

Technical Interfaces for Quantum Readers: Detectors, Finite References and Calibration

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Abstract

This technical companion preserves distinct operational interfaces that should not be forced into one physical apparatus theorem: absorbing matched-template instruments; complete detector-output comparison and finite microscopic closure repairs; finite charge-reference measurement; preparation costs of nonlinear product-response interpretations; and task-specific versus general-effect calibration and online monitoring. Results are conditional mathematical constructions using ordinary quantum probabilities. Model scope, output omissions and preparation/reference costs remain explicit. The note is comprehensive technical support, not a new Born-rule or consciousness paper.

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1 Operational contracts and scope

This companion develops detector, reference and calibration interfaces within Shadow Theory’s pre-existing source–readout programme [9]. A complete record law, a quantum output channel and a conditional pointer statistic are different interfaces. Every section states which it covers. The optical mode models use introduced interactions, and finite-reference costs apply to a separately declared nondegenerate charge encoding. Established antecedents include Davies–Lewis instruments, mode-selective conversion, Stinespring continuity, WAY/asymmetry constraints, symmetric-state discrimination, polynomial multi-copy estimation and martingale calibration. They are credited in the source text and bibliography [1, 3–6]. The source’s scalar event labels do not supply perspective-admission or conscious-incorporation conditions.

1.1 Complete absorbing readers

M3: one reader, null output and retained reference. The normalized matched-template prescription in [7, pp. 18–21] and [8, Section 32, pp. 42–43] informs the template choice; the absorbing dynamics and complete instrument are specified here. Let $|r\rangle$ be a normalized template in a finite source space and $\Pi = |r\rangle\langle r|$. Fix a duration $T \geq 0$ in seconds and a rate $\kappa \geq 0$ in s^{-1} , and set the dimensionless $\alpha = \kappa T$, $\eta = 1 - e^{-\alpha}$. Adjoin an orthogonal absorbing flag f , choose zero interaction-picture Hamiltonian, and a fixed jump $J = \sqrt{\kappa}|f\rangle\langle r|$. The source-input instrument has

$$K_0 = (I - \Pi) + e^{-\alpha/2}\Pi, \quad K_1 = \sqrt{\eta}|f\rangle\langle r|, \quad K_0^\dagger K_0 + K_1^\dagger K_1 = I. \quad (1)$$

Thus

$$E_{\text{click}} = \eta\Pi, \quad p_{\text{click}} = (1 - e^{-\alpha}) \text{tr}(\Pi\rho), \quad (2)$$

which differs from $1 - e^{-\alpha}$. For arbitrary inert reference R and joint input ρ_{SR} ,

$$\mathcal{I}_1(\rho_{SR}) = \eta|f\rangle\langle f| \otimes \langle r|\rho_{SR}|r\rangle, \quad (3)$$

$$\mathcal{I}_0(\rho_{SR}) = (K_0 \otimes I_R)\rho_{SR}(K_0^\dagger \otimes I_R). \quad (4)$$

The inert reference is not traced out: its correct conditional assemblage is retained. The measurement back-action can nevertheless change or destroy original source–reference correlations; retention is not preservation of the input joint state.

Proof and time record. The no-jump propagator solves $\dot{K}_0(t) = -(\kappa/2)\Pi K_0(t)$. A jump at t has operator density $L_t = \sqrt{\kappa}e^{-\kappa t/2}|f\rangle\langle r|$. Integrating $L_t\rho L_t^\dagger dt$ over $[0, T]$ yields the click branch; an absorbing flag prevents a second click. Completeness follows by integration or Eq. (1). Tensoring Kraus operators with I_R proves the reference formula. Retaining exact click time keeps $L_t\rho L_t^\dagger dt$ as a separate classical record. Time bins integrate over each declared bin and retain its label. Coarsening that record is an explicit forgetting operation. $\square$

M4: finite bank and tight simplification. For normalized templates r_j , fixed rates $\kappa_j \geq 0$, orthogonal absorbing flags f_j and $S = \sum_j \kappa_j |r_j\rangle\langle r_j|$, choose $J_j = \sqrt{\kappa_j}|f_j\rangle\langle r_j|$. Then

$$K_0(T) = e^{-ST/2}, \quad L_{j,t} = \sqrt{\kappa_j}|f_j\rangle\langle r_j|e^{-St/2}, \quad (5)$$

$$E_0(T) = e^{-ST}, \quad E_j(T) = \int_0^T e^{-St/2} \kappa_j |r_j\rangle\langle r_j| e^{-St/2} dt. \quad (6)$$

Each coarse click map integrates $L_{j,t}\rho L_{j,t}^\dagger$ and is CP even without commutation. Differentiating e^{-St} proves $E_0 + \sum_j E_j = I$. In general the effects are not independent saturated projectors.

For $\sum_j w_j |r_j\rangle\langle r_j| = I$ and $\kappa_j = \kappa w_j$, the exact instrument is

$$K_0 = e^{-\alpha/2}I, \quad K_j = \sqrt{\eta w_j}|f_j\rangle\langle r_j|, \quad E_j = \eta w_j |r_j\rangle\langle r_j|, \quad E_0 = e^{-\alpha}I. \quad (7)$$

The null probability is state independent and recorded. There is no separate irreversible loss sink in this bank model. Tensoring the operators with I_R retains any inert reference. Conditional click time has density $\kappa e^{-\kappa t}/\eta$, independent of source in this tight case; this need not hold for a non-tight bank.

Physical contract and experimental antecedents. This ordinary Markovian one-excitation model assumes preparable modes, fixed calibrated couplings, initially unexcited absorber/bath, a specified interaction picture and clock, persistent orthogonal flag records, and retained nominated source/controller and null/loss records. These are introduced physical assumptions. An optical Hamiltonian alternative, $H_{\text{int}} = \hbar g(a_r b^\dagger + a_r^\dagger b)$ with $a_r = \sum_x r_x^* a_x$ and vacuum output b , gives $E_{\text{converted}} = \sin^2(gT)\Pi$ on the one-photon sector. Coherent conversion and irreversible exponential absorption are different controlled models.

Ansari et al. [2], Sec. II Eqs. (1)–(7), supply an actual shaped-pump conversion Hamiltonian and measurement operator $\sum_k \eta_k |k\rangle\langle k|$. Serino et al. [10], Secs. II–III and V–VI, demonstrate a five-dimensional multi-output decoder and six MUB settings. This is a calibration platform, not evidence for a microtubule or ontological identification. No new fundamental dynamics is proposed. The instrument here declares registered classical time/label outputs and the residual source and nominated inert reference. It does not identify the microscopic coherent bath outputs of two different implementations. A stronger retained-output comparison must include the bath in both laws and is not established here.

2 Complete retained-output comparisons and microscopic detector closure

2.1 A complete one-transition model and a reference-uniform certificate

Fix a finite input Hilbert space $\mathcal{H}_0$, horizon T , and common terminal output spaces. All changes of basis used for comparison are declared in advance; an arbitrary environment unitary chosen afterwards would change the meaning of retaining the physical output. For detector $X = 0, 1$, let $H_X(t) = H_X(t)^*$ and a jump-amplitude map $B_X(t) : \mathcal{H}_0 \rightarrow \mathcal{O}$ be bounded, measurable coefficients. The post-transition propagation and every retained mark are included in B_X ; this propagation must be an isometry, or its lost channels must be included. Set

$$\begin{aligned} Q_X(t) &= B_X(t)^* B_X(t), & G_X(t) &= -iH_X(t)/\hbar - Q_X(t)/2, \\ \dot{N}_X(t) &= G_X(t)N_X(t), & N_X(0) &= I. \end{aligned}$$

Here H_X is in joules, Q_X and G_X in s^{-1} , B_X in $\text{s}^{-1/2}$, and N_X is dimensionless. This is an absorbing, one-transition model: there are no later transitions back into the untriggered sector. A finite resonance or continuum response may be used, but its amplitude map must actually be supplied.

Define the complete amplitude isometry

$$V_X \psi = N_X(T) \psi \oplus [t \mapsto B_X(t) N_X(t) \psi] \quad \text{in} \quad \mathcal{H}_0 \oplus L^2([0, T]; \mathcal{O}). \quad (8)$$

The first summand is the retained null/no-transition outcome, not a discarded population. The corresponding time-resolved instrument has Kraus density $K_X(t) = B_X(t)N_X(t)$ and null operator $K_X(\emptyset) = N_X(T)$. It may retain the output in $\mathcal{O}$, coarsen time into bins, or apply a fixed common decoder. Its normalized law is the ordinary quantum law of this specified instrument. It is not a derivation of that law from a source theory. For a registered exact time, cq outputs are integrable trace-class conditional kernels over the null atom and time interval, with integrated trace norm. This classical output space is distinct from the coherent temporal L^2 field in (8); no trace-class diagonal density operator on that field is assumed. Coarsening to finite time bins gives ordinary cq blocks.

Proposition 2.1 (Full-output amplitude and kernel mismatch certificate). *Let*

$$\begin{aligned} e(t) &= \int_0^t \|(G_1(s) - G_0(s))N_0(s)\| ds, \\ A_T^2 &= e(T)^2 + \int_0^T (\|(B_1(t) - B_0(t))N_0(t)\| + \|B_1(t)\|e(t))^2 dt. \end{aligned} \quad (9)$$

Then $V_X^* V_X = I$ and $\|V_1 - V_0\| \leq A_T$. For every input state on $\mathcal{H}_0 \otimes R$, with arbitrary finite inert reference R ,

$$\frac{1}{2} \|(V_1 \otimes I_R) \rho (V_1^* \otimes I_R) - (V_0 \otimes I_R) \rho (V_0^* \otimes I_R)\|_1 \leq \min\{1, A_T\}. \quad (10)$$

The same bound applies to the complete time/null/mark cq instrument after a common readout. With a fixed input-independent original stock this is also a half-diamond upper bound for the two declared output channels.

Proof. $G_X^* + G_X = -Q_X$ gives

$$\frac{d}{dt}(N_X^* N_X) = -N_X^* Q_X N_X.$$

Integration proves $N_X(T)^* N_X(T) + \int K_X(t)^* K_X(t) dt = I$, hence isometry and conservation of the null population. The two-parameter propagator generated by G_1 is a contraction. Variation of constants gives

$$N_1(t) - N_0(t) = \int_0^t N_1(t, s)(G_1(s) - G_0(s))N_0(s) ds, \quad \|N_1(t) - N_0(t)\| \leq e(t).$$

Since

$$K_1 - K_0 = (B_1 - B_0)N_0 + B_1(N_1 - N_0),$$

the direct-sum norm in (8), followed by the triangle inequality, proves (9). For normalized pure inputs, the trace distance of two output vectors is at most their vector distance. Tensoring an operator with I_R does not change its operator norm; mixtures follow by convexity or purification and partial trace. Any common time measurement, mark decoder or CPTP readout contracts trace distance. This proves the reference-uniform statement without dividing by a branch probability or charging a reference dimension. The fixed-stock condition supplies actual channels on an unknown input; arbitrary input-correlated preparations supply only their stated statewise comparisons. $\square$

This certificate includes a term absent from a rate-only comparison: $(B_1 - B_0)N_0$. Even $Q_1 = Q_0$ does not make it vanish. Conversely, matching the declared complex amplitude maps, null dynamics and terminal propagation makes the full output equal. Under exact H, Q matching, the computable cross-Gram operator

$$S = V_0^* V_1 = N(T)^* N(T) + \int_0^T N(t)^* B_0(t)^* B_1(t) N(t) dt \quad (11)$$

retains the omitted phase and mark data. For the *coherent* complete channels their exact half-diamond distance is

$$\sqrt{1 - \nu(S)^2}, \quad \nu(S) = \min_{\rho \geq 0, \text{Tr } \rho = 1} |\text{Tr}(\rho S)|. \quad (12)$$

Indeed the overlap of the two pure outputs of any reference-extended input is $\text{Tr}(\rho S)$, where ρ is its input marginal; every such ρ has a finite purification. Mixed inputs cannot improve the optimum, by purification and contraction. Equation (12) is the standard numerical-range description of isometric-channel discrimination, here made explicit for the proposed detector lift. Time measurement can reduce this distance, so a time-record comparison cannot be substituted for it.

2.2 Exact scalar kernel formula: what intensity matching pays for

For a prepared, untriggered scalar detector write

$$\Lambda_X(t) = \int_0^t \Gamma_X(s) ds, \quad p_X(\emptyset) = e^{-\Lambda_X(T)}, \quad p_X(t) = \Gamma_X(t) e^{-\Lambda_X(t)}.$$

Let $|b_X(t)\rangle$ be the normalized retained quantum mark following a transition at time t ; assume the null ray is common. On the common time/null space the positive output kernel is

$$\sigma_X(\emptyset) = p_X(\emptyset)|0\rangle\langle 0|, \quad \sigma_X(t) = p_X(t)|b_X(t)\rangle\langle b_X(t)|.$$

The mark may include outgoing electron energy, polarization, vibrational state, source recoil and other specified outputs; it is not merely a rendered color. Put $c(t) = |\langle b_0(t)|b_1(t)\rangle|$.

Proposition 2.2 (Exact full time/mark discrepancy). *The retained cq half trace distance is exactly*

$$D_{\text{cq}} = \frac{1}{2}|p_1(\emptyset) - p_0(\emptyset)| + \frac{1}{2} \int_0^T \sqrt{[p_1(t) - p_0(t)]^2 + 4p_1(t)p_0(t)[1 - c(t)^2]} dt. \quad (13)$$

For equal rates it reduces to

$$D_{\text{cq}} = \int_0^T \Gamma(t) e^{-\Lambda(t)} \sqrt{1 - c(t)^2} dt. \quad (14)$$

For unequal rates one has the constructive calibration budget

$$D_{\text{cq}} \leq \text{TV}(P_1, P_0) + \int_0^T \min(p_1(t), p_0(t)) \sqrt{1 - c(t)^2} dt, \quad (15)$$

where P_X is the complete time-plus-null probability law.

Proof. For unit vectors u, v and nonnegative p, q , the nonzero eigenvalues of $p|u\rangle\langle u| - q|v\rangle\langle v|$ lie in their span, have sum $p - q$ and product $-pq(1 - |\langle u, v \rangle|^2)$. Their absolute sum is $\sqrt{(p - q)^2 + 4pq(1 - |\langle u, v \rangle|^2)}$. Integrate this identity on the time/null space. For the last bound split off $|p - q|$ times the larger-weight ray and compare the remaining equal weights. Common unchanged output factors, including a product inert reference in this scalar-input comparison, may be tensored on without altering the result. For input-correlated quantum outputs the corresponding full mark states must instead be included explicitly in the compared kernels. $\square$

Thus exact intensity/rate matching really does fix all waiting-time statistics *in this one-transition model*, not only one terminal probability. Its extension cost is determined by output-state overlaps. For constant Γ , orthogonal marks give

$$D_{\text{bit}} = 0, \quad D_{\text{cq}} = 1 - e^{-\Gamma T}.$$

At $\Gamma T = 1$, this is 0.6321205588. The reduced detector equivalence in [7, pp. 10–15] concerns a smaller output interface. The calculation here quantifies the additional condition required for complete retained-output comparison.

2.3 A coherent-field discriminator despite identical count records

A time-resolved measurement has already discarded coherence between different emission times. Retaining the unmeasured complete field is a stronger promise. A declared, broadband absorbing jump coupling can carry a time-dependent phase while leaving its reduced dissipator unchanged. With constant rate $\Gamma > 0$, compare the two normalized complete vectors

$$|\Psi_X\rangle = e^{-\Gamma T/2} |0, \text{vac}\rangle + \int_0^T \sqrt{\Gamma} e^{-\Gamma t/2} e^{i\theta_X(t)} |1, 1_t\rangle dt. \quad (16)$$

Here $|1_t\rangle$ denotes the continuum single-quantum temporal-mode basis; the integral is an L^2 wavepacket, not a distribution-valued state evaluated at a point. This is a signal-marker dilation of the abstract absorbing detector, not a microscopic model of photoelectron emission or a microtubule. The symbols 0, 1 label ready/triggered states, not their energy ordering. An explicit Markov dilation uses the coupling $L_X(t) = \sqrt{\Gamma} e^{i\theta_X(t)} |1\rangle\langle 0|$ with vacuum input; once triggered it cannot jump again. The phase schedule and its source are externally prescribed resources. An autonomous realization would have to include and retain that controller and clock and reprove the comparison.

For $\theta_1(t) - \theta_0(t) = \Delta\omega t$,

$$z_T = \langle \Psi_0 | \Psi_1 \rangle = e^{-\Gamma T} + \frac{\Gamma}{\Gamma - i\Delta\omega} [1 - e^{-(\Gamma - i\Delta\omega)T}], \quad D_{\text{coherent}} = \sqrt{1 - |z_T|^2}. \quad (17)$$

The reduced detector evolution and the entire count-time distribution agree exactly. Nevertheless, at $T = 1/\Gamma$, $\Delta\omega = 2\Gamma$, $D_{\text{coherent}} = 0.5551876958$. In the long-time limit,

$$D_{\text{coherent}} = \frac{|\Delta\omega|}{\sqrt{\Gamma^2 + \Delta\omega^2}}. \quad (18)$$

Consequently a claimed complete coherent-output error at most $\varepsilon < 1$ requires, for this family,

$$|\Delta\omega| \leq \Gamma\varepsilon/\sqrt{1-\varepsilon^2}. \quad (19)$$

For one percent error the permissible detuning is approximately 0.0100005Γ . This is a usable tolerance, not a proposed new physical law. Interference of the outgoing temporal mode with a calibrated reference mode, or a phase-sensitive field measurement, can probe the missing phase; ideal equal-prior Helstrom discrimination succeeds with probability $(1 + D_{\text{coherent}})/2$. That optimal measurement is an in-principle statement. Its apparatus is not included for free in the detector model. Counting and coherent-field measurements are alternative output experiments; one cannot first measure an exact click time and then claim its erased temporal coherence was retained.

2.4 When is the binary reduction microscopically closed?

Let the true finite detector have an untriggered subspace $P_0\mathcal{H}$, a triggered subspace $P_1\mathcal{H}$, a block-preserving Hamiltonian, and absorbing jumps $J_a = P_1 J_a P_0$. Define $Q = \sum_a J_a^* J_a$. The binary population readout is $C(\rho) = (\text{Tr } P_0 \rho, \text{Tr } P_1 \rho)$.

Proposition 2.3 (Exact closure and finite spectral repair). *For a time-independent generator, the binary populations obey the same absorbing scalar rate law $\dot{p}_0 = -\Gamma p_0$ for every microscopic input if and only if*

$$Q = \Gamma P_0. \quad (20)$$

This statement concerns populations; fixed output marks and a two-dimensional quantum-coherence law require additional assumptions.

If instead $Q = \sum_{i=1}^k \gamma_i |i\rangle\langle i|$ on $P_0\mathcal{H}$, $[H, Q] = 0$, consider the declared controlled-rate model whose complete jump preserves the label and is

$$J = \sum_{i=1}^k \sqrt{\gamma_i} |1, i\rangle\langle 0, i| \quad (21)$$

A k -label register is an exact finite repair of this specified model, including label coherence and its inert purifying reference; additional output marks or phases, if present, must be retained in the repaired jump as well. The register must be prepared once as part of the original stock, not redrawn after each use.

Proof. The dual generator gives $\mathcal{L}^*(P_0) = -Q$ and $\mathcal{L}^*(P_1) = Q$. Equality of the scalar population derivative for all states forces $Q = \Gamma P_0$ by polarization; this condition also makes the two-dimensional population observable space invariant under the dual generator, proving sufficiency for all times. Under the second hypotheses, each label has no-jump amplitude $e^{-\gamma_i t/2}$, with its specified commuting Hamiltonian phase, and jump amplitude $\sqrt{\gamma_i} e^{-\gamma_i t/2}$. Equation (21) and its common no-jump dynamics give exactly these amplitudes, also when the label is entangled with an inert reference. Full comparison still retains any additional microscopic channels; none are deleted by naming a label. $\square$

For a diagonal original label stock with probabilities $w_i > 0$, survival is $S(t) = \sum_i w_i e^{-\gamma_i t}$, not generally an exponential. Its no-transition posterior is $w_i(t) = w_i e^{-\gamma_i t}/S(t)$. Therefore a reset wave or a reset binary flag does not reset the actual label stock. An arbitrarily prescribed time-dependent hazard $-S'(t)/S(t)$ can imitate the survival of this one specified mixture; it does not establish preparation-independent microscopic closure, restore the retained label correlations, or license independent successive samples.

For k distinct rates, the number of labels in a time-homogeneous linear classical realization cannot in general be reduced below k . Indeed its $k \times k$ derivative Hankel matrix is

$$\mathcal{H}_{ab} = (-1)^{a+b} S^{(a+b)}(0) = \sum_i w_i \gamma_i^{a+b}, \quad 0 \leq a, b < k, \quad \det \mathcal{H} = \left(\prod_i w_i \right) \prod_{i < j} (\gamma_i - \gamma_j)^2 > 0.$$

Any n -state time-homogeneous linear realization $S(t) = c^* e^{At} b$ has Hankel rank at most n , by factoring each derivative as $c^* A^a A^b b$. Thus $n \geq k$. This is a standard realization-rank argument applied constructively here, not a lower bound on every possible quantum or time-dependent surrogate.

An approximate repair can cluster rates while retaining their original label. For a rate γ replaced by $\kappa > 0$, the exact affinity of its complete time/null law is

$$a_T(\gamma, \kappa) = \frac{2\sqrt{\gamma\kappa}}{\gamma + \kappa} + \left(1 - \frac{2\sqrt{\gamma\kappa}}{\gamma + \kappa}\right) e^{-(\gamma+\kappa)T/2}. \quad (22)$$

This follows by adding the null overlap $e^{-(\gamma+\kappa)T/2}$ to the integral of $\sqrt{\gamma\kappa}e^{-(\gamma+\kappa)t/2}$. Cauchy–Schwarz applied to $|p - q| = |\sqrt{p} - \sqrt{q}|(\sqrt{p} + \sqrt{q})$ gives $\text{TV}(P_\gamma, P_\kappa) \leq \sqrt{1 - a_T^2}$. The same bound holds for their coherent positive-amplitude wavepackets by the pure-state identity. With original label weights w_i the cq cost is at most $\sum_i w_i \sqrt{1 - a_T(\gamma_i, \kappa_{g(i)})^2}$. For an unknown label distribution use the maximum. More strongly, for the fully specified controlled model with the same retained labels, Hamiltonian phases and marks on both sides, the coherent cross-Gram matrix is diagonal with entries $a_T(\gamma_i, \kappa_{g(i)}) > 0$. Equation (12) then gives the exact reference-uniform coherent-channel distance

$$\max_i \sqrt{1 - a_T(\gamma_i, \kappa_{g(i)})^2}.$$

This bound admits label superpositions and arbitrary inert references; it does not replace their amplitudes by a classical mixture. Different retained marks or phase schedules must instead be charged with the full amplitude certificate. Thus a finite approximate repair has an explicit worst-case tolerance, beyond a match to one mean rate.

3 Finite charge references and complete heralded phase measurement

3.1 Physical encoding and admitted experiment

Let the source S have basis $|x\rangle$, $x = 0, \dots, N - 1$, $N \geq 2$, and dimensionless charge $Q_S = \sum_x x|x\rangle\langle x|$. For an energy implementation take $H_S = \hbar\omega Q_S$, with $\omega > 0$ in radians per second. A fixed, known chirp setting a defines the orthonormal basis

$$|a, b\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \exp[2\pi i(ax^2 + bx)/N] |x\rangle, \quad b = 0, \dots, N - 1. \quad (23)$$

No primality assumption is required for this single basis. Its diagonal quadratic mask commutes with Q_S ; the noncommuting resource is the Fourier mixing/readout, not the diagonal phase notation.

The active reference C has integer charges n and a fixed prepared state τ_C , independent of the unknown source and its arbitrary inert reference R . At most M charge sectors are populated. The ideal pure reference is

$$|u_M\rangle = M^{-1/2} \sum_{n=0}^{M-1} |n\rangle.$$

Admissible readout effects on SC commute with $Q = Q_S + Q_C$. All other accessible apparatus states are invariant under the same $U(1)$ symmetry and pointer records carry zero charge. Equivalently one may use charge-preserving joint unitaries with invariant ancillas and a neutral pointer satisfying the Yanase condition. A neutral-pointer Naimark dilation realizes every invariant POVM used here.

This is an explicit conditional resource model. It applies, for example, to coherent mixing of equally spaced energy modes when no additional phase-bearing drive is admitted. It does *not* impose a phase-reference cost on arbitrary degenerate spatial modes: if Q_S is proportional to I , their basis

measurement commutes with charge and can be exact with no such token. An accessible macroscopic oscillator, pump, hidden phase controller or purifying asymmetry register must be added to C ; its resource cannot be silently treated as free. A clock and fixed setting record are common to the compared experiments; their allowed couplings must obey the declared symmetry. No speed, bandwidth or interaction-strength bound is inferred from this resource count.

The task is an exact state-independent heralded measurement: outcomes b have induced effects $\eta|a, b\rangle\langle a, b|$, and the retained null effect is $(1 - \eta)I$. The theorem below optimizes η in this class, not every possible approximate measurement with errors.

3.2 Optimal exact heralding from finite reference support

Sharp support/loss theorem. For any reference supported on at most M integer charge sectors, the largest possible exact heralding probability is

$$\eta_{\max}(N, M) = \frac{(M - N + 1)_+}{M}. \quad (24)$$

A uniform pure reference on M consecutive charges attains it. For $M \geq N$, equality among pure reference states requires this uniform probability profile on a consecutive interval, up to charge translation and arbitrary known phases compensated by the readout. For $M < N$, every reference has zero exact success; no uniqueness assertion applies.

Proof. For a pure reference $|r\rangle = \sum_n r_n |n\rangle$ set $p_n = |r_n|^2$, with missing charges assigned $p_n = 0$. An invariant measurement sees independently the total-charge blocks

$$|v_{bq}\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} r_{q-x} e_N(ax^2 + bx) |x, q-x\rangle, \quad e_N(t) = e^{2\pi it/N}.$$

Twirling is only a device for calculating invariant record probabilities; it does not authorize discarding an actually retained coherent charge archive.

Within a block containing all N terms the reciprocal vectors

$$|w_{bq}\rangle = \frac{1}{\sqrt{N}} \sum_x \frac{e_N(ax^2 + bx)}{r_{q-x}^*} |x, q-x\rangle$$

satisfy $\langle w_{bq} | v_{b'q} \rangle = \delta_{bb'}$. Averaging a measurement over the cyclic relabelings b preserves its mean success, invariance and zero-error property. Every averaged zero-error effect therefore has form $t_q |w_{bq}\rangle\langle w_{bq}|$. Its normalization requires $t_q \sum_x p_{q-x}^{-1} |x, q-x\rangle\langle x, q-x| \leq I$, so $t_q \leq \min_x p_{q-x}$, which is attainable. A block with fewer than N terms cannot give positive symmetric zero-error success: otherwise all N vectors would be linearly independent, since applying their zero-error effects to a linear relation isolates each coefficient. There are fewer than N available dimensions. Thus the exact optimal average success is

$$\eta(r) = \sum_q \min_{0 \leq x < N} p_{q-x}. \quad (25)$$

An exact state-independent heralded measurement must in particular discriminate this basis with mean success η , so this is an upper bound for the nominated task.

For a finite subset A of integer charges, the number of contained length- N consecutive windows is $\sum_{\text{runs}} (|\text{run}| - N + 1)_+ \leq (|A| - N + 1)_+$. Apply the layer-cake identity to $A_t = \{n : p_n > t\}$:

$$\sum_q \min_x p_{q-x} = \int_0^\infty \#\{\text{length-}N \text{ windows in } A_t\} dt \leq \frac{(M - N + 1)_+}{M} \int_0^\infty |A_t| dt = \eta_{\max}.$$

For $M \geq N$, the inequality is strict at every nonempty superlevel set of size below M , and at a nonconsecutive set of size M . Equality therefore forces the stated uniform consecutive profile. Uniform $p_n = 1/M$ has $M - N + 1$ complete windows and attains the bound. Mixed references cannot improve it: decompose the reference into pure states within the same charge support. Zero-error probabilities are nonnegative, so a zero-error mixture measurement is zero-error on each component, and its success is bounded by the same number on every component. An invariant ancillary state decomposes into pure fixed-charge states. On each component its addition merely shifts the reference charge labels, with degeneracy vectors orthogonal between sectors; the same bound applies. Such ancillas therefore do not improve it. The construction below attains the full state-independent effects, not just basis-state average success. $\square$

For any allowed loss ceiling $0 < \epsilon < 1$, exact heralding requires and permits

$$M \geq \left\lceil \frac{N - 1}{\epsilon} \right\rceil. \quad (26)$$

At the hypothetical $N = 180$ energy-mode encoding, one-percent null probability requires at least 17,900 populated reference charges. This is a conditional resource requirement for the declared charge encoding; angular divisions alone do not specify physical reference states. An attaining reference on charges $0, \dots, M - 1$ has energy span $(M - 1)\hbar\omega$ and mean energy $(M - 1)\hbar\omega/2$. No numerical energy is inferred without a physical ω .

3.3 Full retained instrument: coherent versus classical charge record

Take $M \geq N$, $K = M - N + 1$, $\eta = K/M$, and $\mathcal{I} = \{N - 1, \dots, M - 1\}$. Define normalized vectors within total-charge sectors

$$|\phi_{bq}\rangle = \frac{1}{\sqrt{N}} \sum_x e_N(ax^2 + bx)|x, q - x\rangle, \quad q \in \mathcal{I}.$$

The mutually orthogonal invariant projectors $F_b = \sum_{q \in \mathcal{I}} |\phi_{bq}\rangle\langle\phi_{bq}|$ and $F_? = I - \sum_b F_b$ form a projective measurement on SC . Outside the occupied input sector the identity may be extended arbitrarily without affecting this experiment.

Retain a charge archive A by the invariant coherent copying isometry $C_Q|\xi_q\rangle = |\xi_q\rangle|q\rangle_A$. Its basis states are neutral; this records q without changing the charge of SC . Prepare C in $|u_M\rangle$ and let $J\psi = \psi \otimes u_M$. The full operators from the logical input S to SCA are

$$K_b = C_Q F_b J = \sqrt{\eta} |\chi_b\rangle\langle a, b|, \quad |\chi_b\rangle = K^{-1/2} \sum_{q \in \mathcal{I}} |\phi_{bq}\rangle|q\rangle_A, \quad K_? = C_Q F_? J. \quad (27)$$

Direct substitution gives

$$K_?^\dagger K_? = (1 - \eta)I, \quad \sum_b K_b^\dagger K_b + K_?^\dagger K_? = I. \quad (28)$$

Consequently, for every joint input ρ_{SR} ,

$$\mathcal{I}_b(\rho_{SR}) = \eta |\chi_b\rangle\langle\chi_b| \otimes \langle a, b | \rho_{SR} | a, b \rangle, \quad \mathcal{I}_?(\rho_{SR}) = (K_? \otimes I_R) \rho_{SR} (K_?^\dagger \otimes I_R). \quad (29)$$

This retains the active phase reference, source residual, coherent charge archive, neutral outcome pointer and arbitrary inert reference. The normalized null operator is an isometry into the joint output. It preserves the logical source information there; it does not restore the initial hardware source or reference state, provide a free symmetry-preserving inverse, or make the reference reusable.

A genuinely classical charge record is a different instrument: apply charge dephasing to A and retain the value q . Then the conclusive output state becomes the fixed mixed state

$$\sigma_b = K^{-1} \sum_{q \in \mathcal{I}} |\phi_{bq}\rangle \langle \phi_{bq}| \otimes |q\rangle \langle q|_A,$$

still multiplying $\eta \langle a, b | \rho_{SR} | a, b \rangle$. The null channel is now a sum of charge-resolved Kraus branches and is not generally an isometry. Indeed their $K_{?,q}^\dagger K_{?,q}$ are different partial diagonal projectors divided by M ; the Knill–Laflamme scalar condition fails. Thus the coherent-null conclusion is not applied to a classically measured charge archive.

Exact complete-output comparison. Declare the completed ideal basis-measurement channel $\mathcal{J}$ to have output state $|\chi_b\rangle \langle \chi_b|$ in outcome b and zero null branch. For the classical-charge version use σ_b instead. These are specified completions of the ideal readout, not a microscopic identification with an arbitrary detector. The actual channel $\mathcal{I}$ has exactly

$$\frac{1}{2} \|\mathcal{I} - \mathcal{J}\|_\diamond = 1 - \eta = \frac{N - 1}{M}. \quad (30)$$

In fact every normalized input, including every inert-reference extension, attains this distance.

Proof. Each actual conclusive branch is precisely η times its ideal branch on every ρ_{SR} by Eq. (29). The conclusive difference is negative with total trace $1 - \eta$. The orthogonal actual null block is positive with the same trace. Adding the block trace norms gives $2(1 - \eta)$. The same argument survives classical charge recording, since the conclusive factorization and null trace remain exact. This is an exact full-output identity, not a scalar probability bound substituted for an instrument error. $\square$

The isometry $\sum_y F_y \otimes |y\rangle$ with a neutral outcome pointer commutes with total charge and extends to a blockwise charge-preserving unitary. The coherent charge-copy map has the same property. Hence the model has a finite unitary realization; it has not hidden a noninvariant Fourier gate in the apparatus. A common fixed readout of the outcome pointer supplies the declared instrument. Reading additional microscopic pointer/environment degrees of freedom requires declaring their common states and couplings as well; no universal detector equivalence follows.

The finite resource ledger of this explicit dilation is: source dimension N , active reference dimension M , neutral charge-archive dimension $N + M - 1$, and neutral outcome-pointer dimension $N + 1$, plus the nominated fixed controller/clock and arbitrary inert R . A coherent charge archive is quantum memory; a classical one still retains its entire record. The optimum in Eq. (24) counts populated phase-reference charge sectors only. It is not a bound on total apparatus or complete-archive dimension, and it does not identify a single-use Choi rank with retained history capacity.

3.4 Reference dephasing: exact error–erasure boundary

The exact-success boundary is fragile under a specified physical noise model. Let the accessible reference state be

$$\tau_\nu = (1 - \nu) |u_M\rangle \langle u_M| + \nu I_M / M, \quad 0 \leq \nu \leq 1. \quad (31)$$

This is partial dephasing in charge, not an unspecified decoherence rate. Its environment can be included and retained as a fixed passive purification, but is unavailable to the nominated decoder for measurement, conditioning, postselection or feedback. Retaining that quantum output does not grant this access. If a later reader can use a classical noise flag or measure the purification, that is extra accessible side information or a phase resource and the optimization must be recomputed. The following frontier is not asserted optimal for that enlarged complete-archive readout.

Give the N basis letters equal priors and retain every result, including nulls and charge records. For total charge q let

$$X_q = \{x : 0 \leq x < N, 0 \leq q - x < M\}, \quad m_q = |X_q|, \quad w_q = \frac{m_q}{NM}, \quad c_q = \frac{\nu + (1 - \nu)m_q}{N}.$$

The weights w_q sum to one. The subnormalized input block is

$$\rho_{bq} = \frac{(1 - \nu)|u_{bq}\rangle\langle u_{bq}| + \nu I_{m_q}}{NM}, \quad |u_{bq}\rangle = \sum_{x \in X_q} e_N(ax^2 + bx)|x, q - x\rangle. \quad (32)$$

Its prior average is $I_{m_q}/(NM)$.

Exact fixed-token frontier. For a specified conclusive probability $P \in [0, 1]$, the maximum unconditional correct-letter probability is exactly

$$C_{\max}(P) = \max_{\substack{0 \leq t_q \leq 1 \\ \sum_q w_q t_q = P}} \sum_q w_q c_q t_q. \quad (33)$$

It is obtained by accepting the highest m_q blocks first, with fractional acceptance of the final tied block if needed. The minimum conditional label error is $1 - C_{\max}(P)/P$ for $P > 0$. This is globally optimal over all invariant POVMs at this fixed reference state; it is not claimed optimal over all reference profiles at intermediate P .

Proof with a full POVM upper certificate. For arbitrary conclusive effects E_{bq} with $\sum_b E_{bq} \leq I$, the largest eigenvalue of ρ_{bq} is $[\nu + (1 - \nu)m_q]/(NM)$. Thus correct probability in that block is at most c_q times its conclusive probability. The latter is at most w_q because the prior average is $I_{m_q}/(NM)$. This proves the linear-program upper bound without assuming a covariant candidate.

It is attained by the invariant effects

$$E_{bq} = \frac{t_q}{N}|u_{bq}\rangle\langle u_{bq}|, \quad \sum_b E_{bq} = t_q I_{m_q},$$

and the retained complementary null effect. Each effect occupies a maximal-eigenvalue direction of its matching input block. The linear objective therefore sorts the c_q . For a dual certificate at multiplier λ , choose $Y_q = \max(0, c_q - \lambda)I_{m_q}/(NM)$. It obeys $Y_q \geq \rho_{bq}/N - \lambda I_{m_q}/(NM)$ for all b , and its trace sum equals the attained Lagrange objective. This independently certifies the boundary against arbitrary effects. $\square$

For $M \geq N$, there are $M - N + 1$ blocks of size N , total weight $\eta_0 = (M - N + 1)/M$; every smaller size $m = 1, \dots, N - 1$ occurs twice, weight $2m/(NM)$. Therefore for $0 < P \leq \eta_0$ the sharp error floor is

$$1 - C_{\max}(P)/P = \nu(1 - 1/N). \quad (34)$$

Beyond η_0 , the additional slopes are $[\nu + (1 - \nu)m]/N$ for $m = N - 1, \dots, 1$. This is a closed piecewise-linear correct-probability boundary.

At the interior-block operating point $P = \eta_0$, the effects on every unknown logical input, not only the basis test states, are

$$E_b = \eta_0[(1 - \nu)|a, b\rangle\langle a, b| + \nu I/N], \quad E_{\text{?}} = (1 - \eta_0)I.$$

Thus for the declared classical pointer channel, retaining an arbitrary inert reference, the exact half-diamond discrepancy from the pure-token heralded pointer channel is $\eta_0\nu(1 - 1/N)$. It is a uniform

randomization of the ideal letter with probability ν ; the maximum column total variation of that classical postprocessing is $1 - 1/N$, attained by a basis input. This pointer certificate is not a full active-token output bound.

Any $\nu > 0$ makes exact zero-error heralding impossible at every positive conclusive probability: all letter states have the same full support within each block. More generally, any strictly positive reference density matrix on its populated charge space has the same obstruction: every relevant principal submatrix is strictly positive, so each letter's total-charge block has the same full support. This is an exact-zero-error obstruction, not a claim that a small noise produces a large ordinary statistical error. The nearby faithful repair is not to discard the reader but to keep the complete null/error archive and use the exact controlled error floor. For example $N = 5$, $M = 9$, $\nu = 0.03$ permits $P = 5/9$ with minimum conditional error 0.024; no lower-confidence filtering can beat that value. At all-conclusive readout,

$$C_{\max}(1) = (1 - \nu) \left[1 - \frac{N^2 - 1}{3NM} \right] + \frac{\nu}{N}, \quad M \geq N.$$

All these formulas concern declared label statistics. A full noisy retained-output comparison additionally uses the explicit Kraus maps and retained purification; it is not inferred from label error alone. The effects and their positive square roots already give such a normalized finite instrument, tensored with I_R for an arbitrary inert reference.

3.5 Different objective: best no-erasure reference

Exact finite eigenvalue optimization. For a pure reference on charges $0, \dots, M-1$ with amplitude magnitudes $r_n \geq 0$, the maximum equal-prior correct-letter probability with no inconclusive output is

$$P_{\text{correct}}(r) = \frac{1}{N^2} \sum_q \left(\sum_{x=0}^{N-1} r_{q-x} \right)^2 = \frac{r^\top T_{N,M} r}{N^2}, \quad (T_{N,M})_{nm} = (N - |n - m|)_+. \quad (35)$$

Consequently the best reference has $P_{\text{correct}}^{\text{opt}} = \lambda_{\max}(T_{N,M})/N^2$, with its nonnegative principal eigenvector as amplitude profile.

Proof. Within each total-charge block, cyclic symmetrization makes the seed effect have diagonal entries $1/N$ and hence off-diagonal magnitudes at most $1/N$ by positivity. This upper-bounds correct probability by the first expression, attained by a phase-aligned rank-one seed. Summing q counts $N - |n - m|$ pairs when $|n - m| < N$, proving the matrix formula. The variational principle and Perron positivity give the optimum. Mixed references cannot improve a maximization over all pure states and measurements. Compressing any nonconsecutive set of M integer charges decreases every pairwise distance, hence increases this nonnegative kernel entrywise; consecutive charges are sufficient for the global optimum at that cardinality. $\square$

At $N = 2$ this gives the known sine-state result

$$r_n = \sqrt{\frac{2}{M+1}} \sin \frac{(n+1)\pi}{M+1}, \quad P_{\text{error}}^{\text{opt}} = \frac{1 - \cos[\pi/(M+1)]}{2}.$$

For the binary *classical outcome channel*, this is also the exact minimum half-diamond error to the ideal basis measurement, uniform over inert references: the achieved effects are $(I \pm vX)/2$ in target-basis coordinates after the known diagonal chirp mask, with $v = \cos[\pi/(M+1)]$, and any channel comparison is lower-bounded by its equal-prior basis discrimination error. This assertion does not compare retained active-token outputs. The quadratic M^{-2} label-error scaling and the M^{-1} exact-heralding loss law answer different questions.

4 Preparation cost of a product-response reconstruction

Take a finite mode register or a finite spatial bin, a normalized state ρ , a dimensionless intensity effect $0 \leq A \leq I$, and the normalized template projector $B = |R\rangle\langle R|$. For a pure field $\Psi = \sqrt{N}\psi$ with fixed known total intensity $N > 0$, the normalized template coherence used in [7, 8] is $\langle\psi|B|\psi\rangle$ and the bin intensity is a fixed intensity scale times $N\langle\psi|A|\psi\rangle$. Absorb these fixed scales and the susceptibility into $\kappa \geq 0$ with units s^{-1} . This gives the precise proposed quantum-state interpretation

$$x = \text{Tr}(A\rho), \quad y = \text{Tr}(B\rho), \quad \Gamma(\rho) = \kappa xy, \quad p_{\text{src}}(\rho) = 1 - e^{-\eta xy}, \quad \eta = \kappa \Delta t.$$

A state-dependent total intensity or susceptibility must be treated as a further source/controller variable; it cannot be silently absorbed into a state-independent detector. Point evaluation on an unrestricted continuum is not a bounded effect, so the finite bin is also an explicit addition.

Proposition 4.1 (Single-copy obstruction and two-copy effect). *The product $\text{Tr}(A\rho)\text{Tr}(B\rho)$ is affine on the full density-matrix state space if and only if at least one of A, B is scalar. In general it is not a single-copy click effect. Nevertheless, for $0 \leq \eta \leq 1$ the two-copy effect*

$$E_c = \eta A \otimes B, \quad E_0 = I - E_c$$

is positive and complete, and on independently prepared copies it gives $p_c = \eta xy$. Its difference from the source frame rule obeys

$$0 \leq p_c - p_{\text{src}} \leq \frac{1}{2}\eta^2 x^2 y^2 \leq \frac{1}{2}\eta^2.$$

Proof. Write $\rho = I/d + \epsilon X$ for arbitrary traceless Hermitian X and sufficiently small ϵ . Affinity requires $\text{Tr}(AX)\text{Tr}(BX) = 0$ for every X . Two nonzero linear functionals cannot have their kernels cover the whole vector space: choose a vector off the first kernel and, if necessary, add a multiple of a vector off the second. Thus one functional vanishes on all traceless matrices, equivalently its effect is scalar. The converse is immediate. For an explicit obstruction even to an ensemble-independent extension of pure-state rates, take $A = B = |0\rangle\langle 0|$. The equally weighted $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$ ensembles both prepare $I/2$, but their averaged pure-state product rates are $1/2$ and $1/4$. Positivity and completeness of the two-copy effects follow from $0 \leq A \otimes B \leq I$. The bound uses $0 \leq z - (1 - e^{-z}) \leq z^2/2$ for $z \geq 0$. $\square$

An explicit retained-reference instrument is $\mathcal{I}_c(\omega) = \sqrt{E_c}\omega\sqrt{E_c}$ and $\mathcal{I}_0(\omega) = \sqrt{E_0}\omega\sqrt{E_0}$, with the identity on an arbitrary reference, and orthogonal recorded flags. Its probabilities on a correlated two-register input are $\text{Tr}(E_c\omega_{12})$; they factor as ηxy only under the declared product preparation. This needs two preparations, not a machine cloning an unknown input. The standard isometry $V = |c\rangle\sqrt{E_c} + |0\rangle\sqrt{E_0}$ provides a finite apparatus dilation, but a material coupling remains to be supplied. The square roots specify a state update not determined by the scalar rate alone.

Proposition 4.2 (Exact exponential needs an unbounded stock or more input). *For $\eta > 0$, no fixed finite number m of identically prepared copies and a fixed quantum effect implements $1 - e^{-\eta p^2}$ for every unknown qubit pure state with $p = |\langle 0|\psi\rangle|^2$. An exact protocol using a random number of independent two-copy tests exists: sample $K \sim \text{Poisson}(\eta)$ independently, perform the effect $A \otimes B$ on each independently prepared pair, and report a click if any test clicks, performing all K tests. Then $p_c = 1 - e^{-\eta xy}$ and the expected number of consumed copies is 2η (early stopping can reduce consumption but is not used in this count). If the protocol instead has at most m pairs and retains a separate stock-loss flag when $K > m$, its classical report law is within $\text{Pr}\{K > m\}$ in total variation of the exact law (padded with a zero-probability loss flag). For $m + 1 > \eta$,*

$$\text{Pr}\{K > m\} \leq e^{-\eta} \left(\frac{e\eta}{m+1} \right)^{m+1}.$$

Proof. Apply a putative m -copy effect to $|\psi_{p,\phi}\rangle = \sqrt{p}|0\rangle + e^{i\phi}\sqrt{1-p}|1\rangle$. Since the desired result is independent of ϕ , average the effect over the common phase. On these symmetric inputs its expectation becomes $\sum_{k=0}^m \binom{m}{k} p^k (1-p)^{m-k} e_k$, with fixed $0 \leq e_k \leq 1$. This is a polynomial in p , whereas $1 - e^{-\eta p^2}$ is not. Extra state-independent ancillas or randomization do not change the polynomial obstruction. For the random-stock construction the no-click probability is $\mathbb{E}(1 - xy)^K = \exp(-\eta xy)$. Couple the exact and truncated protocols using the same K and tests; their classical reports differ only on $K > m$. The tail bound is the usual Poisson exponential bound, optimized at $e^t = (m+1)/\eta$. $\square$

For $\eta = 0.8$, $xy = 0.24$ and at most six tests (twelve copies), the exact click probability is 0.1746931315; the truncated value is 0.1746753459, with retained stock-loss probability 2.0746838×10^{-5} . The analytic tail certificate is 1.2547853×10^{-4} . Standard-library code `supplement/operational/10_verify_product_reader.py` reproduces these values. This is an exact preparation-cost interface for one interpretation of the source, not a claim that a classical driven field needs duplicated unknown quantum states. It supplies neither fresh independence from a reset nor a primitive Born-event selection. The comparison bound concerns the declared classical reports; it does not identify quantum residual outputs of two incompatible apparatuses. All loss flags, controller draws, input stock, clock and retained references remain part of the complete model.

5 Symmetric-family calibration with explicit confidence and drift

5.1 Source, trusted assays and all nulls

Use $\mathcal{H}_S = \text{span}\{|\text{vac}\rangle, |0\rangle, |1\rangle\}$. The qubit marginal block is $w\rho$, with

$$\rho = (I + tX + vY + uZ)/2, \quad u^2 + t^2 + v^2 \leq 1, \quad p = (1 + u)/2.$$

For hypothesis $h = -$, apply the stipulated qubit Z phase flip to the $h = +$ preparation, reversing the transverse components. The source/controller relation is an assumption. Its hidden true label is not available to the classifier. Sequential use must also obey its declared conditional-state conditions; calibration does not establish them. Vacuum/qubit coherence may exist, since dark-free label effects annihilate vacuum and hence do not see that coherence.

Predetermine N_Z, N_X calibration trials using independently prepared copies of one stationary source marginal. A common trusted calibrator has the same state-independent efficiency $\eta_c > 0$ in both settings and known contrast $0 < \gamma \leq 1$:

$$Z : \quad E_0 = \eta_c |0\rangle\langle 0|, \quad E_1 = \eta_c |1\rangle\langle 1|, \quad (36)$$

$$X : \quad E_{\pm} = \eta_c (P_q \pm \gamma X)/2, \quad E_{\emptyset} = I - \eta_c P_q. \quad (37)$$

Every null is counted and retained. The positive square roots with a neutral pointer give the finite isometry $V = \sum_j \sqrt{E_j} \otimes |j\rangle$, retaining output source and any inert reference. For degenerate modes a balanced rotation and a complete basis absorber realize $\gamma = 1$; known unsharpness permits other values. Nondegenerate energy modes require the phase-reference stock from the finite-reference model. No calibration phase resource is free.

A hit has probability $w\eta_c$, independent of setting. Its conditional Bernoulli parameters are p and $q = (1 + \gamma t)/2$. This conditioning is justified by independent preparations and state-independent loss, not by a reset wave. Let n_Z, n_X be hit counts and K_Z, K_X the true counts of label $0, +$. An adversary may change at most k_Z, k_X accepted labels, even after seeing all data, producing $\widetilde{K}_Z, \widetilde{K}_X$. Nulls cannot be forged, suppressed or selected. This is robustness of the nominated reported-count readout; no optimum over hidden corruption-controller records is asserted.

5.2 One global finite confidence event

For binomial count k in n trials, let $[L_\beta(k, n), U_\beta(k, n)]$ be the two-sided Clopper–Pearson interval of failure at most β . In the interior,

$$L_\beta = B^{-1}(\beta/2; k, n - k + 1), \quad U_\beta = B^{-1}(1 - \beta/2; k + 1, n - k).$$

Use $L(0, n) = 0, U(n, n) = 1$, and $[0, 1]$ if $n = 0$. Define the corruption envelope

$$\mathcal{C}_\beta(\widetilde{K}, n, k_f) = [L_\beta((\widetilde{K} - k_f)_+, n), U_\beta(\min(n, \widetilde{K} + k_f), n)]. \quad (38)$$

Allocate source failure α_S , set $\beta = \alpha_S/3$, and take

$$I_p = \mathcal{C}_\beta(\widetilde{K}_Z, n_Z, k_Z), \quad I_q = \mathcal{C}_\beta(\widetilde{K}_X, n_X, k_X), \quad I_t = [(2I_q^L - 1)/\gamma, (2I_q^U - 1)/\gamma] \cap [-1, 1].$$

Let I_w be the ordinary CP interval for total hits $n_Z + n_X$ in $N_Z + N_X$ trials. For trusted $\eta_c \in [\eta_L, \eta_U]$, use

$$I_w = [I_w^L/\eta_U, I_w^U/\eta_L] \cap [0, 1],$$

with upper endpoint one when $\eta_L = 0$. Then

$$\mathcal{G}_S = \{p_0 \in I_p, t_0 \in I_t, w_0 \in I_w\}, \quad \Pr(\mathcal{G}_S) \geq 1 - \alpha_S. \quad (39)$$

A statistical efficiency calibration requires its own allocated failure.

Proof. Conditional on a hit count the true label count is binomial; CP coverage remains valid after averaging over that random count. Monotonic endpoints and $|K - \widetilde{K}| \leq k_f$ make the envelope contain the true count's interval on every corrupted transcript. Total hits are binomial with parameter $w_0\eta_c$. A union bound proves coverage; independence of the three intervals is unnecessary. CP coverage itself follows by binomial-tail inversion, each excluded tail having probability at most $\beta/2$. Thus this is finite-sample coverage for each fixed underlying state, not a posterior probability on a parameter. $\square$

Hits alone identify $w\eta_c$, not its factors. Without efficiency information the safe bound $w \geq I_w^L$ still follows from $\eta_c \leq 1$. One event constrains the source for every reader, allowing subsequent reader or sign-orientation choice without another source-confidence penalty. A transverse interval straddling zero does not become a positive-direction certificate merely by clipping its lower endpoint.

5.3 Exact source-class minimax certificate

For every future actual conditional input, given the preceding record, assume

$$|p - p_0| \leq d_p, \quad t \geq t_0 - d_t, \quad w \geq w_0 - d_w.$$

Inputs may otherwise be adversarial and correlated through history. These are future admission conditions, not consequences of finite calibration. Trace-distance qubit drift at most δ suffices for $d_p = \delta, d_t = 2\delta$, but the componentwise class below may be larger.

On $\mathcal{G}_S$, enlarge the population interval to I'_p , put $l = I'_t - d_t \geq 0$, $w_L = (I'_w - d_w)_+$, and define

$$I = [2(I'_p)^L - 1, 2(I'_p)^U - 1] \cap [-R, R], \quad R = \sqrt{1 - l^2}. \quad (40)$$

If $l < 0$ this positive-direction theorem is not used; an empty set means model/confidence inconsistency, not a vacuous certificate. Use the real symmetric reader

$$E_\pm = \frac{1}{2} \begin{pmatrix} x & \pm z \\ \pm z & y \end{pmatrix}, \quad E_\emptyset = I - \text{diag}(x, y), \quad 0 \leq x, y \leq 1, \quad 0 \leq z \leq \sqrt{xy}. \quad (41)$$

The reader acts as null on vacuum. Its transverse axis can differ from the calibration axis by any $|\theta| \leq \Delta \leq \pi/2$. Only differential phase matters; global phase convention is irrelevant. The actual coefficients must be known or calibrated. Set $d = (x + y)/2, e = (x - y)/2$, with $A(u) = d + eu > 0$ on I .

C2: sharp error and success. For $a = l \cos \Delta, b = \sin \Delta$,

$$\epsilon_* = \frac{1}{2} \left[1 - z \min_{u \in I} \frac{a - b\sqrt{R^2 - u^2}}{d + eu} \right], \quad (42)$$

$$Q_* = w_L \min_{u \in I} (d + eu). \quad (43)$$

These are the exact worst conditional error and guaranteed unconditional conclusive probability over the stated physical source/drift/axis class. The conditional-error statement applies only to members with positive conclusive probability. In particular, $Q_* = 0$ supplies no positive-rate guarantee and assigns no value to a 0/0 conditional error. They hold for every inert-reference extension, as statements about marginal labels, not full output-state equality.

Proof. At fixed u , Bloch positivity gives $t^2 + v^2 \leq 1 - u^2$. The smallest reader component is

$$X_{\min}(u) = l \cos \Delta - \sqrt{1 - u^2 - l^2} \sin \Delta.$$

Choose the adverse sign of v , saturate its allowed magnitude, and take $t = l, |\theta| = \Delta$: the expression increases in $t \geq 0$ and decreases in $|\theta| \leq \pi/2$. A pure state attains it. For either hypothesis the exact error is $[1 - zX_{\text{reader}}/A(u)]/2$; minimizing over u proves the first claim. Conclusive probability is $wA(u)$, minimized by w_L and an endpoint. References do not change these marginals. $\square$

The finite minimization evaluates endpoints and the admissible interior roots of

$$b(du + eR^2) = ea\sqrt{R^2 - u^2}. \quad (44)$$

Squaring gives

$$(b^2d^2 + e^2a^2)u^2 + 2b^2deR^2u + b^2e^2R^4 - e^2a^2R^2 = 0. \quad (45)$$

Keep the original sign/equality check to reject extraneous roots. This follows by differentiating the ratio. Cases $b = 0$ or $R = 0$ reduce directly. Endpoint-only checking fails, for example, at $I = [-0.8, 0.8], l = 0.6, \Delta = 0.3, (x, y, z) = (0.9, 0.3, \sqrt{0.27})$: the worst point is $u = 0.3685962371$, error 0.3671399216.

C3: sharper coherence interface. Let u_* be the point of I closest to zero, $r_{\max} = \sqrt{1 - u_*^2}$. For $r_{\max} > 0$, define $q = l/r_{\max}$. The exact source-only combined visibility/phase lower bound is

$$m_{\text{cal}} = q \cos \Delta - \sqrt{1 - q^2} \sin \Delta. \quad (46)$$

Divide $X_{\min}(u)$ by $\sqrt{1 - u^2}$. The result increases with $l/\sqrt{1 - u^2}$, hence is minimized at u_* , proving the claim. Supply this directly to the robust frontier; it already includes differential phase and must not be multiplied by another cosine. An independent p, m box is valid but can be looser than Eq. (43), which retains their correlation. If $r_{\max} = 0$, the admitted states have zero transverse coherence; their conditional error is 1/2 for every positive conclusive output. Use C2 directly and do not divide by $r_{\max}$.

The positive-contrast robust frontier uses this interface only when $m_{\text{cal}} \geq 0$. If it is negative, use the general admissible-error certificate C2 or another appropriately enlarged problem; do not insert it into a theorem assuming nonnegative contrast.

At $\Delta = 0$, direct calibration gives $m \geq q$. Separately inferring visibility at least q and phase cosine at least q , then multiplying, gives only q^2 . Those extrema occur at different states: visibility q with zero phase, or unit visibility with phase $\arccos q$. Keeping the measured product removes this unnecessary loss. No source Y assay is needed for this objective: the unmeasured component is retained adversarially within the physical Bloch disk.

5.4 Actual-reader probes and a measured phase-axis bound

A held uncertain loss rate may alter a tuned amplitude ratio, so a worst scalar transfer factor is not an actual-effect certificate. Instead freeze the source-data-selected pulse, then calibrate

$$E_{\pm} = \frac{1}{2} \begin{pmatrix} x & \pm c \\ \pm c^* & y \end{pmatrix}, \quad |c|^2 \leq xy. \quad (47)$$

This structural family follows from the actual two-tap model: a balanced flag matrix times $\text{diag}(a_0, a_1)$ gives $x = |a_0|^2, y = |a_1|^2, c = a_0^* a_1$. Input-independent mixtures preserve it by Cauchy–Schwarz. Unknown path loss and phase change the coefficients but preserve the family. This is a physical-model admission; finite probes cannot prove exact symmetry or dark-free behavior for an unrestricted device.

Use independent identically prepared copies within each trusted-probe batch $|0\rangle, |1\rangle, |+\rangle, |+i\rangle$, with a fixed input-independent reader preparation and stationary pointer probabilities during calibration. Every plus, minus and null is counted:

$$x = P_{\text{con}}|0\rangle, \quad y = P_{\text{con}}|1\rangle, \quad \text{Re } c = P_+|+\rangle - P_-|+\rangle, \quad \text{Im } c = P_-|+i\rangle - P_+|+i\rangle.$$

The fourth probe prevents an imaginary-offdiagonal blind spot. Six marginal CP intervals, with failures summing to α_D , give I_x, I_y, I_r, I_i . Corruption envelopes apply to bounded accepted-label errors. For probe preparation trace error δ_{prep} , widen each event interval by δ_{prep} , since effects are trace-distance contractive. That probe-preparation accuracy is independently trusted.

If $I_r^L > 0$, infer

$$z_L = \sqrt{(I_r^L)^2 + \text{dist}(0, I_i)^2}, \quad \Delta_D = \arctan \frac{\max(|I_i^L|, |I_i^U|)}{I_r^L}.$$

Future hardware allowances widen x, y , lower z_L , and add to Δ_D . They must bound actual conditional instruments, not merely averaged dynamics. The global event is

$$\mathcal{G} = \mathcal{G}_S \cap \mathcal{G}_D, \quad \Pr(\mathcal{G}) \geq 1 - \alpha, \quad \alpha = \alpha_S + \alpha_D. \quad (48)$$

Device coverage is conditional on the preceding source-based choice. Accepting or rejecting the frozen pulse afterward is allowed. Retuning requires fresh calibration or simultaneous catalogue coverage; source-confidence uniformity does not calibrate every hardware setting.

When all admitted source reader components are nonnegative, use x_U, y_U, z_L in Eq. (43) for a valid device-box error certificate. It is sharp for that outer Cartesian coefficient class when the corner is physical. Use x_L, y_L for a safe success lower bound, which need not be attained: positivity or raw-data correlations may exclude that lower corner. No joint optimum over the original multinomial confidence region is claimed. If the source component can be negative, this monotone device-box rule is not used.

Effects certify nominated labels on arbitrary inert-reference inputs; they do not identify the retained field, recoil, controller or bath. The complete operational construction still supplies those maps. Their agreement cannot be inferred from count calibration.

6 Continuum validation of a phase-grid design

Proposition 6.1 (Continuum validation of a phase-grid reader). *Let $\rho_{\pm}(\theta)$ be the pure states of $U_z(\theta)(a|0\rangle \pm b|1\rangle)$, where $a, b \geq 0, a^2 + b^2 = 1, s = a^2 - b^2$, and U_z is a rotation by angle θ about the Bloch z axis. Let a fixed reader and bounded nonnegative decision loss give operators $0 \leq C_{\pm} \leq C_{\max} I$ and risk*

$$R(M, \theta) = \frac{1}{2} \text{Tr}[C_+(M)\rho_+(\theta)] + \frac{1}{2} \text{Tr}[C_-(M)\rho_-(\theta)].$$

For phase differences at most π ,

$$|R(M, \theta) - R(M, \theta')| \leq C_{\max} \sqrt{1 - s^2} \sin(|\theta - \theta'|/2).$$

If a finite grid G has covering radius $h \leq \pi$ in a compact phase interval Θ , and the admissible reader class is the same for both optimizations, then

$$V_G \leq V_\Theta \leq V_G + C_{\max} \sqrt{1 - s^2} \sin(h/2), \quad V_A := \inf_M \sup_{\theta \in A} R(M, \theta).$$

Proof. The pure-state trace distance is exactly

$$D(\rho_\pm(\theta), \rho_\pm(\theta')) = \sqrt{1 - s^2} |\sin((\theta - \theta')/2)|.$$

For trace-zero Hermitian X and $0 \leq C \leq C_{\max} I$, $|\text{Tr}(CX)| \leq C_{\max} \|X\|_1/2$ by the positive and negative parts of X . Apply this to each state difference, average, and choose a grid point within h of each phase. Take infima over the identical admissible reader class. If the grid infimum is not attained, use an arbitrarily close reader and pass to its limiting risk bound. $\square$

This is a standard trace-distance metric-net argument newly specialized to this phase reader, not a new general minimax theorem. If constraints themselves are imposed only on grid points, they need their own modulus margin. In particular, equality of an inconclusive rate at grid points is not proof of that equality between grid points. For an n -copy joint reader one may use the safe modulus $C_{\max} n \sqrt{1 - s^2} \sin(h/2)$, capped by $C_{\max}$, from telescoping trace distance; the one-copy formula must not be reused unchanged.

7 One calibration event and a finite adaptive record

The calibration certificate can be used over a finite sequence without charging its failure probability at every use. This is an application of established martingale concentration, not a new probability law or an independence theorem. It makes the batch-calibration and robust-reader constructions compatible at the level of the complete declared classical record.

Let $\mathcal{F}_0$ contain all original calibration settings, counts, nulls, chosen reader settings and admitted calibration metadata. For the fixed true calibration parameters, let $G \in \mathcal{F}_0$ be the simultaneous confidence event with $\Pr(G) \geq 1 - \alpha$. The drift, access and preparation conditions of the calibration section are physical assumptions: on G , every future actual input conditional on its preceding declared history must remain in the certified set. Neither an unchanged wave, a reset flag nor a null outcome proves this condition for a reused source.

For use t , let C_t indicate a conclusive flag and W_t a wrong conclusive flag; hence $0 \leq W_t \leq C_t \leq 1$. The source-label truth needed to score W_t must be defined in the test model and recorded in a validation experiment; an unknown production label does not become observable from this theorem. For $t \geq 1$, take $\mathcal{F}_t$ to contain the completed-trial outcomes, scoring truths and original calibration record. The physical controller is adapted only to its permitted accessible-record subfiltration. The conditional contracts below are assumed on this augmented mathematical past; they do not follow merely by enlarging an observed-record filtration. Scoring access in the proof grants the reader no access to an unread preparation label. Let $\epsilon_t \in [0, 1]$ and $q_t \in [0, 1]$ be predictable, calibrated bounds for the selected reader. On G suppose

$$\mathbb{E}[W_t \mid \mathcal{F}_{t-1}] \leq \epsilon_t \mathbb{E}[C_t \mid \mathcal{F}_{t-1}], \quad \mathbb{E}[C_t \mid \mathcal{F}_{t-1}] \geq q_t. \quad (49)$$

The constructed symmetric robust reader supplies these inequalities label by label under the calibrated conditional-state contract. Consequently its certificate permits arbitrary, history-dependent label

priors when both label-conditioned source marginals satisfy that contract. This does not transfer an optimality claim from a different prior-weighted task. Arbitrary inert quantum references may be retained. Their additional measurement or feedback is outside the contract unless the augmented conditional input bounds have also been established.

Theorem 7.1 (Finite-use certificate with a single calibration fee). *For any predetermined horizon N and $0 < \beta < 1$, put $b_N = \sqrt{(N/2)\log(2/\beta)}$. With probability at least $1 - \alpha - \beta$, simultaneously*

$$\sum_{t=1}^N W_t \leq \sum_{t=1}^N \epsilon_t C_t + b_N, \quad (50)$$

$$\sum_{t=1}^N C_t \geq \sum_{t=1}^N q_t - b_N. \quad (51)$$

No independence of the future trials is required. With fixed $\epsilon_t = \epsilon$ and $q_t = q$, if $Nq > b_N$ this implies

$$\frac{\sum_t W_t}{\sum_t C_t} \leq \epsilon + \frac{b_N}{Nq - b_N}.$$

Every null and loss event remains in the record and in the horizon cost.

Proof. On each calibration outcome in G , set $Z_t = W_t - \epsilon_t C_t$. It has nonpositive conditional mean and its conditional range is contained in $[-\epsilon_t, 1 - \epsilon_t]$, of length one. The conditional Hoeffding lemma therefore gives $\mathbb{E}[e^{\theta Z_t} | \mathcal{F}_{t-1}] \leq e^{\theta^2/8}$ for $\theta > 0$. Induction and exponential Markov give

$$\Pr\left(\sum_t Z_t \geq b \mid \mathcal{F}_0\right) \leq e^{-2b^2/N}.$$

Apply the same argument to $q_t - C_t$, whose conditional mean is nonpositive and whose range also has length one. Taking $b = b_N$ bounds each failure by $\beta/2$. The union bound supplies β on G ; the original calibration failure adds α once. This proof permits adaptation to all admitted preceding classical outcomes, not retroactive selection of a better calibration confidence event. $\square$

The range-dependent lemma can be checked directly: among bounded variables with fixed mean, convexity maximizes the moment generating function on the two endpoints. The centered endpoint Bernoulli log-mgf has second derivative at most $1/4$; integrating twice from zero bounds it by $\theta^2/8$. This proves the concentration constant used above.

An anytime version is also available. For any fixed $\theta > 0$, with failure at most $\alpha + \beta$,

$$\sum_{t=1}^n (W_t - \epsilon_t C_t) < \frac{\theta n}{8} + \frac{\log(1/\beta)}{\theta} \quad \text{for every } n \geq 1.$$

Indeed $\exp[\theta \sum_{t \leq n} Z_t - n\theta^2/8]$ is a nonnegative supermartingale starting at one; stopping it at its first crossing and using the expectation bound proves the displayed probability inequality. This supports an admitted stopping policy without falsely applying a fixed-horizon bound to a data-selected horizon.

8 General qutrit effect monitoring under a conditional drift contract

8.1 Operational contract and units

The input space is $\mathcal{H} = \text{span}\{|v\rangle, |0\rangle, |1\rangle\}$. It may include physical vacuum. The nominated terminal labels are $\{+, -, \emptyset\}$. At clock slot t , conditional on the preceding declared classical history, there is a common linear measurement

$$E_+(t) \geq 0, \quad E_-(t) \geq 0, \quad E_\emptyset(t) = I - E_+(t) - E_-(t) \geq 0.$$

It is independent of the current probe's classical preparation tag; the probe influences it through its input quantum state only. All conditional source/controller, ready-stock, preparation and independence of the joint input and inert reference from the ready stock conditions needed for such an effect representation are admitted. Correlation of an unknown input with an uncontrolled device stock cannot be cured by tomography.

The device may otherwise depend on the entire past. A known pathwise bound is required for each nominated conclusive effect:

$$\|E_y(t) - E_y(s)\|_\infty \leq \nu|t - s|, \quad y = +, -. \quad (52)$$

Here effects and $\nu|t - s|$ are dimensionless probabilities; ν has units inverse clock slot. If an external clock measures seconds, replace ν by the corresponding rate per second and use actual elapsed times. Calibration uses, production uses, delays, controller changes and finite reference depletion all count toward elapsed time. A bound for an averaged master equation does not prove Eq. (52) for actual conditional instruments. A known deterministic drift envelope $d(t, s)$ can replace $\nu|t - s|$ in the confidence construction.

Every $+, -, \emptyset$, time, probe setting and deployment decision is retained. Neither missed clicks nor accepted labels are selected away. This section certifies the declared labels on inert-reference extensions and the explicit pointer/reference distance stated below. It does not identify the postmeasurement source, coherent field, bath or complete quantum instrument. Those require the operational completion and separate retained-output proofs.

8.2 Nine probes and a general confidence body

Use the three basis projectors and, for each $i < j$, the two projectors onto $(|i\rangle + |j\rangle)/\sqrt{2}$ and $(|i\rangle - |j\rangle)/\sqrt{2}$. Number these nine trusted states σ_j , $j = 0, \dots, 8$. In every trial their actual conditional preparation has trace distance at most δ_{prep} from its nominated state, and the conditional joint input/reference is independent of the device's ready stock. This trusted preparation condition is not inferred from the same unknown detector.

When $|v\rangle$ is physical vacuum and the other modes contain one particle, the four vacuum/occupied superpositions require a phase reference and compatible preparation access. The finite-reference resource model applies when appropriate. If number superselection and block-diagonal source/effects are separately admitted, five probes suffice for the five-dimensional real algebra $\mathbb{C} \oplus M_2$; unrestricted vacuum coherences cannot be certified with that reduced access.

Let $X_{y,j,k}$ be the indicator of label y on the k th use of probe j . Its true conditional mean $p_{y,j,k}$ is predictable and may change with every preceding record. Define the retained empirical frequency $\hat{p}_{y,j,n} = n^{-1} \sum_{k=1}^n X_{y,j,k}$. Suppose at most K monitoring epochs will be opened. Within each epoch use any predictable probe schedule, with a finite total budget; our example uses round-robin cycles. Set

$$\delta = \frac{\alpha}{2JK}, \quad J = 9, \quad u(n) = \sqrt{\frac{\log(\pi^2 n^2 / (3\delta))}{2n}} \quad (n \geq 1). \quad (53)$$

A probe with no observations gives no constraint and cannot trigger a complete certificate.

M1: finite-budget, all-prefix coverage. With probability at least $1 - \alpha$, simultaneously for every opened epoch, both conclusive labels, every probe, and every observed prefix n ,

$$\left| \hat{p}_{y,j,n} - \frac{1}{n} \sum_{k=1}^n p_{y,j,k} \right| \leq u(n).$$

Consequently, at any data-selected calibration completion time T , for probe times $s_{j,k} < T$, the actual next-slot effect lies in the physical confidence body

$$\mathcal{P}_T = \{(E_+, E_-) : E_{\pm} \geq 0, E_+ + E_- \leq I, \quad (54)$$

$$|\text{tr}(E_y \sigma_j) - \hat{p}_{y,j,n_j}| \leq b_j(T), \quad y = \pm, j = 0, \dots, 8\}, \quad (55)$$

$$b_j(T) = u(n_j) + \delta_{\text{prep}} + \frac{\nu}{n_j} \sum_k (T - s_{j,k}). \quad (56)$$

If the body is empty, decline deployment; do not call it a vacuous success. Future time $T + a$ may use the same formula with every radius enlarged by νa , or the operator-ball construction below.

Proof. For a chosen stream, $X - p$ is a martingale difference of conditional range one. The conditional Hoeffding lemma gives the exponential supermartingale with compensator $n\theta^2/8$, also when sampling times are predictable and chosen from the preceding record. At a fixed observed sample index n , optional stopping, or extension of a stopped stream, gives

$$\Pr \left(\left| \sum_{k \leq n} (X_k - p_k) \right| > nb \right) \leq 2e^{-2nb^2}.$$

At $b = u(n)$ the right side is $6\delta/(\pi^2 n^2)$, summing to δ over all $n \geq 1$. Union over $2JK$ streams proves the claim, conditional on every history at which an epoch is opened and then averaged over those histories. No stationarity or fresh statistical independence has been used. Conditional probe-preparation error changes an event probability by at most δ_{prep} ; pathwise drift changes it by at most $\nu(T - s_{j,k})$. Average these inequalities to obtain the body. The same simultaneous event permits arbitrary inspection and stopping within the stated epoch budget. $\square$

The construction remains conservative if only a predetermined finite set of prefixes is inspected. If just one fixed terminal sample size per epoch is used, the simpler width

$$u_{\text{fixed}}(n) = \sqrt{\log(4JK/\alpha)/(2n)}$$

is sufficient; it must not be substituted after selecting a favorable sample size. Tighter established confidence sequences may replace Eq. (53); the displayed one was chosen for a self-contained proof and transparent implementation.

8.3 An exact finite reconstruction constant

For any Hermitian H , write its nine probe expectations as v . The inverse map $\mathcal{R}$ is

$$H_{ii} = v_i, \quad (57)$$

$$\text{Re } H_{ij} = v_{ij,R} - (v_i + v_j)/2, \quad (58)$$

$$\text{Im } H_{ij} = (v_i + v_j)/2 - v_{ij,I}. \quad (59)$$

This fixes the sign convention for the $+i$ probe. The exact rectangular uncertainty radius is the finite computation

$$C(b) = \max_{s \in \{-1, +1\}^9} \|\mathcal{R}(s_0 b_0, \dots, s_8 b_8)\|_{\infty}. \quad (60)$$

Convexity gives equality: the norm of every point of a box is bounded by its largest vertex value, and the vertices are included. Thus $\|E_y(T) - \mathcal{R}(\hat{p}_y)\|_{\infty} \leq C(b)$ on M1. This is sharp for an unconstrained Hermitian box; enforcing the physical joint confidence body can make it smaller.

M2: exact constant for a uniform box. For $b_j = b$,

$$C(b) = (1 + 2\sqrt{5})b. \quad (61)$$

A short reproducible algebraic proof accompanies the code. For every one of 512 sign vertices, form $H = \mathcal{R}(s)$. Its entries are Gaussian integers. All principal minors of $(1 + 2\sqrt{5})I \pm H$ are nonnegative, checked in exact integer arithmetic in $\mathbb{Z}[\sqrt{5}]$. For clarity, with diagonal entries d_i , $t = \text{tr } H$, $c_2 = \sum_{i < j} (d_i d_j - |H_{ij}|^2)$, and $d = \det H$, the determinant is

$$(61 - 21t + c_2 - d) + (46 - 4t + 2c_2)\sqrt{5}.$$

A 2×2 minor for i, j is

$$21 - d_i - d_j + d_i d_j - |H_{ij}|^2 + (4 - 2d_i - 2d_j)\sqrt{5}.$$

One-dimensional minors are immediate. Their signs are decided exactly by integer squares for $A + B\sqrt{5}$; no floating-point tolerance is used in these 7168 checks. The vertex $s = (1, 1, 1, -1, -1, -1, 1, -1, -1)$ gives

$$H = \begin{pmatrix} 1 & -2 + 2i & -2 \\ -2 - 2i & 1 & -2 + 2i \\ -2 & -2 - 2i & 1 \end{pmatrix},$$

whose characteristic polynomial $(\lambda - 1)[(\lambda - 1)^2 - 20]$ attains the claimed radius. The finite sign checks plus this witness prove equality. They do not claim that every sign vertex is itself a physical effect.

Set $\hat{E}_\pm = \mathcal{R}(\hat{p}_\pm)$ and $\hat{E}_\emptyset = I - \hat{E}_+ - \hat{E}_-$. If these are not physical, define

$$\lambda = \max(0, -\min_y \lambda_{\min}(\hat{E}_y)), \quad F_y = \frac{\hat{E}_y + \lambda I}{1 + 3\lambda}.$$

Then F is a physical center. On M1, for $y = \pm$,

$$\|E_y(T + a) - F_y\|_\infty \leq r_y(T) + \nu a, \quad r_y(T) = C(b) + \|F_y - \hat{E}_y\|_\infty. \quad (62)$$

This explicit repair need not be an optimal fit; it costs the computed shift. Using $r = \max_y r_y + \nu a$ gives a physical operator-ball outer class, denoted $\mathcal{B}(F, r)$. It is not the same class as the full body $\mathcal{P}_T$, so sharpness for that outer class is not a sharp multinomial-data claim.

8.4 Sharp conditional error for general asymmetric effects

Fix known normalized source marginals ρ_+, ρ_- and conditional label priors π_+, π_- , summing to one. Let

$$\bar{\rho} = \pi_+ \rho_+ + \pi_- \rho_-, \quad W_F = \pi_+ \text{tr}(F_- \rho_+) + \pi_- \text{tr}(F_+ \rho_-), \quad Q_F = \text{tr}[(F_+ + F_-)\bar{\rho}].$$

Wrong outcomes remain a subset of conclusive outcomes, including dark-count labels if the input is vacuum. Put

$$A_+(\epsilon) = \pi_- \rho_- - \epsilon \bar{\rho}, \quad A_-(\epsilon) = \pi_+ \rho_+ - \epsilon \bar{\rho}.$$

M3: operational support certificate and its sharp regime. For every $E \in \mathcal{B}(F, r)$,

$$W_E - \epsilon Q_E \leq B(\epsilon) := W_F - \epsilon Q_F + r(\|A_+(\epsilon)\|_1 + \|A_-(\epsilon)\|_1), \quad Q_E \geq Q_F - 2r. \quad (63)$$

If $Q_F > 2r$, any $\epsilon \in [0, 1]$ with $B(\epsilon) \leq 0$ certifies conditional error at most ϵ . No source/excitation postselection is used. These statements hold on every inert-reference extension having the nominated marginals.

Both support and success bounds are sharp over the physical operator-ball class whenever

$$r \leq r_{\text{int}} := \min\{\lambda_{\min}(F_+), \lambda_{\min}(F_-), \tfrac{1}{2}\lambda_{\min}(F_{\emptyset})\}. \quad (64)$$

If the radius exceeds this sufficient condition, the bounds remain valid; their physical attainment is no longer asserted.

Proof. Write $E_{\pm} = F_{\pm} + H_{\pm}$. Then $W_E - \epsilon Q_E = W_F - \epsilon Q_F + \sum_y \text{tr}(H_y A_y)$. The duality of operator and trace norms bounds each term by $r\|A_y\|_1$, with equality at $H_y = r \text{sign}(A_y)$. Condition (64) makes these effects positive and keeps $F_{\emptyset} - H_+ - H_- \geq 0$, so they are a real physical POVM. For success use $H_+ = H_- = -rI$, which is also physical in that regime. The denominator is positive when $Q_F > 2r$; the first inequality then implies the conditional-error statement. References do not alter nominated label probabilities. $\square$

There is also a complete pointer/reference guarantee. For the channel $\mathcal{M}_E(\rho) = \sum_y \text{tr}(E_y \rho)|y\rangle\langle y|$,

$$\tfrac{1}{2}\|\mathcal{M}_E - \mathcal{M}_F\|_{\diamond} \leq 2r.$$

For every joint input, each partial-trace block for a Hermitian effect perturbation H has trace norm at most $\|H\|_{\infty}$, by trace-norm duality. The two conclusive blocks and their negative sum give the bound. In the interior regime it is sharp: choose $H_+ = H_- = rI$, with null perturbation $-2rI$. All three blocks are scalar multiples of the reference marginal, giving half trace distance exactly $2r$ on every input. This is a bound for the declared pointer plus an arbitrary inert reference; it makes no statement about retained output source or bath.

For unequal known radii, replace $r\|A_y\|_1$ by $r_y\|A_y\|_1$ and $2r$ by $r_+ + r_-$; the analogous margin condition applies. The proof supplies a finite implementation $V = \sum_y \sqrt{E_y} \otimes |y\rangle$, retaining the output source and arbitrary inert reference. It demonstrates attainability within ordinary quantum instruments, not a derivation of Born statistics or identity with the physically nominated detector's full bath.

Balanced pure hypotheses: a quadratic. Let $\pi_{\pm} = 1/2$, $\rho_{\pm} = |\psi_{\pm}\rangle\langle\psi_{\pm}|$, and $s = |\langle\psi_+|\psi_-\rangle| \in [0, 1]$. In their two-dimensional span,

$$\|A_+(\epsilon)\|_1 + \|A_-(\epsilon)\|_1 = \sqrt{1 - 4\epsilon(1 - \epsilon)s^2}.$$

This follows from $\text{tr} A_y = (1 - 2\epsilon)/2$ and $\det A_y = -\epsilon(1 - \epsilon)(1 - s^2)/4$. A zero eigenvalue in unused qutrit directions causes no difficulty. Thus the sharp error in the interior regime is the root in $[0, 1]$ of

$$W_F - \epsilon Q_F + r\sqrt{1 - 4\epsilon(1 - \epsilon)s^2} = 0. \quad (65)$$

Equivalently,

$$(Q_F^2 - 4r^2s^2)\epsilon^2 + (4r^2s^2 - 2W_FQ_F)\epsilon + W_F^2 - r^2 = 0,$$

retaining the unsquared sign $\epsilon Q_F - W_F \geq 0$. Since $Q_E > 0$ for every member, the physical support function is strictly decreasing in ϵ , and the root is the exact worst ratio. Orthogonal hypotheses give $\epsilon_* = (W_F + r)/Q_F$; identical hypotheses have no informative balanced-label advantage. Outside the interior regime use the support inequality as a conservative certificate, not a minimax claim.

Source uncertainty and arbitrary priors. The example fixes source marginals. If each actual conditional ρ'_h has trace distance at most δ_S from ρ_h , the valid repair is $B(\epsilon) + \delta_S \leq 0$ and $Q_E \geq Q_F - 2r - \delta_S$. Indeed $E_{\text{wrong}} - \epsilon(E_+ + E_-)$ lies between $-rI$ and $(1 - \epsilon)I$; its spectral oscillation is at most one. This term bounds source error without mistaking a measured detector for a measured source. The combined radius/source certificate need not be sharp.

For priors that may vary arbitrarily with history, certify each label separately. If $w_h = \text{tr}(F_{-h}\rho_h)$, $q_h = \text{tr}[(F_+ + F_-)\rho_h] > 2r$, then

$$\epsilon_h \leq \min\{1, (w_h + r)/q_h\}, \quad Q_h \geq q_h - 2r. \quad (66)$$

The support is $w_h - \epsilon q_h + r$, attained by $H_{\text{wrong}} = rI$, $H_{\text{correct}} = -rI$ in the same interior regime. Using $\max_h \epsilon_h, \min_h Q_h$ covers arbitrary label priors. The sharper balanced expression requires balanced conditional priors; neither calibration nor a reset establishes them.

8.5 Finite-budget deployment and certificate expiry

A complete protocol is now explicit. Predetermine α, K , a maximum number of probe trials per epoch, trusted preparations, the drift contract, target error $\bar{\epsilon}$, and target conclusive probability $\bar{q} > 0$. Open an epoch, retain all outcomes, and inspect M1 at any prefixes. Construct F, r ; deploy only if $B(\bar{\epsilon}) \leq 0$ and $Q_F - 2r \geq \bar{q}$ (or their source-uncertain/labelwise versions). Continue only while the aged radius in Eq. (62) passes the same gate. Otherwise take more probes, open the next budgeted epoch, or stop. A different physical pulse requires its own calibration or a separately established transformation bound; an online choice among uncalibrated pulse settings is not licensed.

For balanced pure inputs the largest admitted error radius at a chosen threshold is

$$r_{\text{err}} = \frac{\bar{\epsilon}Q_F - W_F}{\sqrt{1 - 4\bar{\epsilon}(1 - \bar{\epsilon})s^2}}, \quad (67)$$

when numerator and denominator are positive. Also impose $r \leq (Q_F - \bar{q})/2$. The allowed monitoring gap follows by subtracting $r(T)$ and dividing by $\nu > 0$; if sharpness is desired rather than just a valid certificate, additionally impose $r \leq r_{\text{int}}$. This is an operational expiry time, not a detector change-point localization theorem.

There is a real statistical/drift tradeoff even before deployment. For n complete round-robin cycles with probe slots $j + 9k$, $j = 0, \dots, 8$, and next-slot anchor $T = 9n$, the largest mean age is $(9n + 9)/2$. A transparent sufficient radius is

$$r_{\text{unif}}(n) = (1 + 2\sqrt{5})[u(n) + \delta_{\text{prep}} + \nu(9n + 9)/2]. \quad (68)$$

For a fixed terminal sample size set $u(n) = A/\sqrt{n}$, $A = \sqrt{\log(4JK/\alpha)/2}$. The continuous optimum of this displayed design bound is

$$n_{\text{opt}} = (A/(9\nu))^{2/3} \quad (\nu > 0).$$

Check neighboring integers and the declared finite budget. Thus blindly collecting older data eventually worsens a current detector certificate. This is an exact optimum of the stated bound, not of every tomography schedule or confidence method. For the anytime width the optimum is an elementary one-dimensional minimization, provided in the code.

8.6 Online calibration events and adaptive record guarantees

The simultaneous M1 event spans future calibration records; it need not be measurable at the initial time. Conditioning on that future event and reusing the symmetric-family initial-event proof would be invalid. A stopped-process repair gives the correct sequential conclusion.

For analysis let τ be the first completed calibration prefix at which the true current effects leave the certified body. It is a stopping time for the full mathematical filtration, even though an operator is unknown to the experimenter. M1 gives $\Pr(\tau < \infty) \leq \alpha$. At each nominated production opportunity

let D_t be the predictable deploy gate, C_t, W_t its conclusive and wrong indicators, and q_t, ϵ_t its reported bounds. A skipped opportunity has all three indicators zero. Prior to τ , drift plus the gate gives

$$\mathbb{E}[W_t - \epsilon_t C_t \mid \mathcal{F}_{t-1}] \leq 0, \quad \mathbb{E}[q_t D_t - C_t \mid \mathcal{F}_{t-1}] \leq 0.$$

Use zero increments after the first failed prefix in the analysis. The stopped increments have nonpositive conditional mean and range length at most one. For any fixed $\theta > 0$, a nonnegative exponential supermartingale and its first-crossing inequality yield, with failure at most $\alpha + \beta$,

$$\sum_{t \leq n} (W_t - \epsilon_t C_t) < \frac{\theta n}{8} + \frac{\log(1/\beta)}{\theta} \quad \text{for every } n \geq 1. \quad (69)$$

The identical construction applies to $\sum_{t \leq n} (q_t D_t - C_t)$. For simultaneous two-sided objectives split β between them. No conditional law on the future success event was used; on $\{\tau = \infty\}$ the stopped and actual production records agree. The fee α is paid once for the entire finite monitoring budget, not once per production use. This is standard stopped-supermartingale reasoning newly supplied for the online calibration interface.

Here the proof filtration includes completed-trial scoring truths and outcomes, as in Section 7; the controller sees only the permitted accessible-record subfiltration. The admitted conditional source, ready-stock and drift promises must hold on that augmented past. Such a promise is not implied by calibration on a smaller observable history. An inert retained quantum reference does not change the label bounds; measuring that reference or a bath and feeding it back changes the conditional contract unless separately included. Physical resets and fresh source independence are not inferred.

9 Two-template monitoring and task-relative conditional-error certification

9.1 The same conditional-device contract, a smaller task

Let $E_+(t), E_-(t), E_0(t)$ be any conditional three-outcome POVM on a finite-dimensional input space, summing to identity. Qutrit vacuum and occupied modes are an example. The POVM is independent of the current probe's classical preparation tag; the probe affects it through its quantum state only. The actual conditional input and inert reference are independent of the device's ready stock. Uncontrolled input/stock correlation is outside the linear effect representation.

Nominate normalized density matrices ρ_+, ρ_- and trusted probe preparations within trace distance δ_{prep} of them in every calibration trial. Future actual label-conditioned marginals must lie within $\delta_{S,h}$ of ρ_h . These are independently admitted preparation conditions, not conclusions of the unknown detector. Effects may depend on all preceding accessible history. Admit pathwise operator-norm drift envelopes

$$\|E_y(t) - E_y(s)\|_\infty \leq d_y(t, s), \quad y = +, -.$$

A common rate model is $d_y(t, s) = \nu|t - s|$, with ν in inverse clock slots (or use actual seconds and rates per second). Every calibration, production and waiting slot consumes that clock. The envelopes must constrain actual conditional effects, not only an averaged master equation. Neither probes nor resets establish the envelopes.

In a trial of template h , record all three outcomes. Let W indicate a wrong conclusive label, V a correct conclusive label, and $C = W + V$ any conclusive label. Nulls have $W = V = C = 0$. For threshold $e \in [0, 1]$,

$$L_e = W - eC = (1 - e)W - eV \in [-e, 1 - e].$$

Thus it has range length one, even if clicks are rare. No clicked-only sample selection or Bernoulli assumption on thinned clicks is used. All raw outcomes, settings, times and calibration/deploy decisions are part of the physical retained record. The numerical supplement supplies aggregated synthetic count tables, not an ordered experimental archive.

9.2 One event certifies all thresholds and all inspected prefixes

Permit at most K epochs, opened adaptively. Each epoch may use any predictable template schedule and a declared finite budget. For $n \geq 1$ uses of one template, let hats denote empirical means and bars the averages of their true conditional means. Set

$$u(n) = \sqrt{\frac{\log(2\pi^2 n^2 K/\alpha)}{2n}}. \quad (70)$$

With probability at least $1 - \alpha$, simultaneously for all epochs, both templates, all observed prefixes, and $X \in \{W, V, C\}$,

$$|\hat{X}_n - \bar{X}_n| \leq u(n).$$

Indeed the centered indicators are martingale differences of range one. Conditional Hoeffding and exponential Markov give each two-sided failure at most $2e^{-2nu(n)^2} = \alpha/(\pi^2 n^2 K)$. Summing over n gives $\alpha/(6K)$ for each stream; union over $6K$ streams proves the claim. Predictable sampling can be handled by stopping each stream at its observed sample index, or appending zero centered increments after its finite budget ends. No iid outcome law is required.

Crucially, the same event gives

$$|\hat{L}_e - \bar{L}_e| \leq (1 - e)|\hat{W} - \bar{W}| + e|\hat{V} - \bar{V}| \leq u(n) \quad \text{for every } e \in [0, 1]. \quad (71)$$

There is no threshold-grid union cost and an error target can be chosen after observing the retained data. The third counter C supplies a separate range-one conclusive-rate bound rather than the looser sum of two counter errors.

At a future deployment time t after template- h probes at times $s_{h,k}$, define

$$D_{h,y}(t) = \frac{1}{n_h} \sum_k d_y(t, s_{h,k}), \quad b_h = u(n_h) + \delta_{\text{prep}} + \delta_{S,h}.$$

Write $D_w = D_{h,-h}$ and $D_c = D_{h,h}$.

Theorem 9.1 (Direct general-reader deployment certificate). *On the simultaneous event, every actual label-conditioned future input obeys*

$$w_h(t) - eq_h(t) \leq \hat{W}_h - e\hat{C}_h + b_h + (1 - e)D_w + eD_c, \quad (72)$$

$$q_h(t) \geq \hat{C}_h - b_h - D_w - D_c. \quad (73)$$

If the latter lower bound is positive, define

$$e_h^{\text{cert}} = \frac{\hat{W}_h + b_h + D_w}{\hat{C}_h + D_w - D_c}. \quad (74)$$

When $e_h^{\text{cert}} \leq 1$, the conditional error w_h/q_h is at most e_h^{cert} . Otherwise the trivial bound one applies. The maximum of the two label bounds and minimum of the two rate bounds cover arbitrary history-dependent label priors. No balanced prior is required.

Proof. The score effect is $A_{h,e} = (1 - e)E_{-h} - eE_h$, whose spectral range lies within $[-e, 1 - e]$. Each probe-preparation error changes its expectation by at most δ_{prep} , and future source deviation changes it by at most $\delta_{S,h}$. Its drift is at most $(1 - e)d_{-h} + ed_h$ by operator-norm duality. Combine with Eq. (71) and average probe ages to prove Eq. (72). The conclusive effect $E_+ + E_-$ lies between zero and identity, so its preparation/source deviations cost the same trace-distance margins, and its drift costs $D_w + D_c$. The C counter proves Eq. (73).

Positivity of that rate lower bound ensures $\hat{C}_h + D_w - D_c > 0$. Setting the right side of Eq. (72) to zero then gives exactly Eq. (74). Its threshold can be selected from the data because the coverage event is uniform over e . Every statement is about the nominated label marginal and holds for any inert-reference extension of that input marginal. It is not equality of reference assemblages or of complete quantum instruments. $\square$

For the common rate model, with mean probe age $A_h(t)$,

$$e_h^{\text{cert}} = \frac{\widehat{W}_h + u(n_h) + \delta_{\text{prep}} + \delta_{S,h} + \nu A_h(t)}{\widehat{C}_h}, \quad q_h \geq \widehat{C}_h - u(n_h) - \delta_{\text{prep}} - \delta_{S,h} - 2\nu A_h(t).$$

The absence of a subtracted confidence width in the ratio's denominator is justified by the uniform score argument; it is not a ratio of two independent one-sided confidence intervals. The coefficients ν and 2ν are sharp for their respective local drift tasks: at an interior POVM change the wrong effect by $+dI$ and the correct effect by $-dI$ to increase every score by d , or decrease both effects by dI to decrease conclusive probability by $2d$. These different witnesses establish individual bounds, not simultaneous physical attainment of all statistical and drift corners.

9.3 Budget, expiry and a quantified comparison

Alternate the two templates for n cycles and anchor deployment after slot $T = 2n$. Their mean ages are $n + 1$ and n , hence at further age a the worst age is $n + 1 + a$. The common-rate numerator budget is

$$B(n, a) = u(n) + \delta_{\text{prep}} + \delta_S + \nu(n + 1 + a).$$

For fixed data and target $\bar{e}, \bar{q}$, deploy only while $\widehat{W}_h + B \leq \bar{e}\widehat{C}_h$ and $\widehat{C}_h - u - \delta_{\text{prep}} - \delta_S - 2\nu(n + 1 + a) \geq \bar{q}$ for both labels. The smaller of the two resulting gap bounds is an explicit expiry time. A negative gap means stop or take new probes, not extrapolate stability.

At a predetermined terminal sample size the fixed-time width is $u(n) = A/\sqrt{n}$, $A = \sqrt{\log(12K/\alpha)/2}$. The displayed bound $A/\sqrt{n} + \nu n$ has continuous optimum $n_* = (A/(2\nu))^{2/3}$. This is an exact optimum of this specified bound and schedule; check neighboring integers and the finite budget. For the all-prefix width use one-dimensional minimization. Neither claim optimizes over every possible calibration protocol.

For a separate specified stationary asymmetric/dark-count qutrit detector, use the known occupied inputs $\sqrt{0.8}|0\rangle \pm \sqrt{0.2}|1\rangle$. Keep $\alpha = 0.01$, $K = 3$, $\nu = 10^{-11}$ per clock slot, $\delta_{\text{prep}} = 5 \times 10^{-5}$, $\delta_S = 0$, and deployment age 10^8 slots. Both procedures pay actual calibration time in the same drift clock. The nine-probe example used 36 million trials and gave the arbitrary-prior certificate $e \leq 0.169748239$, $q \geq 0.268980153$.

The new predetermined two-template count-table diagnostics give:

Total probe trials	arbitrary-prior error bound	conclusive lower bound
200,000	0.1641071006	0.2820958913
1,000,000	0.1440878798	0.2913177448
2,000,000	0.1387493968	0.2921346968
36,000,000	0.1292163991	0.2948488929

These are synthetic count tables, not experimental data or a proof of sample optimality. Their protocols have different output tasks: the nine-probe method additionally estimates a general qutrit POVM and a pointer/reference channel bound, which two templates do not identify.

There is also a non-simulation sufficiency statement for this stationary benchmark law with exactly nominated calibration preparations. Write its true template means as w_h, q_h . The same simultaneous event guarantees that the deployment gate succeeds if

$$2u(n) + \delta_{\text{prep}} + \delta_S + \nu(n+1+a) \leq \bar{e}q_h - w_h, \quad q_h - 2u(n) - \delta_{\text{prep}} - \delta_S - 2\nu(n+1+a) \geq \bar{q}.$$

This follows by bounding the empirical score and conclusive frequency with their opposite sides on that same absolute-confidence event. For the full-POVM arbitrary-prior targets above, $n = 500,000$ per template satisfies both inequalities: one million probes suffice to guarantee deployment and the stated future conditional performance with probability at least 99 percent for this specified stationary benchmark. This is 36 times fewer trials than that nine-probe count-table demonstration, not a universal 36-fold improvement over all tomography methods or a guarantee for every drift/preparation law in the larger admitted class.

9.4 What two probes can and cannot identify

For arbitrary unknown effects, exact response on a set of target states can be determined from probe expectations if and only if the target states lie in the real linear span of the probe density matrices. Sufficiency follows by linearity. If a target lies outside that span, there is a Hermitian H orthogonal to every probe and not to that target. Starting with a strictly interior POVM, perturb one effect by ϵH and the null effect by $-\epsilon H$. For sufficiently small nonzero ϵ this remains physical, leaves every measured probe response unchanged and changes the target response. This is an exact indistinguishability counterexample, not a failed numerical search. Two distinct template density matrices are independent, so two probes are minimal for full label-response identification in this unrestricted effect model. This is not a minimum for every loose one-sided claim.

Our occupied templates require only the already nominated source's relative-mode preparations. No vacuum/occupied superposition probe is needed. Their unmeasured vacuum response remains unidentified: adding $d|v\rangle\langle v|$ to a conclusive effect and subtracting it from the null is invisible to both templates. It can change another input's response and the full qutrit measurement-channel distance. For a mixed target, an invisible traceless effect direction can likewise change a purifying reference's conditional assemblage while preserving its label probabilities. No full-POVM or arbitrary-input diamond guarantee is claimed here. Future source deviations are charged through $\delta_{S,h}$ rather than assumed harmless.

9.5 Adaptive monitoring and retained records

Open at most K epochs; retain complete outcomes and apply the gate at any covered prefix. Expire it as the drift clock advances. Retuning the physical reader requires new calibration or an independently justified transformation rule. Neither template counts nor old calibration identify an unknown bath law or path-blind Hamiltonian.

Let $D_t \in \{0, 1\}$ be the predictable production/deploy gate at global clock slot t , with $W_t = C_t = 0$ outside production. Let e_t and q_t be its predictable certified error upper bound and conclusive lower bound; values outside production are immaterial. Write $N_n = \sum_{t \leq n} D_t$ for the number of deployed production trials. The global confidence event includes future calibration and must not be conditioned on as initial information. For analysis stop at the first completed probe prefix whose true counter means miss their intervals; its probability is at most α . Before that stopping time, every predictable production gate has the certified conditional error and rate. Stop the production scores there. They

have nonpositive conditional mean and conditional range length at most D_t , so the exponential supermartingale with compensator $N_n\theta^2/8$ gives, for any fixed $\theta > 0$, a uniform record bound

$$\sum_{t \leq n} (W_t - e_t C_t) < \frac{\theta N_n}{8} + \frac{\log(1/\beta)}{\theta} \quad \text{for every } n$$

with failure at most $\alpha + \beta$. The corresponding conclusive lower bound uses $q_t D_t - C_t$; split β if both are required. On no confidence failure the stopped and actual production records agree. Thus calibration risk is paid once across the finite budget, without conditioning on future success or inferring fresh independence. The proof filtration includes scoring truths for completed trials, while controller choices use only the permitted accessible-record subfiltration. Conditional device, source and drift promises are assumed on this augmented past, not inferred by enlarging an observed-history filtration. Known truth labels are required to score wrong outcomes in validation; an unknown production label is not made observable by this theorem.

The code independently checks the continuum score identity, predictable non-iid exponential moments, the blind-direction example, exact drift coefficients and all benchmark budgets. Numerical tables corroborate the proofs and synthetic protocol; they do not certify physical drift or trusted state preparation. All source/controller, clock, null/loss, reference and correction contracts remain. No Born-selection, biological channel or consciousness mechanism is established.

10 Preservation and scientific status

The companion’s proof interfaces differ in observable task and access permissions. Their numerical examples are synthetic; tests corroborate the proofs rather than constitute empirical evidence. Original source/controller, reference, clock, preparation, null/loss and retained-output conditions survive every reuse. The general qutrit method estimates a full effect family; the two-template method certifies a smaller task and does not identify the same quantum output. Their sample counts are therefore not a universal tomography comparison. No physical horizon-to-biological channel has been supplied; no actual-event affinity is inferred from an averaged density equation. The source-readout completion framework remains compatible with unrestricted operational surrogates, so a constrained detector distinction does not establish unique ontology.

Research assistance. AI assistance was used in reconstruction, checking and exposition. Supporting code and verification instructions accompany the draft. These are checked drafts with declared scope, not externally peer-reviewed empirical papers.

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