

Robust Phase-to-Record Conversion with Retained Loss Outputs

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Abstract

We specify a two-mode phase source, bounded coherent reader and terminal stable detector flags in one complete instrument, retaining all null/loss outputs and arbitrary inert references. A solved finite-lifetime control problem gives the sharp unambiguous-success frontier. A distinct bounded-jump-hazard problem has a different attaining instrument and finite collision realization. We derive an analytic minimax conditional-error/conclusive-rate frontier for amplitude, phase and visibility uncertainty, and realize it with the same bounded coupling. A calibrated entrance filter separates the reader ratio from common uncertain conversion; a complete controlled-path overlap certificate then gives a sharp imperfect-matching tolerance and a pointer/reference distance. The results concern declared apparatus and output-access classes. They neither derive Born statistics nor establish a biological or consciousness mechanism.

1 Task, antecedents and operational scope

This paper develops phase-sensitive source-readout constructions within Shadow Theory's pre-existing programme [9]. A source distinction may be invisible in separate mode intensities yet accessible to a phase-sensitive interaction. To turn that observation into a physical statement one must specify the source stock, interaction, committed records, null/loss outputs and admitted decoder. We study the common-sign phase distinction in two degenerate one-excitation modes. The preparation controller is unread by the reader, failed trials are not replaced by freshly sampled sources, and all complete instruments act with identity on an arbitrary inert reference. Unambiguous discrimination and optical filtering are established techniques. Uzdin–Gat's norm-action bound supplies a close unitary-resource antecedent; Chefles–Barnett's state-separation and symmetric-state discrimination results supply the underlying pure-state geometry. The added results here solve the specified independently lossy stable-flag control problem, a source-uncertainty minimax and a restricted common-conversion/imperfect-matching filter problem. These antecedents are credited explicitly [1–6, 10]. No worldwide priority claim is made. Ordinary quantum probabilities and the prescribed preparation/control laws remain premises.

Trial, outputs and notation. One trial here uses one original source preparation and retains its conclusive, null and loss branches. A conclusive-rate guarantee Q and conditional error ε therefore describe a decision problem with inconclusive outcomes. They are not a fixed-copy deadline task in which every record must receive a sign. In particular, the preparation visibility v and transfer ceiling η below have different physical meanings from the detector visibility and erasure efficiency of a repeated equatorial-reader model. Identifying them requires a derived common instrument, not equality of their symbols. A coherent direct-sum output, a classically recorded instrument and its pointer-only marginal are also different channels. We state which is used in each norm or resource claim; none of these choices gives the decoder access to an otherwise unread retained bath or reference.

The Hamiltonians, uncertainty sets and control theorems below specify the apparatus and access conditions for the stated tasks. Specifying and optimizing one reader does not establish a unique underlying source, and retaining a loss bath does not automatically grant permission to measure it. Reduced detector equivalence and complete retained-output comparison are separate claims, treated explicitly in the technical companion.

Organization. Section 2 treats bounded coherent conversion and the distinct bounded-hazard model. Section 3 develops preparation uncertainty and fixed upstream attenuation. The remaining sections separate a calibrated entrance filter from common conversion, then quantify departures from exact matching. The companion contains absorbing-reader banks, detector comparison, finite references and calibration/monitoring interfaces under their own stated contracts.

2 Bounded coherent conversion and complete detector records

2.1 Source and declared apparatus

Use two degenerate source modes, with a single excitation spanning $\mathcal{R} = \text{span}\{|0\rangle, |1\rangle\}$. The two possible source states are

$$|\psi_{\pm}\rangle = a|0\rangle \pm b|1\rangle, \quad a = \sqrt{(1+s)/2}, \quad b = \sqrt{(1-s)/2}, \quad 0 < s < 1. \quad (1)$$

They have the same separate mode intensities and overlap s . Their relative phase differs by π . Using normalized phase-sensitive templates, informed by [7, pp. 18–21, Eqs. (58)–(70)] and [8, Section 32, pp. 42–43], define the reciprocal reader states

$$|\phi_{\pm}\rangle = b|0\rangle \pm a|1\rangle, \quad \langle\phi_{\pm}|\psi_{\mp}\rangle = 0, \quad |\langle\phi_{\pm}|\psi_{\pm}\rangle|^2 = 1 - s^2.$$

The readout has two orthogonal, stable flag modes $\mathcal{F}$ labelled $+$ and $-$. There are also orthogonal lost-output modes, initially vacant. Every mode is included in the one-excitation description; no new actual sample is loaded after a null result. Source/reference inputs can be any $\rho_{\mathcal{R}R}$ with arbitrary finite inert reference R . The reader and vacant output modes have one common input-independent preparation. The loss bath starts in its vacuum, independently of that original preparation; the scalar Markov loss law and absence of re-excitation are physical model assumptions of the declared apparatus. The claim of unambiguity is for the designated pair (1); the instrument itself is defined for every reference-extended input.

The apparatus Hamiltonian on the ready-plus-flag subspace is

$$H(t) = \hbar \begin{pmatrix} A(t) & C(t)^* \\ C(t) & D(t) \end{pmatrix}, \quad A = A^*, \quad D = D^*, \quad \|C(t)\| \leq G. \quad (2)$$

A, C, D are piecewise continuous on the finite control interval; the stated bound holds almost everywhere. This regularity ensures that the propagator below exists. $G > 0$, A, C, D have units s^{-1} and H has units joules. Uniform irreversible loss acts only on the ready subspace at rate $\ell \geq 0$ in s^{-1} . Its complete amplitude map is $L_{\text{loss}} = \sqrt{\ell} W P_{\mathcal{R}}$, where W is an isometry to the lost-output space. It isometrically transports the instantaneous ready component, including its conditional reference assemblage. Earlier filtering and conditioning may have changed the original source/reference correlations; no preservation of those original correlations is asserted. The outgoing state is not replaced by a vacuum or a classical label. Once lost, the excitation never returns. There are no other protected storage subspaces, fresh inputs, gain channels or readers of the lost field in the allowed conclusive-record catalogue. Block Hamiltonians are allowed in the upper bound; they cannot be used to add an uncounted protected

ancilla. The optical/cavity coupling and its timing are prescribed controls; this is not an autonomous clock construction.

The no-loss propagator on the four-dimensional ready-plus-flag space is

$$\dot{U}_{\text{eff}}(t) = [-iH(t)/\hbar - \ell P_{\mathcal{R}}/2] U_{\text{eff}}(t).$$

Let $I_{\mathcal{R}}$ embed the two-dimensional source. A complete terminal instrument consists of

$$\begin{aligned} K_{\pm} &= P_{\pm} U_{\text{eff}}(T) I_{\mathcal{R}}, \\ K_0 &= P_{\mathcal{R}} U_{\text{eff}}(T) I_{\mathcal{R}}, \quad K_{\text{loss}}(t) = \sqrt{\ell} W_t P_{\mathcal{R}} U_{\text{eff}}(t) I_{\mathcal{R}}. \end{aligned} \quad (3)$$

Here W_t retains the outgoing quantum mode. The coherent temporal output is an L^2 amplitude sector; a registered loss time is instead a classical record, with the output-space convention specified below. The identity

$$K_0^* K_0 + K_+^* K_+ + K_-^* K_- + \int_0^T K_{\text{loss}}(t)^* K_{\text{loss}}(t) dt = I \quad (4)$$

follows by differentiating $U_{\text{eff}}^* U_{\text{eff}}$. It makes the sum of the instrument branches completely positive and trace preserving on every $\rho_{\mathcal{R}R}$ with identity on R . Before readout, the direct-sum amplitude map is an isometry retaining coherent quantum output sectors. A classically recorded instrument is obtained by the specified dephasing or measurement of those sectors (and time bins, when used). For an exact-time classical record the output is an integrable trace-class-valued kernel on the null/flag atoms and $[0, T]$, with norm $\sum_c \|\omega_c\|_1 + \int_0^T \|\omega(t)\|_1 dt$. It is not a trace-class diagonal density operator on the coherent temporal L^2 space. Finite time bins give ordinary cq direct sums. These channels share the displayed effects but are not identical channels; their dilation and storage costs must be counted separately. This uses the stipulated ordinary quantum probability law; it does not derive that law.

Only the terminal flags are conclusive. They are committed after the active coupling is switched off. Before that time, flag amplitudes can coherently return to the source, so no promise of an earlier immutable event is made. After decoupling, later controls act on other systems and leave all flag and loss modes untouched; the final records then persist. Reading or dephasing the flags during the active pulse is a different physical model.

2.2 Sharp finite-lifetime success frontier

For a two-dimensional input, error-free conclusive effects necessarily have

$$E_{\pm} = K_{\pm}^* K_{\pm} = \frac{q_{\pm}}{1-s^2} |\phi_{\pm}\rangle \langle \phi_{\pm}|, \quad \bar{q} = (q_+ + q_-)/2. \quad (5)$$

This follows because a positive effect with zero expectation in $|\psi_{\mp}\rangle$ annihilates that vector. In particular,

$$\langle 1 | (E_+ + E_-) | 1 \rangle = \frac{\bar{q}}{1-s}. \quad (6)$$

A bound on coherent transfer of the input $|1\rangle$ therefore bounds the average success of every allowed unambiguous reader, including asymmetric ones.

Define t_* by

$$t_* = \begin{cases} \frac{\arctan(4\Omega/\ell)}{\Omega}, & 0 < \ell < 4G, \quad \Omega = \sqrt{G^2 - \ell^2/16}, \\ 1/G, & \ell = 4G, \\ \frac{\text{artanh}(4\chi/\ell)}{\chi}, & \ell > 4G, \quad \chi = \sqrt{\ell^2/16 - G^2}, \\ \pi/(2G), & \ell = 0. \end{cases} \quad (7)$$

Let $\tau = \min(T, t_*)$ and

$$f_G(t) = \begin{cases} (G/\Omega)e^{-\ell t/4} \sin(\Omega t), & \ell < 4G, \\ Gte^{-Gt}, & \ell = 4G, \\ (G/\chi)e^{-\ell t/4} \sinh(\chi t), & \ell > 4G. \end{cases} \quad (8)$$

The $\ell = 0$ first case has $\Omega = G$.

Proposition 2.1 (Optimal phase-to-flag conversion with an independent loss sink). *Under the declared apparatus and stable-flag readout catalogue,*

$$\boxed{\bar{q} \leq (1-s)f_G(\tau)^2.} \quad (9)$$

The bound is attained with $q_+ = q_-$. For a deadline at least t_* it is

$$q_{\max} = (1-s)e^{-\ell t_*/2}. \quad (10)$$

It remains valid when arbitrary inert references and all original loss/null outputs are retained. It is not an optimum over new measurements of those retained outputs or of the reference.

Proof. First consider any normalized source input and its no-loss vector, split into ready and flag components with norms x and y . Write $x = r \cos \theta$, $y = r \sin \theta$, $0 \leq \theta \leq \pi/2$. Loss gives $\dot{r}/r = -\ell \cos^2 \theta/2$. Cauchy–Schwarz applied to the cross Hamiltonian gives an effective real transfer control g_{eff} with $|g_{\text{eff}}| \leq G$ and

$$\dot{\theta} = g_{\text{eff}} + (\ell/2) \sin \theta \cos \theta.$$

Block Hamiltonians cancel from these norm equations. At zero components the same conclusions follow by continuous limits.

Define

$$F(\theta) = \int_0^\theta \frac{\ell \cos^2 u}{G + (\ell/2) \sin u \cos u} du.$$

For every trajectory, including controls that reverse direction,

$$\frac{d}{dt} [\log r^2 + F(\theta)] = \ell \cos^2 \theta \left[\frac{g_{\text{eff}} + (\ell/2) \sin \theta \cos \theta}{G + (\ell/2) \sin \theta \cos \theta} - 1 \right] \leq 0.$$

Thus $y^2 \leq e^{-F(\theta)} \sin^2 \theta$. This upper envelope increases on $0 < \theta < \pi/2$, since its logarithmic derivative is

$$2 \cot \theta - \frac{\ell \cos^2 \theta}{G + (\ell/2) \sin \theta \cos \theta} = \frac{2G \cot \theta}{G + (\ell/2) \sin \theta \cos \theta} > 0.$$

Scalar comparison also bounds $\theta(t)$ by the trajectory with $g_{\text{eff}} = G$ until that trajectory first reaches $\pi/2$. Hence constant maximal coupling, stopped at that first time, maximizes flag transfer at every deadline. Its two scalar amplitudes are

$$u_G(t) = e^{-\ell t/4} [\cos(\Omega t) - \ell \sin(\Omega t)/(4\Omega)], \quad v_G(t) = -i f_G(t),$$

with their continuous critical and hyperbolic forms. Its first ready zero is (7); at that time $f_G(t_*)^2 = e^{-\ell t_*/2}$. This proves the transfer bound $y(T)^2 \leq f_G(\tau)^2$. Apply it to $|1\rangle$ in (6) to obtain (9).

For attainability let $\mathbf{H}$ be the real balanced two-mode matrix $2^{-1/2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Set $A = D = 0$ and, during the active time τ ,

$$C = \mathbf{H} \text{diag}(g_0, G), \quad f_{g_0}(\tau) = \sqrt{\frac{1-s}{1+s}} f_G(\tau). \quad (11)$$

For $g = 0$ the transfer amplitude is zero and for $g = G$ it is $f_G(\tau)$. Continuity in g therefore guarantees a solution $0 \leq g_0 \leq G$ to this intermediate target. No monotonicity assumption is needed. These are two independent damped taps in the balanced flag basis. Their terminal flag map is

$$-i\mathbf{H} \text{diag}(f_{g_0}(\tau), f_G(\tau)).$$

Equation (11) makes each wrong flag amplitude exactly zero for its excluded source state, and each correct probability equals $(1-s)f_G(\tau)^2$. Switch off the coupling after τ ; the stable flags then remain unchanged until T . Residual ready amplitudes may subsequently be lost, and are included in (3). That proves attainability and preserves (4) on all inputs and their references. $\square$

For $s = 0.6$, $G = \ell$ and $T = 2/G$, one has $t_* = 1.3613444250/G$, $g_0 = 0.3764245988G$, and $q_+ = q_- = 0.2025106210$. The loss-limited transfer factor is 0.5062765525. Requiring success at least 0.2 with that overlap and a sufficiently long deadline gives the sharp threshold $\ell/G \leq 1.0211025506$ in this apparatus class. Above it no permitted pulse sequence achieves the target. At the critical damping point $\ell = 4G$, the transfer factor is e^{-2} ; for large ℓ/G it is asymptotic to $4G^2/\ell^2$. For $\ell = 0$ the frontier reduces to ideal unambiguous discrimination after a finite coherent pulse. These results are parameter requirements for a possible experiment, not measured biological lifetimes or coupling strengths.

2.3 Bounded coherent action and minimum output ranks

The loss-free implementation can be specified without a continuum bath. For any desired $0 \leq q \leq 1-s$, set

$$n_0 = \sqrt{1 - q/(1+s)}, \quad n_1 = \sqrt{1 - q/(1-s)}, \quad \theta_j = \arccos n_j.$$

A constant four-mode Hamiltonian (2) with $C = \mathbf{H} \text{diag}(\theta_0, \theta_1)/\tau$ gives exactly

$$K_0 = \text{diag}(n_0, n_1), \quad K_{\pm} = -i\sqrt{\frac{q}{1-s^2}} |\pm\rangle\langle\phi_{\pm}|. \quad (12)$$

This is a single finite unitary interaction with blank detector modes, not a concatenation of statistical bounds. Excitation-preserving couplings between a source mode and a detector mode are ordinary beam-splitter or resonant Jaynes–Cummings interactions in the one-excitation sector. Their matrix entries have units $\hbar\theta_j/\tau$. After the pulse, decoupled detector excitations are the final stable records. No independent samples or erased failure populations are assumed.

If the resource is instead the cross-subspace action $\mathcal{A} = \int \|P_{\mathcal{F}} H P_{\mathcal{R}}\| dt/\hbar$, the same norm equations at $\ell = 0$ give

$$\bar{q} \leq (1-s) \sin^2(\min\{\mathcal{A}, \pi/2\}).$$

The construction saturates it, with $\mathcal{A} = \arcsin \sqrt{q/(1-s)}$. This is a specialization of known unitary measurement resource bounds, included to make the physical resource contract explicit. Bounded Hamiltonian coupling is not the same constraint as a bounded Markov jump hazard; neither bound may be substituted into the other.

For $0 < q < 1-s$ the three terminal POVM effects have ranks 1, 1, 2 for $+, -, 0$. Any orthogonal-record amplitude dilation of these exact effects requires at least $1 + 1 + 2 = 4$ output dimensions: the amplitude map in each orthogonal record sector has rank equal to its effect. Equation (12) attains this count. At $q = 1-s$ the null effect has rank one and the abstract minimum is three. The four-mode Hamiltonian is still a sufficient implementation; we do not claim it is a minimum-level Hamiltonian at that endpoint. Original input references tensor onto this count; they are not newly consumed reader ancillas. These are reachable orthogonal amplitude-sector dimensions. They do not include an extra pure environment needed to realize a fully dephased classical record channel; classicalizing a coherent flag requires its own retained-output accounting.

2.4 Why a static dark matched reader fails, and a continuous repair

Consider a static absorbing implementation of the matched-template prescription [7, pp. 18–21], with fixed jump readers dark to the other source: $L_{\pm} = \sqrt{\gamma} |\pm\rangle \langle \phi_{\pm}|$. Initially the excluded click amplitude is zero. However the full hazard is $Q = \gamma \text{diag}(1-s, 1+s)$, so its null evolution changes the source. For $|\psi_{-}\rangle$, the wrong + density at time t is

$$p_{+|-}(t) = \gamma a^2 b^2 [e^{-\gamma(1-s)t/2} - e^{-\gamma(1+s)t/2}]^2. \quad (13)$$

It is positive for $s, t > 0$. Its integrated infinite-time value is $s^2/2$; at $s = 0.6$, $\gamma T = 2$, it is 0.0394994846. This is a tested obstruction to calling a fixed dark instantaneous kernel an error-free continuous instrument.

A faithful repair keeps the reciprocal geometry and changes the reader to account for its own null back-action. Write

$$q(t) = (1-s)(1 - e^{-\kappa t}), \quad N(t) = \text{diag} \left(\sqrt{1 - q(t)/(1+s)}, \sqrt{1 - q(t)/(1-s)} \right),$$

where $\kappa > 0$ is a declared maximum signal hazard. Use

$$L_{\pm}(t) = \sqrt{\frac{q'(t)}{1-s^2}} |\pm\rangle \langle \phi_{\pm}| N(t)^{-1}. \quad (14)$$

At finite times these coefficients are bounded and known in advance. The Hamiltonian is zero; the reader does not acquire an actual sign to decide its next setting. Direct calculation gives

$$\sum_{\pm} L_{\pm}^* L_{\pm} = \text{diag} \left(\frac{q'}{1+s-q}, \frac{q'}{1-s-q} \right) \leq \kappa I, \quad L_{\pm}(t) N(t) = \sqrt{\frac{q'(t)}{1-s^2}} |\pm\rangle \langle \phi_{\pm}|.$$

Thus every conclusive time is unambiguous. A null result filters the surviving quantum source, and the predetermined reader compensates exactly for that filter. No fresh independence is inferred.

This bounded-hazard construction is also optimal in its own model class. For any absorbing two-port unambiguous instrument with $Q(t) \leq \kappa I$, the no-transition matrix satisfies $N(t)^* N(t) \geq e^{-\kappa t} I$, since $\frac{d}{dt}(N^* N) = -N^* Q N \geq -\kappa N^* N$. With average cumulative success $\bar{q}(t)$, (6) gives

$$1 - \bar{q}(t)/(1-s) = \langle 1 | N(t)^* N(t) | 1 \rangle \geq e^{-\kappa t}.$$

Therefore $\bar{q}(t) \leq q(t)$, attained above, including when the two successes are allowed to be asymmetric.

Add independent uniform ready loss at rate ℓ . The no-event matrix becomes $e^{-\ell t/2} N(t)$, the signal Kraus density is $e^{-\ell t/2} L_{\pm}(t) N(t)$ and the loss density is $\sqrt{\ell} e^{-\ell t/2} W_t N(t)$. The three full effects plus the null effect sum to identity, as in (4). The optimal conclusive probability is

$$q_{\text{hazard,loss}}(T) = (1-s) \frac{\kappa}{\kappa + \ell} [1 - e^{-(\kappa + \ell)T}]. \quad (15)$$

To see optimality rather than only attainability, consider any competing schedule without loss and its cumulative success $q_c(t) \leq q(t)$. Uniform loss multiplies its conclusive time density by $e^{-\ell t}$. Integration by parts gives

$$\int_0^T e^{-\ell t} dq_c(t) = e^{-\ell T} q_c(T) + \ell \int_0^T e^{-\ell t} q_c(t) dt \leq \int_0^T e^{-\ell t} dq(t),$$

which is (15). This argument requires the same independent scalar loss law and the same stable conclusive-output catalogue; it does not combine unrelated noise models. At $s = 0.6$, $\ell = 0.25\kappa$ and $\kappa T = 2$, the conclusive, null and loss probabilities are respectively 0.2937328004, 0.3967523953 and 0.3095148043.

2.5 A finite collision implementation and retained retry records

The continuous model has infinitely many temporal modes. A finite version need not pretend to store such a continuum in a finite register. Choose M stages, $t_k = k\Delta$, $\Delta = T/M$, $q_k = q(t_k)$ and $N_k = \text{diag } \sqrt{1 - q_k/(1 \pm s)}$. At stage k , first attenuate both ready modes by $e^{-\ell\Delta/2}$ into two fresh, retained loss modes. Then use a two-mode tap with local null matrix $N_k N_{k-1}^{-1}$ and two fresh conclusive modes. Its tap angles are

$$\vartheta_{kj} = \arccos \frac{(N_k)_{jj}}{(N_{k-1})_{jj}}.$$

The same finite Hamiltonian construction as (12) implements these taps and their balanced output mixing. Excitations already in old output modes never interact again. Consequently the cumulative null map is exactly $e^{-\ell t_k/2} N_k$, not a redraw. The conclusive Kraus maps in bin k are

$$K_{\pm,k} = e^{-\ell t_k/2} \sqrt{\frac{q_k - q_{k-1}}{1 - s^2}} |\pm, k\rangle \langle \phi_{\pm}|,$$

and the loss map is

$$K_{\text{loss},k} = \sqrt{1 - e^{-\ell\Delta}} e^{-\ell t_{k-1}/2} W_k N_{k-1}.$$

Every original mode, source remainder and detector register is retained. These finite Kraus maps satisfy completeness exactly, for arbitrary inputs and inert references. They have success

$$q_M = \sum_{k=1}^M e^{-\ell t_k} (q_k - q_{k-1}), \quad e^{-\ell\Delta} q_{\text{hazard,loss}} \leq q_M \leq q_{\text{hazard,loss}}. \quad (16)$$

The bounds follow by bounding $e^{-\ell t}$ on each interval in the exact integral. This is a probability convergence guarantee; no equality of the unmeasured continuum bath and the finite bath is asserted. For the numerical example above $M = 8$ gives $q_M = 0.2842794177$, and $M = 128$ gives 0.2931579860.

For positive loss and finite nonzero stages, each of the $2M$ conclusive effects has rank one, each of the M loss effects rank two, and the null effect rank two. An orthogonal time-bin record dilation thus needs at least $4M + 2$ reachable output dimensions; the one-excitation mode construction has exactly that count. With zero loss it is $2M + 2$. This counts reachable orthogonal amplitude sectors for the specified record isometry, not all physical hardware, an autonomous controller, or an additional environment implementing a fully dephased classical record channel. Arrival-time labels are histories of different possible single absorptions, not independent trials on one reused source.

3 Preparation uncertainty and a sharp conditional-error frontier

3.1 Uncertainty and the unchanged access contract

Use two degenerate one-excitation source modes $|0\rangle, |1\rangle$. The unknown label is $h \in \{+1, -1\}$. Its source marginal belongs to

$$\rho_h(p, \delta, v) = \begin{pmatrix} p & hv\sqrt{p(1-p)}e^{-i\delta} \\ hv\sqrt{p(1-p)}e^{i\delta} & 1-p \end{pmatrix}, \quad p \in [p_-, p_+], \quad |\delta| \leq \Delta, \quad v \in [v_-, 1], \quad (17)$$

where $0 < p_- \leq p_+ < 1$, $0 \leq v_- \leq 1$ and $0 \leq \Delta \leq \pi/2$. Both labels have the same mode intensities. The width in p is source-amplitude uncertainty; v is source visibility, which may arise from correlations with a retained inaccessible environment. An unknown reader phase offset can be included in Δ by adding its bounded offset to the source phase interval, provided the resulting halfwidth is at most

$\pi/2$. Separate finite calibration procedures can supply these bounds, but they are assumptions of this theorem. No future sample or statistical independence is inferred from a calibration trace.

The physical apparatus is exactly the coherent conversion model's ready subspace plus two stable flags. Its Hamiltonian is

$$H(t) = \hbar \begin{pmatrix} A(t) & C(t)^* \\ C(t) & D(t) \end{pmatrix}, \quad \|C(t)\| \leq G,$$

with Hermitian block controls. Uniform ready-subspace loss at rate ℓ transfers the source state isometrically into orthogonal retained output modes. The bath starts in its vacuum, all preparations are independent of the unknown input, and there is no re-excitation or additional protected storage. G and ℓ have units s^{-1} ; the deadline T is in seconds. Source plus arbitrary inert reference R may be in any joint state with marginal (17). The reference is retained and unread. The lost bath, surviving source and null flag are also retained but are inconclusive to this fixed stable-flag readout catalogue. A new coherent bath reader or a procedure reporting guesses on a null outcome changes the allowed decision problem. The two physical stable flags, observed only after the active coupling is off, are the only conclusive outcomes here.

Let $\eta \in (0, 1]$ be the coherent conversion model's optimal transfer ceiling:

$$\eta = f_G(\tau)^2, \quad \tau = \min(T, t_*). \quad (18)$$

For $\ell < 4G$,

$$f_G(t) = \frac{G}{\Omega} e^{-\ell t/4} \sin(\Omega t), \quad \Omega = \sqrt{G^2 - \ell^2/16}, \quad t_* = \frac{\arctan(4\Omega/\ell)}{\Omega}.$$

At $\ell = 0$ take $t_* = \pi/(2G)$. At $\ell = 4G$, use $f_G(t) = Gte^{-Gt}$, $t_* = 1/G$. For $\ell > 4G$, replace sine by hyperbolic sine and Ω by $\chi = \sqrt{\ell^2/16 - G^2}$, with $t_* = \text{artanh}(4\chi/\ell)/\chi$. The previously proved amplitude barrier implies, for every allowed control,

$$E_+ + E_- \leq \eta I, \quad (19)$$

because each input's flag probability is at most η . For completeness, its proof can be stated briefly. Split any no-loss vector into ready/flag norms $r \cos \theta, r \sin \theta$. Then $\dot{r}/r = -\ell \cos^2 \theta/2$ and $\dot{\theta} = g_{\text{eff}} + (\ell/2) \sin \theta \cos \theta$ with $|g_{\text{eff}}| \leq G$. For

$$F(\theta) = \int_0^\theta \frac{\ell \cos^2 u}{G + (\ell/2) \sin u \cos u} du,$$

$\frac{d}{dt} [\log r^2 + F(\theta)] \leq 0$. The envelope $e^{-F(\theta)} \sin^2 \theta$ is increasing up to $\pi/2$, since its logarithmic derivative is $2G \cot \theta / [G + (\ell/2) \sin \theta \cos \theta]$. Scalar comparison bounds θ by the constant-maximal-coupling path. That path first has zero ready amplitude at t_* and flag amplitude $f_G(t)$; switching it off after τ attains the bound. Hence every input has flag probability at most η , proving (19). This is the coherent conversion result's proof, repeated as a dependency, included for completeness. The original loss, deadline and access hypotheses are retained. No unrelated noise bounds are combined with it. The ceiling is an effect inequality; it does not assert that all retained-output channels with those effects are equal.

For a candidate reader define its conclusive probability and wrong-label probability at each admitted state by

$$q_h = \text{Tr}[(E_+ + E_-)\rho_h], \quad w_h = \text{Tr}[E_- \rho_h].$$

The objective is to maximize a guaranteed Q such that, for every h, p, δ, v ,

$$q_h \geq Q, \quad w_h \leq \varepsilon q_h. \quad (20)$$

Thus ε bounds the error *conditioned on a conclusive record*; Q explicitly pays for that selection. No label prior is needed. The error condition is linear in the effects even on a zero-probability branch, where it is simply $0 \leq 0$.

3.2 Exact minimax solution

Put

$$m = v_- \cos \Delta, \quad \xi_{\pm} = \sqrt{p_{\pm}/(1-p_{\pm})}, \quad w = \frac{1}{2} \log(\xi_+/\xi_-), \quad \varepsilon_{\min} = \frac{1 - m \operatorname{sech} w}{2}. \quad (21)$$

All these quantities are dimensionless. The logarithm is a log-amplitude-odds coordinate, not a new physical geometry.

Proposition 3.1 (Sharp robust conditional-error/conclusive frontier). *If $\varepsilon < \varepsilon_{\min}$, no positive guaranteed conclusive probability satisfies (20). The zero-conclusive reader is still allowed. For $m > 0$ and $\varepsilon_{\min} \leq \varepsilon < 1/2$, define*

$$h_{\varepsilon} = \operatorname{arcosh} \frac{m}{1-2\varepsilon}, \quad L = -h_{\varepsilon} - \log \xi_-, \quad U = h_{\varepsilon} - \log \xi_+. \quad (22)$$

The interval $[L, U]$ is nonempty. Let z_* be its point nearest zero and put $r = e^{z_*}$. Then the exact maximum is

$$Q_{\max} = \eta \frac{\min_{p \in \{p_-, p_+\}} [pr^2 + 1 - p]}{\max(1, r^2)}. \quad (23)$$

It is attained by the effects

$$x = \frac{\eta r^2}{\max(1, r^2)}, \quad y = \frac{\eta}{\max(1, r^2)}, \quad E_{\pm} = \frac{1}{2} \begin{pmatrix} x & \pm \sqrt{xy} \\ \pm \sqrt{xy} & y \end{pmatrix}. \quad (24)$$

For $\varepsilon \geq 1/2$, $x = y = \eta$ attains $Q_{\max} = \eta$. If $m = 0$, positive conclusive probability requires $\varepsilon \geq 1/2$. These are optima within the declared stable-flag catalogue, not over readings of every retained output.

Proof. First enlarge the physical set to all positive effects satisfying (19). This can only raise the optimum. The source set is invariant under $Z = \operatorname{diag}(1, -1)$ with label exchange and under complex conjugation with $\delta \mapsto -\delta$. Averaging the candidate effects with these two transformations preserves all linear constraints (20). Thus an optimum in this enlarged set may be taken as

$$E_{\pm} = \frac{1}{2} \begin{pmatrix} x & \pm z \\ \pm z & y \end{pmatrix}, \quad 0 \leq x, y \leq \eta, \quad z^2 \leq xy, \quad z \text{ real}.$$

This averaging is an upper-bound argument about effects; it introduces no unaccounted physical coin or protected register. Because $\cos \delta \geq 0$ and $v \geq 0$, replacing z by $\sqrt{xy}$ reduces every wrong probability without changing any conclusive probability. Therefore the rank-one form (24) suffices in the relaxation.

For positive x, y set $r = \sqrt{x/y}$. The worst phase and visibility are $|\delta| = \Delta$, $v = v_-$. At a specified p ,

$$q(p) = px + (1-p)y, \quad \frac{w(p)}{q(p)} = \frac{1}{2} \left[1 - \frac{2m\sqrt{p(1-p)xy}}{px + (1-p)y} \right] = \frac{1}{2} [1 - m \operatorname{sech}(\log r + \log \xi(p))]. \quad (25)$$

A reader with $x = 0$ or $y = 0$ has conditional error $1/2$ whenever it clicks; it cannot improve the case $\varepsilon < 1/2$. The least contrast in (25) occurs at an endpoint of the log-odds interval. Balancing its two endpoint distances gives the unique minimum possible worst distance w , proving (21). At the floor the optimal ratio obeys $\log r = -\frac{1}{2} \log(\xi_- \xi_+)$.

For an error tolerance at or above the floor, the requirement is precisely

$$|\log r + \log \xi(p)| \leq h_\varepsilon \quad \text{for all } p,$$

which is (22). At fixed r , error is independent of the common scale. Maximal conclusive probability therefore uses the largest scale allowed by $x, y \leq \eta$, as in (24). For $r \leq 1$ its guaranteed probability is $\eta[1 - p_+ + p_+ r^2]$, strictly increasing with r . For $r \geq 1$ it is $\eta[p_- + (1 - p_-)/r^2]$, strictly decreasing with r . The point of the admissible log interval nearest zero is thus optimal, proving (23) in the relaxation. The following construction attains those effects in the actual apparatus, so the upper bound is sharp for the physical model as well. The $m = 0$ and $\varepsilon \geq 1/2$ boundary cases follow directly from (25) and (19). $\square$

This yields a precise obstruction and its useful repair. Robust zero error with positive success is possible only when $m = 1$ and the amplitude interval is a single point. With any genuine amplitude width, visibility loss or phase width, one must either abstain always or allow a positive error. For the point case $p_- = p_+ = p$, $m = 1$, $\varepsilon = 0$,

$$r^2 = (1 - p)/p, \quad Q_{\max} = 2\eta \min(p, 1 - p),$$

recovering the coherent conversion model's exact unambiguous reader. With $p_- = p_+ = 1/2$, the error floor is $(1 - m)/2$ and no amplitude filter improves it.

3.3 Attaining interaction, complete outputs and the loss frontier

Let $H = 2^{-1/2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and use, during τ ,

$$A = D = 0, \quad C = H \text{diag}(g_x, g_y), \quad f_{g_x}(\tau) = \sqrt{x}, \quad f_{g_y}(\tau) = \sqrt{y}. \quad (26)$$

The coherent conversion model's damped transfer amplitude at coupling g is continuous on $[0, G]$, equals zero at $g = 0$ and equals $\sqrt{\eta}$ at $g = G$. Since $x, y \leq \eta$, both required couplings exist by the intermediate-value theorem. The coupling is fixed before the unknown input, uses no actual-sign feedback, has $\|C\| \leq G$, and hence has cross-subspace Hamiltonian norm at most $\hbar G$. Switch it off after τ ; the flags then remain untouched until the deadline T . The terminal conclusive maps are exactly

$$K_+ = -\frac{i}{\sqrt{2}}|+\rangle(\sqrt{x}\langle 0| + \sqrt{y}\langle 1|), \quad K_- = -\frac{i}{\sqrt{2}}|-\rangle(\sqrt{x}\langle 0| - \sqrt{y}\langle 1|). \quad (27)$$

They reproduce (24) for every source-reference input, not only for the designated family.

For full bookkeeping let $V(t) = \exp[(-iH/\hbar - \ell P_{\mathcal{R}}/2)t]I_{\mathcal{R}}$ on the active interval. The surviving-source map is $K_0 = e^{-\ell(T-\tau)/2}P_{\mathcal{R}}V(\tau)$. Lost quantum outputs have maps $K_{\text{loss}}(t) = \sqrt{\ell}W_t P_{\mathcal{R}}V(t)$ for $t \leq \tau$, and the freely decaying ready remainder gives the corresponding maps for $\tau < t \leq T$. They obey

$$K_0^* K_0 + K_+^* K_+ + K_-^* K_- + \int_0^T K_{\text{loss}}(t)^* K_{\text{loss}}(t) dt = I.$$

The isometries W_t transport the instantaneous lost two-mode component, including its conditional correlations with R . Earlier filtering and conditioning need not preserve the original input correlations. No failure is removed or redrawn. Tensoring each map with I_R defines the complete retained instrument. Coherent bath output and a measured time archive are alternative output choices; extra physical classicalization environments are not included in a four-mode count for the active Hamiltonian.

The attained reader has phase-independent conclusive, null and loss probabilities at specified p :

$$q(p) = px + (1 - p)y, \quad n(p) = pn_x + (1 - p)n_y, \quad l(p) = 1 - q(p) - n(p),$$

where $n_j = e^{-\ell(T-\tau)}|u_{g_j}(\tau)|^2$ and u_g is the ready amplitude of the damped tap. Thus loss and null remain separately predictable and retained. The theorem optimizes conclusive probability under an error constraint; it does not additionally minimize lost mass among every possible realization of the same effects.

Write $B = Q_{\max}/\eta$, with B determined entirely by the source uncertainty and chosen error budget. A required guaranteed rate Q_* is feasible exactly when

$$\varepsilon \geq \varepsilon_{\min}, \quad \eta(G, \ell, T)B \geq Q_*. \quad (28)$$

This is a sharp physical loss/control requirement within the same model. It keeps the preparation penalty B separate from the independently specified loss/control factor η . It does not join a visibility parameter from one noise model to an incompatible apparatus bound. For a deadline at least the first transfer peak, $\eta = e^{-\ell t_*/2}$ gives an explicit one-variable loss threshold.

3.4 An exact extension for uncertain mode efficiencies

A calibrated efficiency interval can be treated without pretending that it is uncertainty in the entire Hamiltonian. Put an unknown but fixed two-mode attenuator *before every reader control*, with

$$A_{\alpha\beta} = \text{diag}(\sqrt{\alpha}, \sqrt{\beta}), \quad B_{\alpha\beta} = W \text{diag}(\sqrt{1-\alpha}, \sqrt{1-\beta}),$$

where W maps into two retained upstream-loss modes and $A_{\alpha\beta}^* A_{\alpha\beta} + B_{\alpha\beta}^* B_{\alpha\beta} = I$. Assume Cartesian bounds $0 < \alpha_- \leq \alpha \leq \alpha_+ \leq 1$ and $0 < \beta_- \leq \beta \leq \beta_+ \leq 1$. There is no statistical independence assumption on these device parameters. They are not refreshed following loss, and there is no compensating input-dependent control before the attenuator or conclusive reader of its lost modes.

Proposition 3.2 (Sharp uncertainty in upstream path transmission). *The preceding solution remains exact after replacing the log-odds interval $[\log \xi_-, \log \xi_+]$ by*

$$t_- = \log \xi_- + \frac{1}{2} \log(\alpha_-/\beta_+), \quad t_+ = \log \xi_+ + \frac{1}{2} \log(\alpha_+/\beta_-).$$

Its conditional-error floor is $[1 - m \text{sech}((t_+ - t_-)/2)]/2$. For a feasible $\varepsilon < 1/2$, choose $\log r$ nearest zero in $[-h_\varepsilon - t_-, h_\varepsilon - t_+]$. The exact guaranteed probability on the original trial population is

$$Q_{\max}^{\text{eff}} = \frac{\eta}{\max(1, r^2)} \min_{p \in \{p_-, p_+\}} [p\alpha_- r^2 + (1-p)\beta_-]. \quad (29)$$

Proof. The actual conclusive probability is $p\alpha x + (1-p)\beta y$. Its conditional contrast is $m \text{sech}(\log r + \log \xi(p) + \frac{1}{2} \log(\alpha/\beta))$. The possible argument is exactly the interval just stated, so the same endpoint minimax calculation gives the conditional-error constraint. Conclusive probability is smallest at α_-, β_- and an endpoint of p . At maximum allowed scale its value is increasing in r for $r \leq 1$ and decreasing for $r \geq 1$, for each p separately and hence also for their minimum. This proves (29). The relaxed-POVM upper-bound proof and the actual attaining post-attenuation pulse are unchanged. Every downstream Kraus map is now multiplied on the right by $A_{\alpha\beta}$, and $B_{\alpha\beta}$ is a further retained quantum branch. Completeness therefore remains exact on arbitrary references. No normalization by the probability of upstream survival has been silently performed. $\square$

For the example below, adding $\alpha, \beta \in [0.95, 1]$ raises the conditional-error floor from 0.05764696773 to 0.06144954979. At $\varepsilon = 0.08$, it reduces guaranteed conclusive probability from 0.1809098506 to 0.1656759538, attained by gains $x = 0.1375200951$, $y = 0.5062765525$. This is a sharp reader with uncertain path transmission, not an approximate bound on arbitrary gain miscalibration.

Uncertain scalar lifetime or coherent coupling is a different interface and is held open here. The physical-threshold curve (28) permits redesign at each *known* ℓ ; it is not one pulse certified for an

unknown range of ℓ . Indeed the fixed eight-percent pulse designed at $\ell = 1$ below has worst conditional error 0.08016304046 if the actual rate is 0.8. Less loss changes the relative transfer amplitudes and can increase the conditional error. Substituting a worst-case η while keeping its nominal optimal ratio would therefore be invalid. A joint calibrated effect region or an additional robust-control optimization is needed for that task.

4 A calibrated filter with common uncertain conversion

4.1 The original source population and the two physical stages

Keep the preparation-uncertainty construction's one-excitation source family

$$\rho_h(p, \delta, v) = \begin{pmatrix} p & hv\sqrt{p(1-p)}e^{-i\delta} \\ hv\sqrt{p(1-p)}e^{i\delta} & 1-p \end{pmatrix}, \quad p \in [p_-, p_+], \quad |\delta| \leq \Delta, \quad v \in [v_-, 1].$$

Here $0 < p_- \leq p_+ < 1$, $0 \leq \Delta \leq \pi/2$ and $h = \pm 1$. An arbitrary finite inert reference R is retained and unread. The apparatus and its internal controller have one input-independent original preparation; no failed trial is replaced by a fresh source.

For a feasible error ceiling $\varepsilon < 1/2$ and $m > 0$, put $m = v_- \cos \Delta$, $\xi_{\pm} = \sqrt{p_{\pm}/(1-p_{\pm})}$ and $h_{\varepsilon} = \text{arcosh}[m/(1-2\varepsilon)]$. The previous source theorem, or direct calculation of the two rank-one reader effects, gives the allowed amplitude-ratio interval

$$a = e^{-h_{\varepsilon}}/\xi_-, \quad b = e^{h_{\varepsilon}}/\xi_+. \quad (30)$$

We develop the case $0 < a \leq b < 1$, where mode zero should be attenuated; interchanging source modes treats the reflected case. The interval exists at or above the previously proved conditional-error floor. For any effective ratio R the worst conditional error is

$$\max_{p \in \{p_-, p_+\}} \frac{1 - m \operatorname{sech}[\log R + \frac{1}{2} \log(p/(1-p))]}{2},$$

so the error ceiling is equivalent to $R \in [a, b]$.

First place a calibrated, static passive filter at the entrance:

$$F_r = \text{diag}(r, 1), \quad B_r = \sqrt{1-r^2}|f\rangle\langle 0|, \quad F_r^* F_r + B_r^* B_r = I, \quad 0 < r \leq 1. \quad (31)$$

The reflected mode f is retained and never interacts with the subsequent cell. This is a unitary beam-splitter isometry with one additional vacant mode. For example an exchange Hamiltonian with angle $\arccos r$ implements it; its interaction entries have units $\hbar(\arccos r)/T_F$ for a finite calibrated prefix duration T_F . Its phase must also be calibrated or charged to Δ . This prefix is an additional declared resource. It is outside the uncertain conversion cell; its own losses are included in its calibrated amplitudes, not presumed absent in a real device. The comparison uses the same prefix interval with a bypass for the unfiltered apparatus. The quoted internal time uncertainty concerns the following cell; total elapsed time also includes T_F . No instantaneous filter or unchanged total hardware is asserted.

Next use identical path couplings. In the bright-flag basis the ready/flag Hamiltonian and all internal interactions are path independent:

$$H_{\text{cell}}(t) = I_{\text{path}} \otimes H_{\text{port,controller}}(t). \quad (32)$$

For the simple resonant cell this is the four-mode model with $C(t) = g(t)\mathbf{H}$, where $\mathbf{H} = 2^{-1/2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Uniform ready-subspace loss preserves the path state while transferring its excitation to retained lost modes. The full loss dynamics must be path blind; equal population decay rates alone do not exclude additional path dephasing. The bath and controller are retained. The physical $+$ and $-$ flags read $\langle \pm_x | = 2^{-1/2}(\langle 0 | \pm \langle 1 |)$ once the cell is off. Null, filter failure, source remainder and lost modes are inconclusive; no additional reader of them or of R is included.

4.2 Exact factorization under unknown common internal dynamics

Proposition 4.1 (Internal uncertainty changes exposure, not reader ratio). *For any path-blind cell with its stated original input-independent stock, there is a positive retained controller/bath output τ_F on the flag branch, independent of the path input, with $\text{Tr } \tau_F = \eta \in [0, 1]$. The conclusive instrument on any ρ_{SR} has the exact form*

$$\mathcal{I}_{\pm}(\rho_{SR}) = |\pm\rangle\langle\pm| \otimes \tau_F \otimes \langle\pm_x|F_r\rho_{SR}F_r^*|\pm_x\rangle. \quad (33)$$

Consequently

$$E_{\pm} = \frac{\eta}{2} \begin{pmatrix} r^2 & \pm r \\ \pm r & 1 \end{pmatrix}.$$

All conclusive conditional errors and normalized reference assemblages are independent of unknown common gain, detuning, loss, waveform or stopping time, whenever that branch has nonzero probability. Their common conclusive scale η can change and can even vanish. A positive rate requires a separate lower bound on η .

Proof. Purify the apparatus stock with its inert original purifier. In the transmitted branch, the full path-independent cell isometry acts as identity on the path and reference and as one common isometry on the ready-port/controller stock. Projecting the latter to the flag port gives a subnormalized apparatus vector whose projector, or the corresponding mixture, is τ_F . Projecting the path into the physical $\pm_x$ flag modes gives (33). Its trace supplies the stated effects. The apparatus state conditional on either conclusive flag is the same τ_F/η , independent of the input and reported label. The reference factor is the displayed matrix element; after normalization it has no η dependence.

The filter-failure amplitude B_r and every remaining ready/lost-port amplitude are retained orthogonal output sectors. Combining the filter isometry with the full cell isometry makes their complete sum trace preserving and retains coherences if the output is left unmeasured. This does not claim that full outputs for two different internal cells are equal: τ_F , lost temporal modes and cross-sector phases can differ. The invariance is the exact specified reader/reference interface. No reset or redraw is used, and tensoring with I_R incurs no reference-dimension cost. $\square$

For an original ensemble of unknown internal parameters, retain that ensemble label and its apparatus state. The same formula holds with the corresponding mixture of τ_F , since the stock is independent of the unknown input. There is no fresh statistical independence across uses: one unknown common cell can correlate their success flags. Pointwise error guarantees do not turn those flags into independent trials.

Among calibrated diagonal filters followed by a common cell known only to obey $\eta \geq \eta_{\min} > 0$, the sharp maximin rate is

$$Q_{\text{filter}} = \eta_{\min}[1 - p_+ + p_+b^2], \quad r = b. \quad (34)$$

Indeed the error constraint is $r \in [a, b]$, and for $r < 1$ the original population's conclusive rate is $\eta[1 - p + pr^2]$, smallest at $p = p_+$ and at $\eta_{\min}$ and increasing with r . Allowing an extra common attenuation cannot improve it. This is a sharp optimum in this filter/common-conversion subclass; it is not an unrestricted robust-control optimum.

4.3 Sharp gain/loss/time box bounds

For a constant positive resonant coupling g , scalar ready loss ℓ and interaction duration t , define the positive flag amplitude before its first maximum by

$$f(g, \ell, t) = ge^{-\ell t/4} \begin{cases} \sin(\Omega t)/\Omega, & \Omega^2 = g^2 - \ell^2/16 > 0, \\ t, & g^2 = \ell^2/16, \\ \sinh(\chi t)/\chi, & \chi^2 = \ell^2/16 - g^2 > 0. \end{cases} \quad (35)$$

Its first ready-amplitude zero occurs at

$$t_*(g, \ell) = \int_0^{\pi/2} \frac{d\theta}{g + (\ell/2) \sin \theta \cos \theta}. \quad (36)$$

This is the arctangent/critical/hyperbolic formula of the coherent conversion result. Let actual constants obey

$$g \in [g_L, g_U], \quad \ell \in [\ell_L, \ell_U], \quad t \in [t_L, t_U], \quad 0 < g_L \leq g_U, \quad 0 \leq \ell_L \leq \ell_U, \quad 0 < t_L \leq t_U,$$

and assume

$$t_U \leq t_*(g_U, \ell_U). \quad (37)$$

The gain and loss are in s^{-1} ; the duration is in seconds. This is uncertainty in one rectangular pulse's constants and clock duration, not a bound for arbitrary time-varying coefficients taking values in those intervals.

Proposition 4.2 (Exact transfer and ratio corners). *In this box, f increases with g and t and decreases with ℓ . Thus*

$$\eta_{\min} = f(g_L, \ell_U, t_L)^2, \quad \eta_{\max} = f(g_U, \ell_L, t_U)^2. \quad (38)$$

For a fixed relative coupling $0 < d < 1$ define $R_d(g, \ell, t) = f(dg, \ell, t)/f(g, \ell, t)$. Its exact extrema are

$$R_{\min}(d) = R_d(g_L, \ell_U, t_L), \quad R_{\max}(d) = R_d(g_U, \ell_L, t_U). \quad (39)$$

It increases with g, t and decreases with ℓ , and is increasing in d .

Proof. Equation (36) decreases in both g and ℓ , so the entire box precedes each pulse's first ready zero. Since $\dot{f} = gu$ for its nonnegative ready amplitude u , f increases in time. For the gain and loss derivatives put $a_0 = \ell/4$. In the oscillatory case $x = \Omega t$,

$$\partial_g \log f = \frac{1}{g} + \frac{g}{\Omega^2} (x \cot x - 1), \quad \partial_{a_0} \log f = -t + \frac{a_0}{\Omega^2} (1 - x \cot x).$$

For $t \leq t_*$, $x \leq \arctan(\Omega/a_0)$. The decreasing function $x \cot x$ is at least $\arctan z/z \geq 1/(1+z^2) = a_0^2/g^2$, $z = \Omega/a_0$, which proves the gain derivative is nonnegative. Also $1 - x \cot x \leq x \tan x$ and $a_0 \tan x \leq \Omega$, which prove the loss derivative is nonpositive. In the hyperbolic case $y = \chi t$ the derivatives are

$$\partial_g \log f = \frac{1}{g} + \frac{g}{\chi^2} (1 - y \coth y), \quad \partial_{a_0} \log f = -t + \frac{a_0}{\chi^2} (y \coth y - 1).$$

Here $y \leq \text{artanh}(\chi/a_0)$, $y \coth y \leq \text{artanh } z/z \leq 1/(1-z^2) = a_0^2/g^2$, and $y \coth y - 1 \leq y \tanh y$, $a_0 \tanh y \leq \chi$. These prove the same signs. The critical and zero-loss limits follow by continuity, establishing (38).

For the ratio use the entire sine product, which also covers imaginary Ω :

$$R_d = d \prod_{n=1}^{\infty} \left[1 + \frac{(1-d^2)g^2t^2}{\pi^2n^2 - g^2t^2 + \ell^2t^2/16} \right]. \quad (40)$$

All denominators are positive in the prepeak box. Each factor increases in g, t and decreases in ℓ . The product converges uniformly on compact subsets of this box, so its limits preserve those monotonicities. This gives (39). Increasing d increases its numerator amplitude $f(dg, \ell, t)$ by the gain result; the denominator is fixed. $\square$

4.4 A solved, optimized comparison with unequal taps

The earlier rectangular-pulse subclass has no calibrated input filter and uses $C = g\mathbf{H} \text{diag}(d, 1)$ with one fixed d chosen before unknown g, ℓ, t . It is the actual unequal-tap architecture, retuned for the whole uncertainty box rather than frozen at its nominal value. Its conditional error is controlled by the effective ratio R_d . For $0 < a \leq b < 1$, all pulses in the box are valid exactly when

$$R_{\min}(d) \geq a, \quad R_{\max}(d) \leq b. \quad (41)$$

Both functions increase from zero to one. The feasible coupling interval is $[R_{\min}^{-1}(a), R_{\max}^{-1}(b)]$, if nonempty. Its sharp maximal rate is attained at

$$d_* = R_{\max}^{-1}(b), \quad Q_{\text{unequal}} = \eta_{\min}[1 - p_+ + p_+ R_{\min}(d_*)^2]. \quad (42)$$

To prove the rate statement, both individual transfer amplitudes increase with gain and time and decrease with loss. Thus their weighted conclusive sum is smallest at (g_L, ℓ_U, t_L) and $p = p_+$. It increases with d , giving the largest feasible value (42). This is a solved restricted minimax, not a sampled optimizer. If the feasible interval is empty, this rectangular unequal-tap subclass has no positive-rate valid design.

Whenever the box creates strict ratio variation, $R_{\min}(d_*) < R_{\max}(d_*) = b$, comparison of (34) and (42) gives $Q_{\text{filter}} > Q_{\text{unequal}}$. The calibrated filter fixes the desired ratio outside the uncertain conversion dynamics. It does not cancel loss or recover a missing input sample. That improvement consumes the extra resource specified at the start. A shaped pulse or larger controller could outperform either restricted subclass; no bound on such a protocol is claimed.

5 Sharp protection against imperfect controlled-path matching

5.1 Physical model and a certificate weaker than exact matching

The transmitted input path is a two-mode one-excitation register. The original apparatus stock is independent of that register and any inert source reference. After purification, let the actual cell obey

$$|i\rangle|\text{ready}\rangle \mapsto |i\rangle|v_i\rangle, \quad \|v_i\| = 1, \quad i = 0, 1.$$

The path is preserved; all port, controller and bath variables belong to v_i . This allows path-dependent coherent interactions and marking, but not arbitrary path exchange. Let P_F be the common conclusive port and $P_N = I - P_F$ its complete inconclusive complement, including surviving source and all lost outputs. Only the terminal $+$ and $-$ flags supply decisions; the complementary path, bath and original reference are retained but unread. In this solvable class the two actual efficiencies remain equal:

$$\|P_F v_0\|^2 = \|P_F v_1\|^2 = \eta, \quad 0 < \eta_L \leq \eta \leq \eta_U \leq 1.$$

Equal efficiencies do not imply equal apparatus states or phase. Write $f_i = P_F v_i / \sqrt{\eta}$, and admit the full-response certificate

$$\text{Re}\langle v_0, v_1 \rangle \geq \gamma = \cos \Theta, \quad 0 \leq \Theta \leq \pi. \quad (43)$$

It is a real overlap in a calibrated relative-phase convention, not an absolute fidelity: $|\langle v_0, v_1 \rangle| = 1$ alone permits an unknown relative phase of π and cannot protect this reader. The certificate is a physical premise or an independently justified calibration result. Reading all lost ports to estimate it would require additional coherent access, not granted by retaining them.

A sufficient microscopic consumer uses two controlled-path Hamiltonians $H_i(t)$ on the same finite dilation, with a common initial stock and $\int \|H_0 - H_1\|_\infty dt / \hbar \leq b$. Duhamel's identity gives $\|v_0 - v_1\| \leq b$, hence $\text{Re}\langle v_0, v_1 \rangle \geq 1 - b^2/2$. It requires a bound on the actual full dilation difference; equal averaged decay rates do not establish it. Sharpness below is for the declared Gram class, not for every Hamiltonian norm ball. The common conversion and uniform loss model can supply η_L when its own hypotheses still hold. Otherwise η_L needs a new certificate.

5.2 Sharp postselected overlap and phase response

Put $\chi = \langle f_0, f_1 \rangle$. Orthogonality of the port sectors gives

$$\text{Re}\langle v_0, v_1 \rangle = \eta \text{Re} \chi + \text{Re}\langle P_N v_0, P_N v_1 \rangle \leq \eta \text{Re} \chi + 1 - \eta.$$

Thus

$$|\chi| \leq 1, \quad \text{Re} \chi \geq k(\eta) := 1 - \frac{1 - \gamma}{\eta}. \quad (44)$$

When $k < -1$ the latter supplies no additional restriction. This is exact: any point in the intersection of that disk and half-plane can be completed by normalized null vectors having a suitable real overlap, whenever the global inequality requires it. For $-1 \leq k(\eta) \leq 1$, the boundary used in the positive-contrast optimization below has

$$v_0 = \sqrt{\eta}|F\rangle + \sqrt{1 - \eta}|N\rangle, \quad v_1 = \sqrt{\eta}e^{i\alpha}|F\rangle + \sqrt{1 - \eta}|N\rangle, \quad \cos \alpha = k(\eta).$$

It saturates the global certificate in that domain. A conventional common conversion pulse followed by the controlled flag-only Hamiltonian $H_{\text{mark}} = -\hbar\alpha|1, F\rangle\langle 1, F|/T_M$ for duration $T_M > 0$ realizes it. Its finite action and timing are declared; there is no claim of a free or instantaneous phase pulse. This uses the same two paths and two ready/flag ports, plus the common-conversion filter-failure port. Null/loss amplitudes are retained, not renormalized away.

Let the source family have population $p \in [p_L, p_U]$, visibility $v \in [v_L, 1]$, with $0 \leq v_L \leq 1$, and relative phase $|\delta| \leq \Delta$. For $k_L = k(\eta_L)$ satisfying

$$k_L > \sin \Delta, \quad 0 \leq \Delta < \pi/2, \quad (45)$$

the exact least real conclusive contrast is

$$m = v_L \left[k_L \cos \Delta - \sqrt{1 - k_L^2} \sin \Delta \right] = v_L \cos(\arccos k_L + \Delta) \geq 0. \quad (46)$$

It is strictly positive when $v_L > 0$. If $v_L = 0$, no correctly labelled reader in this family with positive guaranteed conclusive probability can guarantee conditional error below one half. Indeed, for fixed phase the minimizing unit-disk point is $\chi = k_L + i\sqrt{1 - k_L^2}$, with adverse phase sign; increasing η increases k and improves the contrast. The minimum over δ is at an endpoint. Visibility is worst at v_L in the positive regime. The explicit phase-marking dilation above attains every stated minimum. Outside this regime the minimum is $\cos[\min\{\pi, \arccos(\max\{-1, k_L\}) + \Delta\}]$ before visibility minimization. A

negative contrast cannot certify the present correctly labelled reader below one-half error; it is not silently clipped positive.

The limiting cases expose the operational gain. At $\Theta = 0$ the general-POVM matched-reader contrast is recovered. At $\eta = 1$ the phase budget is the ordinary sum $\Theta + \Delta$. At fixed nonzero Θ and decreasing efficiency, conditioning eventually destroys the positive-contrast guarantee. This is a quantified consequence of mismatch concentrated in rare flag events, not an additive generic channel-error estimate.

5.3 Exactly optimal filter with bounded calibration error

Place the same entrance filter before the cell. Its actual transmitted matrix is $F = \text{diag}(re^u, 1)$, where the chosen nominal $r > 0$ has fixed unknown log-gain error $|u| \leq u_0$, with $u_0 \geq 0$, and $r \leq e^{-u_0}$. Its phase error is charged to Δ ; failure remains an orthogonal retained flag. These are amplitude constraints on the actual passive prefix, not a statistical independence assumption between errors. The conclusive effects are

$$E_{\pm} = \frac{\eta}{2} \begin{pmatrix} r^2 e^{2u} & \pm re^u \chi \\ \pm re^u \bar{\chi} & 1 \end{pmatrix}.$$

The complex conjugation convention is immaterial to the symmetric phase set. For either label the conditional wrong probability is

$$\epsilon(p, u, \eta, \chi, \delta, v) = \frac{1}{2} \left[1 - \frac{2re^u v \sqrt{p(1-p)} \text{Re}(\chi e^{i\delta})}{pr^2 e^{2u} + 1 - p} \right].$$

Thus it depends on the normalized apparatus overlap as well as path counts.

Theorem 5.1 (Sharp Gram-robust filter frontier). *Assume Eq. (45), $0 < p_L \leq p_U < 1$ and an admitted error ceiling $0 \leq \varepsilon < 1/2$. Put $\xi_L = \sqrt{p_L/(1-p_L)}$, $\xi_U = \sqrt{p_U/(1-p_U)}$. If $m < 1 - 2\varepsilon$, no positive conclusive reader in this subclass meets the ceiling. Otherwise define*

$$h = \text{arcosh} \frac{m}{1-2\varepsilon}, \quad L = -h - \log \xi_L + u_0, \quad U = h - \log \xi_U - u_0.$$

A feasible filter exists precisely when $L \leq \min\{U, -u_0\}$. Its maximin conclusive probability is attained at

$$r_* = \exp \min\{U, -u_0\}, \quad Q_* = \eta_L [1 - p_U + p_U r_*^2 e^{-2u_0}]. \quad (47)$$

The optimum is among these calibrated attenuate-mode-zero filters and controlled-path cells with the stated Gram and efficiency uncertainty. It is not an unrestricted optical-control optimum.

Proof. The exact worst contrast is m . With $x = \log r + u + \log \xi(p)$, the conditional error is $(1 - m \text{sech } x)/2$. Its ceiling is equivalent to $|x| \leq h$ for every p, u , giving $L \leq \log r \leq U$. Passivity gives the additional cap. At fixed r the original-population success is $\eta[1 - p + pr^2 e^{2u}]$, minimized at $\eta_L, u = -u_0, p = p_U$; it increases with r . This proves the displayed choice and rate. The worst error and worst success can occur at different admitted parameters; each is an individually attained uniform guarantee. The explicit port vectors establish physical attainment of the error bound, and no null or prefix-failure outcome is postselected away. $\square$

Exact admitted mismatch threshold. There is a closed symmetry-breaking tolerance when the balancing filter is passive. Assume $r_c = (\xi_L \xi_U)^{-1/2} \leq e^{-u_0}$, and put

$$W = \frac{1}{2} \log(\xi_U/\xi_L) + u_0, \quad c_{\varepsilon} = \frac{(1-2\varepsilon) \cosh W}{v_L}.$$

For $0 < c_\varepsilon \leq \cos \Delta$, define $\phi = \arccos c_\varepsilon - \Delta \geq 0$. A positive-rate filter meeting the error ceiling exists exactly when

$$\Theta \leq \Theta_{\max} := \arccos[1 - \eta_L(1 - \cos \phi)]. \quad (48)$$

To prove it, the largest absolute log-ratio over the uncertain source and gain interval has minimum W , attained by r_c ; passivity does not obstruct it by assumption. Thus feasibility is exactly $m \geq (1 - 2\varepsilon) \cosh W$. Insert Eq. (46) and invert cosine on its positive branch. At the boundary the admissible filter interval collapses to the balancing choice. If $c_\varepsilon > \cos \Delta$, no mismatch budget is feasible, including zero. If $v_L = 0$ this positive-error-ceiling theorem is not used. For the worked example below, $\Theta_{\max} = 0.1001806624$ radians, about 5.73993° . This threshold applies to the full-response Gram budget; it is not the same numerical budget as a microscopic detuning angle.

5.4 An exact pointer/reference distance, distinct from full output

At fixed actual $r_a = r e^u$ and η , compare with the matched reader $\chi = 1$ having the same filter and efficiency. The aggregate inconclusive pointer effect, summing filter failure, loss and surviving-ready outcomes, is identical. These sectors are coarsened only for the present pointer/reference comparison; their separate retained quantum outputs need not agree. The two conclusive effect differences are A and $-A$, where

$$A = \frac{r_a \eta}{2} \begin{pmatrix} 0 & \chi - 1 \\ \bar{\chi} - 1 & 0 \end{pmatrix}.$$

For the nominated classical pointer channels, including an arbitrary inert reference,

$$\frac{1}{2} \|\mathcal{M}_\chi - \mathcal{M}_1\|_\diamond = \|A\|_\infty = \frac{r_a \eta}{2} |\chi - 1| \leq \frac{r_a \sqrt{\eta}}{2} d, \quad d = \sqrt{2(1 - \gamma)}. \quad (49)$$

In the positive regime the last bound is sharp for the phase-marking vectors. To prove the reference-uniform upper bound, trace-norm duality gives $\|\text{Tr}_S[(A \otimes I)\rho_{SR}]\|_1 \leq \|A\|_\infty$; the two classical blocks have opposite signs. An eigenvector input of A attains equality. Finally $|\chi - 1|^2 \leq 2(1 - \text{Re } \chi) \leq d^2/\eta$ proves the last inequality. This certifies pointer plus reference, not the surviving source, full bath, unknown controller state or equality of their purified outputs. Retaining those outputs does not grant permission to read them.

6 Operating examples, compatibility and reproducibility

For overlap $s = .6$, $G = \ell = 1 \text{ s}^{-1}$ and deadline 2 s , the sharp coherent optimum has active time 1.3613444250 s , weaker coupling $.3764245988 \text{ s}^{-1}$, and each correct conclusive probability $.2025106210$. The static reciprocal absorbing bank instead has wrong-click probability $.0394994846$ by time $2/\gamma$; the moving bounded-hazard reader repairs this back-action in its different model. For source box $p \in [.7, .9]$, $v \geq .95$, $|\delta| \leq 10^\circ$, the calibrated-loss model has error floor $.05764696773$. At conditional-error ceiling $.08$ its exact conclusive guarantee is $.1809098506$. Unknown internal loss cannot be handled by inserting a worst-case transfer factor into a fixed nominal unequal pulse: the nominal pulse can exceed its error ceiling even when loss decreases. For common rectangular uncertainty $g \in [.9, 1.1]$, $\ell \in [.8, 1.2]$ in inverse seconds and $t \in [.9, 1.1]$ seconds, the optimized unequal-tap guarantee is $.09490563225$, while the calibrated-filter/common-cell subclass gives $.1120453274$. Its 18.0597% improvement costs one extra filter-failure port and a calibrated prefix. It is not universal hardware superiority. With additional full-response mismatch $\Theta = .05$ radians and log-filter error $.01$, the sharp protected design gives $.1018845501$ at conditional error $.08$; the formerly matched filter fails at $.09010468261$. This compares protection under a larger uncertainty set, not a rate increase over perfect matching. The analytical proofs establish the quantifiers. Replayed NumPy/SciPy calculations independently reconstruct Hamiltonians,

Gram matrices, retained-reference normalization and numerical optima; their decimals are binary64 diagnostics, not rigorous enclosures or physical data. The clean supplement preserves actual inputs and the exact code used. Calibration of terminal effects does not identify a full residual instrument; finite-reference, detector-comparison and monitoring interfaces are developed separately in the technical companion. The bounded coherent-coupling, bounded hazard, common-cell and imperfect-matching theorems use different declared physical parents. They may not be concatenated without proving common preparation, controller, clock, reference, loss and access conditions. All future uses require fresh stock or a separately established reuse law. No reset alone supplies independence, no flag constitutes a perspective, and the original measurement obligations remain open.

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References

- [1] E. Bagan, R. Munoz-Tapia, G. A. Olivares-Renteria, and J. A. Bergou. Optimal discrimination of quantum states with a fixed rate of inconclusive outcomes. *Physical Review A*, 86:040303, 2012. doi: 10.1103/PhysRevA.86.040303.
- [2] Anthony Chefles and Stephen M. Barnett. Optimum unambiguous discrimination between linearly independent symmetric states. *Physics Letters A*, 250(4–6):223–229, 1998. doi: 10.1016/S0375-9601(98)00827-5.
- [3] Anthony Chefles and Stephen M. Barnett. Quantum state separation, unambiguous discrimination and exact cloning. *Journal of Physics A: Mathematical and General*, 31(50):10097–10103, 1998. doi: 10.1088/0305-4470/31/50/007. URL <https://arxiv.org/pdf/quant-ph/9808018>.
- [4] Miloslav Dušek, Mika Jahma, and Norbert Lütkenhaus. Unambiguous state discrimination in quantum cryptography with weak coherent states. *Physical Review A*, 62:022306, 2000. doi: 10.1103/PhysRevA.62.022306. URL <https://muj.optol.cz/dusek/clanky/pr62-022306.pdf>.
- [5] B. Huttner, A. Muller, J. D. Gautier, H. Zbinden, and N. Gisin. Unambiguous quantum measurement of nonorthogonal states. *Physical Review A*, 54:3783–3789, 1996. doi: 10.1103/PhysRevA.54.3783.
- [6] Kenji Nakahira, Kentaro Kato, and Tsuyoshi Sasaki Usuda. Generalized quantum state discrimination problems. *Physical Review A*, 91:052304, 2015. doi: 10.1103/PhysRevA.91.052304. Read together with errata *Physical Review A* 92, 059901 (2015) and 103, 019902 (2021).
- [7] Jason D. Padgett. Formal operator-level equivalence between photoelectric detection and a qih-inspired microtubule readout model, . Supplied unpublished manuscript, 35 pages. Relevant reader construction pp.18–21; reduced equivalence pp.10–15.
- [8] Jason D. Padgett. Exact reduced-operator equivalence between the photoelectric effect, a QIH horizon readout, and a microtubule quantum readout, . Supplied unpublished manuscript, 74 pages, filename Photoelectric Origin Microtubules.pdf. Matched-reader construction, Section 32, pp.42–43.
- [9] Jeremy Rodgers. Source-readout non-equivalence: Descent and equivariant reconstruction obstructions, 2026. URL <https://doi.org/10.5281/zenodo.21369967>.

- [10] Raam Uzdin and Omri Gat. Time-energy trade-off in unambiguous-state-discrimination positive operator-valued measures. *Physical Review A*, 88:052327, 2013. doi: 10.1103/PhysRevA.88.052327. URL <https://journals.aps.org/prabstract/10.1103/PhysRevA.88.052327>.