized posterior state  $S_{jk}$ . Thereafter (17) supplies exactly the prescribed conditional probability and posterior state at every step. Induction proves the full word law, not only its first few marginals.

Finally

$$r(h)^2 = \frac{h^3(2-h)}{2(h+1)(3h+2)}, \quad \frac{d}{dh} r(h)^2 = -\frac{3h^2(h+2)(2h^2-h-2)}{2(h+1)^2(3h+2)^2}.$$

The unique maximum in  $(0, 2)$  is at  $h_* = (1 + \sqrt{17})/4$ , where  $r^2 = (71 - 17\sqrt{17})/16$ . On  $(0, h_*]$  the function is strictly increasing from zero, so every  $0 < r \leq r_{\max}$  has a unique parameter in this subinterval. A bracketed root finder constructs it. The observed rank-nine cut proves that the admitted dimension-three realization is minimal.  $\square$

There are exactly three Kraus operators per branch. Explicitly, for the positive rank-one  $k$ th block  $\begin{pmatrix} \alpha_k & c_k \\ \bar{c}_k & \beta_k \end{pmatrix}$ , its Choi vector is

$$v_k = \sqrt{\alpha_k} f_k \otimes |1\rangle + \frac{\bar{c}_k}{\sqrt{\alpha_k}} f_{k+1} \otimes |2\rangle.$$

Unvectorize  $v_k$  to obtain one Kraus operator; rotate by covariance for other emitted symbols. Their completeness follows from the trace-preserving proof. This is a defined admitted quantum interaction, preparation and readout law using standard Born-form instruments; it does not derive the Born rule or provide an autonomous collapse mechanism.

#### 4.4 The complete arbitrary-phase class and a sharp scoped boundary

The family is not merely a numerical choice among unspecified phases. Replace  $\zeta$  by  $x + iy$ ,  $t^2 = x^2 + y^2$ , while retaining  $0 < a < 1$  and  $0 < t^2 < q$ . The same exact-reset definition remains unique. The three blocks now have diagonals  $ae_k, (1-a)e_{k+1}$ , cyclically, where

$$e_0 = \frac{1}{3} - \frac{rx}{3t^2}, \quad e_1 = \frac{1}{3} + \frac{r(x - \sqrt{3}y)}{6t^2}, \quad e_2 = \frac{1}{3} + \frac{r(x + \sqrt{3}y)}{6t^2}. \quad (27)$$

Their upper off-diagonal entries are

$$c_k = \frac{t^2 M/D + rG_k}{6\bar{\zeta}}, \quad M = -1 + i\sqrt{3}(2a - 1), \quad (G_0, G_1, G_2) = (x + \sqrt{3}y, -2x, x - \sqrt{3}y). \quad (28)$$

These formulas follow by the same direct Fourier transform, not by imposing complete positivity after a fit.

**Lemma 4.2** (No singular one-step effect in this Weyl class). *At  $r > 0$ , every completely positive exact-reset source of this class has strictly positive effects  $E_j = \Phi_j^*(I)$ .*

*Proof.* The effects have eigenvalues  $e_0, e_1, e_2$ . If  $e_0 = 0$ , positivity of the blocks forces  $c_0 = c_2 = 0$ . Their difference is  $r\sqrt{3}y/(3\bar{\zeta})$ , so  $y = 0$ . Then  $e_0 = 0$  gives  $r = x > 0$ . The numerator of  $c_0$  is  $t^2 M/D + rx$ . Its imaginary part vanishes only if  $a = 1/2$ , in which case  $D = 1/4$ ,  $t^2 = x^2$  and the numerator is  $-4x^2 + x^2 = -3x^2 \neq 0$ , a contradiction. For a zero  $e_1$  or  $e_2$ , the two forced zero off-diagonals instead give  $y = \mp\sqrt{3}x$ ,  $r = -2x = t > 0$  and again the numerator  $t^2 M/D + t^2$ . Its imaginary part forces  $a = 1/2$  and its real part is  $-3t^2 \neq 0$ . Positive semidefiniteness of the effects therefore makes them positive definite.  $\square$

**Theorem 4.3** (Exhaustion of the minimum-Kraus Weyl class). *Among all nonsingular rank-two coordinate-plane Weyl exact-reset sources above, with arbitrary seed diagonal weight and phase, branch Choi rank three is attained exactly by (25), up to phase rotation by  $2\pi/3$  and exchange of  $a$  with  $1-a$ . Thus (26) is the sharp noise boundary within this minimum-Kraus Weyl class. It is not established as the boundary for arbitrary qutrit sources, nor for members of the same Weyl class with higher Kraus rank.*

*Proof.* Put  $\zeta = te^{i\phi}$ ,  $h = r/t$ ,  $v = t^2$ , and

$$u_0 = x/t, \quad u_1 = (-x + \sqrt{3}y)/(2t), \quad u_2 = (-x - \sqrt{3}y)/(2t).$$

These are three cosines spaced by  $2\pi/3$ ; their sum is zero and their squares sum to  $3/2$ . By Lemma 4.2, every block has positive diagonal and hence rank at least one. Choi rank three therefore means that all three block determinants vanish. Directly from (27)–(28), each determinant, indexed by its opposite cosine, has the form

$$9d(u) = A + Bu + Cu^2, \quad A = q(1 - 3h^2/4) - v/D, \quad B = h(q - v/D), \quad C = h^2(q - v). \quad (29)$$

If all three cosines are distinct, a quadratic zero at all three is zero identically. As  $h > 0$ ,  $B = C = 0$  implies  $v = qD = q$ , impossible for  $q > 0$  and  $D = 1 - 3q < 1$ . Thus two cosines coincide and  $\phi$  is an integer multiple of  $\pi/3$ .

For an even multiple the cosine set is  $\{1, -1/2, -1/2\}$ . Positivity of the effect requires  $h < 1$ . Eliminating  $v$  from  $d(1) = d(-1/2) = 0$  gives  $q = h/(3h - 2)$ ; the excluded factors are  $h = 0$ ,  $q = 0$  or  $q = 1/3$ . For  $0 < h < 1$ , this value is either nonpositive or exceeds  $1/4$ . Thus it cannot be a seed with  $0 < q \leq 1/4$ .

For an odd multiple the cosines are  $\{-1, 1/2, 1/2\}$ . Positive effects require  $0 < h < 2$ . Elimination instead gives  $q = h/(3h + 2)$ , and either remaining determinant yields  $v = h(2 - h)/[2(h + 1)(3h + 2)]$ . These are precisely (25). There are no omitted solutions at  $h = 2$ : an effect would be singular, already excluded. The maximum of  $r$  on this curve was proved in Theorem 4.1.  $\square$

## 4.5 What the improved source does and does not settle

For every member of this class, effects are positive definite and every pair posterior has rank two. With  $F_{jk} = \Phi_j^*(E_k)$ ,  $E_k \succeq c_k I$  for some  $c_k > 0$ , so  $F_{jk} \succeq c_k E_j \succ 0$ . Basis-complete reset implies

$$\Phi_k \Phi_j(A) = \text{Tr}(F_{jk} A) S_{jk} \quad \text{on all } M_3.$$

Therefore every length- $n \geq 2$  word branch has Choi rank  $\text{rank } F_{jk} \text{rank } S_{\ell m} = 3 \cdot 2 = 6$ ; all intervening transition weights are positive. The complete  $n$ -record cq channel has rank  $6 \cdot 3^n$ . For the single stationary preparation its output state instead has rank  $2 \cdot 3^n$ . These are different operational tasks.

This remains true for arbitrary phases and higher branch Kraus ranks in this coordinate-plane class. Consequently phase and diagonal freedom can remove one-step excess without reducing the globally compressed finite-horizon channel below  $6 \cdot 3^n$ . In particular it cannot improve its finite-horizon cost merely by changing these seed parameters. Any reduction below  $6 \cdot 3^n$  requires a different encoding. The stronger universal lower bound  $437 \cdot 3^{n-4}$  for  $n \geq 4$ , proved in Section 8.5, leaves a gap below this construction; in particular the simpler  $4 \cdot 3^n$  bound is not attainable at those horizons.

The general result in Appendix A that each branch of a minimal qutrit source for this process has Choi rank at least three makes  $k = 9$  optimal among *all* qutrit realizations, not just this ansatz. Combined with the pure-latitude two-qubit source of  $(d, k) = (4, 3)$  and the three-output floor  $k \geq 3$ , the one-step Pareto frontier is now exactly  $\{(3, 9), (4, 3)\}$  throughout (26). This invocation uses the full CP rank lemma in Appendix A; it is not derived from the more limited exhaustion theorem above.

The algebraic identities and all-word induction, rather than finite numerical fits, establish the result. The singular endpoints  $h = 0, 2$  are not admitted finite encodings. No global monotonicity of the unrestricted qutrit-admissible set is asserted.

## 5 The exact memory boundary remains partially open

Let  $\mathcal{F}_3 \subset [0, 1]$  be the noise parameters admitting a stationary qutrit source with the full word law. This set is compact: three positive  $9 \times 9$  Choi matrices with summed partial trace  $I$  form a compact set, as do initial density matrices. Stationarity and every finite-word equality are continuous closed constraints. Their intersection is closed in that compact parameter space, and its projection to  $\lambda$  is compact. The explicit high-noise family makes it nonempty; the low-noise exclusion shows that its attained minimum  $\lambda_c$  lies in  $(3 \cdot 10^{-8}, \lambda_\dagger]$ . Below  $\lambda_c$ , four dimensions are necessary and sufficient; at  $\lambda_c$  the rank-nine cut makes the minimum three. This does not prove  $\mathcal{F}_3$  is an interval. Adding noise to emitted symbols need not preserve the delayed-process family, so global monotonicity cannot be inferred that way.

## 6 A four-dimensional source attaining the recorded-channel floor

Phase-enhanced pure-state stochastic encodings have established antecedents. The following explicit specialization realizes the delayed-trine law while attaining the smallest possible recorded-channel rank. It supplies the four-dimensional endpoint of the Pareto comparison.

For  $r = 1 - \lambda$ ,  $z = \sqrt{1 - r^2}$  and  $\theta_j = 2\pi j/3$ , define

$$|\ell_j\rangle = \sqrt{\frac{1+z}{2}}|0\rangle + e^{i\theta_j}\sqrt{\frac{1-z}{2}}|1\rangle, \quad |e_j\rangle = (|0\rangle - e^{i\theta_j}|1\rangle)/\sqrt{2}.$$

The rank-one effects  $E_j = (2/3)|e_j\rangle\langle e_j|$  sum to identity. With  $\rho_j = |\ell_j\rangle\langle \ell_j|$  their response is

$$\text{Tr}(E_j \rho_i) = \frac{1 - r \cos(\theta_i - \theta_j)}{3} = T_{ij}.$$

On two qubits let  $S$  swap factors and put

$$K_j = S(\sqrt{2/3}|\ell_j\rangle\langle e_j| \otimes I), \quad \Phi_j(X) = K_j X K_j^\dagger.$$

Then  $\sum_j K_j^\dagger K_j = I_4$  and

$$\Phi_j(\rho_a \otimes \rho_b) = T_{aj} \rho_b \otimes \rho_j.$$

Let  $\tau = \frac{1}{3} \sum_j \rho_j = (I + z\sigma_z)/2$ . The stationary preparation is  $\tau \otimes \tau$ . Since  $\text{Tr}(E_j \tau) = 1/3$ , the first two emitted symbols are independent uniform trits; thereafter the displayed update proves every finite word of the delayed law by induction. This stationary preparation differs from that of the qutrit source and is explicitly budgeted.

Each symbol has one nonzero Kraus operator, so the complete registered one-step channel has Choi rank exactly three. No registered three-symbol channel with three nonzero outputs can have smaller rank, since its orthogonal output blocks each contribute at least one. At  $0 < \lambda < 1$  all words are positive and every word Kraus operator is rank one after two steps. The full registered  $n$ -step channel therefore has Choi rank  $3^n$  for  $n \geq 2$ , attaining the alphabet-imposed floor with retained final memory. It streams with one fresh three-level record port and one fresh three-level record-copy bath per step; old baths never return. These resources are not an uncanceled fresh source or a correlated tape. The initial mixed preparation and its rank-four purifier are separate stock.

For  $\lambda = 0$  the exact-law support floor changes to  $9 \cdot 2^{n-2}$ . At  $\lambda = 1$  the process is iid and memory one suffices; this redundant four-dimensional implementation is not a memory optimum. All claimed two-point frontiers exclude that iid endpoint. A coherent archive or an initially mixed bath defines a different optimization task.

## 7 Universal qutrit lower bounds

### 7.1 Process, observability and the complete-channel task

Let  $0 < \lambda < 1$ ,  $r = 1 - \lambda$ , and

$$T_{aa} = \lambda/3, \quad T_{ab} = (3 - \lambda)/6 \quad (a \neq b).$$

The stationary delayed-trine law is

$$P(w_1 \cdots w_n) = \frac{1}{9} \prod_{t=1}^{n-2} T_{w_t, w_{t+2}}, \quad n \geq 2.$$

All words are positive. Let CP maps  $\Psi_j : M_3 \rightarrow M_3$ , with  $\sum_j \Psi_j$  trace preserving, and stationary  $\tau$  realize this entire law. Write  $\sigma_{ab} = 9\Psi_b\Psi_a(\tau)$  for the normalized pair state,  $E_j = \Psi_j^*(I)$ , and  $F_{ab} = \Psi_a^*\Psi_b^*(I)$ .

The eigenvalues of  $T$  are  $1, -r/2, -r/2$ . Consequently the two-past/two-future matrix

$$H_{ab,cd} = \frac{1}{9} T_{ac} T_{bd} = \frac{1}{9} \text{Tr}(\sigma_{ab} F_{cd})$$

has ordinary rank nine. The nine pair states and nine pair effects are therefore separately bases of the Hermitian qutrit operator space. The same cut rules out internal dimension below three, so every admitted qutrit realization in this parameter range is automatically memory-minimal; this is not an extra restriction on the source class. In particular  $\tau = \frac{1}{9} \sum_{ab} \sigma_{ab}$  is faithful.

The next-two probabilities after any longer history ending in  $b, j$  agree with those after  $b, j$ . Since the future-pair effects span the entire operator space, these predictive equalities force actual memory-state equalities. Thus

$$\Psi_j(\sigma_{ab}) = T_{aj} \sigma_{bj}, \quad \Psi_k \Psi_j(X) = \text{Tr}(F_{jk} X) \sigma_{jk}. \quad (30)$$

Every single branch has linear rank three: its image is precisely the three-dimensional span of  $\{\sigma_{bj} : b = 0, 1, 2\}$ . This observable-state argument does not permit an extra inaccessible history variable to be added to a nominal three-dimensional source.

For a finite horizon define the complete classical-record channel

$$\mathcal{C}_n(X) = \sum_w |w\rangle\langle w|_A \otimes \Psi_w(X), \quad \Psi_w = \Psi_{w_n} \cdots \Psi_{w_1}.$$

Its output retains final memory and the classical word archive, as well as an arbitrary inert reference to the input. Put  $R_n = \text{rank } J(\mathcal{C}_n)$ . This is the minimum initially pure Stinespring bath dimension for this specified final channel. It is not a peak-hardware count or a product of independently minimized time cuts.

The one-step analysis's universal one-step bound remains applicable: a CP qutrit map of linear rank three needs at least three Kraus operators, so  $R_1 \geq 9$ . The full elementary proof is reproduced in Appendix A; the general apparatus sum-of-Kraus-ranks rule is established measurement-model theory. The present improvement uses additional dynamics and support geometry, beyond that one-step lemma.

## 7.2 A rank-three CP map cannot make a rank-two input pure

**Lemma 7.1** (No plane-to-ray output at linear rank three). *Let  $\Phi : M_3 \rightarrow M_3$  be CP with complex-linear rank three. If  $X \geq 0$ ,  $\text{rank } X \geq 2$ , and  $\Phi(X) \neq 0$ , then  $\text{rank } \Phi(X) \geq 2$ .*

*Proof.* If  $X$  is faithful and its output is pure, positivity places the image of every Kraus operator on the same output line; the entire map then has rank one. It remains to treat a rank-two input supported on a plane  $P$ . Suppose its output lies on the ray  $v$ . Each Kraus operator maps all of  $P$  into that ray. Choose  $u$  orthogonal to  $P$  and a basis  $p_1, p_2$  of  $P$ . The responses on  $PBP$  are multiples of  $V = |v\rangle\langle v|$ , and the remaining responses are

$$W = \Phi(|u\rangle\langle u|) \geq 0, \quad C_i = \Phi(|p_i\rangle\langle u|) = |v\rangle\langle z_i|, \quad C_i^\dagger.$$

The nonzero pure response implies  $V$  belongs to the image. If the whole output support is the line  $v$ , the map has rank one. Otherwise  $W$  has a nonzero positive block on  $v^\perp$ , since it is the only response capable of supplying such a block. If some  $z_i$  has a component on  $v^\perp$ , the two cross directions  $C_i, C_i^\dagger$ , modulo  $V$ , are independent; their opposite off-diagonal blocks distinguish them. Together with  $V$  and  $W$  they give at least four independent image directions. If no  $z_i$  has such a component, the image is contained in  $\text{span}\{V, W\}$  and has rank at most two. Neither alternative has rank three.  $\square$

The missing rank is substantive: CP maps of linear ranks two and four can both collapse a positive plane input to a ray. The code gives explicit examples. A rank-three map with two-dimensional input and output supports also exists, so one cannot simply declare all branch supports faithful.

**Corollary: every pair state and effect has rank at least two.** The first-symbol state  $\rho_a = 3\Psi_a(\tau)$  has the same support as  $\Psi_a(I)$  because  $\tau$  is faithful. That support cannot be a line, as the branch has linear rank three. Hence  $\text{rank } \rho_a \geq 2$ , and the lemma applied to  $\Psi_b(\rho_a)$  proves

$$\text{rank } \sigma_{ab} \geq 2 \quad \text{for every } a, b. \tag{31}$$

For the effects, use the stationary adjoint reverse instrument

$$\tilde{\Psi}_j(X) = \tau^{1/2} \Psi_j^*(\tau^{-1/2} X \tau^{-1/2}) \tau^{1/2}.$$

It is CP, its summed map is trace preserving and stationary at  $\tau$ , and its word probabilities are the reversed original word probabilities. Since  $T$  is symmetric, it realizes the same delayed-trine law. Its

pair state for  $b, a$  is  $9\tau^{1/2}F_{ab}\tau^{1/2}$ . Congruence preserves rank, so applying (31) to this output source gives

$$\text{rank } F_{ab} \geq 2 \quad \text{for every } a, b. \quad (32)$$

This reverse is a constructed CP realization; microscopic time-reversal symmetry is not assumed.

After two steps each word map in (30) is measure-and-prepare. For longer words its input effect is a positive scalar times  $F_{w_1 w_2}$ , and its output state is  $\sigma_{w_{n-1} w_n}$ . Its Choi rank is their rank product, at least four. Therefore

$$R_n \geq 4 \cdot 3^n \quad (n \geq 2).$$

This conclusion applies word by word and already holds at horizon two.

### 7.3 Support incidence and preliminary horizon estimates

Let  $Q_j = \text{ran } \Psi_j(I)$  and  $P_j = \text{ran } E_j$ . Their dimensions are two or three; an input or output support of dimension one would give branch rank at most one. Let

$$A = \{j : \dim Q_j = 3\}, \quad B = \{j : \dim P_j = 3\}, \quad a = |A|, \quad b = |B|.$$

Pair states in column  $j$  are supported on  $Q_j$ , and pair effects in row  $j$  are supported on  $P_j$ . No two two-dimensional output supports are the same plane, because six independent pair states cannot all lie in its four-dimensional Hermitian operator space. The analogous input-plane statement follows from independence of the pair effects.

Suppose  $Q_j$  and  $P_k$  are both planes. If their overlap projection  $P_k|_{Q_j}$  has rank two, compression by this invertible 2-by-2 matrix induces an invertible map on the four-dimensional plane-operator space. The composition  $\Psi_k \Psi_j$  then factors through two maps of rank three and an invertible map on that four-dimensional space. Sylvester's rank inequality gives composition rank at least  $3 + 3 - 4 = 2$ , contradicting (30). Thus the projection has rank one, or equivalently

$$\ker E_k \subseteq Q_j \quad \text{whenever } \dim P_k = \dim Q_j = 2. \quad (33)$$

Two distinct output planes intersect in a line. They cannot contain two distinct input-kernel lines. Therefore there cannot be at least two planes on both sides:  $a \leq 1$  and  $b \leq 1$  are impossible. There is a further extreme-case obstruction. If all three input supports are planes ( $b = 0$ ) and any output plane exists, all three input-kernel lines lie in that output plane. Their orthogonal input planes share a common line  $L$ . Hermitian operators supported on three planes sharing  $L$  span at most eight dimensions: one for the projector onto  $L$ , four for cross terms between  $L$  and  $L^\perp$ , and at most three for the remaining plane directions. This contradicts the nine independent effects. Hence  $b = 0$  forces  $a = 3$ . The dual argument gives  $a = 0 \Rightarrow b = 3$ . These obstructions imply the necessary condition

$$a + b \geq 3. \quad (34)$$

No sufficiency or CP attainability is claimed for a support pattern satisfying this inequality. At least three of the six branch input/output support matrices are faithful. The argument is independent of any cyclic covariance.

Put  $S_\sigma = \sum_{ij} \text{rank } \sigma_{ij}$  and  $S_F = \sum_{ij} \text{rank } F_{ij}$ . If  $i \in A$ , the first-symbol state  $\rho_i$  is faithful; its output under  $\Psi_j$  has support exactly  $Q_j$ . Thus  $\sigma_{ij}$  has rank three whenever  $i, j \in A$ . Similarly  $F_{ij}$  has rank three whenever  $i, j \in B$ , since a faithful  $E_j$  satisfies  $\Psi_i^*(E_j) \geq cE_i$ . Therefore

$$S_\sigma \geq 18 + a^2, \quad S_F \geq 18 + b^2. \quad (35)$$

For  $n \geq 4$  the initial and final pairs can be freely combined, giving the exact identity

$$R_n = 3^{n-4} S_F S_\sigma.$$

The minimum allowed product in (35) under  $a + b \geq 3$  is  $19 \cdot 22 = 418$ . Thus

$$\boxed{R_n \geq 418 3^{n-4} \quad (n \geq 4).} \quad (36)$$

The use of branch support geometry strengthens a bound based only on individual pair ranks. The pure-output argument below gives the sharper coefficient 437.

At the shorter horizons let  $h = |A \cap B|$ . Counting the same forced rank three pairs in a word gives

$$\begin{aligned} R_2 &\geq 36 + 2a^2 + 2b^2 + h^2, \\ R_3 &\geq 108 + 6a^2 + 6b^2 + abh. \end{aligned}$$

Minimizing with  $a + b \geq 3$  and  $h \geq \max\{0, a + b - 3\}$  yields

$$\boxed{R_2 \geq 46, \quad R_3 \geq 138.} \quad (37)$$

For example the boundary profiles give the following relaxed minima:

| $(a, b)$             | $R_2$ lower bound | $R_3$ lower bound | $S_\sigma S_F$ lower bound |
|----------------------|-------------------|-------------------|----------------------------|
| $(0, 3)$ or $(3, 0)$ | 54                | 162               | 486                        |
| $(1, 2)$ or $(2, 1)$ | 46                | 138               | 418                        |

The finite enumeration is reproduced exactly with integer counts in the code; the constraints are necessary, not constructions of the mixed profiles.

Equality in (36) would require mixed profiles  $(a, b) = (1, 2)$  or  $(2, 1)$  with rank sums  $(19, 22)$  or its reverse. The construction in the main text realizes  $(19, 24)$  and improves the attainable coefficient to 456 at  $\lambda = .99$ ; the new main-text obstruction raises the lower coefficient to 437; attainment of that value remains unresolved.

## 8 A mixed-support qutrit source with lower complete-history cost

Mixed input/output support profiles are compatible with a completely positive, stationary, full-law source. At  $\lambda = 99/100$  the construction below has one faithful output branch and two faithful input branches, and generates the entire delayed-trine law. Its complete retained-memory/classical-history channel has

$$R_2 = 50, \quad R_3 = 152, \quad \boxed{R_n = 456 3^{n-4} \quad (n \geq 4).}$$

The coordinate-plane Weyl sources attain  $54, 162, 486 3^{n-4}$  instead. This reduces the long-horizon coefficient by  $30/486 = 5/81$ , about 6.173%. A separate universal obstruction below raises the lower coefficient from 418 to 437, leaving the interval  $[437, 456]$ . The specified source uses fifteen one-step Kraus operators, compared with the separate optimum nine: the improvement is in the complete final channel task, not in every resource simultaneously.

### 8.1 Task and baseline identities

Write  $\Phi_j : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$  for autonomous CP symbol maps, with  $\sum_j \Phi_j$  trace preserving and stationary preparation  $\tau$ . The target transition matrix and stationary word law are

$$T_{aa} = \lambda/3, \quad T_{ab} = (3 - \lambda)/6 \quad (a \neq b), \quad P(w_1 \cdots w_n) = \frac{1}{9} \prod_{t=1}^{n-2} T_{w_t, w_{t+2}}.$$

All symbols range over  $\{0, 1, 2\}$ . At the parameter considered here,  $T_{aa} = 33/100$  and  $T_{ab} = 67/200$  for  $a \neq b$ . The complete channel is

$$\mathcal{C}_n(X) = \sum_w |w\rangle\langle w|_A \otimes \Phi_w(X), \quad R_n = \text{rank } J(\mathcal{C}_n).$$

It retains final memory and the classical archive, including any inert input reference.  $R_n$  is the minimum pure Stinespring remainder dimension for this specified final channel, rather than peak causal hardware or the number of append-only records. The comparisons below use this same task and the same three-dimensional internal input/output type.

For a full-law minimal qutrit source the basis-complete reset proof establishes that the nine normalized pair states  $\sigma_{ab}$  are an operator basis and

$$\Phi_j(\sigma_{ab}) = T_{aj}\sigma_{bj}, \quad \Phi_k\Phi_j(X) = \text{Tr}(F_{jk}X)\sigma_{jk}, \quad F_{jk} = \Phi_j^*\Phi_k^*(I). \quad (38)$$

Consequently, with  $s_{ab} = \text{rank } \sigma_{ab}$  and  $f_{ab} = \text{rank } F_{ab}$ ,

$$\begin{aligned} R_2 &= \sum_{ab} s_{ab}f_{ab}, \\ R_3 &= \sum_{abc} f_{ab}s_{bc}, \\ R_n &= 3^{n-4} \left( \sum_{ab} s_{ab} \right) \left( \sum_{ab} f_{ab} \right) \quad (n \geq 4). \end{aligned} \quad (39)$$

The final identity counts freely chosen first/last pairs; the intervening strictly positive weights do not change Choi ranks. We explicitly verify (38) for the source below, rather than assuming that a PSD factorization alone yields a recurrent source.

## 8.2 Explicit algebraic source

Let  $(e_0, e_1, e_2)$  be an orthonormal qutrit basis and  $\omega = (-1 + i\sqrt{3})/2$ . Set

$$p = \frac{32}{35}, \quad c = \frac{7}{20}, \quad u_1 = \frac{e_1 - e_2}{\sqrt{2}}, \quad u_2 = \frac{e_1 - \omega e_2}{\sqrt{2}}.$$

The two output planes are  $Q_b = \text{span}\{e_0, u_b\}$ ,  $b = 1, 2$ . Define the unit vectors and their projectors

$$w_1 = \sqrt{\frac{3}{35}}e_0 + \sqrt{\frac{32}{35}}e_1, \quad w_2 = \sqrt{\frac{3}{35}}e_0 - \sqrt{\frac{32}{35}}e_1, \quad w_3 = e_2, \quad W_l = |w_l\rangle\langle w_l|.$$

The pair states in column zero are

$$\sigma_{00} = \frac{W_1 + W_2 + W_3}{3}, \quad \sigma_{10} = \frac{W_2 + W_3}{2}, \quad \sigma_{20} = \frac{W_1 + W_3}{2}. \quad (40)$$

Put  $q_0 = 33/35$  and  $q_1 = q_2 = 67/70$ . The remaining states are

$$\sigma_{ab} = (1 - q_a)|e_0\rangle\langle e_0| + q_a|u_b\rangle\langle u_b| + z_{ab}|e_0\rangle\langle u_b| + \bar{z}_{ab}|u_b\rangle\langle e_0|, \quad b = 1, 2. \quad (41)$$

Here  $z_{ab} = \sqrt{2}(A_{ab} + B_{ab}\omega)$ , and all coefficients are exact:

| $(a, b)$ | $10^4 A_{ab}$ | $10^4 B_{ab}$ |
|----------|---------------|---------------|
| (0, 1)   | 104           | -112          |
| (0, 2)   | 319           | 226           |
| (1, 1)   | -439          | -1028         |
| (1, 2)   | -299          | 54            |
| (2, 1)   | 269           | 597           |
| (2, 2)   | 1119          | 714           |

All source matrices have entries in the algebraic field  $K = \mathbb{Q}(\sqrt{6}, \omega)$ : the  $\sqrt{2}$  in  $z$  cancels the denominator of  $u_b$ , and  $w_1, w_2$  projector cross terms are  $\pm 4\sqrt{6}/35$ .

Let  $\text{vec}$  stack matrix rows consistently, and form the nine-by-nine matrices

$$S_{:(a,b)} = \text{vec } \sigma_{ab}, \quad (B_j)_{:(a,b)} = T_{aj} \text{vec } \sigma_{bj},$$

with columns indexed lexicographically by  $(a, b) \in \{0, 1, 2\}^2$ . Define the entire arbitrary-input maps by

$$\text{vec } \Phi_j(X) = B_j S^{-1} \text{vec } X. \quad (42)$$

The inverse exists by the exact certificate. Equation (42), together with the small rational table, specifies each map exactly and independently of horizon.

**The first branch has a direct measure–prepare realization.** Let  $v_l = (e_1 + \omega^l e_2)/\sqrt{2}$ ,  $l = 0, 1, 2$ . Then

$$\Phi_0(X) = \frac{2c}{3} \sum_{l=0}^2 \langle v_l | X | v_l \rangle W_{l+1}, \quad E_0 = \Phi_0^*(I) = c(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|). \quad (43)$$

The three Kraus operators are  $\sqrt{2c/3}|w_{l+1}\rangle\langle v_l|$ . Their vectorizations are independent because the  $w_l$  are independent. Thus this branch has Choi rank exactly three, plane input support and full output support. The antipodal relations  $\langle v_0 | u_1 \rangle = 0$  and  $\langle v_1 | u_2 \rangle = 0$  explain the two rank-two states  $\sigma_{10}, \sigma_{20}$ , while the common entrance line  $e_0$  is killed. This gives a concrete mixed-support branch mechanism.

### 8.3 Exact certificate and full-law proof

**Lemma 8.1** (Exact algebraic admissibility). *For the explicit data above,  $S$  is nonsingular; all nine  $\sigma_{ab}$  are positive semidefinite of trace one. The maps in (42) are CP. Branch zero is precisely (43); branches one and two have Choi matrices positive definite on  $\mathbb{C}^3 \otimes Q_1$  and  $\mathbb{C}^3 \otimes Q_2$ , respectively, and zero on the orthogonal output complements. Their Choi ranks are six each.*

*Certificate verification.* The supplied exact verifier certifies this lemma with arithmetic in  $\mathbb{Q}(\sqrt{6}, \omega)$ , not floating-point eigenvalue signs. It checks nonsingularity, symbolic equality with (43), exact Hermiticity/support and positive leading principal minors of the two six-dimensional Choi compressions. State positivity follows from  $q_a(1 - q_a) - |z_{ab}|^2 > 0$  for the six plane states, and from independence of the explicit  $w_l$  in (40). All certificate values and verification instructions accompany the source. Floating-point reconstruction, useful only as a diagnostic, gives a smallest supported Choi eigenvalue approximately  $3.8332 \times 10^{-4}$ . This diagnostic is not used to establish positivity.  $\square$

**Theorem 8.2** (Stationary full-word attainment at  $\lambda = 99/100$ ). *The maps in (42), initialized at  $\tau = \frac{1}{9} \sum_{ab} \sigma_{ab}$ , generate every finite word of the stationary delayed-trine process at  $\lambda = 99/100$ .*

*Proof.* CP is supplied by the certificate lemma. On each basis state,

$$\text{Tr} \sum_j \Phi_j(\sigma_{ab}) = \sum_j T_{aj} = 1 = \text{Tr } \sigma_{ab}.$$

Since the states are an operator basis, the summed map is trace preserving on all inputs. The initial state is faithful because it includes a positive multiple of the faithful  $\sigma_{00}$ . Double stochasticity of  $T$  gives

$$\sum_j \Phi_j(\tau) = \frac{1}{9} \sum_{abj} T_{aj} \sigma_{bj} = \frac{1}{9} \sum_{bj} \sigma_{bj} = \tau.$$

Moreover

$$\Phi_j(\tau) = \frac{1}{9} \sum_b \sigma_{bj}, \quad \Phi_k \Phi_j(\tau) = \frac{1}{9} \sum_b T_{bk} \sigma_{bj} = \frac{1}{9} \sigma_{jk}.$$

Every adjacent pair therefore has probability  $1/9$  and the designated posterior state. Repeatedly applying the exact basis relation  $\Phi_j(\sigma_{ab}) = T_{aj} \sigma_{bj}$  yields the stated product for all longer words. It also proves (38) on each basis operator and hence on arbitrary operators. No fitted finite word list is used to infer the full law.  $\square$

#### 8.4 Exact ranks and improved horizon boundary

**Theorem 8.3** (Attainable mixed-support resources). *This fixed source has*

$$[s_{ab}] = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \quad [f_{ab}] = \begin{pmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

In particular  $S_\sigma = 19$ ,  $S_F = 24$ , branch support counts  $(a, b) = (1, 2)$ , and one-step recorded-channel Choi rank  $k = 15$ .

*Proof.* The state ranks follow from the certificate lemma and (40). Branches one and two have full input support because their supported Choi matrices are strictly positive; their output supports are exactly  $Q_1, Q_2$ . Thus  $E_1, E_2$  are faithful, whereas  $E_0 = cP$  has plane support  $P = \text{span}\{e_1, e_2\}$ . For every  $k$ , (43) gives

$$F_{0k} = \frac{2c}{3} \sum_l \langle w_{l+1} | E_k | w_{l+1} \rangle |v_l\rangle \langle v_l|.$$

All three coefficients are positive: for  $k = 0$  their weights are  $cp, cp, c$ , and for  $k = 1, 2$  positivity follows from faithfulness. Therefore every  $F_{0k}$  has rank two on  $P$ . For  $j = 1, 2$  and each nonzero input vector  $x$ , strict Choi positivity on  $\mathbb{C}^3 \otimes Q_j$  implies  $\langle x | \Phi_j^*(E_k) | x \rangle > 0$  whenever the compression of  $E_k$  to  $Q_j$  is nonzero and positive semidefinite. That compression is faithful for  $k = 1, 2$ , and nonzero rank one for  $k = 0$ . Hence all six  $F_{jk}$  are faithful. Finally the Choi supports of the three classically recorded branches are orthogonal, so their ranks sum to  $3 + 6 + 6 = 15$ .  $\square$

Substituting these ranks into (39) gives the claimed exact  $50, 152, 456 3^{n-4}$ . Together with the universal bounds, the improved attainable intervals at this parameter are

$$48 \leq R_2^{\min} \leq 50, \quad 145 \leq R_3^{\min} \leq 152, \quad 437 3^{n-4} \leq R_n^{\min} \leq 456 3^{n-4} \quad (n \geq 4).$$

The old relaxed profile  $(S_\sigma, S_F) = (19, 22)$  is excluded by the universal theorem below. Attainment of the improved relaxed profile  $(19, 23)$ , or its reverse, remains open. Our source has one further faithful effect beyond that new bound. At  $n = 4$  the final certified upper-minus-lower gap is nineteen.

**Local persistence, with no claimed certified endpoints.** For  $\lambda$  near  $99/100$ , replace

$$p \mapsto \frac{4\lambda - 3}{6 - 5\lambda}, \quad c \mapsto \frac{\lambda}{2p + 1}, \quad q_0 \mapsto \frac{2p + 1}{3}, \quad q_1 = q_2 \mapsto \frac{p + 1}{2},$$

keep the rational  $A, B$  table fixed, and use the same definitions. Branch zero remains exactly (43), as direct substitution into the trine probabilities verifies. The basis determinant, six state determinants and two supported Choi matrices vary continuously, and all have strict nonzero/positive values at  $99/100$ . Thus there is an open nonempty parameter neighborhood on which this same rank profile and 456 coefficient are attained. This existence statement supplies no explicit certified endpoints and does not move the known qutrit memory threshold. The finite-parameter theorem above is the fully specified certificate.

## 8.5 A universal pure-output obstruction: the lower coefficient is 437

The construction above establishes that mixed support profiles exist. A separate positivity argument shows that the former relaxed lower value 418 cannot occur for any admitted qutrit source at any  $0 < \lambda < 1$ . No numerical search or symmetry ansatz enters this obstruction.

**Lemma 8.4** (A singular composite effect forces a pure output). *Let  $U_j = \text{im } \Phi_j$ . For any full-law minimal qutrit source,  $U_0 \oplus U_1 \oplus U_2 = M_3(\mathbb{C})$  as vector spaces: the three states in each column form a basis of  $U_j$ , and all nine states are independent. Suppose  $E_k$  has rank two, with kernel line  $L$ . Among indices  $j$  with faithful  $E_j$ , at most one effect  $F_{jk} = \Phi_j^*(E_k)$  can be singular.*

*Proof.* If  $F_{jk}$  is singular, choose  $0 \neq x \in \ker F_{jk}$ . The positive operator  $Y = \Phi_j(|x\rangle\langle x|)$  is nonzero because  $\text{Tr } Y = \langle x|E_j|x\rangle > 0$ . Moreover  $\text{Tr}(E_k Y) = \langle x|F_{jk}|x\rangle = 0$ . Positivity therefore places  $\text{ran } Y$  in  $\ker E_k = L$ , so  $Y$  is a positive multiple of the rank-one projector  $P_L$ . Thus  $P_L \in U_j$ . Two such distinct indices would put the same nonzero projector in two disjoint branch images, a contradiction.  $\square$

The faithful  $E_j$  hypothesis matters. If  $E_j$  has a kernel, the chosen input might instead be annihilated; no pure-output conclusion follows. Likewise, this lemma does not assert that all singular composite effects are impossible: one faithful-input branch may supply the ray.

**Theorem 8.5** (Universal complete-history lower bounds). *For every stationary autonomous qutrit realization of the entire delayed trine process at any feasible  $0 < \lambda < 1$ ,*

$$\boxed{R_2 \geq 48, \quad R_3 \geq 145, \quad R_n \geq 437 \cdot 3^{n-4} \quad (n \geq 4).} \quad (44)$$

*Proof.* Retain the preliminary notation  $A = \{j : \text{rank } \Phi_j(I) = 3\}$ ,  $B = \{j : \text{rank } E_j = 3\}$ , and  $a = |A|$ ,  $b = |B|$ . The baseline support-incidence proof gives  $a + b \geq 3$ , and every pair state/effect has rank at least two. There are forced rank-three states in  $A \times A$  and forced rank-three effects in  $B \times B$ . If  $b = 2$ , let  $k$  be the sole index outside  $B$ . The lemma implies that at least one of the two additional effects  $F_{jk}$ ,  $j \in B$ , has rank three. Consequently  $S_F \geq 18 + 4 + 1 = 23$ , improving the old 22. Stationary adjoint reversal preserves the full passive law and exchanges state/effect ranks with transposed indices. Thus  $a = 2$  similarly implies  $S_\sigma \geq 23$ . The complete rank-sum bounds are therefore

$$S_\sigma \geq g(a), \quad S_F \geq g(b), \quad (g(0), g(1), g(2), g(3)) = (18, 19, 23, 27).$$

Minimization under  $a + b \geq 3$  gives  $\min g(a)g(b) = 19 \cdot 23 = 437$ , at profiles (1, 2) or (2, 1). The alternatives (0, 3), (3, 0) cost at least 486; (2, 2) costs at least 529; all other allowed profiles are higher. Equation (39) proves the long-horizon bound.

For completeness the short-horizon counting retains the labels, not just rank sums. Consider  $(a, b) = (1, 2)$ , put  $h = |A \cap B| \in \{0, 1\}$ , and let  $k \notin B$ . The old forced pairs yield  $R_2 \geq 46 + h^2$  and  $R_3 \geq 138 + 2h$ . The new faithful  $F_{jk}$  adds at least  $s_{jk} \geq 2$  at horizon two. At horizon three it adds  $\sum_l s_{kl}$ , which is at least seven when  $k \in A$  ( $h = 0$ ), and at least six when  $k \notin A$  ( $h = 1$ ). Thus

$$R_2 \geq 48 + h^2 \geq 48, \quad R_3 \geq 145 + h \geq 145.$$

Stationary reversal gives the same bounds for (2, 1): its word Choi matrices are invertible congruences of adjoints of reversed word maps, so total word Choi ranks are unchanged. The extreme profiles give 54, 162, while the remaining profiles already exceed 48, 145 from the old forced counts alone. This proves all three inequalities.  $\square$

The exact integer verifier `supplement/memory/six/resources/verify_support_bound.py` enumerates all 93 minimal labelled patterns after these forced additions and reproduces 48, 145, 437.

It is a check of the finite counting argument, not a test of CP attainability. The analytical lemma supplies that necessary dynamical constraint; no claim that 437 is attained follows.

A necessary condition for attaining 437 is now more concrete. In the  $(1, 2)$  case, exactly one faithful-input branch must map some pure input to the kernel ray of the sole plane-supported effect, while the other branch must avoid that ray in its image. The dual statement applies to  $(2, 1)$ . Our strictly supported-positive two-plane branches avoid both pure-output degeneracies, explaining their stable 456 coefficient.

## 8.6 Resources and limits

The source's minimum internal memory dimension is three: the nine-by-nine past/future predictive matrix has ordinary rank nine, so any  $d$ -dimensional quantum realization needs  $d^2 \geq 9$ . An implementation of the specified one-step instrument can use a fresh pure fifteen-dimensional port and a three-symbol classical record per clock tick. The independently initialized ports do not carry preinstalled history, and the stationary initial memory and any retained purifier are declared stock. No returning environment, feedback archive or history-dependent external preparation is used. A time-homogeneous Stinespring isometry exists for the CP instrument; the construction here is an operational quantum-source model, not a specified laboratory Hamiltonian, energy budget or clock design.

$R_n$  minimizes the pure remainder of the complete final channel, allowing a global dilation/compression for that channel. It is not the product of one-step minima and does not prove a causal apparatus can store its entire working bath in dimension  $R_n$  at every intermediate time. The simple fresh-port implementation uses raw port dimension  $15^n$  before any compression. The prior  $k = 9$  source therefore remains better on the separate one-step objective, while this source is better on the declared complete history objective. Both retain final memory and a sharp classical archive; neither comparison gives a hardware, speed, work or thermodynamic advantage.

The rational coherence table, algebraic CP maps and resource theorem extend the operator-support analysis to the mixed-support source defined here. Standard quantum probabilities, admitted preparation/intervention class, fresh stock, stationary clock, reference and archive conditions remain assumptions. The result supplies neither a new Born-selection principle nor an ontology or consciousness mechanism.

## 8.7 Two typed horizon tasks do have exact qutrit optima

**Stationary memory–archive state, initial purifier omitted.** For the fixed stationary input and no retained original purifier, the state is

$$\Omega_n = \sum_w P(w) |w\rangle\langle w|_A \otimes \sigma_{w_{n-1}w_n}.$$

Orthogonal archive blocks give

$$\text{rank } \Omega_n = 3^{n-2} S_\sigma \geq 2 \cdot 3^n \quad (n \geq 2). \quad (45)$$

The known orbit source has all nine conditional states of rank two, and attains equality throughout its certified feasible range. This is a complete fixed-input state-rank optimum among the declared qutrit sources. It is a minimum dimension of a global purifier for that state, not a proof of the smallest causal apparatus implementing every intermediate operation.

**Archive-only channel, original input reference retained.** If final memory is deliberately omitted, the channel is

$$\mathcal{D}_n(X) = \sum_w \text{Tr}(\Psi_w(X)) |w\rangle\langle w|_A.$$

Its word Choi blocks are transposed effects, so

$$\text{rank } J(\mathcal{D}_n) = 3^{n-2} S_F \geq 2 \cdot 3^n \quad (n \geq 2). \quad (46)$$

The stationary adjoint reverse of the orbit source attains this lower bound: its pair effects are congruent to the forward pair states and therefore all have rank two. The reverse is an explicit stationary source with the same complete passive word law. At the known one-step optimum its Kraus ranks are unchanged by adjunction, so its reusable memory is three and its one-step pure apparatus cost is nine as before.

The pure Stinespring remainder in (46) includes the omitted final memory. It must not be called an additional external bath independently of that memory. In contrast  $R_n$  retains final memory and counts an external purifying remainder. If the original stationary purifier is retained in (45), faithful  $\tau$  makes the resulting joint-state rank equal to  $R_n$ , by an invertible input congruence of the Choi matrix; the smaller marginal-state rank cannot be substituted for it.

## 9 A certified non-Weyl source with the same passive law

Coordinate-plane Weyl orbits do not exhaust the source encodings. We construct an explicit continuum of such encodings, give a computable sufficient complete-positivity condition, and certify a sizable example with exact rational principal minors. Its full passive word law is unchanged, while an admitted memory intervention separates it from its coordinate-plane parent. This construction extends the source models defined above. It improves neither the known memory threshold nor the environmental upper bound. Its benefit is a certified noncovariant realization method and an explicit limitation of passive source identification. General completely-positive realization methods and quantum reductions of hidden Markov models are established antecedents [7, 15]. The claim here is the particular certified full-law deformation and its admitted intervention witness, not the first noncovariant quantum realization.

### 9.1 State-basis construction and an open positivity neighbourhood

Let  $S_{ab}$  be nine normalized Hermitian states forming an operator basis, and let a stationary qutrit instrument obey

$$\Phi_j(S_{ab}) = T_{aj} S_{bj}, \quad T_{aj} > 0, \quad \sum_j T_{aj} = \sum_a T_{aj} = 1.$$

For the construction below,  $S_{ab}$  has a rank-two support plane  $Q_b$  depending only on its second index. Assume each branch Choi matrix is strictly positive on  $\mathbb{C}^3 \otimes Q_j$ , of dimension six, and zero on its orthogonal complement. This is a nonempty condition: an exact example follows below.

Choose three unitaries  $U_b$  and define

$$S'_{ab} = U_b S_{ab} U_b^\dagger.$$

They are normalized rank-two states. Define the  $9 \times 9$  matrix  $E$  with columns  $\text{vec } S_{ab}$  and  $E'$  with columns  $\text{vec } S'_{ab}$ , using one fixed row-major vectorization. Let  $O_j$  have columns  $T_{aj} \text{vec } S_{bj}$ . When  $E'$  is invertible, the new maps are defined on all operators by

$$\Psi_j = \text{Ad}_{U_j} \circ O_j (E')^{-1}. \quad (47)$$

Here the right-hand matrix is a superoperator. Equivalently,  $\Psi_j(S'_{ab}) = T_{aj} S'_{bj}$ . The  $U_b$  are encoding parameters, not gates selected at runtime by an uncounted classical copy of  $b$ . The instrument consists of the fixed three CP maps in Eq. (47); its finite Choi factors implement those maps without archive feedback.

**Theorem: explicit CP neighbourhood and full-law attainment.** Put  $s = \sigma_{\min}(E) > 0$ , let  $m > 0$  be the smallest positive Choi eigenvalue across the original branches, and set  $L = \max_j \|O_j\|_F$ . If

$$\delta = \|E' - E\|_{\text{op}} < s, \quad \frac{L\delta}{s(s-\delta)} < m, \quad (48)$$

then the new maps form a stationary CP instrument with the same complete delayed word law. They have branch Choi rank six. A sufficient unitary condition is

$$\max_b \|U_b - I\|_{\text{op}} < \frac{ms^2}{6(L + ms)}.$$

*Proof.* Resolvent inversion gives  $\|(E')^{-1} - E^{-1}\|_{\text{op}} \leq \delta/[s(s-\delta)]$ . Undo the output conjugation  $U_j$ . Its superoperator differs from the old one by  $O_j[(E')^{-1} - E^{-1}]$ , of Frobenius norm at most the second quantity in Eq. (48). Choi reshuffling preserves Frobenius norm and operator norm is no larger. Both maps have output on the same fixed plane  $Q_j$  after undoing  $U_j$ ; therefore the corresponding Choi perturbation is supported on the same six-dimensional subspace, not on an uncontrolled former zero eigenspace. It is Hermitian, because the map sends a Hermitian basis to Hermitian matrices. The positive eigenvalue gap thus proves CP and full rank six on this support. Each changed state column has Frobenius error at most  $2 \max_b \|U_b - I\|$ , so  $\delta \leq 6 \max_b \|U_b - I\|$  proves the sufficient condition.

Trace preservation holds on the basis because all states have trace one and  $\sum_j T_{aj} = 1$ . Set  $\tau' = \frac{1}{9} \sum_{ab} S'_{ab}$ . It is faithful because a positive average of an operator basis cannot have a common kernel. Column stochasticity gives

$$\Psi_j(\tau') = \frac{1}{9} \sum_b S'_{bj}, \quad \Psi_k \Psi_j(\tau') = \frac{1}{9} S'_{jk}.$$

Summing the first identity over  $j$  proves stationarity. The first two outputs are uniform with pair probability  $1/9$ ; the reset rule thereafter proves every finite word by induction. No finite truncation, postselection, extra controller memory or freshly redrawn source is used.  $\square$

This theorem is a sufficient neighbourhood criterion, not a necessary CP test. In particular its simple norm estimate can be conservative. Exact finite Choi certificates admit larger rotations, as in the following point.

## 9.2 An exact nonorthogonal-plane realization

Use the coordinate-plane state formula with

$$\alpha = 2/3, \quad t = 9/50, \quad r = 1/10, \quad \lambda = 9/10, \quad \zeta = -t\omega^2, \quad \omega = e^{2\pi i/3}.$$

The seed  $\chi$  is  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \zeta \\ 0 & \bar{\zeta} & 1-\alpha \end{pmatrix}$  and  $S_{ab} = Z^a X^b \chi X^{-b} Z^{-a}$ . Here  $X|p\rangle = |p+1 \bmod 3\rangle$  and  $Z = \text{diag}(1, \omega, \omega^2)$ . Let

$$U_0 = \begin{pmatrix} 99/101 & -20/101 & 0 \\ 20/101 & 99/101 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_1 = U_2 = I.$$

The nine new states and maps are given by Eq. (47). The exact certificate in the supplement verifies their basis determinant is nonzero, all reset identities, trace preservation, stationarity and the first two outputs. Undoing each output rotation and restricting its Choi matrix to the six nonzero output-plane directions, all six leading principal minors are strictly positive rational numbers. Sylvester's criterion proves CP, without a numerical eigenvalue threshold. The eighteen exact minors are included

verbatim in `supplement/memory/expected_outputs/six/encoding/nonweyl_results.json` and are recomputed by the code.

The original support-plane normals are the orthonormal vectors  $|0\rangle, |1\rangle, |2\rangle$ . The new first normal is  $(99|0\rangle + 20|1\rangle)/101$ , while the other two stay fixed. Its inner product with the second normal is  $20/101 \neq 0$ . A common unitary, antiunitary or relabelling preserves normal orthogonality. This source therefore is not merely another common-unitary coordinate-plane Weyl representation. It is an exact, noncovariant qutrit source outside that class. No assertion of classification outside that class is made.

### 9.3 A common intervention exposes the lost distinction

After a recorded pair ending in symbol zero, admit one projective memory probe with click effect  $P_0 = |0\rangle\langle 0|$  in the same physical qutrit basis for both sources. The coordinate-plane parent gives zero. The new source gives, for every preceding symbol  $a$ ,

$$\text{Tr}(P_0 S'_{a0}) = \frac{2}{3} \left( \frac{20}{101} \right)^2 = \frac{800}{30603} \simeq 0.02614.$$

This is an explicit standard-quantum intervention discriminator between two sources with identical passive words. It is not a test separating Shadow Theory from ordinary quantum mechanics. It tests the admitted physical source models after expanding the readout catalogue.

No history needs to be discarded: retain all first two symbols and the probe outcome, and use the joint event “second symbol zero and click”. Its unconditional probability is zero in the parent and  $800/91809$  in the new source. A controller may schedule the same fixed probe after two ticks; selection of the joint event is data analysis, not feedback memory hidden inside the autonomous source. The intervention requires physical memory access, a clock and a calibrated projector. Those are additional admitted operations; passive archives alone cannot supply them.

The separation has a simple controlled tolerance. If each nominated conditional memory state has trace-distance error at most  $d_S$  and the actual probe effect has operator-norm error at most  $d_P$ , put  $d = d_S + d_P$ . The parent conditional click probability is at most  $d$  and the new one is at least  $800/30603 - d$ . Their admitted intervals are disjoint whenever  $d < 400/30603 \simeq 0.01307$ . This follows from contraction of effect probabilities under trace distance and  $\|\Delta P\|_{\text{op}}$ ; it does not infer those calibration premises from passive word equality. The joint-event gap is then at least  $(800/30603 - 2d)/3$  under the same exact pre-intervention word law. Complete reference/bath equivalence is neither assumed nor inferred from this pointer-level discriminator.

The example has one-step record Choi rank eighteen, rather than the minimum nine. Its pair states have rank two and pair effects are strictly positive: writing  $E_j = \Psi_j^*(I)$ , the certificate checks  $E_j > 0$  and the ordered pair effect obeys

$$F_{jk} = \Psi_j^*(E_k) \succeq \lambda_{\min}(E_k) E_j > 0.$$

The full-word reset factorization therefore gives complete horizon channel rank

$$\sum_w \text{rank } S'_{\text{last}(w)} \text{rank } F_{\text{first}(w)} = 6 \cdot 3^n, \quad n \geq 2.$$

It thus supplies a certified alternative encoding and an access-dependent distinction, not a better resource frontier or a lower noise threshold. The construction is useful because noncovariant search can now start inside a rigorously admitted open set rather than at a numerical PSD factorization with unverified dynamics.

**Verification and limits.** `supplement/memory/six/encoding/verify_nonweyl.py` imports no previous producer code. Exact SymPy arithmetic verifies 243 reset entries, nine normalization entries, ninety stationary/initial entries and eighteen positive principal minors. It then enumerates every word through length seven independently in floating point: 3,279 checks pass, with maximum probability error below  $7 \times 10^{-18}$ . These finite word checks are diagnostics; the basis-reset induction is the full-law proof. The numerical neighbourhood estimate, search failures and output-state geometry establish no global source optimum. Conventional Born probabilities, stationary preparation, pure dilation accounting, independently fresh ports and the complete declared access catalogue remain assumptions. No measurement-selection mechanism is inferred.

## 10 Access-dependent distinctions at a common resource optimum

Two parameters  $h_1 \neq h_2$  on opposite sides of  $h_* = (1 + \sqrt{17})/4$  can have identical  $r(h)$  and thus identical full passive laws, dimension three and one-step rank nine. Their common-input pointer channels nevertheless obey  $\frac{1}{2}\|\mathcal{M}_{h_1} - \mathcal{M}_{h_2}\|_\diamond = |h_1 - h_2|/3$ . The effects are  $E_j(h) = (2 - h)I/6 + hP_j/2$  for fixed orthogonal Fourier projectors. A known Fourier input attains the distance; so does the common maximally entangled purifier of stationary local state  $I/3$ . Here  $\mathcal{M}_h(X) = \sum_j \text{Tr}[E_j(h)X]|j\rangle\langle j|$  retains only the pointer, not the final memory. To see the distance directly, write  $\mathcal{M}_h = (h/2)\mathcal{M}_P + (1 - h/2)\mathcal{D}$ , where  $\mathcal{D}(X) = \text{Tr}(X)I_A/3$ . For any positive joint input, let  $q_j \succeq 0$  be the subnormalized reference blocks after the projective Fourier measurement, and let  $q = \sum_j q_j$ . The triangle inequality gives  $\sum_j \|q_j - q/3\|_1 \leq 4/3$ . A Fourier eigenstate attains equality, so  $\frac{1}{2}\|\mathcal{M}_P - \mathcal{D}\|_\diamond = 2/3$  and the displayed result follows by scaling. On the common maximally entangled input the pointer/reference blocks are  $E_j(h)^T/3$ ; their difference has the same trace norm, proving the second attainment claim. The technical companion gives an exact algebraic parameter pair and the operational decoder. Physical readability of the purifier is additional access, and passive dimension optimality does not identify a unique generator.

## 11 Reproducibility and claim status

The supplement supplies the exact rational/algebraic source definitions and certificates. Positivity for the mixed-support source is checked in  $\mathbb{Q}(\sqrt{6}, \omega)$  by Hermiticity, supported principal minors and exact basis inversion; all-word attainment follows by induction from the reset identity. The Weyl family is proved analytically and independently reconstructed symbolically. Integer enumeration checks the finite support-count argument. Floating word enumeration diagnoses implementation mistakes but is not substituted for universal proofs. The intermediate minimum-memory boundary and the exact complete-history optimum remain unresolved. Specifying a stationary CP instrument does not establish a laboratory interaction, clock, preparation procedure or energy advantage. The technical companion preserves additional bounds and alternative realizations under their separate output contracts.

**Assistance disclosure.** AI assistance was used for computational checks and review.

## A A universal one-step qutrit dilation bound

The branch-image dimension forces a lower bound on Choi rank. The following elementary proof treats every complex qutrit CP map with the stated linear rank, without a covariance or source-encoding assumption.

## A.1 A CP map of qutrit linear rank three needs at least three Kraus operators

**Lemma A.1** (Kraus rank from three image directions). *If a nonzero CP map  $\Phi : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$  has complex-linear rank three, its Choi rank is at least three.*

*Proof.* A one-Kraus map  $X \mapsto AXA^\dagger$  has linear rank  $(\text{rank } A)^2$ , which cannot equal three. Suppose two Kraus matrices  $A, B$  suffice.

First consider input support of dimension at most two. Dimension one gives rank at most one. On a two-dimensional support, an invertible input congruence normalizes  $A^\dagger A + B^\dagger B$  to identity without changing linear rank. If the resulting trace-preserving map had rank three, its Hermitian kernel would contain a nonzero traceless  $X$ . Diagonalizing  $X$  gives orthonormal  $u, v$  with equal outputs  $\Phi(|u\rangle\langle u|) = \Phi(|v\rangle\langle v|) = \rho$ . This common output has rank at most two. Positivity places all four Kraus vectors  $Au, Bu, Av, Bv$  in its support, reducing the output to dimension two. If  $\rho$  has rank one the image is one-dimensional. Otherwise write the two output-vector matrices as  $M_0 = \sqrt{\rho}U_0$  and  $M_1 = \sqrt{\rho}U_1$ , where  $U_0, U_1$  are 2-by-2 unitaries. The off-diagonal response is  $C = \sqrt{\rho}U\sqrt{\rho}$ ,  $U = U_0U_1^\dagger$ , and trace preservation gives  $\text{Tr}(\rho U) = 0$ . A positive weighted average of the two unit-modulus eigenvalues of  $U$  can vanish only if they are antipodal. Hence  $U$  is Hermitian up to a common phase, and  $C^\dagger$  is proportional to  $C$ . The entire image then lies in  $\text{span}\{\rho, C\}$ , contradicting rank three. The same argument applied to the adjoint handles output support of dimension at most two.

It remains to treat full input and output support. If the pencil  $A + tB$  contains an invertible matrix, a unitary change of Kraus basis makes the first Kraus matrix invertible. An invertible output congruence then reduces the map to  $X \mapsto X + CXC^\dagger$ . Its superoperator is  $I + C \otimes \bar{C}$ . Among three eigenvalues  $c_1, c_2, c_3$  of  $C$ , no diagonal product  $|c_i|^2$  equals  $-1$ , and at most two unordered pairs can satisfy  $c_i \bar{c}_j = -1$ : all three pairs would force one product to be positive. The algebraic multiplicity of  $-1$  is therefore at most four, also for defective  $C$ . The superoperator has rank at least five.

Finally suppose every pencil matrix is singular. Some linear combination has rank two; otherwise two rank-one matrices either have common image or common kernel, contradicting full supports. For invertible  $L, R$ , replacing both Kraus matrices by  $LAR, LBR$  composes the map with invertible input/output congruences. This preserves CP and complex-linear rank, although it need not preserve trace normalization. Such transformations put

$$A = \text{diag}(1, 1, 0), \quad B = \begin{pmatrix} D & c \\ q & b \end{pmatrix}.$$

The coefficients of  $t, t^2$  in  $\det(A + tB) \equiv 0$  give  $b = 0$  and  $qc = 0$ . Full supports give  $q, c \neq 0$ . A matched invertible two-dimensional change of basis preserving  $A$  puts  $c = e_1, q = e_2^T$ , so

$$B = \begin{pmatrix} a & b' & 1 \\ s & d & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The cubic coefficient gives  $s = \det B = 0$ . Subtracting  $d$  times row three from row two, then  $a$  and  $b'$  times column three from columns one and two, preserves  $A$  and gives  $B = E_{13} + E_{32}$ . Its CP map is

$$X \mapsto \begin{pmatrix} x_{11} + x_{33} & x_{12} & x_{32} \\ x_{21} & x_{22} & 0 \\ x_{23} & 0 & x_{22} \end{pmatrix},$$

of linear rank six. Every case contradicts rank three.  $\square$

The qubit-support reduction also follows from the standard two-Kraus qubit canonical form of Wolf and Perez-Garcia [16], Eq.(5); the proof above supplies the elementary argument actually

used. The displayed singular-pencil map has six independent image directions by direct inspection. The supplement also supplies an independent rational row-reduction check of this normal form in `supplement/check_structural_lemmas.py`.

**Universal one-step bath consequence.** By (30), every branch of any minimal qutrit realization needs at least three Kraus operators. Its classical-record channel has Choi rank at least  $3 + 3 + 3 = 9$ . This is a universal lower bound for the declared qutrit realization class. The exact normal measurement-model apparatus rule is an established antecedent: Pellonpää and Tukiainen, *Rep. Math. Phys.* 79, 261–278 (2017), Eq.(22), Corollary 1 and the finite-dimensional apparatus bound, DOI [10.1016/S0034-4877\(17\)30040-X](https://doi.org/10.1016/S0034-4877(17)30040-X), <https://arxiv.org/abs/1509.08886>. Pure initial apparatus and a sharp pointer are retained hypotheses.

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