Shadow Theory and the Wilson Yang–Mills Construction:

A Source–Readout Reconstruction

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Abstract

We construct a source–readout realization of the Wilson Yang–Mills construction within Jeremy Rodgers's Shadow Theory. The framework separates an admitted native support, its literal preparation, the realized physical histories and the readouts that recover configurations, effective quantities, field labels and operator correlations. Reflective mathematics makes the relation between these layers explicit: reverse sections establish coverage, fibres identify redundant or discarded information, operation identities establish compatibility, and residuals quantify departures from exact recovery.

The paper proves normalized finite-measure recovery and classifies the frame fibres; transports activity domains and budgets; recovers the stationary vertical discrepancy from its residual; and establishes the compatibility of coefficient readers, physical equation quotients, Hilbert-space reconstruction, phases and operator products. These interfaces are developed through the complete quadratic, nonlinear, marked-integration, calibration, continuum and stress calculations. One compatible Wilson family carries full fixed-period mesh limits followed by the permitted refined volume limit. Its selected centered trace has the asymptotic $r^8 \log^2(1/r) G_\Theta(r) \to 3 \dim_{\mathbb{R}} \mathfrak{g}/\pi^4$, with the stated finite mass-shell and Maxwell comparison exclusions. No positive mass gap is asserted.

OpenAI's underlying Lean construction is presented through the supplied expanded mathematical translation. The contribution here is the explicit source–readout construction and its recovery proofs, with all substantive Yang–Mills derivations retained. The given gauge group, faithful representation, four-dimensional Wilson law and analytic hypotheses remain the physical inputs to this conditional realization.

Authorship. Jeremy Rodgers originated Shadow Theory, the source–readout architecture and the Source Lift protocol, and directed this research using those methods and examples. AI assisted in carrying out the work.

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1 Introduction

The Wilson Yang–Mills construction joins several kinds of mathematics: finite group-valued configurations and probability measures, effective actions and stationary variables, local coefficient spaces and equation quotients, continuum correlations, Hilbert-space fields and spectral readers. Recovering its conclusion requires these objects and operations to belong to one compatible construction. Equality of final formulas alone does not establish that differentiation, integration, calibration, quotienting and limiting operations preserve the information needed by the next stage.

This paper addresses that problem using Shadow Theory's source–readout architecture and Source Lift protocol, originated by Jeremy Rodgers [3]. Rodgers directed the research through that framework and its worked examples; AI assisted its execution. A Source Lift constructs an admitted native support, its literal preparation and a realized sector containing the physical histories and auxiliary data. Typed readouts then recover the objects of the target derivation. Their sections establish reverse coverage, their fibres identify the distinctions removed, and their operation identities establish how the derivation passes through the realization. Reflective mathematics is the explicit accounting for these maps, domains, residuals and recovery claims. Section 2 introduces this framework before applying it.

The application is OpenAI's underlying Lean Yang–Mills construction, available here through its supplied expanded mathematical translation [2]. The companion paper, *A Wilson Yang–Mills Construction: A Natural-Language Mathematical Exposition*, presents that construction on its own [1]. Both papers are included in the same Zenodo release, DOI 10.5281/zenodo.24173681, and each contains its essential mathematics. The companion emphasizes the construction itself. This paper develops the source–readout realization, the reflective analysis of its interfaces and the recovery proofs throughout the same derivation.

Mathematical contribution. The finite realization has an explicit section, a complete frame-fibre description and an exactly normalized measure pushforward. The family of histories is supported by an admitted fan, with a groupoid equivalence on each finite realized leaf. The reconstruction then proves frame-blindness of flags and budgets, ordered-derivative and activity-norm recovery, and quantitative recovery of an eliminated vertical discrepancy from the stationary residual. At the field stage it gives the curl/radial section and its gauge fibre, the calibrated coefficient reader, and the physical quotient defined by all spectator readings, with inclusions and derivatives descending to that quotient. The Hilbert-space, phase, product and collision-extension interfaces retain their actual domains and choices. These results explain precisely which operations are recovered, rather than treating every reduction as the same kind of inverse.

The full Yang–Mills calculations remain in the paper. They use Wilson's lattice formulation [5], reflection positivity and reconstruction [11, 12, 13, 14, 15, 16], renormalization and cluster methods [17, 18, 19], and the conventional one-loop coefficient [20, 21]. The given physical inputs and the exact public conclusion are stated in Section 4; the assumption and dependency registers collect their scope. The earlier Einstein source–readout reconstruction [4] provides a model of exposition: define the primitives and maps, derive the operation, prove coverage, and identify the residual or information loss. Its gravitational results are not Yang–Mills premises.

Organization and reading conventions. The framework, public conclusion and analytic conventions fix the objects, target and normalizations. Native admission, the finite realization and the interface table then introduce the common witness. The construction follows dependency order: quadratic kernels; nonlinear geometry and budgets; marked integration and reset; stationary graphs, calibrated histories and convergence; labels, shell charges and physical detection; the full

inner continuum limit; phase and Wightman reconstruction; separated and nested products; collision extensions; Gaussian comparison; and the stress, trace and equation-sensor calculations. The common-family assembly collects their witnesses. Complete transport proofs and the assumption, amendment and correspondence registers are included as mathematical appendices.

At each stage the derivation is followed by its source–readout formulation: the realized record, reader, section or quotient, operation compatibility and recovery claim, with the relevant error or choice retained. Original equation tags are kept throughout. Input-line ranges refer to the preserved expanded source and support the complete coverage register. These references locate provenance; the local derivations supply the mathematics.

2 Source–readout and reflective mathematics: the objects and the task

The framework distinguishes the mathematical input from its native antecedent. The *supplied mathematical source* is the expanded translation of OpenAI's underlying Lean Yang–Mills construction. The *native source support* is an admitted object in Rodgers's source–readout architecture: it carries a dependency order and marked finite supports. The physical lattice, group, fields and law enter in the realization over that support. This separation lets the paper construct and compare the actual readout operations while retaining the provenance of their physical assumptions.

In ordinary terms, a readout is a specified way of obtaining a mathematical object from a richer record. A lattice configuration can be read from links written in independently chosen endpoint frames. An effective action can be read from a function of retained and auxiliary variables by evaluating on its stationary graph. A physical field label can be read from a coefficient vector after identifying vectors that every allowed spectator test fails to distinguish. These are different maps, with different fibres. The paper proves their recovery statements on their respective domains; it does not treat them as interchangeable instances of an unrestricted inverse.

The term *reflective mathematics* means mathematics of the readout objects together with explicit accounting for their source–readout relations. In Rodgers's Source Lift protocol this includes the maps and domains, admission conditions, distinctions removed by a map, residuals, available reconstruction routes and the status of each claim. Here those items are made concrete by the following definitions and calculations. Reflection is this source-aware accounting within the mathematics. It is not an additional dynamical law, and it does not assert that a theory selects its own physical inputs [3].

This use of *reflective* is distinct from *reflection positivity*, the positive-pairing property of Euclidean correlations used later to construct a Hilbert space. Reflection positivity is one of the specific mathematical operations accounted for here; it is not the definition of reflective mathematics. Likewise, matching status profiles does not establish an equivalence: an equivalence must preserve the specified objects and operations through actual maps.

2.1 Four layers that must remain distinct

The forward architecture used here has the form

$$X \in \mathsf{S}_{\mathrm{adm}} \;\longmapsto\; Y = J_{\mathrm{real}}(X, m) \;\longmapsto\; (z, \delta) \in \mathcal{K}_{\mathrm{YM}}(Y) \;\xrightarrow{\pi}\; O. \tag{RF1}$$

The arrows do not all have the same meaning. Admission places X in the specified native class. Preparation restricts its inherited source data to the finite support of a marked token m. Realization is a relation $\mathcal{K}_{\mathrm{YM}}$, because a single prepared support can carry many raw histories and discrepancies.

The last arrow is the particular readout currently being studied, defined on its stated part of that raw domain.

More precisely, a source support contains vertices with a dependency preorder, deposition-potential tokens with a distinguished marked subsort, finite marked supports, lineage, source invariants and intrinsic boundary data. Every marked support contains a strict dependency pair. A strong map preserves these data and is injective on each marked support. The admitted class is the least isomorphism-saturated class containing the specified finite witnesses and closed under the prescribed strong images, finite attachments along source faces, presentation-preserving retracts and stabilizing directed colimits. Stabilization means that every finite cell, support, strict pair and inherited datum is eventually unchanged. This is the admission notion used in Proposition 6.1 and the fan construction in Section 8; arbitrary graphs are not admitted merely by being called sources.

The literal preparation remembers the ambient support as well as the chosen marked occurrence:

$$\begin{aligned} Y &= (X, m, S_m, \preceq_m, \ell_m, \mathrm{Inv}_m, \mathrm{Bnd}_m), \\ S_m &= \mathrm{Supp}_X(m). \end{aligned} \tag{RF2}$$

All restricted data are inherited from X. This finite tuple is not the physical lattice. In particular, the strict pair in S_m is not a pair of physical events, and the ambient dependency order is not the scale order or Euclidean time of the Yang–Mills construction.

The raw sector has type

$$\mathcal{K}_{\mathrm{YM}}(Y) \subseteq \mathsf{O}_{\mathrm{YM}}^{\mathrm{raw}} \times \mathsf{D}_{\mathrm{YM}}. \tag{RF3}$$

Here $z \in \mathsf{O}_{\mathrm{YM}}^{\mathrm{raw}}$ retains the group and faithful representation, oriented lattice, cutoff and period, links and frames, action, couplings, local functions, auxiliary variables, calibration choices and later comparison witnesses. The discrepancy column $\delta \in \mathsf{D}_{\mathrm{YM}}$ records failures of the equations or comparisons relevant at that stage. Its components may have different types: a group element, a vector, a distribution tested on a specified test space, or an error measured in an explicitly weighted norm. No addition of unlike discrepancies is intended. Off-shell histories and nonzero discrepancies belong to the raw record before any stationary, equation-null or limiting restriction is imposed.

When two stages are compared, they must map into one declared comparison space. For an interface $e : s \to t$ write

$$\delta_e = d_e(L_e z_s, R_e z_t). \tag{RF4}$$

The maps L_e, R_e include the required coordinate, scale and unit conversions. The comparison d_e is subtraction only when its arguments belong to an appropriate common linear space. Formula (RF4) prevents comparing two expressions in incompatible gauges, norms, phases or regulators merely because their symbols resemble one another.

2.2 Readout, section, fibre and admissible comparison

Let $\pi : \mathcal{Z} \to \mathcal{Q}$ be one readout on a declared domain. Its fibre over q is $\pi^{-1}(q)$: all realized records giving that same readout. A reconstruction section $\iota : \mathcal{Q}_0 \to \mathcal{Z}$ on a stated target class satisfies

$$\pi\iota = \mathrm{id}_{\mathcal{Q}_0}. \tag{RF5}$$

Thus every target in $\mathcal{Q}_0$ is covered. This identity alone does not imply $\iota\pi = \mathrm{id}_{\mathcal{Z}}$: a chosen representative need not recover the original raw representative. The second identity holds only for an actual inverse, or after a proved restriction or quotient. Existence of a section on one class proves no coverage outside that class.

For example, on one finite lattice let V be its vertices and E one stored orientation of its bonds. With $e : x \to y$, the frame realization is

$$\mathcal{Z} = G^V \times G^E, \qquad \pi(r, A)_e = r_x^{-1} A_e r_y, \qquad \sigma(U) = (1, U). \tag{RF6}$$

Then $\pi\sigma(U) = U$ for every $U \in G^E$. If two pairs have the same readout, the unique frame change taking one to the other is $k_x = r'_x r_x^{-1}$, with $A'_e = k_x A_e k_y^{-1}$. Consequently this readout forgets precisely the redundant endpoint frames, once the lattice and other tags are fixed. Its full proof, its physical gauge action and its normalized measure are given in Section 7.

This example also gives a useful check on the scope of the explanation. For a path $\gamma : x \to y$, cancellation of the intermediate frames gives $\pi(r, A)_\gamma = r_x^{-1} A_\gamma r_y$. The trace of a closed holonomy, and hence the Wilson action, is unchanged. For a gauge-invariant observable F, $F \circ \pi(r, A) = F(A)$. On such observables the additional frame is an identity encoding, not a new physical mechanism. The later stationary, coefficient, equation and limit readers supply the other substantive recovery questions of the paper.

Allowed gauge and presentation comparisons can be retained as arrows rather than suppressed. A recovery witness for an object O is then a tuple (X, Y, z, φ), with $\varphi : \pi(z) \simeq O$ an allowed comparison. The lift groupoid consists of these witnesses and comparisons compatible with φ. Section 8 proves that, on the chosen finite frame realization, this groupoid is nonempty and contractible: two witnesses have exactly one compatible arrow. This is uniqueness within that realized branch, up to its specified comparisons. It is not a classification of every admitted source that might realize the same physical theory.

2.3 Recovering operations, rather than only objects

The target includes more than a set of configurations. Its hypotheses, boundary and preparation data, allowed inputs, laws, symmetry actions, observables and sequence of proof operations must be fixed as well. At an individual step let

$$D_j : \mathcal{Q}_{\text{in}} \times \mathcal{U}_j \longrightarrow \mathcal{Q}_{\text{out}}$$

denote the target operation, with its actual domain understood. A realized operation $D_j^{\mathcal{Z}}$ recovers it when

$$\pi_{\text{out}}\big(D_j^{\mathcal{Z}}(z, u)\big) = D_j\big(\pi_{\text{in}}(z), u\big). \tag{RF7}$$

The same u, hypotheses and domain are required on both sides. Where the identity is conditional or approximate, retain its defect in a common comparison space instead of writing (RF7) without qualification. Limits require their own convergence argument; a finite commuting diagram does not justify interchanging an integral, derivative or limit.

Lemma 2.1 (Composition and the criterion for descent)**.** *Suppose the operations in a finite dependency graph satisfy* (RF7) *and cover the inputs and hypotheses used by their consumers. Their composites are recovered in their stated order. For any readout $q : A \to B$ and required output $f : A \to C$, a map $\bar{f} : \operatorname{im} q \to C$ with $f = \bar{f} q$ exists exactly when f is constant on each fibre of q. It is then unique on* $\operatorname{im} q$*. An operation descends to the readout only when its equal-readout relation is preserved by that operation for the same admitted auxiliary inputs.*

Proof. Order the graph so that every producer precedes its consumers. Substitute (RF7) at the first operation, then at each successive operation; coverage permits each substitution on the domain needed by the next step. This proves composition by induction. If $f = \bar{f} q$, equality of q values

implies equality of f values. Conversely set $\bar{f}(q(a)) = f(a)$; fibre constancy makes this independent of the representative a. The same argument applied to the required output of an operation proves the last assertion. This is a set-level argument; continuity or differentiability of $\bar{f}$ needs an additional topological or analytic proof. □

There are two concrete uses of this criterion throughout the construction. First, an activity is not its particular atom presentation. The blocking maps must give the same activity when equivalent presentations are used; their analytic and Lipschitz estimates must then hold in the activity norm, not in an unrelated norm on presentations. Second, the physical coefficient quotient must be stable under inclusions and differentiation. It is insufficient merely to declare a vector invisible to one test and then differentiate it. The calibrated equation image and the complete spectator family establish the required descent in Subsection 18.9.

Variation has the same requirement at the tangent level. Differentiating (RF6), with $k_x = \dot{r}_x r_x^{-1}$, gives

$$\dot{U}_e = r_x^{-1}(\dot{A}_e - k_x A_e + A_e k_y) r_y. \tag{RF8}$$

Every target tangent h_e is covered by choosing $\dot{A}_e = r_x h_e r_y^{-1} + k_x A_e - A_e k_y$. The terms involving the frame rates are essential when the frames vary. Likewise, a background-dependent fluctuation chart needs its actual compensating coordinate change, Jacobian and covariance ratio. The protected-variable construction in Section 15.5 supplies these data. An equality of field values alone would not recover the differentiated integral.

2.4 What is retained, what is removed, and what is selected

The reflective record distinguishes several mathematically different situations. A *redescription fibre* consists of presentations of the same retained object, as in the frame example. An *observational quotient* identifies records indistinguishable by a specified family of readers. A *stationary restriction* selects a solution of an auxiliary equation. A *controlled comparison* retains a nonzero error. A *selection* retains a complement, calibration, phase or extraction witness that the argument is permitted to choose. None is automatically the same as another.

For a partial nonlinear readout, write

$$\mathrm{Eq}(\pi) = \{(z, z') : z, z' \in \operatorname{dom}\pi,\ \pi(z) = \pi(z')\}. \tag{RF9}$$

This is an equivalence relation on the admitted domain. Points outside that domain are a separate record of non-admission, not elements of a linear kernel. A linear kernel is appropriate only for the linear maps for which it is actually defined. Fixed coarse bins similarly determine an equivalence relation; pairwise closeness below a tolerance need not be transitive. No comparison resolution is changed after observing a discrepancy.

For stationary elimination the relevant objects are particularly explicit. In its purchased corridor, the supplied score $f(x, v)$ has retained coordinate x, vertical coordinate v, residual $S(x, v) = D_v f(x, v)$ and a unique analytic graph $v = g(x)$. The readout and section are

$$q(x, v) = x, \qquad \iota_{\mathrm{st}}(x) = (x, g(x)), \qquad q\iota_{\mathrm{st}} = \mathrm{id}. \tag{RF10}$$

The raw vertical coordinate is not reconstructed by x alone. Retaining the residual gives the quantitative estimate

$$\|v - g(x)\|_2 \le \frac{2}{m_{\mathrm{gap}}} \|S(x, v)\|_2, \tag{RF11}$$

where $m_{\mathrm{gap}} > 0$ denotes the local counting-gap parameter called m in Theorem 14.5; it is unrelated to the size of the faithful matrix representation. The theorem also supplies the period-independent maximum-norm bound with its weighted counting constants. The chain rule on the stationary graph gives $D_x(f \circ \iota_{\mathrm{st}}) = D_x f|_{v=g(x)}$, because $D_v f = 0$ there. Thus the equation selects the section and controls the off-shell discrepancy. It is not a bijection from the entire raw corridor to retained coordinates, and no physical particle mass gap is involved in this finite-dimensional Hessian estimate.

At the coefficient stage let E_d be the finite-dimensional coefficient space and let N_d be the subspace annihilated by the complete separated spectator readers at degree horizon d. The physical space is $F_d = E_d/N_d$. Its reader recovers precisely the information visible to those tests. The construction proves the calibrated equation-image identification and operation compatibility needed by this quotient; it does not recover an arbitrary original coefficient representative from its class. Similarly the radial reconstruction in Theorem 16.1 recovers a potential from a closed homogeneous curvature coefficient on the radial slice, while its unrestricted fibre is the linearized gradient freedom. Here the equation and the gauge fibre are distinct: the free Maxwell equation further restricts the curvature space after curl and radial reconstruction have been defined.

At the Hilbert-space stage, reflection positivity removes the null word space. Equality of the positive pairings makes the realized and target quotients isometric; dense range then gives a unitary after completion (Theorem C.1). The information retained is the physical word state and the operators on their common word domain, not a preferred pre-null word representative. The phase reader subsequently evaluates the same compressed algebra element at a character. Averages over characters recover the unselected hierarchy, but an average does not reconstruct a specified individual phase without the retained phase data.

2.5 One family, one endpoint, and the scope of reflection

Every interface in this paper belongs to one compatible realized family. The same calendar, calibration recipe, Wilson family and permitted common refinement must survive the passage from finite insertions to continuum fields. In particular, with $R_{N,p}(t)$ a tested lattice moment, the public statement has the order

$$r_p(t) = \lim_{N\to\infty} R_{N,p}(t), \qquad \lim_{k\to\infty} r_{p(q(k))}(t) = S(t). \tag{RF12}$$

The inner limit is full at each suitable fixed period. The outer limit uses the allowed refinement of an arbitrary increasing period sequence. An auxiliary mesh extraction used to construct other data is not a replacement for the full inner limit. The two-limit argument and the identification of auxiliary tested limits are supplied in the continuum chapter.

The interacting-to-Gaussian comparison has a different scope. A station can be chosen so that its geometric radius equals a requested radius; the finite interacting and Gaussian moments still differ by the stated comparison error. In the trace argument the comparison construction covers a subsequence of *every* sequence of radii tending to zero, on the same physical table and selected phase. Positivity gives a monotone Laplace reader, and the two-sided radius sandwich promotes those comparisons to the full limit

$$\lim_{r\downarrow 0} r^8 \log^2(1/r) G_\Theta(r) = \frac{3\dim_{\mathbb{R}} \mathfrak{g}}{\pi^4}. \tag{RF13}$$

The universal subsequence property and the monotonicity argument, proved in Subsection 24.2, are essential; convergence on one favourable sequence would not prove this statement. The exact

outcome is a recovered short-distance asymptotic with the supplied comparison-class exclusions. It supplies no positive lower bound on the energy spectrum above the vacuum.

The contribution of the reflective organization can now be stated precisely. It identifies which mathematical object a formula reads, whether a reverse section covers the next step, which operation descends, and which discrepancy or choice must survive a reduction. Its claims are proved by the actual maps and estimates, not by assigning a status word. The gauge group, dimension, Wilson action, physical normalization and analytic assumptions remain the supplied realization inputs except for the particular dependencies derived within the construction. Alternative realizations can be placed over the same native support; its admission therefore does not select those inputs. The remaining sections implement this framework through the entire supplied derivation.

3 Provenance and mathematical contract

3.1 Source provenance and attribution

OpenAI is credited for the underlying Lean Yang–Mills construction. The mathematical source used here is the project's expanded natural-language extraction, prepared with AI assistance: *Wilson Yang–Mills Construction: Mathematical Translation* [2]. Its complete text is reproduced unchanged in this release as `sources/Full_pre_source_translation.md`. The 15,788-line source is numbered using universal-newline counting; the package manifest records its full hash. No model, repository, version or generation-date identifier is assigned where the supplied materials do not establish one.

Jeremy Rodgers originated Shadow Theory, the source–readout architecture and the Source Lift protocol, and directed this research using those methods and examples. He is the author of the reconstruction and its mathematical presentation. AI assisted in developing and checking the realization, preparing the exposition and performing the recorded computational checks. The source–readout definitions and proofs needed for this application are contained in this paper.

3.2 Mathematical contract

The physical group and faithful representation, lattice dimension, Wilson law, preparation and analytic conditions are the inputs to the realized sector. The public conclusion is recovered with its original hypotheses, witnesses and order of limits: the full mesh limit at each suitable fixed period precedes the permitted refined volume limit. Native admission and physical realization remain distinct layers, as developed in Section 2.1.

The derivations are retained with their original equation tags. The explicit formula amendments, typography clarifications and local recovery corrections are documented with their supporting calculations in Section I.1. The preserved source remains unchanged. The verification record in Section J distinguishes general recovery proofs, finite computational checks and the scope of the supplied Yang–Mills construction.

4 The supplied public conclusion and its recovery target

The following records the opening theorem's content (input lines 7–61) in typeset notation. Its mathematical assertions are supplied results. They specify the target of the translation, rather than consequences selected by native admission.

Let $G \subseteq U(m)$ be closed in its given faithful representation. Its real Lie algebra is exactly the matrices X with $\exp(tX) \in G$ for every real t, and is assumed nonabelian and simple: every real linear subspace contained in it and stable under commutators with it is either zero or the whole

algebra. No connectedness assumption is added. There is one calibrated Wilson family with strictly increasing positive integer periods L_p and denominators d_N, mesh $h_N = d_N^{-1}$ and lattice side $L_p d_N$, with law

$$d\mu_{N,p} = Z_{N,p}^{-1} \exp(-\beta_{N,p} S(U)) \prod_e d\nu(U_e),$$

$$S(U) = \sum_{x,i<j} \left[1 - \frac{1}{m} \operatorname{Re} \operatorname{tr}(U_{x,i} U_{x+e_i,j} U_{x+e_j,i}^{-1} U_{x,j}^{-1})\right].$$

Its token insertions $I_{N,p,v,x}$ are continuous, gauge invariant and local on a fixed finite offset stencil for each token v, uniformly in N, p, x. A finite set of integer offsets depending only on v controls each insertion at every N, p, x: agreement of configurations on edges whose endpoints belong to the translated offset set implies equality of the insertions. Its positive coupling slope obeys $\beta_{N,p}/\log(h_N^{-1}) \to s > 0$ uniformly in p.

For every strictly increasing period index sequence $p(k)$ there is a strictly increasing refinement $q(k)$ and a single hierarchy on $p(q(k))$. Its graded finite-dimensional complex normed spaces F_d inject into, and jointly cover, the field-label space of one Wightman theory; tokens project onto their disjoint union. A compact metric character space carries a full-support Borel probability measure and a selected character. The separated correlations S_n^χ and unselected S_n are smooth away from collisions, permutation invariant, continuous in the character and satisfy $S_n = \int S_n^\chi \, d\mu(\chi)$.

For every smooth compactly supported collision-free test t and finite token word, the lattice smeared moment is the source's h_N^{4n}-weighted site sum at centered physical positions. Explicitly, with the original centered placement $x \mapsto x^{\mathrm{phys}}$, this is

$$h_N^{4n} \sum_{x_1,\dots,x_n} t(x_1^{\mathrm{phys}}, \dots, x_n^{\mathrm{phys}}) \int \prod_{i=1}^n I_{N,p,v_i,x_i} \, d\mu_{N,p}.$$

At each suitable fixed period it has a full $N \to \infty$ limit r_p. Along the refined periods, r_p converges to the unselected tested correlation. Suitability means that the centered open period box contains the test support, and holds eventually along the refined sequence. The full inner limit is retained before the outer subsequence limit.

The selected theory has normalized vacuum Ω and a common dense invariant domain equal to the span of finite smeared field words on Ω. Fields are complex linear in labels and Schwartz tests and continuous in the test on each domain vector. They satisfy the smeared adjoint identity, spacelike commutativity and proper orthochronous Poincare covariance. Translations are strongly continuous and unitary, with joint momentum projection measure in the closed forward cone. The invariant vacuum is unique up to scalar, every nonzero label gives a nonzero smeared operator, and at least one field acts nontrivially. The public symmetry coordinates cover all proper orthochronous affine Poincaré transformations. Strictly ordered imaginary-time tube values recover selected Euclidean moments; product-Schwartz smeared boundary limits recover vacuum Wightman moments.

For a graded cluster a, write $|a| = \sum_i \deg(a_i)$. One phase- and truncation-independent channel system $O_{j,\lambda}, C_{a,j,\lambda}(x; o)$ has finitely many channels at each degree. For fixed $M \geq 0$, $\eta > 0$, $\varepsilon > 0$, each fixed channel eventually satisfies

$$|C_{a,j,\lambda}(x; o)| \leq r^{j-|a|-\varepsilon}$$

when $|x_i - o| \leq Mr$ and $|x_i - x_{i'}| \geq \eta r$. For a fixed spectator word b, $c > 0$ and $D \geq 0$, the same cluster domain and spectator separations $|o - y_l| \geq c$, $|y_l - y_{l'}| \geq c$ give

$$\left| S^\chi(a(x), b(y)) - \sum_{j=0}^{D} \sum_{\lambda} C_{a,j,\lambda}(x; o) S^\chi(O_{j,\lambda}(o), b(y)) \right| \leq r^{D+1-|a|-\varepsilon}.$$

The assertion is uniform in all phases and includes the unselected hierarchy; its threshold may depend on the fixed words and stated parameters.

Finally a centered degree-four field Θ has selected two-point function $G_\Theta(r)$ at $(\mp r/2, \mathbf{0})$ with

$$\lim_{r\downarrow 0} r^8 \log^2(1/r) G_\Theta(r) = \frac{3 \dim_{\mathbb{R}} \mathfrak{g}}{\pi^4} > 0.$$

It cannot agree at every sufficiently small radius with a finite complex linear combination of finite products of the specified mass-shell integrals. Each factor integrates $E(p)^{t-1} P(p) e^{-rE(p)}/2$ on $\mathbb{R}^3$, where $E(p) = \sqrt{\text{mass}^2 + |p|^2}$, mass ≥ 0, t is a nonnegative integer and P is continuous and positively homogeneous of nonnegative integer degree. This is the stated comparison class, not an exclusion of every generalized free theory. The supplied proof chooses blocking penalty $b = 1$ and projects the single compatible witness family through its common refinement.

Scope of the stated conclusion. The supplied construction proves more than the public statement records: it also supplies stress Ward identities, nested expansions and contact extensions, developed on its stronger common witnesses. *These are not additional assumptions and not additional clauses of the public theorem.* The reconstruction follows this division. The recovery target of the whole paper is the conclusion stated in this section; the further results are lifted where they are constructed and are recorded as such in the claim-status register (Section H). No positive mass gap is asserted in the supplied conclusion, and none is asserted anywhere in this paper.

5 Inherited analytic conventions

A positive-plane plaquette $i < j$ uses the ordered Wilson product in the public statement. All lattices are four dimensional, with cover $\mathbb{Z}^4$ and period-n lattice $(\mathbb{Z}/n\mathbb{Z})^4$. A positive link (x, i) joins x to $x + e_i$; reverse reading takes its inverse. Signed coordinate permutations retain this convention. Distance is the maximum centered-coordinate distance. A support is a finite site set, and its internal links have both endpoints in it.

The physical Lie-algebra inner product and chord are

$$\langle X, Y \rangle_{\mathfrak{g}} = -m^{-1} \operatorname{Re} \operatorname{tr}(XY), \qquad \operatorname{chord}(g) = \|g - 1\|_{\mathrm{F}} / \sqrt{m}.$$

Fix the source's orthonormal real color basis. Matrix Frobenius norm is $\sqrt{m}$ times color norm. The inner product is positive on the skew-Hermitian algebra and conjugation invariant. Inversion and conjugation preserve the chord, and the triangle bound is $\operatorname{chord}(gh) \leq \operatorname{chord}(g) + \operatorname{chord}(h)$. The color projection of $(g - g^{-1})/2$ is the source's Lie-algebra curvature readout. These normalizations are kept in the quadratic action, the coupling slope and the trace calculation.

At scale i, the spacing is $a_i > 0$, the odd integer block ratio $L_i = a_i/a_{i+1} \geq 3$, and $t_i = \log(a_i^{-1})$, so $t_{i+1} - t_i = \log L_i$. Block bounds and coupling-entry delays are chosen before cutoff and physical period. The reference clock is

$$\widehat{\beta}_i = \bar{\beta} + s t_i, \qquad s = \frac{11 C_A}{24\pi^2}, \qquad C_A = -\frac{1}{\dim \mathfrak{g}} \sum_{\alpha} \operatorname{tr}(\operatorname{ad} T_\alpha)^2 > 0.$$

Represented couplings retain their distinct coordinates and explicit comparisons with this clock.

The radius-B halo of X is the union of its radius-B lattice balls. The cover number $\operatorname{cov}_B(X)$ is the least number of such balls whose union is connected by unit lattice steps and contains X; their centers need not belong to X. Periodic wrapping is treated by that same connected-union definition.

For a fixed stage ratio L, retain

$$\beta_F(x, L) = x + s \log L, \quad B(x) = \lceil (\log x)^2 \rceil, \quad \mathrm{temp}(x) = x^{-1/2} e^{(\log\log x)^2}.$$

Rate a means that for every $b > a$, eventually $|f(x)| \le \beta_F(x, L)^b$. Thresholds may depend on the fixed stage and ratio but not the period. Rate zero allows subpower growth and does not assert boundedness. The fine threshold is $\mathrm{fiery}(x) = \mathrm{temp}(\beta_F(x, L))$. Finite sums preserve the larger rate, products add rates, and a positive exponent gap absorbs fixed multiplicities and subpower factors. Later scale-path exponents retain their separately specified paths.

Every local weighted bound retains derivative order, allowed directions and exponential support weight. Countable sums are controlled absolutely, equivalently by a common bound on finite partial sums of the nonnegative majorant. This convention permits countable dictionaries without assuming a finite number of terms. Terms such as a bill, a payment or a paid bound in the source denote these explicit inequalities, not additional probabilistic assumptions. The lift carries these conventions as realized analytic data; conjugation of the finite frames preserves the color inner product, while arbitrary componentwise boxes still require their explicit coordinate transport.

5.1 Notation and the roles of recurring symbols

The construction uses several local alphabets. The following register fixes their roles; later local uses are qualified by their section and indices. In particular, a local parameter called a mass or a gap need not be a spectral gap of the reconstructed physical theory.

Notation	Meaning and convention
$X, Y, \mathcal{K}_{\mathrm{YM}}(Y)$	Native admitted support, literal preparation, and raw realized sector. Physical data occur in the third object, not in the source signature.
π, σ, ι	A readout and its specified sections. The finite frame section is $\sigma(U) = (1, U)$. Other sections are named for their actual domains; none is a global inverse of every readout.
$G \subseteq U(m)$, $\mathfrak{g}$	The given closed matrix group in its faithful representation, its matrix size m, and its simple nonabelian real Lie algebra. The hypothesis does not add connectedness.
U_e, A_e, r_x	Physical link, raw framed link and endpoint frame, $U_e = r_x^{-1} A_e r_y$ for $e : x \to y$. A reversed link is read by inversion.
N, p, i	Mesh cutoff index, period index and scale level. The full $N \to \infty$ limit is taken first at each suitable fixed p; the outer period limit may use the stated common refinement.
h_N, L_p, d_N	Physical mesh $h_N = d_N^{-1}$, physical integer period L_p and positive denominator d_N. The lattice side is $L_p d_N$.
a_i, L_i, t_i	Scale spacing, block ratio $L_i = a_i/a_{i+1}$ and logarithmic scale $t_i = \log(a_i^{-1})$. Some retained calendar passages call this last clock τ_i; it is the same scale clock there. L_i is not the physical period L_p.
$\beta_{N,p,i}, \widehat{\beta}_i$	Actual calibrated action coefficient and reference clock $\widehat{\beta}_i = \bar{\beta} + s t_i$. Their difference is estimated, not set to zero.
s, b_0, C_A	The computed logarithmic slope $s = 2b_0 = 11C_A/(24\pi^2)$. Later occurrences of s as a positive radius are local scale variables, not this slope.

Notation	Meaning and convention
g_i, g_W, g_X	Regulator coupling $g_i = \beta_{N,p,i}^{-1/2}$ and the Wilson and measured position couplings used in the final comparison. Their asymptotic identification is proved; it is not a notation convention.
$S_W,\ S(x,v)$	Wilson action and stationary residual, respectively. The latter is the vertical derivative of the local score and is not an action. The unadorned S in the predictor Fourier identities is yet another locally defined symbol.
C, P	Covariances, precision/comparison matrices or coefficient maps according to their declared types. In a Gaussian change of variables C_0 and C_V denote distinct base and varied covariances. An auxiliary nonsymmetric inverse is not itself a covariance.
E_d, N_d, F_d	Degree-d coefficient space, null subspace for all separated spectator tests, and physical quotient $F_d = E_d/N_d$. Inclusions and derivatives on these quotients are constructed and proved compatible.
$\deg, \lvert a\rvert$	Engineering degree of a label and total engineering degree $\lvert a\rvert = \sum_i \deg(a_i)$ of a cluster. Polynomial position degree and field count are separately specified in the radial-jet chapter.
$S_n, S_n^\chi, \mu_{\mathrm{ph}}$	Unselected and character-dependent separated correlations, and the full-support phase measure, so $S_n = \int S_n^\chi \, d\mu_{\mathrm{ph}}(\chi)$. The source often writes this measure as μ; it is distinct from the finite Wilson laws.
$\Omega, \mathcal{D}$	Selected vacuum and common invariant word domain. The native preparation subscript in $\mathrm{RealPrep}_\Omega$ is a source convention, not a claim that a vacuum has already been derived.
$\Theta, G_\Theta, E(r)$	Centered degree-four physical trace label, its selected Euclidean two-point function, and the positive-time Laplace reader identified with it in the spectral argument.
A_G, c_G	The full trace amplitude is $A_G = 3\dim_{\mathbb{R}} \mathfrak{g}/\pi^4$. Other local scalar or color projection constants called c_G retain their local definitions.

Terms such as *bill*, *payment*, *price* and *purchase* denote the explicitly stated inequalities and the order in which their constants and thresholds are chosen. A *calendar* is the common sequence of scales, stage assignments and entry thresholds. A *beacon* is a specified finite reference view used to read a local coefficient, and a *path book* retains the path-series and insertion data of a chosen history. These words do not supply extra hypotheses beyond their mathematical definitions. Their exact weights, derivative orders and choices remain in the corresponding sections.

6 Native admission, literal preparation and the realized sector

The pre-sectoral source signature consists of a dependency preorder on source vertices, deposition-potential tokens, a marked subsort, finite supports, lineage, invariant data and intrinsic boundary data. A marked support contains a strict dependency pair. Strong maps preserve these structures and are injective on marked supports. No vertex, token or dependency is by definition a spacetime point, gauge variable or coupling.

Proposition 6.1 (An admitted antecedent)**.** *The walking marked dependency $X_\star$, with vertices $0 \prec 1$ and one inherited marked token $m_\star$ supported on $\{0,1\}$, is admitted by the initial witness*

clause of the source-admission construction. Its literal preparation is

$$Y_\star = J_{\mathrm{real}}(X_\star, m_\star) = (X_\star, m_\star, S_{m_\star}, \preceq_{m_\star}, \ell_{m_\star}, \mathrm{Inv}_{m_\star}, \mathrm{Bnd}_{m_\star}).$$

This supplies a native antecedent for the realization below without introducing a physical label into its source signature.

Proof. The walking marked dependency is one of the specified initial witnesses. Its support is finite and nonempty; the displayed pair is strict, and its lineage, invariant and boundary data are inherited from that witness. The least isomorphism-saturated admission closure contains every initial witness, hence contains $X_\star$. Restricting the inherited data to its marked support gives precisely the literal preparation tuple. Neither this admission step nor restriction supplies any physical coordinate or law. □

The realized Wilson sector is a relation

$$\mathcal{K}_{\mathrm{YM}}(Y_\star) \subseteq \mathsf{O}_{\mathrm{YM}}^{\mathrm{raw}} \times \mathsf{D}_{\mathrm{YM}}.$$

A raw tuple retains the supplied group and representation, lattice and orientation, cutoff and period, coupling, frames and links, local functions, supports and all subsequent auxiliary construction data. The discrepancy column retains equation and comparison residuals, including nonzero residuals. Thus admission is not defined by requiring a successful final quantum theory. The supplied construction specifies the physical law and proves its consequences; the present lift proves recovery of that construction on this realized branch. A single marked occurrence supports this whole realized tuple. It is not a map from the two abstract vertices onto every physical lattice endpoint.

For a target configuration O, a recovery witness consists of this antecedent, its literal preparation, a realized history z, and an allowed comparison isomorphism $\varphi : \pi(z) \simeq O$. The corresponding lift groupoid is

$$\mathrm{Lift}(O) = \mathcal{E}^{\simeq}_{\mathrm{YM}} \times^{h}_{\mathsf{Comp}^{\simeq}_{\mathrm{YM}}} \{O\}.$$

Here the arrows and retained data are those of the specified realized branch. The explicit section below proves nonempty recovery fibres for every finite link configuration. It does not classify every possible native antecedent of the same target. Every later observable and limit uses the same realized history and its declared refinements; successful outputs from incompatible histories are not combined.

7 Finite-lattice realization and exact recovery

Let G be the source's compact gauge group and let (V, E) be one of its finite oriented lattices, with one stored orientation for each unoriented bond. Write $\mathcal{U} = G^E$. Introduce the redundant realization

$$\mathcal{Z} = G^V \times G^E, \qquad \pi(r, A)_e = r_x^{-1} A_e r_y \quad (e : x \to y), \qquad \sigma(U) = (1, U).$$

For reverse orientations set $A_{\bar{e}} = A_e^{-1}$ and $U_{\bar{e}} = U_e^{-1}$. All lattice, cutoff and representation tags are retained with these tuples.

Theorem 7.1 (Complete frame fibre and section)**.** *The map π is onto and $\pi\sigma = \mathrm{id}$. Its fibres are exactly the orbits of the free frame action*

$$k \cdot (r, A) = (kr, (k_x A_e k_y^{-1})_e), \qquad k \in G^V.$$

Consequently $\mathcal{Z}/G^V$ is canonically identified with $\mathcal{U}$, with inverse represented by σ.

Proof. Substitution gives $\pi\sigma(U) = U$ and $\pi(k \cdot (r, A)) = \pi(r, A)$. If $\pi(r, A) = \pi(r', A')$, put $k_x = r'_x r_x^{-1}$. The equality of link readouts gives $A'_e = r'_x r_x^{-1} A_e r_y (r'_y)^{-1} = k_x A_e k_y^{-1}$, so the two points lie in the same orbit. The equation $kr = r$ forces $k = 1$, proving freeness. These formulas also prove the inverse identities, not merely a choice of representative for a restricted class of links. □

For a path $\gamma : x \to y$, cancellation of every intermediate frame gives

$$U_\gamma = r_x^{-1} A_\gamma r_y.$$

In particular a closed plaquette has conjugate holonomy, and its represented trace and Wilson action agree exactly. The frame action above is distinct from the physical gauge action on readouts. The latter can be lifted by $r_x \mapsto r_x g_x^{-1}$ with A fixed, giving $U_e \mapsto g_x U_e g_y^{-1}$.

Theorem 7.2 (Normalized measure recovery)**.** *Let S_W denote the supplied finite Wilson action and $d\nu$ normalized Haar measure. The lattice action and invariant group measure use the classical Wilson and Haar constructions [5, 6]. The probability law*

$$d\widetilde{\mu}(r, A) = Z_W^{-1} e^{-S_W(A)} \prod_{x \in V} d\nu(r_x) \prod_{e \in E} d\nu(A_e)$$

has $\pi_* \widetilde{\mu} = \mu_W$. Every integrable target insertion F therefore satisfies $\int F \circ \pi \, d\widetilde{\mu} = \int F \, d\mu_W$.

Proof. At fixed r, change each stored link by $A_e = r_x U_e r_y^{-1}$. Bi-invariance of normalized Haar measure preserves each link factor. The path identity makes $S_W(A) = S_W(U)$. Integrating the remaining independent frame factors gives one. This proves both the partition normalization and the integral identity; no unnormalized gauge volume is introduced. The statement extends from nonnegative measurable functions by monotone convergence and to integrable complex insertions by their real and imaginary parts. □

8 The source support of the realized family, its lift groupoid and its interfaces

The preceding two sections give the lift of one finite lattice: an admitted antecedent (Proposition 6.1), a redundant realization, its readout and its section (Theorems 7.1 and 7.2). The supplied construction, however, is not one lattice. It is a countable family of finite histories, indexed by a cutoff, a period and a scale level, together with comparisons between them, auxiliary reference volumes, coefficient spaces of field labels, an imported free comparison theory and several successive limits. This section develops the source-side structure that is needed to carry that family, and states what that structure does and does not determine. It has four parts: the admitted support; the lift groupoid of a single leaf; the sectors and interfaces of the supplied construction, with the status of each discrepancy; and the separation of what follows from the source side from what is supplied as realization data. Nothing in this section is used as a hypothesis by the supplied derivations that follow; it is the frame in which their source–readout formulations are stated.

8.1 A fan of marked dependencies

Let I be a nonempty countable set. The *fan* Φ_I has vertices $\{\mathsf{r}\} \cup \{\ell_a : a \in I\}$, dependency preorder generated by $\mathsf{r} \prec \ell_a$ for every a, and one marked token m_a for each $a \in I$, with support $\{\mathsf{r}, \ell_a\}$. The lineage, invariant and intrinsic boundary data of m_a are those of the walking marked dependency

$X_\star$, transported along the map $c_a : X_\star \to \Phi_I$, $c_a(0) = \mathsf{r}$, $c_a(1) = \ell_a$. For a subset $F \subseteq I$ write Φ_F for the corresponding subfan.

Proposition 8.1 (Admission of the fan)**.** *For every countable I the fan Φ_I is admitted. More precisely:*

1. *for finite F, Φ_F is obtained from $|F|$ copies of $X_\star$ by successive attachments along the root face $\{0\}$, and is admitted;*
2. *for $F \subseteq F'$ the inclusion $\Phi_F \to \Phi_{F'}$ is a strong map, and Φ_I is the directed colimit of the Φ_F over finite $F \subseteq I$; this colimit is stabilizing;*
3. *distinct leaves are incomparable: for $a \neq b$ neither $\ell_a \preceq \ell_b$ nor $\ell_b \preceq \ell_a$.*

Proof. The admission construction of [3] is the least isomorphism-saturated class containing the initial witnesses, among them $X_\star$, and closed under the specified finite attachments along source faces, presentation-preserving retracts and stabilizing directed colimits. The root $\{0\}$ of $X_\star$ is a source face: it is predecessor-closed, and it is lineage- and boundary-closed because the marked token is not supported in it and the intrinsic boundary is empty. (1) Attaching a further copy of $X_\star$ to Φ_F along the identification of its root with r is a pushout along that face. The attaching map is strong and reflects marks, since no marked support lies in the face. The pushout adds one vertex, one strict pair and one marked token, and leaves every existing support, strict pair and lineage datum unchanged; the candidate conditions (finite nonempty supports each containing a strict pair, well-typed inherited data) therefore hold in $\Phi_{F\cup\{a\}}$. Induction on $|F|$ gives admission. (2) An inclusion of subfans preserves the preorder, the marking, the supports with their strict pairs and the inherited data, and is injective; it is therefore strong. Every finite set of vertices and tokens of Φ_I lies in some Φ_F with F finite, and its preorder, supports and inherited data are the same in every larger subfan; this is the stabilization condition. (3) The preorder is generated by the pairs $\mathsf{r} \prec \ell_a$, and neither a pushout along the root nor the colimit adds a relation between leaves. □

This is the same admission that supports the family of prepared paths in the gravitational reconstruction [4]; only the index set and the realized sector differ. The literal preparation of a leaf is

$$Y_a = J_{\mathrm{real}}(\Phi_I, m_a) \cong Y_\star,$$

the preparation of Proposition 6.1, by restriction along c_a.

Proposition 8.2 (What the support determines)**.**

1. *If $|I| \geq 2$, the strong automorphisms of Φ_I are exactly the maps fixing r and permuting the leaves, with $m_a \mapsto m_{\sigma(a)}$; thus* $\mathrm{Aut}(\Phi_I) \cong \mathrm{Sym}(I)$.
2. *The fold map $\varphi : \Phi_I \to X_\star$, $\varphi(\mathsf{r}) = 0$, $\varphi(\ell_a) = 1$, $\varphi(m_a) = m_\star$, is strong, and $\varphi c_a = \mathrm{id}$ for every a.*
3. *Every finite family of marked tokens of Φ_I lies in a finite admitted subfan.*

Proof. (1) A strong automorphism preserves the preorder, hence fixes the unique vertex that lies strictly below at least two others, and permutes the remaining vertices. It preserves supports, so it sends m_a to the token supported on $\{\mathsf{r}, \ell_{\sigma(a)}\}$. Conversely each permutation of I defines such a map, and it preserves the inherited data because these are transported from the same $X_\star$ along the maps c_a. (2) φ preserves the preorder and the strict pairs $\mathsf{r} \prec \ell_a \mapsto 0 \prec 1$, carries each support bijectively onto $\{0, 1\}$, so is injective on marked supports, and preserves the inherited data by their definition. (3) is Proposition 8.1(2). □

The three statements have direct consequences for the reconstruction. By (1), nothing on the source side distinguishes one leaf from another. Any assignment of cutoffs, periods or scale levels to leaves, any order among them and any relation between two leaves is therefore *realization data*. By (2), the single-cell presentation of Proposition 6.1, in which one marked occurrence supports the whole realized tuple, is the image of the fan presentation under a strong map; the two presentations carry the same realized record, and every statement of the preceding sections applies leaf by leaf. By (3), every estimate of the supplied construction, which always concerns finitely many histories at a time, is a statement over an admitted finite support, and a limit is a readout of a net of realized data over the colimit; it is not a new source object.

8.2 The lift groupoid of one leaf

Fix one finite lattice (V, E), connected as a graph, with $\mathcal{Z} = G^V \times G^E$, $\mathcal{U} = G^E$ and π, σ as in Theorem 7.1. Two groups act on $\mathcal{Z}$: the frame group G^V by $k \cdot (r, A) = (kr, (k_x A_e k_y^{-1})_e)$, and the lifted gauge group G^V by $(r, A) \cdot h = ((r_x h_x^{-1})_x, A)$. The two actions commute. Let $\mathcal{E}$ be the action groupoid of $G^V_{\text{frame}} \times G^V_{\text{gauge}}$ on $\mathcal{Z}$, and Comp the action groupoid of the gauge group on $\mathcal{U}$, $(h \cdot U)_e = h_x U_e h_y^{-1}$. These are the groupoids of realized histories and of allowed comparison isomorphisms on this branch.

Theorem 8.3 (The realization is an equivalence of groupoids)**.** *The assignment $z \mapsto \pi(z)$, $(k, h) \mapsto h$ is a functor $\Pi : \mathcal{E} \to \mathsf{Comp}$, and it is an equivalence: it is surjective on objects, full and faithful. Consequently*

1. *for every $U \in \mathcal{U}$ the lift groupoid $\mathrm{Lift}(U) = \mathcal{E} \times^h_{\mathsf{Comp}} \{U\}$ is nonempty and contractible: any two recovery witnesses (z, ϕ), (z', ϕ') of U are related by exactly one arrow of $\mathcal{E}$ compatible with ϕ, ϕ';*

2. *for $z = (r, A)$ the automorphism group of z in $\mathcal{E}$ is*

$$\mathrm{Aut}_{\mathcal{E}}(z) = \{(k, h) : k_x A_e k_y^{-1} = A_e,\ h_x = r_x^{-1} k_x r_x\} \ \cong\ \mathrm{Stab}_{G^V}(\pi(z)),$$

and for any base vertex x_0 evaluation $h \mapsto h_{x_0}$ identifies $\mathrm{Stab}_{G^V}(U)$ with the centralizer in G of the holonomy group of U at x_0.

Proof. For $z = (r, A)$ and (k, h) one computes $\pi(k \cdot z \cdot h)_e = (k_x r_x h_x^{-1})^{-1} k_x A_e k_y^{-1} (k_y r_y h_y^{-1}) = h_x\, r_x^{-1} A_e r_y\, h_y^{-1} = (h \cdot \pi(z))_e$, so Π is a functor. It is surjective on objects because $\pi\sigma = \mathrm{id}$. Fullness: let $h \cdot \pi(z) = \pi(z')$. Then $z \cdot h$ and z' have the same readout, so by Theorem 7.1 there is k with $k \cdot (z \cdot h) = z'$, and (k, h) is an arrow $z \to z'$ over h. Faithfulness: if (k, h) and (k', h) are arrows $z \to z'$, then $k \cdot (z \cdot h) = k' \cdot (z \cdot h)$, and freeness of the frame action gives $k = k'$. An equivalence of groupoids has contractible homotopy fibres, which is (1): an object of $\mathrm{Lift}(U)$ is a pair (z, ϕ) with ϕ a gauge transformation carrying $\pi(z)$ to U, and an arrow $(z, \phi) \to (z', \phi')$ is an arrow (k, h) of $\mathcal{E}$ with $\phi' h = \phi$; here $h = \phi'^{-1}\phi$ is forced and then k exists and is unique by fullness and faithfulness. (2) An arrow (k, h) fixes $z = (r, A)$ exactly when $k_x r_x h_x^{-1} = r_x$ and $k_x A_e k_y^{-1} = A_e$; the first equation gives $h_x = r_x^{-1} k_x r_x$, and then $h_x \pi(z)_e h_y^{-1} = r_x^{-1} k_x A_e k_y^{-1} r_y = \pi(z)_e$, so $h \in \mathrm{Stab}(\pi(z))$; conversely each such h determines $k_x = r_x h_x r_x^{-1}$, which fixes A. Finally, if h stabilizes U then $h_y = U_\gamma^{-1} h_{x_0} U_\gamma$ for every path $\gamma : x_0 \to y$, so h is determined by h_{x_0}, and applying this to closed paths shows that h_{x_0} commutes with every holonomy at x_0; conversely, for g in that centralizer the formula $h_y = U_\gamma^{-1} g U_\gamma$ is independent of the path and defines a stabilizing gauge transformation. □

The theorem sharpens the statement, made after Proposition 6.1, that the recovery fibres are nonempty: on this realized branch a recovery witness is unique up to unique isomorphism, and

the realization neither adds nor removes symmetry; the reducible configurations, with nontrivial stabilizers, are exactly as reducible upstairs as downstairs. It does not classify other native antecedents of the same target; that question is addressed in the last subsection. The signed lattice permutations may be adjoined to both groupoids, since they act compatibly, $\pi\rho_Z = \rho\pi$ (Section 9); the statement and its proof are unchanged.

Corollary 8.4 (The fibre law)**.** *Under the law $\widetilde{\mu}$ of Theorem 7.2 the frame r and the readout $\pi(r, A)$ are independent, r has law $\nu^{\otimes V}$ and $\pi(r, A)$ has law μ_W. For bounded measurable Φ on $\mathcal{Z}$,*

$$\int \Phi \, d\widetilde{\mu} = \int_{\mathcal{U}} \Big[\int_{G^V} \Phi\big(r, (r_x U_e r_y^{-1})_e\big) \, d\nu^{\otimes V}(r) \Big] d\mu_W(U).$$

Proof. This is the change of variables $A_e = r_x U_e r_y^{-1}$ in the proof of Theorem 7.2, applied to a general integrand: at fixed r it preserves the product Haar measure and carries $e^{-S_W(A)}$ to $e^{-S_W(U)}$, so the joint law of (r, U) is the product $\nu^{\otimes V} \otimes \mu_W$. □

A consequence must be stated plainly. For a gauge-invariant function F one has $F \circ \pi(r, A) = F(A)$: the lift of F is F itself evaluated on the raw links, and the frame is an independent spectator. On gauge-invariant quantities the frame realization is therefore an identity encoding. It carries real content only where the supplied construction uses objects that are not gauge invariant (charts, axial gauges, predictors, tree slices, potentials), and there the covariance identities of the tangent chapter are what is used. The substantive source–readout structure of the construction does not lie in the frame fibre. It lies in the interfaces listed next, whose fibres are not torsors of a symmetry group and whose descent statements are theorems of the supplied construction.

8.3 Sectors, interfaces and discrepancies of the supplied construction

A *realized family* over the fan consists of: an injective labelling λ of the leaves by the index set of the construction; for each leaf a a raw datum z_a in a sector over Y_a, with its discrepancy column; and, for each interface e between sectors s and t, maps L_e, R_e into a common comparison space and the retained discrepancy

$$\delta_e = d_e(L_e z_s, R_e z_t).$$

An interface is *exact* if $\delta_e = 0$ identically, *controlled* if δ_e is bounded in a declared norm on a declared regime, and *selected* if it involves a choice (of a subsequence, a complement or a representative) that is retained as a witness. The index set of the supplied construction consists of the triples (N, p, i) with cutoff index N, period index p and scale level $0 \le i \le N$, together with labels for the reference tori of the beacons and for the free comparison theory. The numbers attached to a label (lattice side, block length, clock value, stage) are read from the one calendar $\mathfrak{c}$ of Theorem 14.16.

The sectors are the following.

(S1) *Finite Wilson history* at (N, p): $z = (r, A) \in \mathcal{Z}_{N,p}$ with the law $\widetilde{\mu}$; readout π.

(S2) *Represented state* at (N, p, i): the tuple Q of Definition 14.19, with readers $\beta(Q)$, $v(Q)$, the layout and the evaluated density.

(S3) *Stationary corridor* of a blocking step: pairs (x, v) of retained and vertical coordinates with the residual $S(x, v)$ (Section 14).

(S4) *Reference views*: the folded reference tori of the beacons and of the reference chart.

(S5) *Label coefficient spaces* E_d with their spectator readers, and the physical quotients E_d/N_d (Section 16, Subsection 18.9).

(S6) *The free comparison hierarchy*: the imported Gaussian theory on the same label algebra.

(S7) *The continuum hierarchy*: correlations on the physical quotient, its reflection Hilbert space and its characters (Theorems C.1 and D.1).

Table 2 lists the interfaces, the discrepancy that each retains, its status, and the statement in which that status is proved. It is the map of the reconstruction: every source–readout formulation in the following sections refines one row.

Table 2: Interfaces of the realized family and the status of their discrepancies.

	Interface	Discrepancy and status	Where proved
I1	Frame fibre within (S1)	None; exact. Fibre a torsor of G^V, lifts unique up to unique isomorphism	Theorems 7.1, 7.2, 8.3
I2	Startup: Wilson law at (N,p) to the represented state at level N	Zero: an identity of densities, with the representative inside the activity ball	Theorem 14.18
I3	Blocking step: level $i+1$ to level i	Zero: evaluation intertwines the forward recipe with block integration; retirement is an exact change of presentation	Theorem 14.20; (RET1)–(RET2)
I4	Stationary elimination within a step, (S3)	Residual $S(x,v)$ retained; vertical distance to the graph bounded by the residual and zero exactly on the graph	Theorems 14.4, 14.5
I5	Presentation of a state to its activity data	None: the step maps are constant on presentations, hence descend, and are analytic and Lipschitz on the normed data	Theorem 14.28
I6	Calibration of a history	Bottom coupling equal to $\bar{\beta}$ by choice of the bare parameter; the fibre is one point within the retained recipe	Theorems 14.24, 14.29
I7	Actual coupling to the reference clock	Bounded, not zero: $\lvert\beta_{N,p,i+1} - \beta_{N,p,i} - s\log L_i\rvert \le d$; summed to the slope	Theorem 14.12; (RUN1)–(RUN5)
I8	Finite period to infinite volume, for one film	Controlled: at most a strip price times $e^{-\sigma n/4}$	Proposition 14.6
I9	Reference views to the interior density, (S4)	The fourth difference annihilates every face contribution; boundary information discarded by design	Lemma 14.25

	Interface	Discrepancy and status	Where proved
I10	Atom families to their moments	Many-to-one; all produced families lie in the fibre over zero, which is stable under the operations used	Proposition 14.10
I11	Histories at different cutoffs	Full-sequence convergence of banked limits by contraction; path books only subsequentially, as retained witnesses	Subsection 15.11; Proposition 15.3
I12	Change of background in the fluctuation chart	Horizontal lift exact on the protected domain; residual retained off it	Section 15.5
I13	Jet to lattice cylinder to slot readout, (S5)	Exact local section; label reading exact through Taylor order d in the mesh	(PF7), (PF10); Lemma 16.2
I14	Transported to read coefficients	Invertible reader with two-sided bounds at each level; not uniform in the level	Theorem 16.3
I15	Coefficients to the physical quotient	Kernel N_d; its finite-cutoff shadow, the calibrated equation image, is silent in separated tests; operations descend	Proposition 18.2; Theorem 18.3
I16	Interacting to free hierarchy, (S6)	Selected scaling maps; readings converge to all derivative orders along radii r_k; residuals retained	Theorems 23.2, 23.1
I17	Word space to reflection Hilbert space, (S7)	Null space; insertions descend to bounded operators	Proposition 19.1; Theorem C.1
I18	Unselected hierarchy to its phases	Rooted bounds descend to every character	Theorem 23.1; Theorem D.1
I19	Lattice symmetry to Euclidean symmetry	Signed permutations exact on the source; $O(4)$ derived on the readout only	Theorem 19.3
I20	Stress and trace	Equation residual retained in the finite candidate; charges representable as fields; full short-distance limit of the trace	Theorems 23.3, 24.1

Two structural conditions of the method are visible in the table. First, all rows are attached to one calendar and, in the limits, to one composite refinement: estimates obtained on different calendars or along incompatible subsequences are never combined. The order of the limits is part of the realized record, with the full limit in the cutoff taken at each fixed period before the limit in the period along the refined sequence. Second, the rows fall into the method's exactness classes without exception being made for convenience: I1–I3, I5 and I12 are exact; I4 and I6 are on-shell or calibrated restrictions with the off-shell data retained; I15, I17 and I18 are quotient equivalences established by congruence statements; I7, I8, I13, I14 and I16 are controlled approximations with their regimes stated; and I9–I11 record deliberate or unavoidable loss.

8.4 What follows from the source side and what is supplied

The division is the following, and it is used without further comment in the formulations below.

From the source side. The admitted fan provides a support with countably many marked cells (Proposition 8.1); no relation between distinct cells and no preferred labelling (Proposition 8.2); and the reduction of every finite statement to an admitted finite support. On each leaf the realized branch has contractible lift groupoids (Theorem 8.3). These are structural facts. They contain no dynamics.

Supplied as realization data. Everything that makes the family a Yang–Mills construction is chosen in the realized sector: the compact group $G \subseteq U(m)$ with simple nonabelian Lie algebra; the four-dimensional periodic lattices and the Wilson action; the blocking maps and the stationary corridors; the calendar $\mathfrak{c}$; the calibration value $\bar{\beta}$ and the blocking penalty one; the label algebra and the free comparison theory; and every estimate. The labelling of leaves by (N, p, i), and hence the notions of finer, larger and coarser, are of this kind.

Derived within the realization. Some quantities are neither chosen nor source-side: they are computed from the realization data by the supplied construction. The slope $s = 11C_A/(24\pi^2)$ is the coefficient of the logarithm in the explicit annular integral (DR5); the amplitude $A_G = 3\dim_{\mathbb{R}} \mathfrak{g}/\pi^4$ is a Wick contraction (Theorem 24.1); the one-dimensionality of the translation Ward kernel of the free comparison theory (FS10) and the $O(4)$ covariance of the hierarchy (Theorem 19.3) are consequences.

Not obtained. No selection theorem is proved, and none is claimed. The method's test for a source-derived premise is to look for inequivalent raw realizations over the same admitted support and the same literal preparations. Here they exist in abundance: replacing the group, the dimension of the lattice, or the Wilson action by another gauge-invariant local action gives a different realized sector over the same fan Φ_I, with the same lift-groupoid theorem. Admission therefore does not select the gauge group, the dimension, the action or the coupling history. The recovery proved in this paper is conditional in exactly this sense: given the supplied construction as realization data, its objects, operations, limits and conclusions are recovered through the readouts, with the stated exactness classes. This distinguishes an explicit mathematical recovery theorem from selection of the physical realization.

9 Activity domains, ordered derivatives and absolute budgets

The supplied construction fixes, before any estimate is used, what an activity estimate means (tags (AB1)–(AB12)). Every later regular, anchored, early or forced-hit budget refers to these objects. This section gives those definitions in full and proves that each is carried, without loss, by the frame realization $\pi : \mathcal{Z} = G^V \times G^E \to \mathcal{U} = G^E$ of Theorem 7.1.

9.1 Local activity domains, derivatives and absolute budgets

We fix the meaning of all activity estimates before using them. At a fixed finite torus $\Lambda_n = (\mathbb{Z}/n\mathbb{Z})^4$, let $\mathcal{C}_n = G^{\Lambda_n \times \{0,1,2,3\}}$ be the configuration space and $\mathcal{A}_n = \mathfrak{g}^{\Lambda_n \times \{0,1,2,3\}}$ its real space of left-link directions. A function is local on a site set X if its value depends only on links with both endpoints in X. Gauge invariance refers to $U_{x,i} \mapsto k_x U_{x,i} k_{x+e_i}^{-1}$.

For a plaquette p, write $V(p)$ for its four visited vertices and U_p for its holonomy. At threshold $0 < \vartheta \leq 1$ define

$$\mathcal{F}_\vartheta(U) = \{x : \exists p,\ x \in V(p),\ \mathrm{chord}(U_p) > \vartheta\}, \qquad \mathcal{H}_\vartheta(X) = \{U : X \cap \mathcal{F}_\vartheta(U) = \varnothing\}. \tag{AB1}$$

Thus equality with the threshold is healthy. Each flag event is open, because it is a finite union of strict inequalities for continuous plaquette functions. Each healthy domain is closed and Borel. The mask is the literal function $\mathbf{1}_{\mathcal{H}_\vartheta(X)}F$; its boundary is not differentiated in any regular budget.

A coefficient which also reads the final flags in X is required to agree whenever both the links internal to X and $X \cap \mathcal{F}_\vartheta(U)$ agree. Its actual link read set is bounded by

$$X^{\mathrm{ins}} = X \cup \bigcup_{p:V(p)\cap X\neq\varnothing} V(p). \tag{AB2}$$

Agreement on internal links of this enlarged set fixes every plaquette incident to X, hence all flags inspected there. This is why flag-dependent payloads are assigned enlarged physical footprints even though compatibility in a gas is tested on the originally assigned supports.

For $A \in \mathcal{A}_n$, set

$$(\mathrm{inc}(U,tA))_e = \exp(tA_e)U_e, \qquad \nabla_A F(U) = \frac{d}{dt}F(\mathrm{inc}(U,tA))\Big|_{t=0}.$$

For an ordered list $\ell = (A_1,\dots,A_q)$, define recursively

$$\nabla_\varnothing F = F, \qquad \nabla_{A::\ell}F(U) = \frac{d}{dt}(\nabla_\ell F)(\mathrm{inc}(U,tA))\Big|_{t=0}. \tag{AB3}$$

The order matters; no commutation of left derivatives is assumed. These derivatives are required to exist where tested. A regular representative on X is a globally Borel, gauge-invariant local function F such that $v \mapsto F(\mathrm{inc}(U,v))$ is C^j near zero for every $U \in \mathcal{H}_\vartheta(X)$, and its recursively specified left derivatives have the derivative identities in (AB3) through order j. In particular the smooth germ is required also at a healthy configuration lying on the threshold boundary. Off-domain values of the representative remain available for locality and measurability, but do not enter its regular derivative price.

A budget point consists of integers $p \geq 2, j \geq 6, B \geq 1$, support exponent $s > 0$, and threshold $0 < \vartheta \leq 1$. We denote it by $\mathfrak{p} = (p, j, s, B, \vartheta)$. For anchor o, its allowed directions satisfy

$$|A_e| \leq \vartheta(1 + \mathrm{dist}(e^-, o))^p \quad \text{for every link } e. \tag{AB4}$$

Here e^- is the initial vertex of the stored positive link, and the distance is the periodic maximum-coordinate distance. A nonnegative number b is a derivative envelope for F on (o, X) if

$$|\nabla_\ell F(U)| \leq b \quad \text{whenever } U \in \mathcal{H}_\vartheta(X),\ |\ell| \leq j, \text{ and every } A \in \ell \text{ satisfies (AB4).} \tag{AB5}$$

Norms replace absolute values for vector-valued coefficients. The support price is

$$\mathrm{mass}_{\mathfrak{p}}(X,b) = e^{s\ \mathrm{cov}_B(X)}b. \tag{AB6}$$

The factor is at least one. Every sum of envelopes below means a nonnegative summable family together with its asserted bound on the sum. Equivalently, every finite partial sum obeys that bound; taking increasing finite partial sums proves summability and the same total bound for countable dictionaries.

A regular address is (o_i, X_i) with $o_i \in X_i$. A regular family E_i has anchored budget δ if it has envelopes $b_i \geq 0$ satisfying (AB5) and

$$\sup_{o\in\Lambda_n} \sum_{i:o_i=o} e^{s\,\mathrm{cov}_B(X_i)}b_i \leq \delta. \tag{AB7}$$

The representation used after the kinetic reset additionally has $E_i(1) = 0$. On a nonwrapping support its identity Hessian

$$D^2[v \mapsto E_i(\mathrm{inc}(1, v))]_0[A, C]$$

vanishes whenever A, C are affine link fields in the canonical lift about o_i. Explicitly an affine field has

$$A_{x,a} = a_a + \sum_b \mathrm{lift}(x - o_i)_b \, d_{ab}. \tag{AB8}$$

The reset chapter constructs this Hessian cancellation; it is not imposed on arbitrary primitive functions before projection.

A forced address consists of a support Y_i, a nonempty inventory $I_i \subseteq Y_i$, and a fixed extra predicate $P_i(U)$. Its coefficient K_i is globally Borel and gauge invariant and vanishes unless

$$Y_i \cap \mathcal{F}_\vartheta(U) = I_i, \qquad P_i(U). \tag{AB9}$$

The predicate is Borel and gauge invariant; both coefficient and predicate may read links and final flags on Y_i, hence have the physical read enlargement (AB2). A forced budget κ consists of numbers $c_i \geq 0$ such that $|K_i(U)| \leq c_i$ on every configuration and

$$\sup_{x \in \Lambda_n} \sum_{i : x \in Y_i} e^{s \, \mathrm{cov}_B(Y_i)} c_i \leq \kappa. \tag{AB10}$$

This is a support-hit budget, in contrast to the single-anchor sum (AB7). Coefficients equal to zero still retain their addresses and eligibility predicates.

For a finite set D of forced addresses, the gas term is zero unless their supports are pairwise disjoint and their inventories have union exactly $\mathcal{F}_\vartheta(U)$; otherwise it is $\prod_{i \in D} K_i(U)$. Summing these terms over finite D defines the forced gas. The regular exponent is the sum of $\mathbf{1}_{\mathcal{H}_\vartheta(X_i)} E_i$. At a fixed finite torus these series are absolutely convergent: assign each regular term to its anchor and each forced term to one site of its nonempty support. Equations (AB7)–(AB10) bound their total envelope sums by $|\Lambda_n|\delta$ and $|\Lambda_n|\kappa$. The sum of absolute products over all finite subsets is at most $\prod_i (1 + c_i) \leq e^{\sum_i c_i}$. Restricting to compatible subsets only decreases it. If there are no flags, only the empty forced set can contribute, since each inventory is nonempty; the gas is then exactly one.

Signed translations are implemented by fixed permutations of the whole regular and forced address lists, carrying anchors, supports, inventories, predicates and coefficient functions together. Thus comparing two activity values in a single chart always means comparing the same lists and symmetry permutations. It does not mean identifying two representations merely because their total densities agree.

An early source budget uses $s + 3$, rather than s, in (AB7). This is the pre-reset reserve. The three additional support-exponent units pay the local splitting and conversion operations explicitly described in the reset and signed-expansion chapters. Reducing the exponent from $s + 3$ to s decreases every nonnegative summand. The rotation of derivative directions costs the finite factor

$$\mathrm{turnFee}(\mathfrak{p}) = (2\,2^p)^j. \tag{AB11}$$

Indeed the anchor itself is transported exactly. A reversed link is stored at its other endpoint, at distance at most one from the transported base, so $1 + d + 1 \leq 2(1 + d)$ costs 2^p in (AB4). Its right increment can be converted to a left increment by the adjoint action of the background link, which preserves color norm. Expanding each pulled-back left derivative into its positive-link and negative-link contributions gives at most 2^q ordered lists at order q; each has length q and direction

inflation at most 2^p. Multilinearity of the iterated derivatives bounds their sum by $(2\,2^p)^q b$, which is at most (AB11) times b for $q \le j$. This expansion follows by induction on the ordered list, using the exact left/right increment identity and the disjoint link supports of the positive and negative parts. Support enlargement is charged separately by its cover-number inequality; it is not included in this factor.

At level i put $x_i = \widehat{\beta}_i$ and $M_i = \exp((\log\log x_i)^2)$. The normalizations used in (LCN1) are the calendar's actual allowances

$$b_{P_i} = x_i^{-11/10}, \qquad k_{P_i} = \exp(-M_i^{1/5}). \tag{AB12}$$

Both are strictly positive. Thus $\|A\|_{P_i} \le t$ means exactly that its regular and forced components obey (AB7) and (AB10) with budgets tb_{P_i} and tk_{P_i}, respectively, in their fixed address layout. The retained physical states have $t \le 1/8$. Triangle inequalities give the norm properties used later: sums of admissible nonnegative envelopes remain envelopes, derivatives are linear, and scalar multiplication multiplies all envelopes by the scalar modulus. Least envelopes, followed by finite partial-sum limits, give attainment of an infimum whenever it is used in (LCN1). The separated completion removes only differences of zero price in these tested germs and forced values.

The definitions (AB1)–(AB12) are the ordinary mathematical content of RG/Masks, RG/Activities, the reset Quotes and Investor definitions, and the activity normalization in FullVolume/Sym/Shorn. They will be used with the same domains and directions in every subsequent reference to a regular, anchored, early or forced-hit budget.

Source–readout formulation. The target objects of this subsection are functions and events on the finite configuration space $\mathcal{C}_n = G^{\Lambda_n \times \{0,1,2,3\}}$, which is the readout space $\mathcal{U}$ of Theorem 7.1 for the torus Λ_n. The realized sector is $\mathcal{Z} = G^V \times G^E$ with readout $\pi(r,A)_e = r_x^{-1} A_e r_y$ and section $\sigma(U) = (1,U)$. An activity datum is lifted by composition: every regular representative, forced coefficient and predicate F is replaced by $F \circ \pi$, while every piece of lattice bookkeeping (anchor, support, inventory, address list, scope) is copied unchanged into the raw tuple. The operations to be lifted are the flag map, the mask, the left increment and its ordered derivatives, the signed lattice permutations, and the sums defining the budgets and the norm. The fibre of π is the frame orbit; the statements below show that all of these operations commute with π and that every number defined in (AB1)–(AB12) is constant on that fibre. The exactness class is an embedding with section: nothing in the budgets is enlarged, sharpened or lost.

Proposition 9.1 (Frame-blind flags)**.** *For every $(r,A) \in \mathcal{Z}$ and every plaquette p based at x,*

$$\pi(r,A)_p = r_x^{-1} A_p r_x, \qquad \operatorname{chord}(\pi(r,A)_p) = \operatorname{chord}(A_p), \qquad \mathcal{F}_\vartheta(\pi(r,A)) = \mathcal{F}_\vartheta(A).$$

Hence $\pi^{-1}\mathcal{H}_\vartheta(X) = G^V \times \mathcal{H}_\vartheta(X)$, a closed Borel set stable under the frame action of Theorem 7.1, and for every gauge-invariant F one has $(\mathbf{1}_{\mathcal{H}_\vartheta(X)} F) \circ \pi(r,A) = \mathbf{1}_{\mathcal{H}_\vartheta(X)}(A) F(A)$.

Proof. The path identity $U_\gamma = r_x^{-1} A_\gamma r_y$ at a closed path gives the conjugation formula. The chord $\|g-1\|_{\mathrm{F}}/\sqrt{m}$ is invariant under conjugation by a unitary matrix, which gives the second equality at every plaquette and therefore equality of the two flag sets. The preimage formula is a restatement. The frame action sends A_p to $k_x A_p k_x^{-1}$, which again preserves every chord. Finally $\pi(r,A)$ is the gauge transform of A by $g_x = r_x^{-1}$, so a gauge-invariant F has $F \circ \pi(r,A) = F(A)$. □

Thus the frame column is a pure spectator for gauge-invariant activities. It is not a spectator for the charts, axial gauges and predictors of the nonlinear chapter, which are not gauge invariant; there the frame is retained and the covariance identities of the tangent chapter are used instead. A

flag-reading coefficient lifted by composition with π reads the raw links internal to X^{ins} of (AB2) and, unless it is gauge invariant, the frames on X^{ins}.

For the increment and the derivative define, for $z = (r, A) \in \mathcal{Z}$ and $B \in \mathcal{A}_n$,

$$\mathrm{inc}_Z(z, tB) = \big(r,\ (\exp(t\, r_x B_e r_x^{-1}) A_e)_{e:x\to y}\big), \qquad \nabla^Z_{B::\ell}\Phi(z) = \frac{d}{dt}(\nabla^Z_\ell \Phi)(\mathrm{inc}_Z(z, tB))\Big|_{t=0},$$

with $\nabla^Z_\varnothing \Phi = \Phi$. The raw direction is the target direction conjugated into the stored frame; the frame itself is not moved.

Theorem 9.2 (Ordered-derivative recovery)**.** *For all $z \in \mathcal{Z}$, $B \in \mathcal{A}_n$ and real t, $\pi(\mathrm{inc}_Z(z, tB)) = \mathrm{inc}(\pi(z), tB)$. If F has the ordered derivatives* (AB3) *through order j at $\pi(z)$, then $F \circ \pi$ has the ordered derivatives ∇^Z_ℓ through order j at z and*

$$\nabla^Z_\ell (F \circ \pi)(z) = (\nabla_\ell F)(\pi(z)) \qquad (|\ell| \le j),$$

with the same ordered list on both sides. The raw-link direction $r_x B_e r_x^{-1}$ has the same color norm as B_e.

Proof. For a link $e : x \to y$, $r_x^{-1} \exp(t\, r_x B_e r_x^{-1}) A_e r_y = \exp(tB_e)\, r_x^{-1} A_e r_y$, since conjugation commutes with the exponential. This is the first identity. The derivative identity is proved by induction on the list. It is trivial for the empty list. If it holds for ℓ on a neighbourhood of the curve, then

$$\nabla^Z_{B::\ell}(F \circ \pi)(z) = \frac{d}{dt}(\nabla_\ell F)\big(\pi(\mathrm{inc}_Z(z, tB))\big)\Big|_0 = \frac{d}{dt}(\nabla_\ell F)(\mathrm{inc}(\pi z, tB))\Big|_0 = \nabla_{B::\ell} F(\pi z).$$

The increment inc_Z leaves the frame fixed, so the induction never leaves the fibre over the varying readout and the order of the list is not changed. The color inner product $-m^{-1} \operatorname{Re}\operatorname{tr}(XY)$ is conjugation invariant, which is the last statement. □

Proposition 9.3 (Budgets are fibre constants)**.** *Let F be a regular representative on X and K_i, P_i forced data. For every frame r:*

1. *b is an envelope* (AB5) *for F on (o, X) if and only if $|\nabla^Z_\ell (F\circ\pi)(r, A)| \le b$ for all $A \in \mathcal{H}_\vartheta(X)$, $|\ell| \le j$ and lists obeying* (AB4)*;*
2. *the prices* (AB6)*, the anchored budget* (AB7)*, the forced bounds $|K_i \circ \pi| \le c_i$, the eligibility condition* (AB9) *and the forced budget* (AB10) *are the same numbers and the same events for the lifted family;*
3. *the identity-value and affine-Hessian conditions lift to every flat raw configuration $A_e = r_x r_y^{-1}$: there $E_i \circ \pi = E_i(1) = 0$ and the second derivative $\nabla^Z_{(A,C)}(E_i \circ \pi)$ vanishes for affine A, C of the form* (AB8)*.*

Proof. (1) By Proposition 9.1, (r, A) lies over the healthy domain exactly when A does, and $\pi(r, \cdot)$ maps $\mathcal{H}_\vartheta(X)$ onto itself by a gauge transformation. Theorem 9.2 equates the two derivative values with the same list, so the two suprema coincide. (2) Supports, anchors, cover numbers and inventories are lattice data untouched by π; coefficients and predicates are gauge invariant, so their lifted values at (r, A) are their values at A; flags agree by Proposition 9.1. (3) A flat raw configuration reads to $U = 1$, and the Hessian is the order-two case of Theorem 9.2. □

Gas and exponent. By Proposition 9.3 each term of the forced gas and of the regular exponent, lifted by composition, equals the corresponding original term at A and has the same majorant. Consequently the absolute convergence statements derived above hold for the lifted series with the same bounds, and

$$\mathrm{Gas}^{Z}(r, A) = \mathrm{Gas}(A), \qquad \mathrm{Exp}^{Z}(r, A) = \mathrm{Exp}(A),$$

termwise and before summation. The address lists are part of the realized tuple: two representations with the same total density but different address layouts are different source records, exactly as in the supplied construction, and the readout does not identify them.

Symmetries and the turn fee. On $\mathcal{Z}$ a signed lattice permutation ρ acts geometrically on frames and raw links, with inverse raw links under orientation reversal; the path identity $U_\gamma = r_x^{-1} A_\gamma r_y$ gives $\pi\rho_Z = \rho\pi$, so the lifted address lists are permuted by the same fixed permutations. In the derivation of (AB11) the conversion of a right increment into a left increment uses, in the realization, the raw background link, $A_e \exp(tA') = \exp(t\, A_e A' A_e^{-1}) A_e$, and the frame conjugations of Theorem 9.2 preserve the color norm. The fee is therefore the same number on both sides of π and is not paid twice.

Theorem 9.4 (Isometric recovery of the activity chart)**.** *Let Act_{P_i} be the space of activities in a fixed address layout with finite norm and $J\mathsf{A} = \mathsf{A} \circ \pi$, applied to each regular representative, forced coefficient and predicate while keeping the address lists. With the lifted envelopes of Proposition 9.3(1), J is a linear isometry of $(\mathsf{Act}_{P_i}, \|\cdot\|_{P_i})$ onto its image; restriction to the section $\sigma(U) = (1, U)$ is its inverse; J descends to the separated completions and intertwines the signed-translation permutations, the reserve reduction $s + 3 \mapsto s$ and the turn-fee estimate* (AB11)*.*

Proof. Linearity is clear. Equality of norms is Proposition 9.3 applied to each address, followed by the same suprema and sums. $(J\mathsf{A}) \circ \sigma = \mathsf{A}$ because $\pi\sigma = \mathrm{id}$. An activity difference has zero price exactly when all its tested germs and forced values vanish, a condition preserved in both directions, so J is well defined and isometric on the separated quotient and extends to the completion. The intertwining statements were proved in the two preceding paragraphs. □

The status of Theorem 9.4 is exact recovery on the declared class. What the realization does *not* supply is a bound on derivatives in frame directions for functions that fail to be gauge invariant; none is used for activities. What is realization data rather than a consequence of admission is everything in (AB1)–(AB12) itself: the threshold, the stiffness and budget points, the envelopes and the calendar normalization (AB12) are chosen in the supplied construction and retained in the raw tuple.

9.2 The finite Wilson law is a probability measure

First the finite Wilson law is an actual probability measure. Every unitary matrix V has $-m \le \mathrm{Re}\,\mathrm{tr}\, V \le m$: each diagonal entry has modulus at most one. Thus every plaquette price lies in $[0, 2]$. If there are F plaquettes and $\beta \ge 0$, its continuous density relative to the product Haar probability measure obeys

$$e^{-2\beta F} \le e^{-\beta S(U)} \le 1, \qquad e^{-2\beta F} \le Z \le 1.$$

In particular normalization is legitimate and bounded measurable cylinders are integrable. A site gauge change multiplies a link on its two ends; by Haar invariance its change of variables preserves the product prior. Its plaquette holonomy is conjugated at the base site, so the trace and the action are unchanged. The normalized law is consequently gauge invariant. A signed lattice permutation merely reindexes plaquettes, possibly reversing their holonomies; inversion preserves their real

traces. It too preserves both the prior and the law. The local representatives in (PF12), and their finite products, therefore have precisely the finite-lattice expectations used below.

Source–readout formulation. In the realization the density of Theorem 7.2 is the same function of the raw links, $e^{-\beta S(A)}$, and the frame factors integrate to one. Therefore

$$Z_Z = Z \in [e^{-2\beta F}, 1], \qquad \pi_* \widetilde{\mu} = \mu_W,$$

with the same two-sided density bounds. A physical gauge transformation g is lifted by $r_x \mapsto r_x g_x^{-1}$ with A fixed, so it does not move the density at all; a signed lattice permutation acts on both columns and preserves the product Haar prior and the raw action for the reasons just given. The local representatives used in the reflection argument and their finite products have exactly the same expectations before and after the lift. This is exact recovery of a normalized probability law, not of an unnormalized weight modulo a gauge volume.

10 Finite reflection positivity and rooted bounds

Cut the periodic lattice perpendicular to a coordinate axis. A leaf X is the set of links tangent to one slice; threads are the links joining successive slices. The supplied finite argument starts with the positive kernel $K(g,h) = \operatorname{Re}\operatorname{tr}(gh^*)$, the real inner product of real and imaginary matrix entries. Finite sums, pullbacks and nonnegative multiples preserve positivity. Products preserve it by tensoring Gram vectors; the exponential preserves it by its nonnegative power series and finite-matrix limits. At identity thread the joining-leaf kernel is

$$W_\beta(X, Y; 1) = e^{-\beta q} E_\beta(X) E_\beta(Y) \exp\left[\frac{\beta}{m} \sum_\ell \operatorname{Re}\operatorname{tr}(X_\ell Y_\ell^*)\right]. \tag{RP1}$$

Here q counts crossing plaquettes and E_β allocates the tangential half-action. For $\beta \geq 0$ it is continuous and positive semidefinite. Endpoint gauge changes and normalized Haar invariance give

$$\int W_\beta(X, Y; u)\, du = \iint W_\beta(u \cdot X, v \cdot Y; 1)\, du\, dv. \tag{RP2}$$

The identity follows by replacing one thread variable by its left translate and combining the two endpoint gauge actions. The right side is positive semidefinite: apply the positive-kernel quadratic inequality to finite linear combinations of the averaged feature vectors, or approximate the integrals by simple functions in the same Gram identity. Compact boundedness justifies Fubini and the continuity of the averaged kernel. Site reflections have shared seam variables and constant conditional cross kernel. Link reflections have two distinct periodic seam threads, including at the shortest allowed half-length, and the product of their averaged kernels, with one argument order reversed, is positive.

The seam/open-link assembly changes variables only by left/right multiplication and inversion, preserving the product Haar prior by iterated invariance. To make the splitting precise, list the seam variables, the two seam threads when present, and the open variables on the two halves. Assemble the original link configuration from these coordinates by their assignments and the endpoint gauge multiplications along the cut direction. Its plaquette reading splits the action into the two halves and seam factor. Gauge-invariant probes strictly in one half read that half and shared seam variables. Conditional on the seam, with half-action factors included in f, the unnormalized pairing is

$$\iint f(X)\overline{f(Y)} K_{\text{seam}}(X, Y)\, dX\, dY \geq 0.$$

The integrated Gram inequality proves its sign. Integrating over the seam preserves nonnegativity. Bounded measurable probes are integrable against the compact Wilson density. Dividing by the strictly positive partition function yields

$$\langle F\,\overline{F\circ\vartheta}\rangle_\beta \geq 0. \tag{RP3}$$

The reflection includes orientation inversions; no crossing link inside the probe is replaced.

On an admissible rectangular tiling with v cells, define

$$\mathcal{R}(F) = \left|\left\langle \prod_c F^{(c)} \right\rangle_\beta\right|^{1/v}.$$

The copied functions retain the prescribed translations, reflections and conjugations. For $V(F_\bullet) = |\langle\prod_c F_c^{(c)}\rangle_\beta|$, reflection Cauchy–Schwarz across each translated cell seam gives

$$V(A)^2 \leq V(A_+)V(A_-), \tag{RP4}$$

where each assignment folds the corresponding half into the other. In a finite alphabet the maximum exists. Both folded values are at most that maximum; if it is positive the inequality forces both to equal it, and if zero equality is immediate. Thus maximizing assignments remain maximizing under both folds. Starting with a one-cell-wide identical strip, successive boundary folds extend it to a half-period and then the full coordinate. A fold in another coordinate preserves equality along the coordinates already filled, so repeating in each coordinate yields a constant maximizing assignment, without a power-of-two hypothesis. Normalize each chamber function by $\mathcal{R}(F_c)+\epsilon$. Every constant assignment has value at most one, hence the mixed maximum does too. Rescaling and $\epsilon \downarrow 0$ prove

$$\left|\left\langle \prod_c F_c^{(c)} \right\rangle_\beta\right| \leq \prod_c \mathcal{R}(F_c). \tag{RP5}$$

Copied scalars give $\mathcal{R}(zF) = |z|\mathcal{R}(F)$: a copied scalar is either z or $\bar{z}$, and the absolute value of their full product is $|z|^v$. Expanding the 2^v assignments in the copied product of $F+H$ and applying RP5 bounds its expectation by $(\mathcal{R}(F)+\mathcal{R}(H))^v$. Taking the root proves subadditivity. This proves all the rooted homogeneity, subadditivity and mixed-product estimates used below directly for the Wilson law.

For the lift, reflect frames and raw links by their geometric vertex/link assignment, with inverse raw links under orientation reversal. The path identity then gives $\pi\vartheta_Z = \vartheta\pi$. Thus $F^Z = F\pi$ has exactly the RP3 pairing by measure recovery. Every copied chamber product likewise has the original expectation, so its rooted seminorm and RP4–RP5 bounds are identical. All seam and support records are retained. This proves preservation of the finite positive form and rooted estimates before the later Hilbert completion.

Positive-form recovery on the retained probe algebra. Let $\mathcal{A}_+$ be the supplied bounded gauge-invariant chamber algebra, with its strict half-space support convention, and let $J_+F = F\circ\pi$. Equip its image $J_+\mathcal{A}_+$ with the reflected form

$$B_Z(J_+F, J_+H) = \int (F\circ\pi)\,\overline{(H\circ\pi)\circ\vartheta_Z}\,d\widetilde{\mu} = \int F\,\overline{H\circ\vartheta}\,d\mu_W = B(F,H).$$

The middle identity uses the geometric intertwining and normalized pushforward together; positivity alone would not identify the form. The section gives $(J_+F)\circ\sigma = F$, so this is an injective

identification of the retained probe algebra. For any finite list F_a, the entire reflected Gram matrix is identical on both sides, not only its diagonal. Consequently J_+ identifies the zero-form subspaces $\mathcal{N} = \{F : B(F,F) = 0\}$ and $\mathcal{N}_Z = \{J_+F : B_Z(J_+F, J_+F) = 0\}$; it induces an isometry of $\mathcal{A}_+/\mathcal{N}$ onto $J_+\mathcal{A}_+/\mathcal{N}_Z$, and hence of their completions. This statement concerns precisely the retained probe image. For each copied chamber product the same identities give $\mathcal{R}_Z(J_+F) = \mathcal{R}(F)$, including zero roots, so normalization by $\mathcal{R}(F) + \epsilon$ and the limit $\epsilon \downarrow 0$ commute with recovery. Strict support is transported with the endpoints of each raw link and its endpoint frames; the original readout support is recovered on the section. Neither a new seam convention nor a positivity assertion on arbitrary raw-frame functions is needed.

11 Quadratic ultraviolet kernels and comparison

The recovery problem at quadratic order. The finite Wilson measure is now fixed. The next task is to control its quadratic approximation under repeated blocking. A quadratic history retains the cochain spaces, block averages, precision forms, covariance symbols and stationary predictors at every intermediate scale. A precision is the quadratic form in the Gaussian exponent; its inverse is taken only on the specified gauge and harmonic complement. The source record must therefore retain that complement, rather than replacing a singular form by an unspecified inverse.

At the tangent level the frame readout is a surjection p with section j, $pj = I$. An operator T is realized by jTp, whereas its quadratic form is pulled back by p^*Qp. These two constructions have different types. In particular the pulled-back form has the vertical frame directions in its radical and cannot be coercive there. The ensuing Fourier divisions, alias sums and positive-margin repairs are performed in the original cochain spaces and are then recovered through these maps. The principal-predictor amendment in Appendix I.1 is essential to cancellation before Fourier extraction; a bound on the separate singular pieces would not replace that identity.

For reporting level k write $T(k,i) = 2 + Z(k) + |\operatorname{scale}(i) - \operatorname{scale}(k)|$. The spacings a_i tend to zero. The supplied $Q_{n,i}$ is the action Hessian on link potentials, with no additional factor one half, obtained from a curvature packet by D^*KD; it does not specify the curvature metric away from curls. The row bound R_s sums absolute entries over input sites at fixed input orientation, with weight $e^{s\,\mathrm{distance}}$, and takes the supremum over output links.

Retain the horizon hypotheses named H.Suffices P and Flat.meshed, including the actual destination response in each preceding history. In particular $y_i = w(J(i))$, $0 < w(j) \le \min(P.\alpha(j), 1)$, and on the admitting tail $y_i \le \min(1, P.\alpha(J(i+1)), P.\alpha(J(i)))$. Retain $2y_i \le \operatorname{block}(i)y_{i+1}$, one-step price at most $\operatorname{block}(i)^{-\theta}$ for $\theta > 0$, and $P.\Omega(J(i)) \le BT(k,i)^l$ uniformly in $k, i \ge e$.

The terminal saddle difference is a curvature packet with cost $\operatorname{topTab}(N) = 2|\mathrm{Faces}|P.\Omega(J(N))$. The exact crossing identity propagates this packet using the two actual histories' responses, retaining neutrality and translation covariance. Width crossing licenses each propagation. In particular, this is an identity for the two actual histories rather than a comparison with a separately chosen limiting history. Its cost satisfies

$$C_N = \operatorname{topTab}(N), \qquad C_i \le \operatorname{block}(i)^{-\theta} C_{i+1}.$$

Before replacing the quote by its block bound, propagation multiplies the packet cost by $\operatorname{quote}(i, y_{i+1})$. The reciprocal Stair.weight constructs a nonnegative majorant whose recurrence is an equality with this actual quote. World.overlap then applies the polynomial-age decay lemma to that majorant, without assuming a continuum limit. The polynomial decay argument below then yields $A \ge 0$, an integer $d \ge 0$ and $c > 0$ such that

$$R_{y_i}(Q_{n,i} - Q_{N,i}) \le AT(k,i)^d (a_N/a_i)^c, \qquad e \le i \le N \le n. \tag{Q1}$$

To recover the cover quadratic kernel, let D be the lattice coboundary and write $W(K) = D^*KD$, interpreting K through its two-point kernel. For the packet-to-kernel step, finite support of D supplies width-one row and column bound L. Weighted convolution gives $R_s(D^*KD) \le |\mathrm{Faces}|^2 L^2 C$ for $s \le 1$ and packet cost C, independently of torus size. Equality of the packet form and kernel difference on all scheduled tori gives cover equality by the following uniqueness argument. In a periodized entry, the indicator selecting each fixed cover site converges to the indicator of the chosen site as torus side tends to infinity. Summable absolute entries dominate this sum; dominated convergence recovers the cover entry. Thus two summable kernels with identical periodizations agree entrywise. This is Flat.resolve, distinct from the matrix inverse named resolve elsewhere.

Q1 makes every entry Cauchy for fixed $i \ge e$. Its limit F_i obeys, by Fatou on weighted absolute rows,

$$R_{y_i}(Q_{N,i} - F_i) \le AT(k,i)^d (a_N/a_i)^c, \qquad R_{y_i}(F_i) \le B'T(k,i)^l.$$

Uniqueness of entrywise limits gives the same family for every reporting level k; before e it may be defined as zero, where no conclusion is imposed.

For completeness retain the scalar majorant proof, with nonnegative arrays, $A \ge 0$, integer $d \ge 0$, $Z \ge 0$ and $\theta > 0$. All constants are uniform over the schedules and terminal indices under consideration. Let $\Delta_i = t_{i+1} - t_i \ge \log 3$ and $\tau_i = 2 + Z + |t_i - t_k|$. If $f_i \le e^{-\theta\Delta_i} f_{i+1} + g_i$, $f_N \le g_N$ and $0 \le g_i \le A\tau_i^d$, put $K = 2(d+1)/\theta$, $q = e^{-\theta \log 3/2}$ and $B = A/(1-q)$. Here $K > 0$, $0 < q < 1$, $\tau_i \ge 2$, and $\tau_{i+1} \le \tau_i + \Delta_i$. The elementary inequality $e^u \ge 1 + u$ gives

$$K + \tau_{i+1} \le (K + \tau_i) e^{\Delta_i/K}.$$

Since $d/K \le \theta/2$,

$$(K + \tau_{i+1})^d e^{-\theta\Delta_i} \le q(K + \tau_i)^d.$$

At the terminal index the bound follows from $A \le B$. If it holds at $i+1$, the recurrence gives $(Bq + A)(K + \tau_i)^d = B(K + \tau_i)^d$. Thus backward induction gives $f_i \le B(K + \tau_i)^d \le B(K+1)^d \tau_i^d$, using $K + \tau_i \le (K+1)\tau_i$. If $g_i e^{c(t_N - t_i)}$ is polynomially bounded, multiply the recurrence by $e^{\mu(t_N - t_i)}$, $\mu = \min(c, \theta/2) > 0$. The transformed forcing is polynomial because $\mu \le c$, and the new decay exponent is $\theta - \mu > 0$. The same argument gives $f_i \le A'\tau_i^{d'} e^{-\mu(t_N - t_i)}$, equal to a positive power of a_N/a_i. A forcing supported only at N automatically satisfies the weighted hypothesis, because its exponential factor is one on its support. For well-founded grades, finite polynomials in earlier-grade bounds provide the forcing; next-scale ages are polynomially controlled when block lengths are polynomial in current age, since $\tau_{i+1} \le \tau_i + \log(\mathrm{block}(i)) \le \tau_i + \mathrm{block}(i)$. Difference forcing retains polynomial absolute bounds of both histories, including the current grade, and difference bounds only at strictly earlier grades. Multiplication by polynomial absolute bounds preserves exponential decay. Finite sums take the minimum positive decay exponent.

Every admissible horizon has the same fixed terminal bare quadratic form. On each scheduled finite torus its stationary response satisfies

$$Q_{\mathrm{coarse}} = T^{\mathsf{T}} Q_{\mathrm{fine}} T + b(I - PT)^{\mathsf{T}}(I - PT), \qquad b > 0.$$

Here P is the common probe and T the actual extracted response. All spaces in this comparison are real with the stated physical metrics; $*$ below denotes their adjoint, represented by transpose in orthonormal coordinates. For two symmetric fine precisions Q, R, symmetric coarse precisions A, B, and responses T, W, the primal and dual equations give

$$QT = P^*A, \quad RW = P^*B, \quad A = b(I - PT), \quad B = b(I - PW),$$

$$W^*(R - Q)T = BPT - (PW)^*A = B(I - A/b) - (I - B/b)A = B - A.$$

Explicitly, transposing $RW = P^*B$ gives $W^*R = BP$. The ordered mixed products cancel without commutation of A and B, even when auxiliary factorizations differ. Common terminal forms therefore yield common evaluated quadratic forms backwards, and summable cover uniqueness yields common kernels. The source's shadow clips change the numerical factors to $\gamma(j), \gamma(j)/32 > 0$ while retaining relaxation, radius, slow selector, schedule, patches, smoothing levels and the blocking penalty, so this comparison applies to those horizons. It asserts equality on link-potential quadratic forms, not equality of higher-order horizon data. The responses T, W and their primal and dual equations are constructed in the harmonic and scaled-exterior arguments below.

Under the tangent readout, the packet is pulled back as a form, while responses use jTp. The exact crossing and comparison identities therefore recover by the operator/form theorem. All row bounds apply on the transported section with the original color norms; the redundant vertical directions are not assigned a positive coercive floor. This identifies the source quadratic convergence mechanism on the realized branch without replacing its horizon assumptions.

Recovery of the ultraviolet limit and common output. At a fixed reporting lattice i, choose the tangent readout p_i and section j_i once, independently of the terminal cutoff n, with $p_i j_i = I$. For the finite-cutoff forms and their limit put

$$\widetilde{Q}_{n,i} = p_i^* Q_{n,i} p_i, \qquad \widetilde{F}_i = p_i^* F_i p_i, \qquad j_i^* \widetilde{Q}_{n,i} j_i = Q_{n,i}, \qquad j_i^* \widetilde{F}_i j_i = F_i.$$

The standard tangent readout is a fixed finite stencil at this level. Every entry of its pulled-back form is therefore a finite linear combination of entries of the physical form. Entrywise convergence commutes with this pullback and with section restriction. On the section, use the transported physical coordinate basis and the original distance and color norm. The weighted row bound of the restricted difference is then exactly $R_{y_i}(Q_{N,i} - F_i)$, with the original polynomial and decay constants; this is not a bound in an arbitrarily chosen norm on the redundant coordinates. The same restriction recovers the Fatou bound and the family independent of reporting level k. On each scheduled torus the readout stencil is its periodic version, so the finite-stencil pullback commutes with periodization of the summable kernel. Consequently the cover-uniqueness argument compares the same physical entries after readout. It requires no inverse of the redundant precision. When two horizon outputs agree at every cutoff, their section restrictions and their entrywise limits agree; this transfers the supplied common-output and entrance conclusions with their original admissibility conditions. It does not identify curvature metrics away from curls or nonlinear data of the two horizons.

11.1 The finite quadratic history contract and its terminal majorant

Before the horizon hypotheses are stated it must be said exactly what a finite quadratic history is. At each $i \le n$ it consists of five real Laurent rows on lattice two-forms: an unamended metric H, its inverse, an amended metric M, its inverse, and a residual Gram root F. Their weighted coefficient bounds have width $\alpha(J(i)) > 0$ and price $\Omega(J(i))$. At every torus period

$$HH^{-1} = MM^{-1} = I, \qquad F^*F + cI = M, \qquad Md_1 = Hd_1, \qquad M(0) = I, \tag{1}$$

where $c > 0$ is the direct clip coefficient, so the full Gram factor is the stack $(\sqrt{c}\, I, F)$. Both metrics are symmetric with quadratic forms between $\eta(J(i))I$ and $\Omega(J(i))I$, $\eta > 0$, and the terminal unamended metric is I. Each transition carries a fine potential response A_0, a curvature response T, and mixed, normal and image rows satisfying the stationary and Gram identities derived in the following subsections. The mixed row has bound Ω/L^2; the normal and image rows have bound Ω.

A word of r carried fine differences of A_0 or of T has bound ΩL^{-1-r} or ΩL^{-2-r} respectively, through order $J(i+1)-2$. A carried difference increments the fine residue, shifts the coarse displacement when the increment crosses its centred seam, and subtracts the original entry: it is the actual fine difference including the carry. These all-period conditions are *constructed* by the metric, positive-margin, auxiliary-tower and pinning arguments below; they are not additional hypotheses of the public theorem.

Proposition 11.1 (Lift of the five-row contract)**.** *Let $p(\phi,a)=a+d_0\phi$ and $j(u)=(0,u)$ be the potential readout and section of Theorem A.1. Two-form rows are read through d_1p, and $d_1p(\phi,a)=d_1a$. Then:*

1. *the identities* (1) *hold verbatim for the realized rows, which act on the same two-form space; in particular $(\sqrt{c}\,I,F)$ is a Gram factor of M, so $M\geq cI$;*
2. *the exterior identity pulls back to $M\,d_1p=H\,d_1p$ on the whole redundant potential space, and both sides vanish on the vertical subspace $\{(\phi,-d_0\phi)\}$;*
3. *the realized potential response $A_{0,Z}=j_fA_0p_c$ satisfies $p_fA_{0,Z}=A_0p_c$, and a word of carried fine differences of $A_{0,Z}$ reads to the same word for A_0, with the same residue increments and seam carries.*

Proof. (1) The frame redundancy acts on potentials only, and $d_1d_0=0$ shows that no two-form coordinate is changed; the rows and their products are literally the supplied ones. The quadratic form of F^*F+cI is $\|Fv\|^2+c\|v\|^2$. (2) Compose $Md_1=Hd_1$ with p and use $d_1p(\phi,-d_0\phi)=-d_1d_0\phi=0$. (3) The first identity is Theorem A.1. A carried fine difference is a fixed finite combination of lattice translations and index carries, which commute with p and j because d_0 is translation covariant; induction on the length of the word gives the claim. □

The loss in this lift is explicit: the amendment $M-H$ is determined by (1) only off the range of d_1, and the realization records which amendment the history actually chose. Two histories with the same potential Hessian $d_1^*Hd_1=d_1^*Md_1$ but different amended metrics are different realized records with the same potential readout.

For the terminal-only recurrence of the flat convergence argument the majorant can be written in closed form. With $t_j=\log(a_j^{-1})$,

$$\widehat{C}_i=\mathrm{topTab}(N)\prod_{j=i}^{N-1}L_j^{-\theta}=\mathrm{topTab}(N)\,e^{-\theta(t_N-t_i)}.$$

The empty product gives the terminal value and each backward step is exactly the required contraction. Put $u=t_N-t_i\geq 0$. The triangle inequality gives $T(k,N)\leq T(k,i)+u$, and $T(k,i)\geq 2$, so $T(k,N)\leq T(k,i)(1+u)$. The polynomial cap on the terminal price yields

$$\widehat{C}_i\leq C\,T(k,i)^l(1+u)^le^{-\theta u}\leq C'\,T(k,i)^le^{-\theta u/2},\qquad C'=C\sup_{u\geq 0}(1+u)^le^{-\theta u/2}<\infty.$$

Since $e^{-u}=a_N/a_i$ this is the asserted decay $C_i\leq AT(k,i)^l(a_N/a_i)^c$ with $c=\theta/2$. It uses no continuum convergence. The general recurrence lemma, which also allows forcing at intermediate levels, is proved with all its steps in the scalar majorant paragraph above (constants $K=2(d+1)/\theta$, $q=e^{-(\theta/2)\log 3}$, $B=A/(1-q)$ and the bound $f_i\leq B(K+1)^d\tau_i^d$). Because the majorant is a scalar function of the realized scale clock and of the tagged terminal price, its lift is the identity on values: $\widehat{C}_i^Z=\widehat{C}_i$, and the recovered kernel bound has the same constant C' and exponent $\theta/2$.

11.2 Width selection and neutral propagation

The entrance construction first fixes the retained-support linear choice, its positive clip curve γ and length demand n. For its price choice X, use $E(j) = \max(\text{tickets}(X, j), n(j))$ and require $E(J(i + 1)) \le \text{block}(i)$, including auxiliary steps before admission. Apply the linear horizon construction to the sheet with rearranged clips. The delay assumption at the admitting index supplies the widths checked below. The supplies construction on the shadow horizon gives one common family F_i, polynomial absolute bounds and decaying difference bounds. The same_shadow identity identifies every original finite-cutoff output with its shadow output; equality of these sequences transfers their limits and all bounds to every original horizon meeting its admissible price conditions. This proves the quadratic-history comparison. The responses themselves are constructed, and their primal and dual equations derived, in the harmonic and scaled-exterior subsections below (Subsections 11.9 and 11.10).

For a fixed price table choose $w(j) = \min(1, P.\alpha(j), P.\alpha(j+1)) > 0$ before the lengths. If $h(j, u)$ is the adjacent smoothing allotment, put

$$B(j, u) = |\text{Faces}|^2 \frac{2}{w(j)} h(j, u)^2, \quad E_P(j) = \left\lceil 2/w(j) + B(j, j)^2 + B(j, j-1)^2 \right\rceil .$$

The smoothing index stays fixed or increases by one at each step. Thus $u = J(i)$ is $j = J(i + 1)$ or $j - 1$. The requirement $L \ge E_P(j)$ gives both $B(j, u)^2 \le L$ and $L \ge 2/w(j)$, and pays $B(j, u)/L \le L^{-1/2}$ and $2w(u) \le 2 \le Lw(j)$. These are the contraction and width-crossing inequalities with $\theta = 1/2$. Adjacency also makes $w(u)$ no greater than either analytic width required at the crossing. The thresholds depend on the already fixed prices, before L is chosen. With $Z(k) = 3 + \log \widehat{\beta}_k$, $\widehat{\beta}_k > 1$, and slope $s > 0$, the affine clock gives $\widehat{\beta}_i \le \widehat{\beta}_k + s|t_i - t_k|$. Divide by $\widehat{\beta}_k > 1$ and use $\log v \le v - 1$ to obtain

$$\log \widehat{\beta}_i \le \log \widehat{\beta}_k + s|t_i - t_k| \le (1 + s)[2 + Z(k) + |t_i - t_k|].$$

Thus the admitting delay $P.\Omega(J(i)) \le \log \widehat{\beta}_i$ gives the required degree-one age bound. Together with the previous inequalities this establishes every spacing condition used in the flat convergence argument.

A nearest-neighbor coordinate path from x to z has length $a = |z - x|_1$ and stays within that distance of x. Such a path is obtained by changing each integer coordinate toward its target one unit at a time. The Atlas one-step data supply a weighted forward directional difference row bounded by c/L^{d+1}. At each intermediate site v, the triangle inequality gives

$$e^{\sigma \delta_L(x,y)} \le e^{\sigma a/L} e^{\sigma \delta_L(v,y)}.$$

The same row bound therefore applies to forward and backward steps, with the factor $e^{\sigma a/L}$. Telescoping yields

$$\sum_y |U((z, i), (y, j)) - U((x, i), (y, j))| e^{\sigma \delta_L(x,y)} \le a e^{\sigma a/L} c/L^{d+1}.$$

The centered estimate uses $d = 2$ and spends half the input exponential width on a. The actual response and its one-step bound are constructed in the atlas, response and exterior-lifting arguments that follow. For a packet K, neutrality means $\sum_z K(x, (z, a)) = 0$ for each held x and each orientation a separately. Its propagated kernel is

$$K'(y, z) = \sum_{x,v} T(x, y) K(x, v) U(v, z).$$

Weighted row/column absolute majorants justify every regrouping: first sum the nonnegative absolute majorant in the output variables, then in the input variables. The resulting finite product of bounds dominates the full multiple series, including regrouping over the finite orientation sets. Before taking absolute values, subtract from the right factor the value at the site of x with the orientation of v; neutrality cancels the removed sum orientation by orientation. Explicitly the centered factor is $U(v,z) - U((x.\mathrm{site}, v.\mathrm{orientation}), z)$. For $\tau > 0$ and $2\sigma \le L\tau$, $ae^{\sigma a/L} \le (2/\tau)e^{\tau a}$: use $\sigma/L \le \tau/2$ and $a \le (2/\tau)e^{\tau a/2}$, which follows from $u \le e^u$ for $u \ge 0$. The degree-two centered right row therefore costs $|\beta|(2/\tau)c/L^3$; the left column costs $|\alpha|qL^2$. A packet of cost p consequently gives

$$\mathrm{folded}_\sigma(K') \le [|\alpha|q|\beta|(2/\tau)c/L]p.$$

The L^{-1} gain requires centering before the norm. Site-independent orientation-wise output sums of U preserve neutrality. Summing $K'(y,(z,b))$ in z then gives a linear combination of the zero orientation-wise sums of K, so it vanishes. Shifting fine summation variables by L times a coarse shift proves covariance. Splitting absolutely convergent sums into residue classes proves exact torus periodization $T^*_{\mathrm{torus}} K_{\mathrm{torus}} U_{\mathrm{torus}}$. Applying the mixed stationary identity identifies this periodized packet with the coarse quadratic difference. These operations lift with the original orientations and absolute majorants. The predictor spatial estimates and constant-curvature transport property remain the stated dependencies; the secured pilot and wire bounds supply the former.

Quadratic comparison and neutral-cost recovery. The scalar recursion source retains each nonnegative forcing value, age and scale increment. Its readout is the identity on these entries, so the majorant, terminal condition and exponentially weighted recurrence recover literally, with the same K, q, B, μ and well-founded grade order. The price table and widths are fixed before the chosen block length; neither the lift nor the shadow quadratic comparison reorders these choices.

For finite physical tangent spaces use sections j_f, j_c and readouts p_f, p_c. Transport T as $j_f T p_c$ and a precision as the pulled-back form $p_f^* Q p_f$. Restriction to the sections then recovers every stationary and mixed comparison identity, including $W^*(R-Q)T = B - A$, in the original metrics. On curvature packet coordinates transport each response and packet with their corresponding section maps. The orientation-wise cancellation of the constant right predictor is retained before forming the absolute majorant. Its scalar cost reader is exactly

$$(|\alpha|qL^2)\,(|\beta|(2/\tau)cL^{-3})\,p = |\alpha|q|\beta|(2/\tau)cL^{-1}p.$$

Thus the width crossing, neutral gain and periodized quadratic difference recover with the same actual histories, and no positive form is imposed on redundant vertical directions. Absolute convergence licenses the same finite-orientation regrouping and residue-class periodization used in the physical comparison.

11.3 Full-space correction and analytic roots

On the whole three-form coordinate space the source symbol is $A = V + lI - mS^*S$, with $m \ge 0$, $V \ge -a^2 CI$ and $S^*S \le a^2 BL^{-2} I$. Choosing $l \ge a^2 C + 2$ before requiring $ma^2 B \le L^2$ gives $A \ge I$. Here m is the fine amendment, l the coarse amendment, and a a fixed orientation count; $C, B > 0$ depend only on the fixed smoothing pair and penalty. At real momentum the symbols are Hermitian, and for every full three-form vector x,

$$\langle x, Ax\rangle \ge [-a^2 C + l - ma^2 B/L^2]\|x\|^2 \ge \|x\|^2.$$

Thus the coarse threshold is chosen before the length, which then absorbs the already fixed fine amendment. For $R = F^*AF$, a uniformly bounded right inverse of the actual square finite matrix F is also a left inverse. Consequently F is invertible and has $\|x\|^2 \le W\|Fx\|^2$, hence $R \ge W^{-1}I$. An independent analytic row estimate supplies the upper bound. This is a full three-form bound, not merely a curl-image bound. The rebased identity is

$$B[Y - T^*XT - b^{-1}d_1d_1^*]B = d_2^*Rd_2.$$

Indeed substitute $Y = B_0 + lR_0^*R_0$, $X = b_0 + mF_0^*F_0$, the unamended defect $R_0^*VR_0$ and $SR_0 = F_0T$, so the bracket is $R_0^*(V + lI - mS^*S)R_0$. Then use $R_0B = Fd_2$ and symmetry of B. The exterior lifts, regular symbols and analytic inverse bounds of this section construct B_0, b_0, S and F with the estimates used here. These are the source's amended inverse metrics and installed response; no arbitrary extension is substituted.

For $M \ge b_0I$ and $0 < c \le b_0/2$, the residual $M - cI \ge b_0I/2$ has a real analytic root row F. Subtracting the constant row cI supplies its analytic strip bound. On each finite torus $F^*F = M - cI$, so the augmented row has the exact Gram identity

$$(\sqrt{c}I, F)^*(\sqrt{c}I, F) = cI + F^*F = M.$$

The allowed upper cap of c is half a uniform positive metric lower bound. To retain the root construction, let Q have strip η, weighted coefficient bound $A \ge v > 0$ and real symbols $vI \le Q \le AI$. With $D = A/(\eta/2)$ and fixed entrywise-to-operator norm factor $C > 1$, choose $\tau = \min(\eta/2, v/(2CD))$. On the real axis, unitary diagonalization gives $\|A^{-1}Q(p) - I\| \le 1 - v/A$. Differentiating the absolutely convergent Fourier series on the smaller strip $\eta/2$ gives the entrywise symbol-difference bound with constant D. Take C to be one plus the norm of the fixed identification from entrywise matrices to operator norm. Comparison with $\operatorname{Re} p$ in the width-τ collar gives $\|Q(p) - Q(\operatorname{Re} p)\| \le v/2$, and therefore $\|A^{-1}Q(p) - I\| \le 1 - v/(2A) < 1$ uniformly. Thus

$$P = \sqrt{A}\sum_{n\ge 0}\binom{1/2}{n}(A^{-1}Q - I)^n$$

is a uniformly bounded holomorphic periodic branch, equal to the positive square root on real momentum by diagonalization of the series. In particular $P(p)^* = P(p)$, $P(p)^2 = Q(p)$ and $\sqrt{v}I \le P(p) \le \sqrt{A}I$ there. Reality of the original Laurent coefficients preserves this branch under conjugation at opposite real momenta. Fourier inversion, taking real parts of the coefficients, thus recovers the same real-axis function. Shift each integration contour within the collar. With output width $e = \tau/3$, a further width e remains to sum absolute coefficients. In four dimensions this gives a weighted coefficient sum at most $K[2(1 + e)/e]^4$, where K bounds the entries of P on the collar. These constants depend only on v, A, η and the fixed matrix size, independently of the finite period. Discrete Fourier diagonalization proves $P_{\mathrm{torus}}^* = P_{\mathrm{torus}}$ and $P_{\mathrm{torus}}^*P_{\mathrm{torus}} = Q_{\mathrm{torus}}$ for every period: the real-axis identity holds at every discrete frequency, and the invertible discrete Fourier matrix transfers it to real coordinate matrices. Thus the factorization and analytic decay hold simultaneously for every period, as required in the bottom construction. This reduces the remaining root step to the analytic square-root construction, rather than assuming a factorization with the desired decay. Intersecting finitely many positive widths and enlarging the common cost pays the metric, inverse, diagonal, dual diagonal and root rows. The resulting bed satisfies the required finite-torus floor identities and bounds for every permitted train. The lift transports these full-space bounds on their declared section, retaining each original inverse and norm.

Full-space correction and root recovery. For each physical exterior degree r retain a section j_r and readout p_r with $p_r j_r = I$. The full three-form bound is recovered on $j_3 H_3$, where H_3 is the entire original three-form space:

$$\langle j_3 x, p_3^* A p_3 j_3 x\rangle = \langle x, Ax\rangle \geq \|x\|^2 \qquad (x \in H_3).$$

Here the pairing denotes the pulled-back form evaluated on the section. Thus every original three-form vector is included, while the redundant vertical kernel is allowed. Transport exterior maps as $j_{r+1} d_r p_r$ and the installed response by its corresponding operator lift. Restrict pulled-back metric forms to their sections before composing the rebased identity. Every intermediate $p_r j_r$ cancels, and the result is exactly $B[Y - T^* X T - b^{-1} d_1 d_1^*]B = d_2^* R d_2$ in the physical metrics. The original inverse bound for F is used on its physical full space; no inverse of $p_3^* A p_3$ on redundant coordinates is introduced.

For the residual metric root, retain the physical output coordinates of the row F and realize it as $F^Z = Fp$. Its augmented row is $H^Z = (\sqrt{c}\, p, Fp)$, so

$$(H^Z)^* H^Z = p^*(cI + F^* F)p = p^* M p, \qquad H^Z j = (\sqrt{c} I, F).$$

This is a factorization of the pulled-back metric form. Separately, if P is the positive analytic square-root operator of Q, its operator realization is $P^Z = jPp$. It satisfies $(P^Z)^2 = jQp$ and $pP^Z j = P$. In the transported section metric it recovers the original positive self-adjoint root; no choice of an ambient redundant-coordinate inner product is implicit. The root branch, collar and Laurent coefficients are the original retained analytic data. At each period the same row factorization and section identities recover the torus Gram identity, with precisely the original widths and costs on the physical section.

11.4 Smoothing histories and actual block averages

For positive odd M, retain the entire functions

$$R_M(z) = M^{-1} \sum_{u=0}^{M-1} e^{iuz}, \qquad P_M(z) = e^{-i\lfloor M/2\rfloor z} R_M(z), \quad A_M(p) = \prod_{a=1}^{4} P_M(p_a).$$

Both line functions equal one at zero. Geometric summation, including the zero set by the polynomial identity, gives $M R_M(z)(e^{iz} - 1) = e^{iMz} - 1$. Splitting the integer index as $Mu + v$ gives $R_L(Mz) R_M(z) = R_{LM}(z)$. For odd L, M, the centers obey $M\lfloor L/2\rfloor + \lfloor M/2\rfloor = \lfloor LM/2\rfloor$, giving $P_L(Mz) P_M(z) = P_{LM}(z)$. Center reflection and conjugation give $P_M(-z) = P_M(z)$ and $\overline{P_M(z)} = P_M(-\bar{z})$.

A train descends through smoothing indices which stay fixed or decrease by one backwards, and its mass is the product of its odd block lengths. A terminal train has mass and smoothing function one. At a hop of block L from index J to j, $j \leq J \leq j+1$, the new mass is $L\,\mathrm{mass}(v)$ and its smoothing function becomes $A_L(\mathrm{mass}(v)p)^J \,\mathrm{smooth}_v(p)$. Equivalently,

$$\mathrm{smooth}_v = A^j_{\mathrm{mass}(v)}\,\mathrm{extra}_v, \qquad \mathrm{extra}_{\mathrm{new}} = A^{J-j}_{L\,\mathrm{mass}(v)}\,\mathrm{extra}_v\,.$$

On $|\operatorname{Im} p_a| < \zeta/H$, finite sums give $|A_M(p)| \leq e^{4\zeta M/H}$. This latter representation separates the fixed stage power from the extra smoothing inserted at stage changes. Masses grow by factors at least three, and the extra exponent is zero or one, so summing the geometric mass series gives $|\mathrm{extra}_v(p)| \leq e^{8\zeta\,\mathrm{mass}(v)/H}$ for mass at most H. The real principal estimate $|P_M(p_a/H)| \geq e^{-cM/H}$ yields the companion extra lower bound $e^{-8c\,\mathrm{mass}(v)/H}$ and hence a positive lower bound

for $|\operatorname{smooth}_v(p/\operatorname{mass}(v))|$ depending only on j, ζ, c, independently of chain length. The normalized finite-sum proof of the scalar lower estimate is retained in the cumulative-filter argument (QF1)–(QF4).

At aliases $q_l = (p + 2\pi \operatorname{lift}(l))/\operatorname{mass}(v)$, inverse-linear coordinate decay of the principal factor and the extra upper bound give

$$|\operatorname{smooth}_v(q_l)| \le C_j \prod_a (1 + |\operatorname{lift}(l)_a|)^{-j},$$

with the source's up-normalization where required. The supremum coordinate weight $\operatorname{grade}(l)$ is at most $\operatorname{mix}(l)$, since all remaining factors are at least one. This pays the high-alias sums; the nonzero-alias reciprocal estimate supplies the corresponding bound for R_M. These are actual patch weights: J independent centered box rolls have symbol A_L^J, counting multiplicities even when displacements coincide. A positive length-L line path has Fourier sum $LR_L(q_a)$, so the defined directional patch mean is exactly

$$LA_L(q)^J R_L(q_a).$$

Applying the mean to each discrete Fourier basis vector proves this formula. The factor L counts edge occurrences. Paths wrapping around a small torus still count every edge occurrence; the Fourier computation does not require that the path visit distinct sites. The realized block operation is the same tagged path sum under readout, and therefore retains both this factor and all roll multiplicities; a normalized point average would be a different operation. For clarity, the smoothing recurrence also has an explicit source–readout recovery. At every train retain the ordered list of box-roll displacements, with their multiplicities, and the ordered edge occurrences of each line path in the source operation. On the physical tangent section let $\mathcal{A}_v$ be the corresponding finite convolution, with lifted response $\widetilde{\mathcal{A}}_v = j\mathcal{A}_v p$. Since $pj = I$, successive lifted responses satisfy

$$(j\mathcal{A}_1 p)(j\mathcal{A}_2 p) = j\mathcal{A}_1\mathcal{A}_2 p, \qquad p\widetilde{\mathcal{A}}_v j = \mathcal{A}_v.$$

Thus Fourier evaluation after recovery gives exactly the scalar smoothing recurrence above, with the same terminal value and mass recursion. Applying this equality to each finite Fourier basis vector recovers the patch mean, including L, even on wrapping paths. The norm used for these smoothing bounds is the transported physical-section norm; the recovery does not add a bound on vertical frame directions. Scalar color-independent multipliers retain the original color identity factor. Hence neither source lifting nor recovery changes the smoothing stage, the alias coordinate, the history length independence of the bounds, or the occurrence counts.

11.5 Uniform reciprocals and carried residue differences

A real lower bound must be extended to a common complex collar before inversion. Suppose f_ξ is holomorphic with $|f_\xi| \le Cw_\xi$ on a complex cube of width η and $|f_\xi| \ge cw_\xi$ on the real cube of width θ, with $C, c > 0$ and $w_\xi > 0$. For

$$r = \min(\eta/4, \theta, c\eta/(8C)),$$

For p in the width-r cube, the radius-$\eta/2$ ball about $y = \operatorname{Re} p$ stays in the original cube. Apply Cauchy's estimate along the line from y to p, using $|f_\xi(z) - f_\xi(y)| \le 2Cw_\xi$ on that ball. This gives $|f_\xi(p) - f_\xi(y)| \le (4Cw_\xi/\eta)|p - y| \le cw_\xi/2$. Thus $|f_\xi(p)| \ge cw_\xi/2$ and the reciprocal is holomorphic and bounded by $2/(cw_\xi)$ on a collar independent of ξ. This proves the reciprocal step used for the scalar factors without inferring complex nonvanishing merely from real positivity.

With $\chi(z) = e^{iz} - 1$, set

$$\lambda_M(p) = \sum_a [M\chi(p_a/M)][M\chi(-p_a/M)], \quad B_{M,L,l}(p) = L^2\lambda_M((p + 2\pi \operatorname{lift}(l))/L).$$

On real momentum $\lambda_M(p) = \sum_a |M\chi(p_a/M)|^2$, and the bounds, uniform in M, are $c_-^2|p|^2 \le \lambda_M(p) \le c_+^2|p|^2$. For $l \neq 0$, choose a coordinate attaining $\operatorname{grade}(l)$. The source lower alias estimate and holomorphic upper estimate are respectively $(c_-\zeta)^2 \operatorname{grade}(l)^2$ and $C \operatorname{grade}(l)^2$, where $\operatorname{grade}(l) = 1 + \max_a |\operatorname{lift}(l)_a|$. The preceding lemma gives a common collar with $|B_{M,L,l}^{-1}| \le C' \operatorname{grade}(l)^{-2}$. The principal alias satisfies $B_{M,L,0} = \lambda_{LM}$ and is not inverted on a collar containing zero.

Put $\operatorname{mix}(l) = \prod_a (1 + |\operatorname{lift}(l)_a|)$ and $\operatorname{decay}_j(l) = \operatorname{mix}(l)^{-j}$. Then $\operatorname{grade}(l) \le \operatorname{mix}(l)$, so

$$\operatorname{grade}(l)^u \operatorname{decay}_{j+u}(l) \le \operatorname{decay}_j(l).$$

Furthermore,

$$\sum_{l \bmod L} \operatorname{decay}_2(l) \le \left[\sum_{n\in\mathbb{Z}} (1 + |n|)^{-2}\right]^4 < \infty.$$

Centered representatives are injective, so each one-dimensional finite sum is bounded by the full integer series, and the finite four-dimensional sum factors coordinatewise. This proves the decay summation used in the earlier feed and high-alias estimates, without paying a factor equal to the number L^4 of residues. Hence no factor L^4 is paid. For $u \ge 0$, the displayed absorption inequality follows by taking the uth power of grade $\le$ mix. A secured fine-difference word u contributes $CL^{-|u|} \operatorname{grade}(l)^{|u|}$, its toll L^{-e}, and a uniformly bounded residue phase, because the residue coordinates have size at most L and $|\operatorname{Im} q_a| \le \zeta/L$. Its symbol contributes $\operatorname{decay}_j(l)$. For $j - |u| \ge 2$ the residue sum costs $C'L^{-(e+|u|)}$. Alias permutation proves periodicity; reflection/conjugation gives real coefficients, and a narrower collar gives the priced Fourier rows. The regular coefficient and symmetry arguments supply these secured bounds for the actual pilot and wire symbols.

The scalar average bounds follow directly because $\chi(z)/z$ extends with value i and is nonzero on $|z| \le 3\pi/2$: there is no zero on that disk because $e^{iz} = 1$ requires z to be a real multiple of 2π. Compactness gives $c_-|z| \le |\chi(z)| \le c_+|z|$. Geometric summation yields $|R_M(z)| \ge a_0 = \min(1, c_-/c_+)$ when $M|z| \le 3\pi/2$. For real z with $H|z| \le 3\pi/2$, $M \le H$, finite centered sums give $|P_M(z) - 1| \le kM/H$, $k = c_+(3\pi/2)$. If $y = |P_M(z)|$, then $y \ge a_0$ and $1 \le y + kM/H$, and $e^{kM/(a_0H)} \ge 1 + kM/(a_0H)$, whence $ye^{kM/(a_0H)} \ge y + kM/H \ge 1$. This proves the principal lower estimate with leading constant one.

For centered nonzero alias $q = (p + 2\pi l)/L$, the chosen cube gives $L|q| \ge \zeta(1 + |l|)$, $L|\operatorname{Im} q| \le \zeta$, $|q| \le 3\pi/2$, with $\zeta = \pi/4$ and odd $L \ge 3$. Separate the real shift $2\pi l$ from $|\operatorname{Re} p| \le \pi + \zeta$ to obtain these inequalities. The identity $\chi(p + 2\pi l) = \chi(p)$ gives $L|R_L(q)|c_-|q| \le c_+(3\pi/2)$, hence inverse-alias decay. At $l = 0$ the finite sum bounds R_L by e^{ζ}; centering costs at most another e^{ζ}. Coordinate products supply the four-dimensional bound. The remaining alias work concerns the Laplacian denominator and the summation of the chosen decay weights.

For actual carried cross-row differences, write a fine site as centered residue r and coarse displacement. A positive step has carry $c_a(r) = e_a$ exactly at $\operatorname{lift}(r)_a = \lfloor L/2 \rfloor$, and zero otherwise. The identity

$$\operatorname{lift}(r + e_a) + Lc_a(r) = \operatorname{lift}(r) + e_a$$

introduces the phase $e^{ip\cdot c_a(r)}$ in the advanced residue. Subtracting the original gives the alias multiplier $e^{iq_a} - 1$; iteration gives the product for any word of fine differences. Fourier inversion identifies the priced symbols with these actual carried differences. This identity follows from the real

sums, rather than treating a residue wrap as an ordinary shift within the same copy. Intersecting the finitely many widths for words of length at most $j-2$ and enlarging their costs gives one table with cross-row bound $CL^{-(e+\mathrm{length}(u))}$, uniform over flights at the fixed smoothing indices. The regular coefficient bounds and inverse-third-power estimate below provide the analytic bound for these secured symbols. The lift retains the same residue and carry records, so it does not turn wrapping into an uncarried shift or invert the excluded principal zero.

11.6 Laurent exterior division

Set $q_s(p) = e^{ip_s} - 1$ and let δ_r be exterior multiplication by q on degree-r forms. For a lattice monomial e_x, E_s sets its s-coordinate to zero. Define $J_s e_x$ by summing the monomials with that coordinate $0, \dots, x_s - 1$ for $x_s \geq 0$, and minus those with coordinate $x_s, \dots, -1$ otherwise, retaining the other coordinates. Telescoping gives

$$q_s J_s e_x = e_x - E_s e_x.$$

The interval defining J_s has $|x_s|$ terms, each with length at most $|x|_1$. A width-η coefficient bound $\sum_x |a_x| e^{\eta|x|_1} \leq C$ is preserved by E_s. For $\theta = \eta/2$, $|x_s| e^{\theta|x|_1} \leq \theta^{-1} e^{\eta|x|_1}$, so J_s has width-$\eta/2$ bound $2C/\eta$. This proves absolute convergence of the transformed rows, which licenses termwise telescoping and the identity $q_s J_s A = A - E_s A$, including $q_s = 0$.

Suppose A maps degree-$(r+1)$ forms to a fixed coordinate space and $A\delta_r = 0$. Let $L_r(s)$ denote multiplication by the s-th exterior basis vector. Define relief $A_s = (J_s A) L_{r+1}(s)^*$ and residual $R_s = (E_s A)[I - L_r(s) L_r(s)^*]$, namely evaluation at $p_s = 0$ followed by deletion of columns whose orientation contains s. The contraction identity

$$L_{r+1}(s)^* \delta_{r+1} = q_s I - \delta_r L_r(s)^*, \qquad \delta_r(p) = \delta_r(p|_{p_s=0}) + q_s L_r(s)$$

gives $A_s \delta_{r+1} + R_s = A$: multiply by q_s and cancel where nonzero, then extend by continuity to its zero hyperplane. Also $R_s \delta_r = 0$ by exterior nilpotency. Apply the four axes in order to the preceding residual. Columns containing a processed axis vanish thereafter, and a previously zero column remains zero. Since every positive-degree orientation contains an axis, the final residual is zero. Adding reliefs gives $B\delta_{r+1} = A$ with positive analytic width and finite explicit losses, uniform over every family with a common input coefficient bound. Evaluation gives $B_{\mathrm{torus}} d_{r+1} = A_{\mathrm{torus}}$ on every period. Conversely all-period identities give the symbol cycle by Fourier uniqueness; one period alone is insufficient.

Transposition gives the corresponding left lift. For the two-sided lift of a two-form row retain both all-period cycle conditions $Ad_1 = 0$ and $d_1^* A = 0$. First choose $d_2^* W = A$, put $F = W d_1$, and lift $d_2^* F = 0$ to $d_3^* T = F$. Here T has degree-four target. The top-degree left-cycle argument gives $T d_0 = 0$, since $d_3^* T d_0 = W d_1 d_0 = 0$ and the next exterior space is zero. Right-lift T to $X d_1 = T$. Then $K = W - d_3^* X$ obeys $K d_1 = 0$, $d_2^* K = A$. A final right lift $B d_2 = K$ gives $d_2^* B d_2 = A$. Every operation is a finite composition of these analytic lifts, exterior derivatives, sums and transposes, hence preserves familywise exponential coefficient decay. This is the exterior factorization used for the fund; refined block-length powers remain separate quantitative estimates. Under the source readout, exterior operators and their ordered compositions use the transported section. No pointwise quotient by q_s at zero is introduced, and the two-sided correction is retained rather than replacing it by an arbitrary one-sided lift. The recovery can be expressed degree by degree. For the original degree-k coordinate space V_k, use its declared source section $j_k : V_k \to Z_k$ and readout $p_k : Z_k \to V_k$, with $p_k j_k = I$. On a form space with no redundant frame coordinates

these can be the identity; on the potential space they are the tangent maps already constructed. Put

$$\widetilde{d}_k = j_{k+1} d_k p_k, \qquad \widetilde{B} = j_3 B p_3, \qquad \widetilde{A} = j_2 A p_2.$$

Use the transported section inner products for adjoints. Then the ordered source composition satisfies

$$\widetilde{d}_2^{\dagger} \widetilde{B} \widetilde{d}_2 = j_2 (d_2^* B d_2) p_2 = \widetilde{A}, \qquad p_2 \widetilde{A} j_2 = A.$$

Here † is the section adjoint, and the equality is on the declared sections. The coefficient operations E_s, J_s act before realization on the same Laurent coefficient arrays, retaining every orientation and signed interval. Recovery therefore returns their termwise telescoping identities, including the zero hyperplanes. The section norm is transported from the physical coordinate norm, so the coefficient-width losses and their familywise uniformity are unchanged. This provides an explicit recovery of the two-sided factorization and does not assert coercivity or an adjoint identity in unused vertical directions.

11.7 Regular two-form metrics and precision-preserving amendment

For a train v of scale M, the real-momentum factors z_a below are nonnegative. Set $z_a = \mathrm{seedQ}_M(p,a)\,\mathrm{seedQ}_M(-p,a)$ and $\lambda_M = \sum_a z_a$. Retain $h_0 = \mathrm{halo}(v)$, $l_a = \mathrm{lid}(v,a)$, $r_a = \mathrm{drain}(b,v,a)$ and

$$g_a = l_a + \lambda_M r_a, \qquad h = h_0^{-1} - \sum_a \frac{z_a^{(1)} r_a}{l_a g_a}.$$

Here $z_a^{(1)}$ is the same factor at scale one. The identity $z_a^{(1)} h_0 = z_a l_a$ gives, by common denominators,

$$\lambda_M h = \sum_a z_a^{(1)} / g_a. \tag{Q2}$$

The vault diagonal on orientation $\{a,c\}$ is $h g_a g_c$ and pulse is its reciprocal, with off-diagonal entries zero. Their analytic definitions give $g_a(0) = h(0) = 1$, without a separate harmonic replacement. On the real cube $l_a > 0$, $r_a \geq 0$ imply $g_a \geq l_a$. If $g_a \leq C$, Q2 and the Laplacian comparisons imply $h \geq c_-^2/(C c_+^2)$ away from zero, and at zero $h = 1$. The joint lower bound is $h \geq \min(1, c_-^2/(C c_+^2)) > 0$. Thus both metrics are real diagonal positive mutual inverses. Uniform bounds and positive lower bounds for g_a, h give bounds for their reciprocals and finite products on a common smaller holomorphic collar. Taking Fourier coefficients gives exponentially decaying metric and inverse rows.

Away from zero, $D_a = g_a/\lambda_M$ and $\rho = \sum_a z_a^{(1)}/D_a = \lambda_M^2 h$ give vault $= \rho\,\mathrm{mask}_2(D_a)$ and pulse $= \rho^{-1}\,\mathrm{mask}_2(D_a^{-1})$. Agreement under reciprocal-lattice shifts on overlaps of real coordinate cubes follows from the periodicity of D_a and ρ in these representations; the filled g, h expressions extend through the apparent pole. At the top train $D_a = \lambda_1^{-1}$ and $\rho = \lambda_1^2$, giving identity two-form symbols and the bare Wilson precision.

The centered-average, nonzero-alias and cumulative-filter estimates supply the smoothing, remainder and scale-uniform Laplacian comparison bounds used in Q2. The part of the analytic bound that accounts for arbitrarily long histories is the nozzle weight. Let $t(v)$ be the sum of mass^{-2} over the nozzle list. Each old mass is multiplied by L under a hop and a new mass-one train is added. Thus long histories are controlled by the nozzle weight recurrence $t(\mathrm{top}) = 0$, $t(\mathrm{hop}(v)) = 1 + L^{-2} t(v)$. Since $L \geq 3$, induction gives $t(v) \leq 2$: from $t(v) \leq 2$ the next value is at most $1 + 2/9 \leq 2$. Each feed costs at most $C\,\mathrm{mass}^{-2}$, so levelNoise costs at most $2C$ independently of depth; in particular, the constants do not accumulate one unweighted term per renormalization

step. For $j \geq 4$ high-alias smoothing decay times the inverse-square grade divisor is dominated by the summable order-two reference decay. The finite alias sum has a bound independent of its side length, using the alias rail, inverse-divisor and reference decay-sum estimates; their scalar formulas are derived in the cumulative-filter construction (QF1)–(QF9) below. This gives the feed mass bound and, with b^{-1} levelNoise, the drain bound. The lid is the product of the drive at p/M and $-p/M$. Positive smoothing and rail lower bounds therefore give a positive lid lower bound, and similarly a positive halo lower bound; bounded drain and λ_M bound g_a above. The rational expression for h then has only uniformly separated denominators. No bound on $1/\lambda_M$ at zero is used.

For the unamended metric H and $B = H^{-1}$ set

$$X = d_2 B, \quad A = B + lX^* X, \quad G = A^{-1}, \quad R = I + l d_2 B d_2^*, \quad D = R^{-1}, \qquad l \geq 0.$$

On every finite torus H, B are symmetric positive matrices. The supplied pledge, amend, bail and treble rows represent A, G, R, D, respectively. Positivity gives $A > 0$ and $R \geq I$, so their analytic inverses G and D exist with uniform narrower strip bounds. Exterior nilpotency gives $XHd_1 = 0$, hence $AHd_1 = d_1$, $Gd_1 = Hd_1$ and

$$d_1^* G d_1 = d_1^* H d_1.$$

The amendment changes the extension on two-forms while preserving the potential precision. Also $RX = d_2 A$ implies $XG = Dd_2$. At zero $d_2 = 0$ and $B(0) = I$ give $G(0) = I$.

The inversion applies to both G and D. Uniform inversion is in fixed orientation dimension s, independently of the site count: $vI \leq A(p) \leq CI$ gives $|\det A| \geq v^s$ and $C^{-1}I \leq A^{-1} \leq v^{-1}I$ by diagonalization. Analytic coefficient bounds control the determinant change from the real slice; choose a common smaller collar losing at most half its positive real lower bound. Adjugate over determinant gives a uniformly bounded holomorphic inverse, preserving periodicity and real coefficients. Fourier decay and discrete evaluation give the inverse at every period. All spectral bounds transfer with the discrete Fourier Gram factor n^4, which cancels from both sides, including at side one.

To recover the metric and its inverse without confusing forms with responses, retain the degree-two section maps j_2, p_2. The metric form on redundant coordinates is $\mathsf{H} = p_2^* H p_2$; its null space is $\ker p_2$ because $H > 0$. Its restriction to the section is positive, with $j_2^* \mathsf{H} j_2 = H$. The inverse response on that section is $\widetilde{B} = j_2 B p_2$, so $p_2 \widetilde{B} j_2 = B$. Using the transported section inner product, the amended source responses are

$$\widetilde{A} = j_2 A p_2, \quad \widetilde{G} = j_2 G p_2, \qquad \widetilde{A}\widetilde{G} = \widetilde{G}\widetilde{A} = j_2 p_2.$$

Thus their product is the section identity; it is not an asserted inverse on redundant vertical directions. Recovering the precision identity gives exactly $d_1^* G d_1 = d_1^* H d_1$. The origin value, finite-torus inversion and all physical-section spectral bounds survive because the readout and section compositions recover their original matrices. The analytic and history estimates above continue to concern the original orientation-sized symbols before realization, preserving their period independence.

For constant-curvature transport, an ordinary Laurent row has spatial sum its zero symbol. Installed responses additionally depend on the initial residue. The duo response has at every residue the same zero entry $\mathrm{toll}(L, 2)I$: nonzero aliases vanish and the principal alias is normalized. For orientation-valued total sums s, t of rows S, T, absolute convergence gives total sum st for their convolution, in that order. Sandwiching this response by the fine inverse and coarse metric therefore preserves residue-independent spatial sums, establishing travels_route, the travels property used

in neutral propagation. The regular wire formulas and their explicit harmonic values supply the alias vanishing and normalization. For an installed cross row, the starting site also has a residue modulo the block length. The lift retains these orientation/residue records and the exact precision identity; equality on curls is not promoted to equality of arbitrary two-form extensions.

11.8 Defect factorization and finite-horizon assembly

On a coarse torus set $V = B_1 - T^* B_2 T - b^{-1} d_1 d_1^*$ and $M = H_1 V H_1$, where $B_i = H_i^{-1}$, the subscript one denotes coarse data, two denotes fine data, T is the installed response, and $b > 0$. The actual stationary identities

$$TH_1 d_1 = H_2 e_1 A_0, \quad T^* e_1 = d_1 Q, \quad QA_0 = I - b^{-1} d_1^* H_1 d_1$$

give the explicit intermediate identity

$$T^* B_2 T H_1 d_1 = d_1 [I - b^{-1} d_1^* H_1 d_1],$$

and hence $V H_1 d_1 = 0$. Thus $M d_1 = d_1^* M = 0$, and the two-sided analytic lift gives $d_2^* F d_2 = M$. Replace F by $(F + F^*)/2$, retaining the identity and making its real symbols Hermitian. Multiplication by B_1 on both sides gives

$$V = (d_2 B_1)^* F (d_2 B_1).$$

This is the fund used in the positive correction. Similarly $e_2 B_2 T H_1$ annihilates d_1, because $TH_1 d_1 = H_2 e_1 A_0$ and $e_2 e_1 = 0$; hence a right lift gives $S d_2 = e_2 B_2 T H_1$, or $S d_2 B_1 = e_2 B_2 T$. Analytic losses remain paid through the two exterior lifts, symmetrization and the surrounding bounded rows. The refined fine-difference estimate supplies the inverse-third-power residue price and, after summation, the separate L^{-2} estimate on $S^* S$; algebraic factorization alone does not supply that power.

Choose the retained-support stage data before eventual block lengths. For legal neighbors $j, t \geq 4$ with $t = j$ or $t + 1 = j$, the flight gives a coarse-floor threshold $\delta(j, t)$. Taking $l(i) = \max(\delta(i, i), \delta(i + 1, i))$ pays both incoming possibilities. With fine and coarse parameters $l(j), l(t)$ fixed, each legal pair supplies its positive analytic width, positive coercivity parameter, cost and integer block threshold. At each stage also choose the positive clip cap and bottom metric estimates. For the bottom and two outgoing possibilities take minima of positive widths/coercivities and maxima of costs/length demands. Weakening these bounds preserves each estimate. Backwards from the terminal train construct $R_{N,i}$ as each prescribed landing train, choose its metric bed with $l(J(i))$ and its specified duo tower. The identity $R_{N,i} = \mathrm{land}(\mathrm{destination}(N, i))$ aligns those choices. The fixed penalty is the same in the chosen tower. Tangent flight equations give the primal and dual stationary identities on every scheduled finite torus; the rebased equation gives the associated inverse and error data. Bottom and transition bounds together with terminal normalization form the linear horizon. Pinning arranges the additional horizon conditions for the given sheet profile while retaining the beds and required tower components, hence retains the destination equalities and supplies Present and H.Suffices. This finite assembly uses, rather than replaces, the bottom, flight and pinning analytic estimates; these are proved in the harmonic, positive-margin and auxiliary-tower arguments of this section (Subsections 11.9, 11.11 and 11.12), and the common price table is assembled from the metric, positive-margin and flight estimates established there.

11.9 Harmonic values and Fourier normalization

At $p = 0$ exterior signals vanish, bulb and bend are one, and principal patch multipliers are identity, giving filled pilot and wire value I. At each nonprincipal alias of zero, a nonzero residue coordinate

has $LR_L(q_a)\chi(q_a) = \chi(0) = 0$ with $\chi(q_a) \neq 0$. Its positive-power box filter vanishes. The pilot leading term contains $S = 0$, and its correction contains the zero coarse longitudinal row. The wire's first term contains the zero coarse adjoint derivative, while its second contains the vanishing flux. These explicit values prove the harmonic symbol equations without deleting that fibre.

If $AG = Fa$, $FF^* = F^*F = vI$ and $GG^* = G^*G = uI$, taking adjoints and multiplying gives $A^*F = G[(v/u)a^*]$. Equal representations imply equal real matrices by invertibility of G. Products and differences transfer to products and differences of the symbol matrices whenever the intermediate Fourier matrices coincide. Equality follows because the difference of the represented real matrices annihilates the invertible matrix G. For coarse side c, the Gram scalar is c^4; for fine side $f = cL$, it is f^4. Their ratio is $c^4/f^4 = L^{-4}$. The block mean itself carries L, so its transpose mean alias column is $L^{-3}V_1(q_l)$, while installed A_0, T columns are $L^{-1}\,\mathrm{pilot}_l, L^{-2}\,\mathrm{wire}_l$. The fine derivative is $L^{-1}d^{(L)}$ and precision $L^{-2}K_l$. Restoring these factors gives

$$Q_f A_0 = \mathrm{mean}^*\, Q_c, \quad \mathrm{mean}\, A_0 = I - b^{-1}Q_c,$$

$$T^* d_f = d_c\,\mathrm{mean}, \qquad T H_c d_c = H_f d_f A_0.$$

The residue sum represents the alias Fourier matrix times the stacked column, and evaluation at each coarse discrete frequency, followed by invertibility of the finite Fourier matrices, proves the real identities at every period. Together with amended-metric preservation and pinning, these supply the linear horizons for the flat entrance estimate. The cumulative-filter section supplies the scalar covariance feedback and train identities used here. The lift retains harmonic values, residue installation and Fourier Gram scalars together. It therefore recovers these finite-horizon equations with their actual normalization, rather than only a punctured-momentum surrogate.

11.10 Scaled stationary columns

Use $q_a^{(S)} = S(e^{ip_a} - 1)$, $d_r^{(S)} = q^{(S)\wedge}$ and the analytic adjoint $\delta_r^{(S)}(p) = d_r^{(S)}(-p)^\top$, equal to the Hermitian adjoint on the real slice. The patch identity $F_{r+1}(q_l)d_r^{(L)}(q_l) = d_r(p)F_r(q_l)$ follows coordinatewise from $R_L(q_{l,a})q_a^{(L)}(q_l) = q_a(p)$. Transposing at $-p$ gives its dual, and exterior nilpotency holds identically.

At regular momentum set $D = \mathrm{diag}(S^{-2}D_a)$, $I_D = D^{-1}$ and $\alpha = S^4\rho$. The two-form metric symbol is $\alpha^{-1}\,\mathrm{mask}_2(\mathrm{diag}\, I_D)$. Expanding its weighted wedge contraction gives

$$K = I_D - \alpha^{-1} I_D d_0^{(S)} \delta_0^{(S)} I_D.$$

The diagonal contraction is $\sum_a q_a^{(S)}(-p)(I_D)_{aa} q_a^{(S)}(p) = \alpha$, and the remaining terms are the displayed rank-one product. With $P = \alpha^{-1}\delta_0^{(S)} I_D$, $C = \alpha^{-1} I_D d_0^{(S)}$ this yields

$$K = I_D - I_D d_0^{(S)} P, \quad KD = I - C\delta_0^{(S)}, \quad DK = I - d_0^{(S)} P,$$

with $K d_0^{(S)} = \delta_0^{(S)} K = 0$.

Write D_l for the fine diagonal covariance at alias q_l, H_l for its two-form pulse and K_l for its potential precision. Let D_c, H_c, K_c denote the coarse analogues, F_r the patch multipliers, V_r their duals and P_c the coarse longitudinal row. Retain

$$E_l = \frac{m(-q_l)\,\mathrm{tr}\, D_l}{Z}, \qquad Z = \sum_l |m(q_l)|^2\,\mathrm{tr}\, D_l.$$

On the punctured real cube the principal summand is strictly positive and all summands nonnegative, so $Z \neq 0$ and $\sum_l E_l m(q_l) = 1$. The covariance recursion is $\sum_l F_1 D_l V_1 = D_c - b^{-1} I$. Thus the actual pilot

$$A_l = D_l V_1 K_c + E_l d_0^{(L)} P_c$$

obeys the averaging equation by multiplying by F_1 and summing. The hinge identity turns the second term into $(\sum_l E_l m(q_l)) d_0 P_c = d_0 P_c$. Consequently

$$\sum_l F_1 A_l = (D_c - b^{-1} I) K_c + d_0 P_c = I - b^{-1} K_c,$$

using $D_c K_c = I - d_0 P_c$. Also $K_l A_l = V_1 K_c$: nilpotency kills the longitudinal term and $\delta_0^{(L)} V_1 \delta_1 = 0$ kills the rotor correction.

Set $C = H_c d_1 D_c \delta_1$ in the following wire calculation. Its identities are $\delta_1(I - C) = 0$ and $(I - C) H_c d_1 = 0$. The actual wire is

$$T_l = H_l d_1^{(L)} D_l V_1 \delta_1 + y_l V_2 (I - C),$$

with the original chain scalar y_l. Multiplying on the left gives $\delta_1^{(L)} T_l = V_1 \delta_1$, using $K_l D_l V_1 \delta_1 = V_1 \delta_1$ and $\delta_1^{(L)} V_2 = V_1 \delta_1$. Multiplication on the right by $H_c d_1$ kills the cloud-complement term by $(I - C) H_c d_1 = 0$; the pilot longitudinal term is killed by $d_1^{(L)} d_0^{(L)} = 0$. Hence $T_l H_c d_1 = H_l d_1^{(L)} A_l$. The filled pilot and wire equal these regular-set formulas away from the puncture. Their continuity extends the equations through zero with the explicitly retained harmonic values. The installed tolls L^{-1}, L^{-2}, L^{-3} and Fourier Gram ratio L^{-4} restore precisely the finite primal, dual and transport identities; the Fourier transfer of Subsection 11.9 restores these factors and proves the asserted real matrix identities at every period. The lift carries these same ordered exterior operations and alias data; it does not reinterpret analytic transpose as complex conjugation away from the real slice.

11.11 Positive extension and the complete tower Gram identity

Suppose the analytic Hermitian three-form symbol Q obeys $\langle d_2 a, Q d_2 a \rangle \geq v \|d_2 a\|^2$ on real momenta. This controls every vector at zero: for a complex three-form x, the real-linear map $n \mapsto (in) \wedge x$ from $\mathbb{R}^4$ to the complex one-dimensional top space has a nonzero real kernel vector n. Exterior contraction gives $x = (in) \wedge a$. Since $q(tn)/t \to in$, the vectors $d_2(tn)(a/t)$ converge to x. Apply the assumed image inequality to these vectors and pass to the limit; continuity yields $Q(0) \geq vI$.

Increase its analytic coefficient bound $C \geq 0$ to a positive number if necessary. With $C > 0$ and N the fixed orientation constant, choose $\delta = \eta v / (2N^2 C)$. The departure estimate bounds each entry of $Q(p) - Q(0)$ by $\delta C / \eta$, and hence gives $Q(p) \geq vI/2$ when every $|q_a(p)| \leq \delta$. Else choose k with $|q_k| > \delta$ and decompose

$$x = y + z, \quad y = d_2(q_k^{-1} \iota_k x), \quad z = q_k^{-1} \iota_k d_3 x, \qquad \|z\|^2 \leq N \delta^{-2} \|d_3 x\|^2.$$

With $M = N^2 C$, the quadratic form first obeys

$$\langle x, Qx \rangle \geq v\|y\|^2 - 2M\|y\|\|z\| - M\|z\|^2.$$

The cross-term estimate then gives $\langle x, Qx \rangle \geq (v/2)\|y\|^2 - (M + 2M^2/v)\|z\|^2$. Since $\|x\|^2 \leq 2\|y\|^2 + 2\|z\|^2$, adding

$$r d_3^* d_3, \qquad r = (M + 2M^2/v + v/2) N \delta^{-2}$$

gives a global floor $vI/4$. Near zero it adds a nonnegative form. This finite-support correction has a common analytic bound and preserves $d_2^* Q d_2$ by $d_3 d_2 = 0$. Increasing the upper cost therefore yields the uniformly positive full-space row with exactly the original curvature contraction on every period, as required in pinning.

On a fixed pair of tori let $d, d\prime$ be the fine and coarse first derivatives and e the coarse second derivative. For the tower retain fine and coarse full Gram factors $W = J^*J$, $W' = J'^*J'$, inverses R, R' and penalty square root $s = \sqrt{b}$. The factorized inverse defect is

$$W'ZW' = e^*P^*Pe, \qquad Z = R' - T^*RT - b^{-1}d'd'^*.$$

Put $Q = J'R'J'^*$, $X = PeR'J'^*$ and

$$M = JRTJ'^*, \qquad N = s^{-1}d'^*J'^*, \qquad O = \mathrm{stack}(I - Q, X).$$

Then $Q^* = Q$, $Q^2 = Q$, $QJ' = J'$, and multiplication gives

$$M^*M = J'T^*RTJ'^*, \quad N^*N = J'b^{-1}d'd'^*J'^*, \quad O^*O = I - Q + J'ZJ'^*.$$

Substitute Z and Q in their sum to get $M^*M + N^*N + O^*O = I$ on the entire stack source, including harmonic coordinates. The actual transport identities $TW'd' = WdA_0$, $T^*d = d'$ mean further give

$$MJ'd' = JdA_0, \quad M^*Jd + N^*(-s\,\mathrm{mean}) = 0,$$
$$sN^* = J'd', \quad NJ'd' = s^{-1}d'^*W'd', \quad OJ'd' = 0.$$

The last uses $(I - Q)J' = 0$ and $ed' = 0$. These are exact roll identities, not spectral estimates.

Weighted convolution at $\gamma = \min(\sigma_f, \sigma_c, \nu) > 0$ supplies the analytic bounds. The full J stacks $\sqrt{\mathrm{clip.\,direct}}I$ and its root row, with bound $\max(\sqrt{\mathrm{clip.\,direct}}, \mathrm{rootCost})$. For derivatives the period-independent bound is the finite sum of $e^{\gamma|z|_1}$ times the absolute stencil coefficients over all orientations and stencil sites. Transposition preserves this bound. Composition of two rows with an intermediate orientation space of size n costs at most n times the product of their entrywise weighted bounds, by the distance triangle inequality and absolute convergence. Apply the convolution rule in order to JR, then its composition with T, then multiplication by $J\prime^*$. The mixed row thus retains its L^{-2} factor, normal and image rows have length-independent bounds, and stacking takes the maximum. Choose one upper price covering these constants and all original donated prices. Keeping A_0, T literally unchanged preserves the donated carried-difference bounds $L^{-(1+\mathrm{order})}$ and $L^{-(2+\mathrm{order})}$ at the smaller width and larger price. This is F08; the exact roll equations above give F09 independently of these estimates. Under readout this entire Gram identity is transported on the same section; no component or harmonic coordinate is dropped to obtain positivity.

11.12 Pinning, coefficient symmetries and the inverse-third-power price

Pinning changes only auxiliary tower rows. At a transition assume the beds H_f, H_c have uniform analytic bounds and positive floors, A_0, T have their fine-difference bounds, and the inverse defect is $d_2^* C d_2$ with C analytic and positive on the d_2 image. Extend the defect's three-form row positively without changing its d_2 contraction, take its analytic root P, and form

$$F_f B_f T F_c^*, \qquad b^{-1/2} d_1^* F_c^*, \qquad \mathrm{stack}(I - F_c B_c F_c^*, P d_2 B_c F_c^*).$$

Here F_f, F_c are the full two-form Gram factors of the beds and B_f, B_c their inverse metrics; A_0, T are retained literally. These formulas are supplied by common cover rows for all periods. The

auxiliary-row construction supplies a positive width and common costs, including the required fine differences. Their analytic prices are chosen from the fixed clip profiles, estimates and penalty before the particular block length. For each smoothing index first apply positive extension and the analytic root, then choose auxiliary-row prices for each of the two adjacent-stage possibilities. Intersect those widths with the original width and maximize costs. Selecting the corresponding tower at each transition gives the full horizon conditions at these new prices. The Gram proof supplies the roll equations, while unchanged beds and A_0, T retain terminal normalization, crossings and floor bounds.

Pilot and wire coefficient bounds use one output cube before L is chosen. The mass filter costs $C\,\mathrm{decay}_j(l)$ with positive principal lower bound; rails have common upper and positive principal lower bounds. At nonzero aliases $D_{l,a}$ and R_l^{-1} cost respectively $C\,\mathrm{grade}(l)^{-2}$ and $C\,\mathrm{grade}(l)^{-4}$; the seed representation bounds S. The inverse-alias bounds use the common-collar reciprocal argument and the bulb and bend bounds. The sums H, T_a contain two smoothing factors, contributing decay_{2j}, while each D contributes another inverse grade squared. Each resulting weight is bounded above by decay_2 for $j \geq 4$, whose uniform finite-alias sum pays both sums. The quantity $U = |m(q_0)|^2 \sum_a f_a + SH$ is bounded above and has a positive real lower bound from its principal filter and one f_a, all summands being nonnegative. The common-collar lemma bounds U^{-1} and hence the source rests, fled and gains in fixed orientation dimension.

For the wire, Λ contains only controlled inverse principal filter and rail factors; B, M, y are bounded products and sums. At nonzero aliases the first stake has one grade-one signal, grade-minus-two covariance and smoothing flux; the second has two smoothing factors, grade-minus-four inverse, bounded coarse notch and another flux. Each is bounded by $C\,\mathrm{decay}_j$. This proves the asserted per-alias wire bound with constants chosen before L. Periodic regular covariances and patch identities show that a coarse shift $2\pi x$ permutes aliases by $l \mapsto l + \mathrm{wrap}_L(x)$. For $x \neq 0$, if both momenta lie in the enlarged real cube neither is zero, so the original regular-set stationary matching formulas apply on both sides. For $x = 0$ the overlap identity is immediate. Reflection follows by even scalar symbols and conjugation of Fourier rows/exterior matrices. At zero use the explicit principal and vanishing nonprincipal branches. Thus $\overline{\mathrm{pilot}(p, l)} = \mathrm{pilot}(-p, -l)$ and likewise wire. These are the relay and reflection properties required for the secured symbols and license the common-collar Fourier export of the residue sums.

Finally buckle is the secured product $\mathrm{vault}(\mathrm{wing}, q_l)\,\mathrm{wire}(p, l)$ with toll L^{-2}. Vault and its overlap identities preserve securedness. Thus one actual carried fine difference has exponential coefficient bound CL^{-3}. Its fine curl sums four directions and fixed two-form incidence signs, costing at most $4|\mathrm{Faces}_2|CL^{-3}$; this constant uses only orientation counts and introduces no residue factor. Multiplying by the bounded coarse metric and applying the coarse-column exterior right lift retains the prefactor, so sale costs $C'L^{-3}$ per residue. Its Gram sum has L^4 residues and therefore costs $C''L^4(L^{-3})^2 = C''L^{-2}$. This is the precise inverse-square scale used in the positive three-form absorption. The fine difference is taken before the lift and norm estimate; applying an ordinary unweighted bound before the fine difference would lose this power. Readout preserves this order and the actual carried stencil, so no lost block power is replaced by an unproved estimate.

Recovery of the assembled and pinned tower. Retain the ordered finite schedule, train at every level, metric beds, penalty and response rows as one realized history. Stagewise minima of positive widths and maxima of costs are scalar readers on this record, so their recovery keeps the parameter order and both adjacent stage possibilities. Pinning acts on the auxiliary rows of that same history and leaves its declared beds, A_0 and T literally fixed. For the full physical source H of a tower stack and its three output spaces H_a, choose sections j, j_a and readouts p, p_a with $pj = I$

and $p_a j_a = I$. Lift its rows $U_a = M, N, O$ by $\widetilde{U}_a = j_a U_a p$. Section adjoints then give

$$\sum_a \widetilde{U}_a^\dagger \widetilde{U}_a = j\Big(\sum_a U_a^* U_a\Big)p = jp, \qquad p\Big(\sum_a \widetilde{U}_a^\dagger \widetilde{U}_a\Big)j = I_H.$$

Thus the entire original stack source, including harmonic coordinates, is recovered. The identity jp acts on the retained section; it need not be the identity on unused redundant directions. Equivalently, keeping physical output rows $U_a p$ yields Gram form $p^* p$. Every roll equation recovers by the same intermediate section cancellations and the same real-slice adjoints. The positive extension $Q \mapsto Q + r d_3^* d_3$ preserves its curvature contraction under recovery since $d_3 d_2 = 0$; its full physical three-form lower bound and root are recovered as in the preceding constructions. Fourier normalizations are evaluated before this transport, retaining each coarse/fine volume factor and each installed toll. Consequently the assembled horizon retains its stationary equations, terminal normalization, crossings and donated fine differences, with the original widths and prices in the transported physical norms.

11.13 Filling the principal-alias pilot

For a flight put $q_l = (p + 2\pi \operatorname{lift}(l))/L$, $q_0 = p/L$, $S = \lambda_{\mathrm{mass(land)}}(p)$, $g_a = \mathrm{bulb}(\mathrm{land}, p, a)$, $h = \mathrm{bend}(\mathrm{land}, p)$. Retain $m(q) = \mathrm{massFilter}(q)$ and $f_a = \mathrm{bulb}(\mathrm{wing}, q_0, a)$. For $l \neq 0$ set $D_{l,a} = \mathrm{bulb}(\mathrm{wing}, q_l, a)/\operatorname{blow}(\mathrm{mass}(\mathrm{wing}), p, l)$ and

$$T_a = \sum_{l \neq 0} |\operatorname{pullLine}_a(q_l)|^2 D_{l,a} + b^{-1}, \qquad g_a = |\operatorname{pullLine}_a(q_0)|^2 f_a + S T_a. \tag{Q3}$$

Absolute squares mean opposite-momentum products off the real slice. Define

$$H = \sum_{l \neq 0} |m(q_l)|^2 \sum_a D_{l,a}, \qquad U = |m(q_0)|^2 \sum_a f_a + SH.$$

Away from zero U is S times the original weighted covariance sum. Its first real term has a positive lower bound and the second is nonnegative. The earlier alias estimates bound H, T_a, U on a common collar, and the reciprocal lemma bounds U^{-1} there. Thus U is the regular denominator replacing the original singular covariance sum.

The regular correction coefficients are

$$G_{0,a} = m(q_0)^{-1}(T_a/g_a - H/U),$$

$$G_{l,a} = m(-q_l)\left[(\sum_c D_{l,c})/U - D_{l,a}|R_L(q_{l,a})|^2/g_a\right] \quad (l \neq 0).$$

Every denominator is regular: $g_a, m(q_0), U$ and the nonzero-alias blow. Bounds are common for G_0 and $C \operatorname{decay}_j(l) \operatorname{grade}(l)^{-2}$ otherwise. Algebraically Q3 gives $E_l - r_{l,a} = S G_{l,a}$, where E_l is the original normalized covariance weight and $r_{l,a}$ its directional rebate; in particular Q3 yields the displayed principal formula for $G_{0,a}$. The isolated longitudinal factor is S^{-1} times $h^{-1} d_0(p)^* \operatorname{diag}(g_a^{-1})$, so S cancels before Fourier extraction. The filled pilot correction is

$$\operatorname{diag}(G_{l,a}) d_0^{(L)}(q_l) h^{-1} d_0(p)^* \operatorname{diag}(g_a^{-1}).$$

Its leading term is $\operatorname{diag}(f_a) \operatorname{flux}_1(q_0) \operatorname{diag}(g_a^{-1})$ at $l = 0$, and $S \operatorname{diag}(D_{l,a}) \operatorname{flux}_1(q_l) \operatorname{diag}(g_a^{-1})$ otherwise. Here the flux is the transpose of the directionwise patch multiplier on one-forms, and $d_0^{(L)}$

has its prescribed scaled normalization. For the bound, the leading term has $\mathrm{decay}_j(l)$, while the correction has $\mathrm{decay}_j(l)\,\mathrm{grade}(l)^{-2}$ times a signal bounded by a constant times $\mathrm{grade}(l)$; all remaining factors are uniformly bounded. Since $\mathrm{grade}(l) \geq 1$, the correction has the same decay_j bound. This formula supplies the analytic value at zero; no holomorphic extension of the uncancelled isolated longitudinal factor is asserted. This establishes analytic regularity of the filled pilot. Its inputs are all derived within the construction: the covariance recursion in Subsection 11.15, the stationary equations in Subsection 11.10, their finite-torus restoration in Subsection 11.9, and the reciprocal-shift and reflection identities in Subsection 11.12. Together these arguments establish the analytic predictor and its actual stationary response, including the harmonic fibre. The lifted statement is accordingly unconditional relative to the supplied construction: the realized predictor row is the supplied one, and its readout through $p_f T_Z = T p_c$ (Theorem A.1) recovers the stationary response on the same domain. The five-row contract of Proposition 11.1 is the form in which these outputs are consumed by the flat convergence argument. The lift retains this cancellation order. In particular, realize the already cancelled predictor A_l^{fill} as $j_f A_l^{\mathrm{fill}} p_c$. Recovery by p_f and j_c returns A_l^{fill}, including its value at zero, because $p_f j_f = p_c j_c = I$. No value for the separate factor S^{-1} at zero is needed in either realization or recovery.

11.14 The filled dual predictor on two-forms

Continue the pilot notation, with H_f, H_c the wing/land pulse, $d = d_1(p)$, $\delta = d_1^{(L)}(q_0)$, $F_r = \mathrm{hinges}_r$ and $V_r = \mathrm{flux}_r$, transposed at opposite momentum. Here d^* means the Fourier adjoint on real momentum and its analytic continuation using $-p$. Define

$$\Lambda = m(q_0)^{-1}\,\mathrm{mask}_2(R_L(q_{0,a})^{-1}), \quad y = |m(q_0)|^2 h_c/h_f(q_0), \quad x_a = T_a/g_a,$$

$$B = H_f(q_0)[\mathrm{diag}_{\mathrm{pairs}}(x_a + x_c) - S\,\mathrm{mask}_2(x)], \qquad M = \mathrm{diag}(g_a).$$

The pair diagonal sums the two coordinate entries for each two-form orientation $\{a, c\}$. The principal wire is

$$W_0 = yV_2(q_0) + [B\Lambda d M - H_f(q_0)\Lambda d\,\mathrm{diag}(T_a)]d^*.$$

The pairwise identity $1 - \mathrm{mask}_2(1 - Sx) = S[\mathrm{diag}_{\mathrm{pairs}}(x_a + x_c) - S\,\mathrm{mask}_2(x)]$, gives, by substituting Q3 in each two-form diagonal entry,

$$SB = H_f(q_0) - yV_2(q_0)H_c(p)F_2(q_0).$$

Together with $F_2(q_0)\Lambda = I$ and $F_2(q_0)\delta = dF_1(q_0)$ this identifies W_0 with the original stationary dual expression off zero. Every displayed regular factor has a common analytic bound, so the filled expression extends through zero.

For $l \neq 0$ put $R_l = \mathrm{blow}_l^2\, h_f(q_l)$, $y_l = |m(q_l)|^2 h_c/R_l$ and $N = S^2 I - SH_c(p)dMd^*$. The actual stake branches are

$$W_l = H_f(q_l)d_1^{(L)}(q_l)\,\mathrm{diag}(D_{l,a})V_1(q_l)d^* + y_l V_2(q_l)N.$$

The identities portD=scaleD, portR=scaleR and $N = S^2(1 - \mathrm{cloud})$ identify them with the original dual on the regular set. Only the separated nonprincipal blow is inverted. The nonzero-alias lower bound gives R_l^{-1} its $\mathrm{grade}(l)^{-4}$ bound and D_l its $\mathrm{grade}(l)^{-2}$ bound. These, with signal and flux factors, yield $C\,\mathrm{decay}_j(l)$. The previously retained relay/reflection proof exports these bounds to real Fourier rows. Source readout preserves this branchwise matching and its opposite-momentum adjoints; regularity of the filled expression does not assert regularity of each uncancelled term in the original formula. The recovery is applied to the complete regular branch. If W_l maps coarse to fine two-form coordinates, set $\widetilde{W}_l = j_{f,2} W_l p_{c,2}$. Then $p_{f,2}\widetilde{W}_l j_{c,2} = W_l$, with the same equality for the

principal value at zero. Every product in the regular matching identities is lifted with the section and readout of its actual intermediate degree; cancellation of $p_k j_k = I$ recovers the displayed order of multiplication. The analytically continued transpose remains the original opposite-momentum operation before realization, while section adjoints recover the real-slice adjoints. Consequently the lift preserves the denominators that are separated, the branchwise distinction between $l = 0$ and $l \neq 0$, and the dependency on the secured relay/reflection bounds. Fourier export is performed after forming these bounded regular branches.

11.15 Cumulative filters and covariance propagation

A finite history beginning at order j is empty or $(L, J; v)$ with odd $L \geq 3$ and $j \leq J \leq j+1$. The tail v begins at order J. Its mass is M_v, the product of its block sides, with empty mass one. For centered box filters write $A_L(p) = \prod_{\nu=0}^{3} p_L(p_\nu)$; odd-side composition is $A_{LM}(p) = A_L(Mp)A_M(p)$. Define

$$E_\varnothing = 1, \quad E_{(L,J;v)} = A_{LM_v}^{J-j} E_v, \qquad S_v = A_{M_v}^{j} E_v.$$

Then

$$S_{(L,J;v)}(p) = A_L(M_v p)^J S_v(p). \tag{QF1}$$

The exponent $J - j$ is zero or one: only stage changes add extra factors at successive cumulative lengths, while the principal filter retains exactly its original exponent j. Finite sums and induction give entire periodic filters, signed-coordinate symmetry, value one at zero and $\overline{S_v(p)} = S_v(-\bar{p})$.

On $|\operatorname{Im} p_\nu| \leq z/L_0$, $|A_M(p)| \leq e^{4zM/L_0}$; on the fixed real principal domain with $M \leq L_0$, $|A_M(p/L_0)| \geq e^{-4c_{\mathrm{box}}M/L_0}$. For a history of total mass $M_v \leq L_0$, cumulative masses increase by at least three, giving

$$|E_v(p)| \leq e^{8zM_v/L_0}, \qquad |E_v(p/L_0)| \geq e^{-8c_{\mathrm{box}}M_v/L_0}. \tag{QF2}$$

Indeed a new factor costs $4zM_v/L_0$ and the prior bound at most $8zM_v/(3L_0)$; their sum is at most $8zM_v/L_0$. The lower exponent has the identical calculation; absent factors use mass monotonicity. The principal alias bound $|A_M(\mathrm{alias}_l(p))| \leq B^4 \operatorname{mix}(l)^{-1}$ therefore yields, on taking $L_0 = M_v$ in QF2,

$$|S_v(\mathrm{alias}_l(p))| \leq B^{4j} e^{8z} \operatorname{mix}(l)^{-j}, \tag{QF3}$$

$$|S_v(p/M_v)| \geq e^{-4jc_{\mathrm{box}} - 8c_{\mathrm{box}}} > 0. \tag{QF4}$$

All constants depend on the initial order, not depth or total dilation, on the common stated neighborhood. In particular all alias-filter functions have a common holomorphic neighborhood and the indicated uniform alias decay.

Retain the longitudinal rail and its opposite-momentum product:

$$D_{v,a} = S_v r_{M_v}(p_a), \quad P_{L,J,a} = A_L^J r_L(p_a), \quad Y(f)(p) = f(p)f(-p).$$

Rail composition gives

$$D_{(L,J;v),a}(p) = P_{L,J,a}(M_v p) D_{v,a}(p). \tag{QF5}$$

On real momentum $D_{v,a}(-p) = \overline{D_{v,a}(p)}$, so $Y(D) = |D|^2 \geq 0$. The propagated alias operator is

$$\mathcal{F}_{v,a} f(p) = M_v^{-2} \sum_{l \in (\mathbb{Z}/M_v\mathbb{Z})^4} Y(D_{v,a})(\mathrm{alias}_l(p)) f(\mathrm{alias}_l(p)). \tag{QF6}$$

Define $\mathcal{L}_{L,J,a}$ by the one-patch formula. The radix bijection of aliases for LM_v and QF5 prove

$$\mathcal{F}_{(L,J;v),a} f = \mathcal{L}_{L,J,a}(\mathcal{F}_{v,a} f). \tag{QF7}$$

The prefactor is exactly M_v^{-2}. The lift retains each history, radix multiplicity and rail in this operation; normalized Haar frames do not modify its dimensional scaling. These finite alias identities precede the remaining covariance recursion and its analytic estimates. The radix proof of QF7 substitutes the nested finite sum, factors the outer $Y(P_{L,J,a})$ by periodicity and uses $(LM_v)^{-2} = L^{-2}M_v^{-2}$. The empty propagated operator is the identity. Every step adds an independent local covariance k^{-1}. Let $\mathcal{N}_{v,a}$ sum the covariances propagated through all prefixes: the empty history has none, and an extension adds the empty prefix and extends every old prefix by the new patch. Retain them through

$$\mathcal{N}_{\varnothing} = 0, \qquad \mathcal{N}_{(L,J;v),a} = 1 + \mathcal{L}_{L,J,a}\mathcal{N}_{v,a}, \qquad \mathcal{D}_{v,a} = \mathcal{F}_{v,a}(\lambda^{-1}) + k^{-1}\mathcal{N}_{v,a}.$$

Here λ is the initial Wilson quadratic divisor. Linearity and QF7 give the exact feedback

$$\mathcal{D}_{(L,J;v),a} = \mathcal{L}_{L,J,a}\mathcal{D}_{v,a} + k^{-1}. \tag{QF8}$$

All sums are finite; for $k > 0$ real nonnegativity holds termwise. Alias permutation proves reciprocal periodicity, evenness and signed-coordinate covariance. This formula accounts for every earlier noise; retaining only the final noise would give a different covariance.

Separating the principal alias gives, on the regular denominator set,

$$\mathcal{D}_{v,a}(p) = \frac{Y(D_{v,a})(p/M_v)}{\lambda_{M_v}(p)} + \mathcal{H}_{v,a}(p) + k^{-1}\mathcal{N}_{v,a}(p), \tag{QF9}$$

$$\mathcal{H}_{v,a}(p) = \sum_{l\neq 0} \frac{Y(D_{v,a})(\mathrm{alias}_l(p))}{M_v^2\lambda(\mathrm{alias}_l(p))}.$$

The principal divisor includes its M_v^2 scaling. Neither auxiliary divided symbol is declared continuous at harmonic momentum; filled regular formulas supply that reconstruction separately. Both remainders are nonnegative on real regular momenta, and QF4 gives the nonvanishing principal numerator.

The filters are literal normalized sums $r_M = M^{-1}\sum_{u=0}^{M-1} e^{iuz}$ and

$$p_M = M^{-1}\sum_{u=0}^{M-1} e^{i(u-\lfloor M/2\rfloor)z}.$$

Partition $0,\dots,LM-1$ as $Mu+v$ to obtain $r_L(Mz)r_M(z) = r_{LM}(z)$. For odd sides, $M\lfloor L/2\rfloor + \lfloor M/2\rfloor = \lfloor LM/2\rfloor$ gives the centered composition. The principal lower bound used in (QF2) has coefficient one in front of its exponential, which is essential for arbitrarily long histories. To retain it, choose $c_\pm > 0$ with $c_-|z| \le |e^{iz}-1| \le c_+|z|$ on a disk excluding other exponential zeros. The quotient extends continuously and is nonzero at zero, so such constants exist. The exact identity $Mr_M(z)(e^{iz}-1) = e^{iMz}-1$ gives $|p_M(z)| = |r_M(z)| \ge c_0 = \min(1, c_-/c_+) > 0$ for real z with $M|z|$ in that disk, including zero. If $M \le L_0$, $L_0|z| \le R_0$, then $|p_M(z)-1| \le k_0 M/L_0$, $k_0 = c_+R_0$. With $u = |p_M(z)|$, $x = M/L_0$, one has $u \ge c_0$, $u + k_0 x \ge 1$, and, using $e^y \ge 1+y$,

$$ue^{k_0x/c_0} \ge u + uk_0x/c_0 \ge u + k_0x \ge 1.$$

Hence $u \ge e^{-(k_0/c_0)x}$, giving $c_{\mathrm{box}} = k_0/c_0$ with no repeated fixed prefactor below one. Bounding each centered exponential by $e^{M|\operatorname{Im} z|}$ proves the complex upper estimate. The source lift retains every prefix noise and this principal/remainder decomposition; neither frame integration nor a change of coordinates discards a covariance contribution.

12 Local nonlinear geometry and Haar realization

From a linear predictor to an actual group-valued operation. Quadratic estimates alone do not define a nonlinear block map. This stage constructs logarithmic charts, local minima and group-valued predictors, and compares their derivatives with the flat cochain operators. The raw record retains the chosen chart, branch, guide tree, support, thresholds and all failed-chart flags. A minimum selected on an allowed branch is a conditional operation; it is not a globally unique logarithm of every configuration.

For a physical predictor P, an arbitrary coarse frame s gives the realized coarse link $B_e = s_x P(\pi(r,A))_e s_y^{-1}$. Its coarse readout is exactly $P(\pi(r,A))$. Differentiating this formula includes the frame response and preserves the physical derivative, as in (RF8). The estimates below control this actual operation on its specified domains. They also retain literal healthy masks: an indicator's boundary is not differentiated as though it were a smooth cutoff. Locality includes the links that determine the flags. The energy, packet and connected-cover estimates subsequently pay for those supports and for the off-chart terms. Discarding a branch, a flag or a path before its estimate is obtained would change the operation being recovered.

Work in the closed matrix image of the compact gauge group. Its real Lie algebra consists of the matrices X for which $\exp(tX)$ belongs to that image for every real t. Let N be the orthogonal complement of this real Lie algebra in the ambient real matrix space. If nonzero $X_j \to 0$, $\exp X_j \in G$ and $X_j/\|X_j\| \to u$, take $k_j = \lfloor t/\|X_j\| \rfloor$ for $t \geq 0$. Then $k_j\|X_j\| \to t$ and $\exp(k_j X_j) = (\exp X_j)^{k_j} \to \exp(tu) \in G$ by closedness; inversion covers negative t. Thus u is in the Lie algebra. Consequently small $n \in N$ with $\exp n \in G$ must be zero: otherwise normalized directions have a unit subsequential limit in both N and the algebra.

Identify the color space linearly with the Lie algebra, writing $C(a)$ for that identification. The map $(a,n) \mapsto \exp(C(a)) \exp n$ has invertible derivative $C(a) + n$ at zero. Its local smooth ambient inverse defines the color component $\log_G$. For a nearby group element its normal component vanishes by the preceding isolation. Thus $\log_G(\exp(C(a))) = a$ for small a. Conversely, if g has inverse coordinates (a,n), then $\exp n = \exp(-C(a))g \in G$ with n small, so $n = 0$ and $\exp(C(\log_G g)) = g$. Shrinking the color ball gives an open exponential image in G, since the continuous inverse sends a group neighborhood of each image point back into the ball. This requires neither connectedness of G nor a global logarithm.

Define $F_u(a) = \log_G(u \exp(C(a)))$ on its chart domain, $A(a) = DF_{\exp(-C(a))}(a)$ and $J(a) = \det A(a)$. Smoothness and the identity germ give $J(0) = 1$. If $b = F_u(a)$, the germ identity $F_{\exp(-C(a))} = F_{\exp(-C(b))} \circ F_u$ yields

$$J(b) \det DF_u(a) = J(a).$$

Shrink the ball S so $1/2 < J < 2$ and put $U = \exp_G(S)$. The pushforward $v = (\exp_G)_*(\mathbf{1}_S J(a)\, da)$ is finite and nonzero. On the overlap F_u is injective and its image is the relevant piece of $A \cap uU$. Positivity of J turns the overlap law into $|\det DF_u(a)| J(F_u(a)) = J(a)$. Change of variables therefore gives $v(u^{-1}A) = v(A \cap uU)$ for measurable $A \subseteq U$. Average over normalized Haar measure. Tonelli and right invariance make the left integral $v(G)\mu_G(A)$. For fixed $g \in A$, the set of u for which $g \in uU$ has measure $\mu_G(U)$ by inversion and right translation, so the right integral is $\mu_G(U)v(A)$. Hence

$$v(G)\mu_G(A) = \mu_G(U)v(A).$$

The set U is nonempty and open, and J is bounded above and below on the finite-radius ball. Thus both constants are positive and finite, so $c = \mu_G(U)/v(G) > 0$ gives

$$(\exp_G)_*(\mathbf{1}_S c J(a)\, da) = \mu_G|_U.$$

Intersecting arbitrary test sets with U proves the full local measure identity. The realized frame construction uses this same G, chart, density and normalization; endpoint conjugations preserve the color metric, but no new globally defined chart is introduced by the readout. Product-chart and fluctuation changes below retain their separate Jacobian factors.

12.1 Primitive Haar charts and fluctuation normalization

Right Haar invariance sends $\mathbf{1}_{B_\rho} cJ(a)\,da$ under $a \mapsto \exp_G(a)V$ to Haar restricted to $\exp_G(B_\rho)V$. The finite product on active links has density $\prod_i cJ(a_i)$, with inactive links fixed. The tree-gauge splitting into active and exterior link coordinates is measure preserving. Fubini permits active integration first, then exterior integration, after insertion of the bounded local ingredients. The powers of c and the Jacobian logarithms come from this product density. Regrouping then produces the quadratic Gaussian density and the explicit vertex factors; the grafted-signal and Gaussian-exponent constructions supply the selector implication and that decomposition. Sector selectors ensure no nonzero contribution is omitted: if $z = U_i V_i^{-1}$, shell access gives $\mathrm{chord}(z) \le Kt$, while on the wider shell $\mathrm{chord}(z) \le 2Kt$ the logarithm satisfies $\|\log_G z\| \le \gamma\,\mathrm{chord}(z)$ and is inverse to the exponential. The active chart is sufficient because the sector selectors force every active link of a nonzero integrand to lie in that chart relative to its replica background. Choose $\gamma K \le a$ and $at < \rho$. Then $U_i = \exp_G(v)V_i$ with $\|v\| \le at$. Outside the chart the cutoff makes the transformed integrand vanish, so no exterior logarithm value is used in the change of variables. The inserted cutoff functions can be handled at all color arguments.

The inner-share cutoff χ_a lies between zero and one, equals one on $\|v\| \le at$ and vanishes outside $2at$. Thus $\chi_{2a} = 1$ on its support. Set $g_a(v) = \chi_{2a}(v)\log J(v)$. If $2at < \rho$, $e^{g_a} = J$ on the active cutoff support. Since $J(0) = 1$ and J is smooth positive, $|\log J(v)| \le C\|v\|$ locally; when $4at$ is in this radius, $|g_a(v)| \le 4Cat$ globally, including the zero-cutoff branch. No exterior logarithm is used where the integrand vanishes.

For p active real coordinates and coupling scale $g > 0$, the affine dilation $D(x) = \mu + gx$ has determinant g^p. Here active coordinates are indexed by active-link/color pairs. Finite-dimensional change of variables pushes $g^p dx$ to active-chart Lebesgue measure. Reindexing these pairs into a color vector at each active link is a coordinate permutation and regrouping, preserving measure first on measurable rectangles and hence on all measurable sets. The active part of the original fluctuation $\mu + gz$ is precisely $D(x)$. The dormant matrix is zero on active and identity on vacant coordinates; its energy is $\|y\|^2/2$, so for vacant dimension d_v its density is

$$C(y) = (\sqrt{2\pi})^{-d_v} e^{-\|y\|^2/2}, \qquad \int C(y)\,dy = 1,$$

including $d_v = 0$. Hence the exact integrable factorization is $\int Y(x) g^p C(y)\,dx\,dy = g^p \int Y(x)\,dx$. The original configuration and cutoff/Jacobian factors depend only on active coordinates. Fubini uses the primitive envelopes; dormant coordinates are integrated, not deleted. This accounts for the active-coordinate Jacobian and the unused Gaussian coordinates in the exact fluctuation integral, rather than discarding them by a dimension convention. The packet and non-packet estimates below supply the vertex envelopes.

If color dimension is r, active free-link count h and total free tree-link count f, then $p = rh$, ambient dimension is $rh + d_v$, and frozen free-link count is $f - h$. With $m_g = c(\sqrt{2\pi}g)^r$,

$$(\sqrt{2\pi})^{-d_v} g^{rh} c^h = (\sqrt{2\pi})^{-(rh+d_v)} m_g^f e^{-(f-h)\log m_g}.$$

The counts come from the actual finite partition of free links into active and frozen sets. Cancel m_g^{f-h} and substitute $m_g = c(\sqrt{2\pi}g)^r$ to prove the identity. The separate blank-form determinant

factor is

$$e^{-\log\det Q_{\mathrm{blank}}/2}.$$

A positive covariance Q in dimension d has density

$$(\sqrt{2\pi})^{-d}\exp[-\log\det Q/2 - z^{\top}Q^{-1}z/2].$$

Determinant factors are not dimension conventions. Thus all determinant and Gaussian-volume factors are explicit before the remainder vertices are estimated.

Finally retain the raw action split $\beta\,\mathrm{cut}_R + E_R = v_R|\Lambda| - \mathrm{reducedAction} - \mathrm{additiveActivity}$. The correction uses the whole gap $\mathrm{cut}_R(U) - \mathrm{coarseCrown}(V)$ and cancels the retained crown already transferred to the raw density exactly:

$$e^{-\beta\,\mathrm{coarseCrown}}e^{\beta\,\mathrm{cut}_R}e^{-\beta(\mathrm{cut}_R\,-\,\mathrm{coarseCrown})} = 1.$$

The sector sum is thus the tagged block kernel times the original marked density pointwise. The activity's summable envelope and bounded sector factors justify integration/summation. The source lift preserves these actual factors under measure readout, including every determinant and sector tag. This is the initial action cancellation in the primitive Mark formula; the marked-integration chapter supplies the local vertex decomposition and primitive-integral estimates completing that formula.

Recovery of the active chart measure. Fix the physical replica background V and an active-link set E_a. For each fixed frame r the raw active-chart parametrization is

$$A_e = r_x\exp_G(a_e)V_e r_y^{-1}, \qquad e: x\to y, \qquad \pi(r,A)_e = \exp_G(a_e)V_e.$$

Left and right Haar invariance followed by the original chart identity therefore gives the same active density $\prod_{e\in E_a} cJ(a_e)\,da_e$ for every r. Integration over normalized frame Haar measure contributes one. For every integrable test supported in this product chart,

$$\int F(\pi(r,A))\,d\nu(r)\prod_{e\in E_a}d\nu(A_e) = \int F((\exp_G(a_e)V_e)_e)\prod_{e\in E_a}cJ(a_e)\,da_e,$$

with the chart restriction understood on the left and all inactive data held fixed at their declared readouts. The logarithmic coordinate reader is $a_e = \log_G(\pi(r,A)_e V_e^{-1})$ on this chart. The identity-frame section recovers the original chart point and each selector, cutoff and Jacobian factor. This requires no logarithm of an arbitrary raw link. The active affine dilation and coordinate permutation are performed on these recovered color coordinates, so their determinant is still g^{rh}. The separately normalized dormant Gaussian still integrates to one. Hence neither frame integration nor readout changes c^h, the active/frozen census, the blank-form determinant, or the crown-gap cancellation in the primitive integral. The same integrable envelopes license these product integrations and subsequent sums.

12.2 The reference quadratic floor before localization

Let Lu be the active link direction, Du the physical stacked motion and P_d the orthogonal dormant-coordinate projection. The reference precision is $Q = F^{\top}F + P_d$, with

$$u^{\top}F^{\top}Fu = \sigma^2\sum|Du|^2.$$

The local geometric estimate is $|Lu_i| \le ZR_i(u) + MC_i(u)$, where R_i is the square root of the sum of squared physical motions within fixed native radius d, and C_i is the square root of the sum of squared active link directions within radius $3B$. Both use actual torus max-distance and the actual link census. Set $W = \sum |Lu_i|^2$, $V = \sum |Du|^2$ and $N_r = (2r+1)^4 L_{\mathrm{block}}^4 |\mathrm{Axis}|$. Squaring and charging occurrences gives

$$W \le 2Z^2 N_d V + 2M^2 N_{3B} W.$$

At the chosen block/stage Z is fixed and M has reference-coupling order $-1/2$ with the stated width factors. Polynomial width growth permits the eventual condition $2M^2 N_{3B} \le 1/2$. Absorbing gives $W \le 4Z^2 N_d V \le aV$ for a positive enlarged a. Orthogonal active/dormant splitting then proves

$$\|u\|^2 = \|Lu\|^2 + u^\top P_d u, \quad u^\top P_d u \ge 0, \qquad u^\top Q u \ge \min(\sigma^2/a, 1)\|u\|^2.$$

This derives the floor from physical motion, conditional on the source's local driving/flattening estimates NF1–NF4, before choosing any covariance; it is not an assumed lower bound for the full matrix. The reference precision is a Gram matrix with an identity completion on dormant coordinates.

Each dormant row and column of the plate is identity and its linear source is zero, including in principal restrictions. The inverse equation therefore gives identity dormant rows and columns in each restricted inverse. Padding a column retains this property because its diagonal lies in its footprint and other entries outside are zero. Symmetrizing the padded inverse gives identity dormant covariance rows/columns, and the mean $-P^* b$ vanishes there. These exact facts justify the dormant Gaussian used above. Positivity and an upper bound for the whole padded covariance still use the localized inverse-window estimates. Under the source lift the floor is on this actual physical/dormant coordinate space and its isometric section, not on an enlarged space containing extra frame-vertical null vectors.

12.3 Localized inverses and positive finite-window covariance

Let $A = A^\top \ge rI$, $r > 0$, be indexed by coordinates at sites a_i, and have width-η ($\eta \ge 0$) absolute row and column prices at most $X > 0$. Principal restrictions inherit the floor by zero extension and the prices by restricting nonnegative sums. For $0 \le \theta \le 1$, $\theta X \le r/2$, take $u \ge 0$, so $A_u = A + uI$ has an actual two-sided inverse. Fix j and conjugate A_u by $s_i = e^{\theta\eta\,\mathrm{dist}(a_i,a_j)}$. If $y = A_u^{-1} e_j$ and $w_i = s_i y_i$, then

$$(A_u + E)w = e_j, \qquad E_{ik} = (A_u)_{ik}(s_i/s_k - 1).$$

Distance and convexity give $|s_i/s_k - 1| \le \theta e^{\eta\,\mathrm{dist}(a_i,a_k)}$. Indeed the log ratio lies between $\pm\theta D$ and $e^{\theta D} - 1 \le \theta(e^D - 1)$; for a negative logarithmic ratio v, use $1 - e^v \le e^{-v} - 1$. The added diagonal contributes zero to E. Its row/column prices are at most $r/2$, so $2|v_i v_k| \le v_i^2 + v_k^2$ gives $|v^\top E v| \le (r/2)\|v\|^2$. Pairing the equation with w yields $(r/2 + u)\|w\|^2 \le w_j \le \|w\|$, with $c = \min(1, r/2)$ this implies $c(1+u)\|w\|^2 \le w_j \le \|w\|$. The zero-vector case is immediate; otherwise $\|w\| \le [c(1+u)]^{-1}$. Hence

$$e^{\theta\eta\,\mathrm{dist}(a_i,a_j)} |(A + uI)^{-1}_{ij}| \le [\min(1, r/2)(1+u)]^{-1}.$$

This is dimension-uniform and applies to each principal restriction.

Take $\theta = \min(1, r/(2X))$, $b = \theta\eta$ and $k = b/3 > 0$ on the positive-width application domain. With at most m coordinate labels per site, summing the remaining $e^{-k\,\mathrm{dist}}$ tail gives a period-independent price P at width $2k$. Indeed the sum of $e^{-(b-2k)\,\mathrm{dist}}$ over torus sites is bounded by the corresponding absolutely convergent lattice sum, independently of the period. The full inverse and

zero-padded principal inverse columns obey this price. Choose each endpoint-defined window W_j to contain the radius-B coordinates and lie within radius $B+2$, and set $K_{ij} = (A|_{W_j})^{-1}_{ij}$ inside, zero outside. Product prices at width $2k$ bound $E = AK - I$ by $XP + 1$. The exact principal equation makes this error zero when $i \in W_j$, in particular within radius B. On all remaining entries the loss of width k supplies e^{-kB}. Its width-k price is therefore at most

$$\varepsilon_B = e^{-kB}(XP + 1).$$

The exact identity $K - A^{-1} = A^{-1}E$ bounds its error price by $P\varepsilon_B$.

For $C = (K + K^\top)/2$, symmetry gives error price price$(C - A^{-1}) \le P\varepsilon_B$ and price$(C) \le P$. Schur bounds convert the error to a quadratic bound: for a matrix E whose absolute row and column sums are at most $r/2$, expanding the double sum and using $2|v_i v_k| \le v_i^2 + v_k^2$ gives $|v^\top Ev| \le (r/2)\|v\|^2$. Since $A \le XI$, $A^{-1} \ge X^{-1}I$; this also follows by expanding the nonnegative form of $w - X^{-1}v$ with $w = A^{-1}v$. Thus $P\varepsilon_B \le 1/(2X)$ yields

$$(2X)^{-1}I \le C \le PI.$$

The covariance vanishes beyond $B + 2$, and $AC - I = A(C - A^{-1})$ has width-k price at most $XP\varepsilon_B$. All choices are uniform in volume. For the physical plate its prior floor, stencil prices and coordinate census provide r, X, m; increasing B at fixed block/stage pays the error. The physical operator bounds and local driving estimate of this chapter establish these inputs for the plate. The lift transports whole-color-block windows and the original endpoint rule, retaining K separately from C for the mean. It does not identify a padded principal inverse with the full inverse or assert positive exponential width from an input width of zero.

Recovery of the window covariance and mean. Retain the endpoint-defined windows, their principal inverses and the padded matrix K separately from $C = (K + K^*)/2$. For a physical linear source b the retained mean is $\mu = -K^*b$. If $E = AK - I$, then for every physical comparison vector v,

$$\mu - v = E^*v - K^*(Av + b).$$

Indeed $E^* = K^*A - I$ since $A = A^*$, and substitution cancels K^*Av in the displayed order. Symmetrization of the covariance therefore does not authorize replacing this mean by $-Cb$. For a section j and readout p with $pj = I$, realize the physical Gaussian $\gamma_{\mu,C}$ by the pushforward $j_*\gamma_{\mu,C}$. Its ambient mean and covariance are $j\mu$ and jCj^*, and its readout is exactly $\gamma_{\mu,C}$. In particular

$$p(j\mu) = \mu, \qquad p(jCj^*)p^* = C.$$

These are covariance pushforward identities, distinct from the pullback p^*Ap of a precision form. The realized Gaussian is supported on the declared section. No ambient Lebesgue density or inverse in frame-vertical directions is introduced. Pulling test functions back by p and restricting through j recovers the original Gaussian integrals and their dormant factors. Window error, finite range, positive floor and upper bounds are read in the original physical coordinates and transported section norm, with the same site census and widths. Thus the finite-window approximation remains the original one, and both its residual and its mean recover exactly.

12.4 Axial gauge and bounded local logarithms

For finite cover support X, let e_X sum internal positive-link chords and f_X sum wholly internal plaquette chords, with boundary/incomplete terms zero. Both are nonnegative and f_X is gauge

invariant. An ordered list of distinct axes defines a slab with those coordinates in $[-r, r]$ and the others pinned. From a root p in the slab, travel first in the first listed axis to the corresponding coordinate of x, then in the next axis, and so on. The remaining coordinates are pinned, so this route ends at x; its ordered link product is $H_p(x)$. Extending the list by axis j makes every internal j-link identity in this gauge, since the routes to its endpoints differ by precisely that final link.

With those links identity, the plaquette and chord triangle inequalities give

$$|\operatorname{chord} U(x + e_j, i) - \operatorname{chord} U(x, i)| \le \operatorname{chord} \operatorname{hol}(\square(i, j), x).$$

Apply the chord triangle inequality to ab^{-1} and its inverse for the two horizontal links. Reversing the axis order only inverts the holonomy and preserves its chord. Summing the four positive axes over a slice bounds the change by four times the adjacent-layer plaquette sum; telescoping within the slab gives $|e_{\text{slice(z)}} - e_{\text{slice(u)}}| \le 4 f_{\text{slab}}$. All layers used by the telescoping sum lie inside the slab; adding the unused nonnegative layers proves this bound even when the two slices are far apart. Inductively a root exists with $e_{\text{slab}}(H_p \cdot U) \le 4k(2r+1) f_{\text{slab}}$ for k listed axes. The base is a single site. Choose a minimum-face-sum slice for the next axis, whose face sum is at most $f_{\text{slab}}/(2r+1)$, since the slice face sets are disjoint subsets of the full slab face set. Apply the prior root construction there, and extend its routes. The chosen slice edge sum is unchanged. The new links are identity and the slice comparison gives the claimed next bound after summing slices. No small-curvature assumption enters root existence.

For four axes this gives $16(2r+1) f_{\text{box}}$. A periodic radius-r patch with plaquette chords at most q has $f_{\text{box}} \le (2r+1)^4 |\text{PlaqAxes}| q$. Lift the patch to that box and apply the preceding construction. When $2r < n$ its box lift is injective; the route gauge descends and extends arbitrarily elsewhere. If $d + 1 \le r$, each radius-d bond lies within the box, and

$$\operatorname{chord}((z \cdot U)(b)) \le 16(2r+1)^5 |\text{PlaqAxes}| q.$$

The lift carries the same patch and gauge, including its nonwrapping condition; abstract frame freedom does not enlarge this geometric domain.

The local analytic logarithm has $\log 1 = 0$, giving $\|\log M\| \le D\|M - I\|$ on a small matrix ball. Shrink the ball for the local exponential inverse. Since $\|\operatorname{mat}(U) - I\| = \operatorname{colorScale} \operatorname{chord}(U)$, this yields constants $C, \delta > 0$ with $\exp(\log U) = U$ and $\|\log U\| \le C \operatorname{chord}(U)$ for chord at most δ. Set the field to this logarithm on the specified interior patch and zero outside. If its chords are at most $h \le \delta$, the global field is bounded by Ch and recovers the patch links; boundary continuity is not required for this finite lattice field.

For an analytic map f, its series identifies the directional derivative L with the bounded derivative $Df(0)$. The first two terms and uniform remainder on one fixed ball give $\|La\| \le K\|a\|$ and $\|f(a) - f(0) - La\| \le K\|a\|^2$. Applied to the exponential they bound its chord linearly. For the four-factor plaquette curvature the derivative is $a_0 + a_1 - a_2 - a_3$, yielding an actual-curvature versus discrete-curl error $K\|a\|^2$ uniformly in position and volume, including repeated edges, because each of the four arguments is bounded by the same field norm. Finally expanding $UXU^* - X = (U-1)XU^* + X(U^*-1)$ and using unitary invariance of the Frobenius norm together with $\|\operatorname{colorMat}(a)\| = \operatorname{colorScale} \|a\|$ gives

$$\|\operatorname{Ad}(U)a - a\| \le 2 \operatorname{colorScale} \operatorname{chord}(U) \|a\|.$$

Induction along a walk gives chord of its transport at most its length times h, with inverse-link chords unchanged. These are the nonlinear/local bounds transported through the actual realized chart and route data.

12.5 Quantitative nonlinear and flat row comparison

Set $\mathrm{widen}(R,h,\tau) = (1+R)h + e^{-\tau R}$; cutoffs read actual path length, including winding. A path slot reads a plaquette after adjoint transport along its walk from the reading site. In a chart field $\|a\| \le h$, link chords are at most $B_{\mathrm{ch}}h$, curl at most $A_{\mathrm{ch}}h$ and curvature-minus-curl at most $A_{\mathrm{ch}}h^2$. A retained transport slot of length at most R changes its curl by at most $2\,\mathrm{colorScale}\,RB_{\mathrm{ch}}A_{\mathrm{ch}}h^2$. Hence slot-minus-curl is bounded by $Jh\,\mathrm{widen}$, $J = A_{\mathrm{ch}}(1 + 2\,\mathrm{colorScale}\,B_{\mathrm{ch}})$. For an omitted slot of length ℓ with $d\ell > R$ and stage width $\sigma > 0$,

$$1 \le e^{\sigma\ell}e^{-\sigma R/d} \le e^{\sigma\ell}\,\mathrm{widen}(R,h,\sigma/d).$$

With $\sum_i |u_i|e^{\sigma\ell_i} \le$ paths, the omitted-slot inequality pays its whole linear curl, and absolute summation of retained errors and omitted linear terms gives

$$\|x_{\mathrm{lin}}(a)_j\| \le Ch, \qquad \|\,\mathrm{plotted}(\exp a)_j - x_{\mathrm{lin}}(a)_j\| \le Ch\,\mathrm{widen}(R,h,\sigma/d),$$

where $C = \sqrt{\mathrm{direct}}A_{\mathrm{ch}} + \mathrm{paths}\,J$. The direct coordinate uses its quadratic face error and $h \le \mathrm{widen}$. The same absolute envelope handles countably many slots; the constants and small chart radius depend on the fixed stage and d, independently of period and R.

For cover row $K_{ab}(u)$ with $\sum_u e^{\eta|u|}|K_{ab}(u)| \le A$, for each pair of finite residue types a, b, route each coefficient by its designated adjoint walk, retaining it when d length $\le R$ and only in its endpoint congruence class. Assume d length $\le D(1+|u|)$. The full flat term is $K_{ab}(u)I$ without cutoff. Unitary transport bounds the routed norm by $|K|$. With link chords at most $C_{\mathrm{ch}}h$, retained transport differs from identity by at most $2\,\mathrm{colorScale}\,C_{\mathrm{ch}}Rh$. For omitted terms the route inequality gives $\eta R/D \le \eta + \eta|u|$, hence $1 \le e^{\eta}e^{\eta|u|}e^{-\eta R/D}$. Thus either error is bounded by

$$J\,\mathrm{widen}(R,h,\eta/D)e^{\eta|u|}|K_{ab}(u)|, \quad J = 1 + 2\,\mathrm{colorScale}\,C_{\mathrm{ch}} + e^{\eta}.$$

Each cover displacement belongs to exactly one endpoint congruence class, even for many windings. Summing absolute majorants first on the cover costs only the finite residue census. Absolute convergence permits interchange of these nonnegative bounds, even on short tori with many windings. For n_a, n_b types, $C_K = (n_a + n_b)AJ$ bounds row and column sums of both matrices and gives difference price C_K widen.

Same-level coordinate routes have length $|u|$, $d = L$; residue-to-Lu routes have length at most $L|u| + 4L$. Both admit $D = 5L$. Replica links are identity or original coarse links and preserve the chord bound. For the F, I, N rows take $A = \max(\max(0,E), E/L^2)$ and $\tau = \eta/(5L)$, then maximize their costs with one and intersect chart radii. Constants are chosen after the fixed L, uniformly in period/cutoff. At arbitrary realized frames the flat identity-color row becomes its endpoint-conjugated row; readout transports the same comparison, rather than comparing to an untransformed identity in unrelated frame coordinates.

12.6 Clean-hull guides and signed stationary cancellation

The guide is the pair of coarse response fields Mv, Mw, compared with flat F, I applied to $x_{\mathrm{lin}}(a)$. The flat N row vanishes on that field by $\mathrm{flatN}\,D_{\mathrm{full}}d = 0$, because $x_{\mathrm{lin}}(a) = D_{\mathrm{full}}da$, including harmonic coordinates. On a clean radius-$B/4$ hull agreeing with $\exp a$, room inequalities contain the row reach $R = \mathrm{till}(x)$ and the plotted field's further $R+4$ sites. Cleanliness makes the retained indicator and clipping factor $\min(1, \mathrm{temp}\,/\,\mathrm{toll})$ equal to one there, and locality replaces the row by its value at $\exp a$.

If the nonlinear row has price X, its difference from flat row D price XW, and the plotted/linear field discrepancy is at most U with linear field bound Z, write $B = Q(\exp a)$, $f = \text{plotted}(\exp a)$ and $g = x_{\text{lin}}(a)$. The exact decomposition is

$$Bf - Dg = B(f - g) + (B - D)g, \qquad \|Bf - Dg\| \le XU + XWZ.$$

These substitutions hold only on that same hull. The kernel-chart and plotted-chart estimates supply positive constants X, Z_0, s, t. Here t is the lower-stage width divided by the positive block length. The source choices are $M(x) = Z_0 H(x)$, $U(x) = M(x)\,\text{widen}(\text{till}(x), H(x), t)$ and

$$e(x) = XU(x) + X\,\text{widen}(\text{till}(x), H(x), s)M(x).$$

With eventual nonnegative rate-$-1/2$ H and the actual radius/cutoff bounds, widening retains that rate because it is a radius factor times H plus an exponentially small cutoff term. The products in e therefore have rate -1. It bounds both guide discrepancies and nonlinear N, whose linear part is zero.

Choose together the supplied quadratic beds, their attached path realizations, positive floors, diagonal/roll identities and invariance under signed coordinate changes. Let W be their installed stationary map and U its extracted fine-field matrix for the chosen forest. Then let $D_0 = I - \text{probe}\,U$. The diagonal identity is

$$K_c = U^\top K_f U + \text{penalty}\, D_0^\top D_0.$$

Pairing each color coordinate proves fine shifted energy plus penalty residual norm squared equals coarse energy, including harmonic modes. Signed covariance of the actual nonlinear predictor and noise energy, applied to exponentiated ta and divided by t^2, transfers this identity to every signed orientation. Bed invariance supplies the other terms; entrywise isotropy is not assumed.

Let w, u, v be the transformed coarse, shifted fine and noise linear moment fields. The attached path realizations and floors identify moment-square sums with bed energies; all norms here are finite counting sums of squared color norms. Thus $\|u\|^2 + \|v\|^2 = \|w\|^2$. Set $U_1 = Fw$, $V_1 = Iw$. Since $Nw = 0$, the roll identities give $w = F^\top U_1 + I^\top V_1 = F^\top u + I^\top v$ in the finite counting inner products. Pairing with w gives $\|U_1\|^2 + \|V_1\|^2 = \|w\|^2$ and $\langle U_1, u\rangle + \langle V_1, v\rangle = \|w\|^2$. Expanding therefore yields

$$\|U_1 - u\|^2 + \|V_1 - v\|^2 = 0.$$

Nonnegative summands prove equality row by row, hence linearCost equals linearGuide in every signed orientation. Subtraction gives the uncut spring cancellation. The source lift preserves the same beds, forest, signed action and counting metric, so this cancellation is recovered without replacing a nonlinear cutoff identity by a global assertion outside its clean hull.

12.7 Nearest group points and analytic focus

Use the real Frobenius inner product. Compactness of the closed unitary matrix group gives a nearest group point to every matrix, and left/right group multiplication preserves that minimization. If 1 is nearest to X, differentiating $\|\exp(tY) - X\|^2$ at zero for every Lie algebra Y shows $X - 1 \perp \mathfrak{g}$. Therefore any nearest u obeys $u^*X - 1 \in \mathfrak{g}^\perp$.

The analytic tubular map $T(a, n) = \exp(\text{colorMat}(a))(1 + n)$ is defined for $n \in \mathfrak{g}^\perp$ and has derivative $\text{colorMat}(a) + n$ at $(0, 0)$, an isomorphism by the orthogonal tangent/normal decomposition. Its local analytic inverse defines focus as the exponential of its tangent coordinate. To identify every nearest point near identity, choose any nearest $b(X)$. Minimality gives $\|b(X) - X\| \le \|I - X\|$ and hence $\|b(X) - 1\| \le 2\|X - 1\|$, so $b(X) \to 1$. Its normal coordinate $n = b(X)^*X - 1$ is perpendicular and tends to zero, as does its local tangent coordinate $a = \log b(X)$. Both coordinates

therefore lie in the inverse-function neighborhood when X is sufficiently near I. Since $T(a,n)=X$, uniqueness of the local inverse forces $b(X)=\mathrm{focus}(X)$. Exceptional nearest choices arbitrarily near identity would contradict this argument. Translation gives a uniform tube about the whole group, including all its components.

Differentiating the inverse gives $D(\mathrm{mat}\circ\mathrm{focus})(1)=\mathrm{proj}_{\mathfrak{g}}$. Indeed the inverse derivative sends $\mathrm{colorMat}(a)+n$ to (a,n), and the exponential derivative discards n. Analyticity makes this a strict derivative, which yields, after shrinking the neighborhood for each $\varepsilon>0$,

$$\|\,\mathrm{mat}(u)-\mathrm{mat}(v)-\mathrm{proj}_{\mathfrak{g}}(X-Y)\|\le\varepsilon\|X-Y\|$$

for the nearest points of X,Y. On the unique tube, unitary norm invariance and uniqueness prove equivariance of focus under left/right group multiplication. Differentiating that equality gives its derivative covariance. These prove the focus properties used in the frame and predictor derivatives from the local group chart and the ordinary inverse function theorem. The source lift uses this actual analytic branch and its radius. Outside it, compactness still gives minimizers but no arbitrary global selection is assigned these uniqueness, analyticity or equivariance properties.

12.8 Analyticity of the actual normalized frame

Fix the period. No volume-independent analytic radius is asserted. For logarithmic input v, retain actual patch frame matrices H_p normalized by $H_{p_0}=1$. At zero all paths are identity; the nonnegative score has identity minimizer of value zero, and uniqueness identifies the selected frame. Its positive cutoff gives an open minimum branch, and selected-minimizer continuity gives $H_p\to 1$, with all tags choosing that branch.

The actual self-consistency cloud is the finite analytic expression

$$\mathrm{cloud}_p(v,H)=\mathrm{average}_{q,o}\,H_q\,\mathrm{travel}_v(p\to q;o)^*.$$

The path products are finite products of exponentials and adjoints, so the cloud is analytic jointly in v,H and equals one at $(0,1)$. Continuity places every cloud in the analytic-focus neighborhood; self-consistency says H_p is its nearest group point. Thus the selected frame solves $F_{p_0}=H_{p_0}-1=0$ and $F_p=H_p-\mathrm{focus}(\mathrm{cloud}_p)=0$ otherwise. A kernel vector of its H derivative has $h_{p_0}=0$ and $h_p=A=\mathrm{proj}(\mathrm{average}_q\,h_q)$ for $p\neq p_0$. For the positive number N of patch points its average is $(N-1)A/N$, and A is already projected. Hence $A=(N-1)A/N$, so $A=h=0$. This finite-dimensional endomorphism is invertible.

Apply the analytic inverse function theorem to $(v,H)\mapsto(v,F(v,H))$. The continuous actual solution lies in the local inverse neighborhood; uniqueness identifies it with the analytic implicit branch. This proves analyticity of the actual normalized frame, using the separately retained nearest-point, self-consistency and minimizer arguments. The realized predictor lift composes this same selected operation with readout; it does not enlarge its radius to a uniform-in-volume statement or differentiate an arbitrary fallback branch.

12.9 First derivative of the actual rooted predictor

Let $\mathrm{fee}_y(p,q,o;a)$ be the tangent link sum of the actual ordered route from p to q embedded at coarse root y, and define $\mathrm{tap}_y(p;a)=-\mathrm{average}_{q,o}\,\mathrm{fee}_y(p,q,o;a)$. Differentiating each finite route product gives its color matrix, with reversed links contributing minus signs. Each color component of tap is exactly the scalar path functional $\tau_y(p)$ applied to the color-coordinate input.

For the actual frame derivative d_p, the cloud derivative at identity is

$$\mathrm{average}_q\,d_q+\mathrm{colorMat}(\mathrm{tap}_y(p;a)).$$

The focus derivative is projection onto the Lie algebra, so differentiation gives

$$d_p = \mathrm{proj}(\mathrm{average}_q\, d_q) + \mathrm{colorMat}(\mathrm{tap}_y(p;a)).$$

Since the distinguished frame is identically one, $d_{p_0} = 0$. Subtract its equation to obtain

$$d_p = \mathrm{colorMat}[\mathrm{tap}_y(p;a) - \mathrm{tap}_y(p_0;a)].$$

This proves tag independence of the derivative on the common actual branch. For a coarse bond the left frame, length-L route and inverse right frame therefore give connector derivative $\mathrm{colorMat}(\mathrm{bill}_p(a))$, where

$$\begin{aligned}\mathrm{bill}_p(a) = {} & \mathrm{tap}_y(p;a) - \mathrm{tap}_y(p_0;a) + \mathrm{linkSum}(\mathrm{bondRoute}_p;a)\\ & - \mathrm{tap}_{y+e_i}(p;a) + \mathrm{tap}_{y+e_i}(p_0;a).\end{aligned}$$

The best-predictor branch is open at zero by zero cost and positive cutoff. Every connector is near one and the weighted connector average equals one at zero. Near zero the actual predictor is focus of this analytic average, whatever fallback tag was supplied. Its derivative is the weighted sum of these Lie-algebra connector derivatives, unchanged by the focus projection. Expanding tap and the original rooted path and probe definitions gives, coordinate by coordinate,

$$\sum_p \mathrm{weight}_p\, \mathrm{bill}_p(a) = \mathrm{probe}(a).$$

Weights sum to one, retaining both center subtractions. Thus the predictor is identity at zero and its literal derivative is $\mathrm{colorMat}(\mathrm{probe}(a))$. The local analytic frame equation and its normalization are the inputs to this differentiation; the subtraction shows explicitly why the derivative is independent of the frame tag. The horizontal derivative formula proved for the lift recovers precisely this rooted probe, rather than an unrooted or independently averaged substitute.

12.10 Uniform predictor linearization and entry bounds

For fixed patch/kernel let S_{pred} contain every ordered interpatch route, its translated endpoint copy and every length-L connector. Equal fields on its rooted links have equal path products, axial/via frames and scores; uniqueness identifies the minimizing frames, while the fallback uses the equal axial frames. Thus actual predictors agree with the original endpoint and connector tags. The maximum route distance of these finitely many roots is predictionReach; agreement on that parent-distance hull suffices.

At fixed period/tags the noise field

$$Z_k(a,b) = \sqrt{\mathrm{penalty}}\ \mathrm{curvature}(\exp b\ \mathrm{prediction}(\exp a, k)^{-1})$$

is analytic near zero, vanishes there and has derivative $z_{\mathrm{lin}} = \sqrt{\mathrm{penalty}}(b - \mathrm{probe}(a))$. This follows from the actual predictor derivative, skew-adjoint inverse derivative and curvature's linear color reading. Taylor gives a linear bound and quadratic remainder. There are finitely many tags and bond positions, so take the maximum of their constants and the positive minimum of their radii on that fixed torus.

For volume uniformity choose a reference coarse side $c_0 = 2M + 2 > 1$ large enough to inject the stencil, with M its largest absolute coordinate. At an input bond root x, transplant reference fine values by

$$y \longmapsto \mathrm{wrap}_f(\mathrm{lift}(y - \mathrm{embed}(0))) + \mathrm{embed}(x).$$

Then set the coarse reference field constantly to the original bond value. Norms do not increase. Copy original endpoint tags to distinct reference sites $0, e_i$ and its connector tag to their bond. All stencil reads agree for every scalar multiple of the inputs, including repetitions identified by a short original torus. Values agree at one and derivatives agree at zero. The reference Taylor bounds therefore give volume/tag-independent C_z, r_z such that

$$\|z_{\mathrm{lin}}(a,b)\| \le C_z \max(\|a\|,\|b\|), \quad \|Z_k - z_{\mathrm{lin}}\| \le C_z \max(\|a\|,\|b\|)^2$$

inside radius r_z. This uses a copied actual branch, not a new minimizer.

The full entry is plotted fine moment direct-summed with Z, and its linear entry is $x_{\mathrm{lin}} \oplus z_{\mathrm{lin}}$. For logarithmic inputs bounded by H, combine the plotted comparison already proved with the preceding inequality for Z: set $M = (C_x + C_z)H$ and $W = C_x H \,\mathrm{widen}(R, H, \mathrm{width}_{\mathrm{top}}) + C_z H^2$. Then each linear entry has norm at most M and each entry-minus-linear-entry has norm at most W, simultaneously in the two kinds of row. For H of rate $-1/2$ and $R = \mathrm{till}(x)$ they have rates $-1/2, -1$. Predictor covariance and curvature conjugation give gauge covariance, and the union of cutoff stencil, predictor reach and coarse root link gives protected-hull locality. These establish the entry-near and entry-locality inputs used next.

A curl costs at most $4\|v\|$; path sums cost $4\,\mathrm{paths}\,\|v\|$ with omitted tail $4\,\mathrm{paths}\,\|v\| e^{-\mathrm{width}_{\mathrm{top}} R}$. Direct and noise rows cost $4\sqrt{\mathrm{direct}}\|v\|$ and $C_z\|v\|$. Thus $U_{\mathrm{st}} = 4(\mathrm{paths} + \sqrt{\mathrm{direct}}) + C_z$, $K_{\mathrm{st}} = U_{\mathrm{st}} + 4\,\mathrm{paths}$ bound full/cutoff rows and their exponentially small difference at every cutoff. The cutoff bound follows by comparison with the full row and $e^{-\mathrm{width}_{\mathrm{top}} R} \le 1$. Absolute summability licenses subtraction. A single-link impulse at parent distance g has zero cutoff reading if $s +$ predictionReach $+4 < g$. For $g >$ predictionReach $+5$, choosing $s = g -$ predictionReach -5 and comparing both cutoffs to the full row yields

$$e^{\mathrm{width}_{\mathrm{top}}\, g} \|\overline{\mathrm{Field}}_R(a_{\mathrm{single}})\| \le 2K_{\mathrm{st}} e^{\mathrm{width}_{\mathrm{top}}(\mathrm{predictionReach}+6)} \|a\|.$$

If R is too short the cutoff row is already zero. Nearby links use the uniform row bound. These bounds also give the spatial impulse estimate used in the plate prices. The lift copies every repeated source value and tag on this stencil, and hence retains both the volume-uniform estimate and its actual locality domain.

12.11 Shell comparison and the tree-slice floor

A differentiable group curve with $U(0) = 1$, $\dot U(0) = \mathrm{colorMat}(a)$ satisfies $\mathrm{chord}(U(t))/|t| \to \|a\|$ because chord is the Frobenius distance from identity divided by colorScale and $\|\,\mathrm{colorMat}(a)\| = \mathrm{colorScale}\,\|a\|$. If w vanishes on forest trunks, $\exp(tw)$ remains on the exact tree slice. Near zero the actual frame branch has positive mass and usable tags. Its nonlinear shell comparison with coarse field one gives

$$\mathrm{chord}(U(t)_e) \le C\,\mathrm{port}(\mathrm{parent}(e), U(t)) + \mathrm{chord}(Q(t)_{\mathrm{parent}(e),\mathrm{axis}(e)}).$$

This upper bound is valid in the internal-block case as well, where adding the last nonnegative term is harmless. Bound the port by the full macro-box plaquette sum, retaining periodic repetitions, divide by $|t|$ and take the finite-sum limit. Plaquette and predictor derivatives give

$$\|w_e\| \le C \sum_{z \in \mathrm{box}(\mathrm{macroR})} \sum_p \|\,\mathrm{curl}(w)_{\mathrm{wrap}\, z + \mathrm{embed}(\mathrm{parent}(e)),p}\| + \|\,\mathrm{probe}(w)_{\mathrm{parent}(e),\mathrm{axis}(e)}\|.$$

This uses the nonlinear shell comparison, not a newly chosen linear gauge.

Let $N = |\operatorname{box}(\text{macroR})|$, $m = |\text{PlaqAxes}|$, $q = L^4$. Macro-sum Cauchy–Schwarz costs Nm. For each fixed offset z, the map $(x,p) \mapsto (\operatorname{wrap} z + \operatorname{embed}(x), p)$ from coarse site and plaquette type is injective, so summing squares costs at most the full fine sum; summing offsets adds N. Each coarse site has q children and four outgoing fine axes. Square the local link estimate using $(a+b)^2 \le 2a^2 + 2b^2$. Therefore

$$\sum_e \|w_e\|^2 \le \beta \sum_p \|\operatorname{curl}(w)_p\|^2 + \gamma \sum_b \|\operatorname{probe}(w)_b\|^2, \quad \beta = 8C^2 q N^2 m, \quad \gamma = 2q.$$

Different offsets need not be injective against each other. If the fine bed has curl floor $a > 0$ and penalty $\kappa > 0$, its vertical tree-chart Hessian dominates $a \sum \|\operatorname{curl} w\|^2 + \kappa \sum \|\operatorname{probe} w\|^2$. Linear inclusion identifies this Hessian with the full germ on the collected field zero on trunks. Hence

$$\eta = \frac{\min(a,\kappa)}{1+\beta+\gamma} > 0$$

gives the floor $\eta \sum_{\text{free}} \|v_e\|^2$, since $\eta\beta \le a$ and $\eta\gamma \le \kappa$. This unfolds the fixed-block counting-coordinate Hessian floor. The nonlinear shell comparison supplying C remains the stated input. Here "vertical" is the source's free tree-chart variation, not the additional frame-fibre kernel of the source–readout realization. Keeping those spaces distinct preserves both this positive floor and the exact redundant-frame null directions.

12.12 Uniform cutoffs, continuity and minimizer uniqueness

The finite-pair small-transport constant $d > 0$ depends only on G. Choose frame cutoff at most $d/\operatorname{colorScale}$. On an all-minimum patch use its center axial frame a and pair transports $W_{xyk} = a_x \operatorname{travel}(x \to y, k) a_y^{-1}$. Their matrix distance from identity is at most d. Transforming a competing frame by $b_x a_x^{-1}$ identifies pair cost and original frame score. Every competitor can be left normalized without changing cost. Compactness of the finite product of G with its center fixed gives a normalized minimizer, pair uniqueness gives uniqueness modulo common left multiplication, and center value one removes it. Thus the cutoff is independent of patch order and size.

For real weights w_i summing to one and $m = \sum_i w_i \operatorname{mat}(f_i)$, completion of the square gives

$$2\operatorname{repDim} \sum_i w_i \operatorname{price}(f_i u^{-1}) = \sum_i w_i \|\operatorname{mat}(f_i)\|^2 - \|m\|^2 + \|m - \operatorname{mat}(u)\|^2.$$

Hence predictor minimization is nearest-point minimization. For nonnegative weights the squared-norm inequality for a weighted average gives the explicit bound

$$\|m - \operatorname{mat}(u)\|^2 \le 2\operatorname{repDim} \sum_i w_i \operatorname{price}(f_i u^{-1}).$$

If this right side is strictly below r^2, the centroid lies in the nearest-point tube and has a unique nearest point. Cutoff at most $r^2/(2\operatorname{repDim})$, where r is the focus tube radius, ensures uniqueness; this choice also depends only on G.

A closed minimizer relation $M \subseteq X \times Y$ with compact Y has continuous arbitrary selections at each singleton fibre: for an open neighborhood V of its unique value, the projection of $M \cap (X \times V^c)$ is closed and omits that input. Its complement forces every selected minimum into V. For frames this relation is the intersection of the closed center-normalization condition and the score inequalities against all normalized competitors; each score is a continuous finite path expression. Predictor relations use the corresponding cost inequalities. Strict finite cutoff tests make the branches open. This supplies the continuity used in the analytic-frame theorem.

For a finite pair family set $\mathrm{term}(h;x,y,k) = \mathrm{price}(h_x W_{xyk} h_y^{-1})$ and average over the three indices for the cost. For self-consistency, reversal exchanges x, y and applies the mirror permutation to k, preserving the sum. Thus reversal symmetry pairs the off-diagonal terms when one frame coordinate h_x is varied; diagonal terms vanish because $W_{xxk} = 1$. Varying only h_x changes the total cost by twice the off-diagonal row change. Global minimality therefore minimizes the off-diagonal row. Adding the frozen self term, nonnegative and zero at h_x, proves whole-row minimality. The normalized whole row is the weighted squared-distance problem to $h_y W_{xyk}^{-1}$, up to a positive constant and a competitor-independent term. Completing its square identifies h_x as nearest to $\mathrm{feedback}(h)_x$.

If $\|W_{xyk} - 1\| \le d$, identity-array comparison bounds mean squared pair separation by $4d^2$. After normalizing so identity is nearest to its centroid, a suitable index gives centroid distance at most $2d$, and feedback differs from the centroid by at most d, hence lies within $3d$ of identity. Choose the focus two-point radius δ for error $1/4$ and $d = \min(\delta/4, 1/4)$. Multiply each of two minimizing arrays h, k on the left by the inverse of a nearest point to its centroid. This preserves minimization and makes identity nearest to both centroids. With $u = \max_x \|h_x - k_x\|$, their feedback difference B at a maximizing index obeys $\|B\| \le u$. Subtracting centroid difference gives error at most ud by factoring each term as $(h_y - k_y)(W_{xyk}^{-1} - 1)$. Both centroid-minus-identity vectors are normal, so $\|\operatorname{proj} B\| \le ud$. Since $3d < \delta$, self-consistency and the focus estimate apply to both feedbacks and imply

$$u \le \|\operatorname{proj} B\| + \tfrac{1}{4}\|B\| \le ud + u/4 \le u/2.$$

Thus $u = 0$. Undoing normalization proves common-left-multiple uniqueness; center normalization makes it literal uniqueness. The source lift retains this group freedom separately from its redundant presentation frames and does not extend the singleton-fibre claims to arbitrary nonunique branches.

12.13 Finite-pair alignment and route census

For reversal-symmetric pair transports W_{xyk} and a minimizing array h, put $\mathrm{income}(x) = \mathrm{average}_{y,k}\,\mathrm{chord}(W_{xyk})$ and $m = \mathrm{average}_x\,\mathrm{income}(x)$. Identity-array comparison bounds minimum price by m, since chord squared is at most twice chord. Factoring the pair separation through its discrepancy and squaring gives

$$\mathrm{average}_{x,y}\,\|h_x - h_y\|_{\mathrm{F}}^2 \le 8\,\mathrm{colorScale}^2\,m.$$

Indeed the squared triangle inequality bounds each squared pair chord by twice the squared discrepancy plus $4\,\mathrm{chord}(W_{xyk})$; averaging with minimality gives the factor eight. Set

$$\mathrm{centroid}(h) = \mathrm{average}_x\,\mathrm{mat}(h_x),$$
$$\mathrm{feedback}(h)_x = \mathrm{average}_{y,k}\,\mathrm{mat}(h_y W_{xyk}^{-1}).$$

Direct subtraction gives $\|\mathrm{feedback}(h)_x - \mathrm{centroid}(h)\|_{\mathrm{F}} \le \mathrm{colorScale}\,\mathrm{income}(x)$. By self-consistency each h_x is nearest to its feedback. Choose a nearest centroid point b and replace h by $b^{-1}h$. Identity is then nearest to the centroid, whose distance from identity is at most $\sqrt{8\,\mathrm{colorScale}^2\,m}$, using a suitable index and the squared-average inequality.

Choose the focus comparison radius δ for error coefficient one, $\sigma = (\delta/2)^2/(8\,\mathrm{colorScale}^2)$ and $a = \delta/(2\,\mathrm{colorScale})$. If $m < \sigma$ and income is below a, both feedback and centroid lie in the comparison domain. The centroid lies within $\delta/2$ of identity and feedback within δ. The focus comparison and projection norm at most one give

$$\|\mathrm{mat}(h_x) - I\| \le 2\|\mathrm{feedback}(h)_x - \mathrm{centroid}(h)\|,$$

and hence $\operatorname{chord}(h_x) \le 2\,\operatorname{income}(x)$. Remaining indices use $2 \le (2/a)\,\operatorname{income}(x)$. Averaging these two index classes gives the constant $\max(2, 2/a)$. If $m \ge \sigma$, choose $b = 1$; the universal chord bound gives $2 \le (2/\sigma)m$. Undoing the common normalization and using conjugation invariance therefore gives an aligning element b with

$$\operatorname{average}_x \operatorname{chord}(h_x b^{-1}) \le C_{\mathrm{align}} m, \qquad C_{\mathrm{align}} = \max(2/\sigma, 2, 2/a).$$

No individual-transport smallness is required for this averaged statement.

For a radius-r box of N sites, replacing selected coordinates of x by those of y gives $d_p(x, y)$. Swapping those coordinates is an endpoint-pair involution, hence $\sum_{x,y} f(d_p(x, y)) = N \sum_x f(x)$. Swapping a site coordinate with an independent height likewise gives

$$\sum_{x,\mathrm{height}} f(\operatorname{update}_j(x, \mathrm{height})) = (2r + 1) \sum_x f(x).$$

A positive straight segment's chord is bounded by its whole internal line sum, by the product triangle inequality and then enlargement to the full line; for a negative segment reverse the walk and use inversion invariance. Appending an axis adds this line cost at $d_p(x, y)$. The two census identities bound its total by $N(2r + 1)\,\operatorname{edgeSum}(U)$. Induction yields $\operatorname{length}(p)N(2r + 1)\,\operatorname{edgeSum}(U)$, with constant four for all axes. The source lift retains these endpoint-pair sums, roots and route multiplicities, so the averaged alignment is not replaced by an unsupported pointwise bound.

12.14 Predictor queries and synchronization

For connector values Y_p with nonnegative weights summing to one, take $q = xy^{-1}$ and $\tau = \sum_p \operatorname{weight}_p \operatorname{chord}(x^{-1} Y_p y)$. Cyclic invariance identifies it with the discrepancy from q. For $m = \sum_p \operatorname{weight}_p \operatorname{mat}(Y_p)$, one has $\|m - \operatorname{mat}(q)\| \le \operatorname{colorScale}\,\tau$. In the best branch its output b is nearest to m, so

$$\|\operatorname{mat}(b) - \operatorname{mat}(q)\| \le \|\operatorname{mat}(b) - m\| + \|m - \operatorname{mat}(q)\| \le 2\,\operatorname{colorScale}\,\tau.$$

In the fallback branch its minimum price is at least predictorCut; since $\text{price} = \text{chord}^2/2 \le \text{chord}$, comparison with q gives $\tau \ge$ predictorCut. Since $0 \le \text{chord} \le 2$, the universal chord bound therefore proves in both branches

$$\operatorname{chord}(x^{-1} \operatorname{predict}(Y) y) \le 2 \max(1, \text{predictorCut}^{-1})\tau.$$

This includes the actual fallback output.

The radius-r route census gives $\sum_{p,q} \operatorname{chord}(\operatorname{travel}(p \to q; o)) \le 4N(2r + 1)\,\operatorname{edgeSum}(U)$ and hence $N\,\operatorname{meanPairChord} \le 4(2r + 1)\,\operatorname{edgeSum}(U)$; the census follows by charging each coordinate segment to its internal positive edges, as in the route-count argument above. In the minimum frame branch, after gauge H, the array $b_p = \operatorname{frame}_p H_p^{-1}$ minimizes the corresponding pair energy; gauge covariance and center normalization preserve competitors. Alignment gives c with $\operatorname{average}_p \operatorname{chord}(b_p c^{-1}) \le C\,\operatorname{meanPairChord}$. Cyclic invariance and the route bound then give

$$\sum_p \operatorname{chord}(c^{-1} \operatorname{frame}_p H_p^{-1}) \le 4C(2r + 1)\,\operatorname{edgeSum}(H \cdot U).$$

In the fallback branch positive frame mass makes the axial tag eligible. Its normalized route gauge d has edge sum at most $\text{axialFactor}(2r + 1)\,\operatorname{faceSum}(U)$. Set $g = dH^{-1}$. Factoring links gives

$\mathrm{edgeSum}(g \cdot 1) \le \mathrm{edgeSum}(d \cdot U) + \mathrm{edgeSum}(H \cdot U)$. Pure-gauge travel is $g_p g_q^{-1}$. Choose a root q whose endpoint sum is at most the average, and apply the route bound. With $c = g_q$ this gives

$$\sum_p \mathrm{chord}(c^{-1}\,\mathrm{frame}_p\, H_p^{-1}) \le 4(2r+1)\big[\,\mathrm{edgeSum}(H \cdot U) + \mathrm{axialFactor}(2r+1)\,\mathrm{faceSum}(U)\big].$$

Thus $A = \max(4C, 4)$ pays both synchronization branches, using the finite-pair cluster inequality and route census as its two inputs.

The patch weight is the distribution of J independent uniform small-box rolls. Conditional on all but the first roll, at most one first roll yields any specified sum. Its box has L^4 points, so $L^4\,\mathrm{weight}_p \le 1$. Coincident roll sums are not assumed distinct. The source lift retains this probability census and actual fallback tags alongside the connector readout; no successful minimizing branch replaces a fallback contribution.

12.15 One macro gauge and predictor price

The planting argument uses the axial estimate already proved and the following frame synchronization inequality. Synchronization on a usable radius-r patch gives a constant p with frame-error sum at most $A(2r+1)[\mathrm{edgeSum}(H \cdot U) + \mathrm{axialFactor}(2r+1)\,\mathrm{faceSum}(U)]$. The constant A depends only on the group. For smoothing index j, put $\ell_0 = 2(j+7)+1$, $D = 16\ell_0$ and $C = A\ell_0(D + \mathrm{axialFactor}\,\ell_0)$. The macro radius is patch radius plus $4L$, so $2\,\mathrm{macroR} + 1 \le \ell_0 L$. Apply the four-axis axial construction to the whole macro box, giving one gauge H with edge sum at most DL port. All patches centered at Ly, $y \in [-2,2]^4$, lie there. Nonnegativity bounds their sums by the macro sums. Applying synchronization to each with that same H gives p_y and error sum at most CL^2 port.

For adjacent roots, a connector factors into left frame error, length-L transport in H and inverse right error. If frame-error sums are at most KL^2 port and macro edge sum at most TL port, the sum of transport chords over starting points is at most L times that edge sum. Each step translation is injective and stays in the macro box; repeated torus edges still count as separate steps. The weight bound $L^4\,\mathrm{weight}_p \le 1$ gives $L^4\tau \le L^2(2K+T)$ port. The query estimate bounds the chord by $2\max(1, \mathrm{predictorCut}^{-1})\tau$. Multiply by L^2 and use $L > 0$ to obtain

$$L^2\,\mathrm{chord}(p_y^{-1} Q_{y,i} p_{y+e_i}) \le 2\max(1, \mathrm{predictorCut}^{-1})(2K+T)\,\mathrm{port}\,.$$

For $\alpha = 2\max(1, \mathrm{predictorCut}^{-1})(2C+D)$, each gauged predictor link costs $\alpha\,\mathrm{port}\,/L^2$. Endpoint factors cancel around the coarse plaquette, yielding $\mathrm{chord}(\mathrm{face}\,Q) \le 4\alpha\,\mathrm{port}\,/L^2$. For any V, the factorization inequality

$$\mathrm{chord}(ab(cd)^{-1}) \le \mathrm{chord}(ac^{-1}) + \mathrm{chord}(bd^{-1})$$

follows by factoring into one discrepancy and a conjugate of the other. Iteration with inversion invariance on the reversed links gives $\mathrm{chord}(\mathrm{face}\,V) \le \mathrm{chord}(\mathrm{face}\,Q) + \mathrm{fuel}(Q, V)$, where fuel retains all four wire discrepancies. The lift uses one common macro witness and these same endpoint gauges; independent favorable gauges on separate patches would not establish this joint cancellation.

12.16 Nonlinear shell-link comparison

On the actual forest slice close a fine link by the reverse tree hike from its first coarse root and the forward hike to its second coarse root. Both hike transports are identity, so this walk reads $U(x,i)$, has

length at most $8L+1$, and has the two embedded coarse parents as endpoints. Lift the whole walk to the cover starting at zero over its first root; every point lies in the same macro box. Planting there supplies H, p_y with edge sum at most DL port and frame-error sum at most CL^2 port. At the distinguished patch point the frame is one, giving $\mathrm{chord}(H(\mathrm{centerRoot}(y))p_y) \le CL^2$ port. The adjacent-root predictor bound is $\mathrm{chord}(p_0^{-1}Qp_{e_i}) \le \alpha L^{-2}$ port, where $\alpha = 2\max(1, \mathrm{predictorCut}^{-1})(2C+D)$.

The triangle inequality along all occurrences of the lifted walk bounds its gauged chord by at most u port, $u = (8L+1)DL$. The period condition $100\,\mathrm{macroR} \le \mathrm{fineSide}$ makes wrapping injective on the macro box. If coarse parents agree, the lifted endpoint is zero, the replica comparison link is identity and conjugation gives the bound directly. If parents differ, the centered parent-hop identity makes the second parent the positive coarse neighbor, and injectivity gives the endpoint $\mathrm{centerRoot}(e_i)$, including across a periodic seam. Factoring the predictor through the same p_0, p_{e_i} bounds its gauged chord by v port, $v = 2CL^2 + \alpha/L^2$. Compare the fine link and predictor in this same macro gauge and use the triangle inequality. Therefore

$$\mathrm{chord}(U(x,i)Q^{-1}) \le (u+v)\,\mathrm{port}\,.$$

Comparison to the desired coarse link V adds exactly its wire discrepancy with coefficient one. The constant is uniform in period and forest on the room domain. The room hypotheses supply its size assumptions, so the same constant is available in the local driving inequality. The source lift retains that exact forest slice, macro gauge and wrapping hypothesis, so the shell estimate used by local driving has the same domain and coefficient; it is not inferred from separate local gauges at the walk endpoints. Explicitly, let $\mathcal{D}$ be this forest-slice room domain and let $f(U,V)$ be the left-hand discrepancy, with the replica interpretation when both parents agree. The recovered scalar is $\widetilde{f} = f \circ \pi$ on $\pi^{-1}(\mathcal{D})$, and the port and wire budgets are pulled back by the same map. Thus the bound pulls back pointwise with the identical constants $u+v$ and one. Evaluation at the section σ recovers the original inequality because $\pi\sigma = I$. Endpoint frame factors on the lifted walk cancel internally and leave precisely its endpoint transforms; their chord is unchanged by the corresponding unitary conjugation. This establishes recovery with the same periodic injectivity and forest hypotheses.

12.17 Local driving from selected motions

Let $\mathrm{lens}(u)$ be the active fine tangent. At coarse o, current is the square root of the sum of squared lens norms within the radius-$3B$ fine-link neighborhood, and response is the square root of the sum of squared selected-motion norms within radius $\mathrm{macroR}+2$. Nonnegativity of all squared terms bounds each summand norm by the corresponding root sum. For a hot link e with parent o, a gauge z making background chords at most h on radius $2B$ permits the transported-to-flat comparison. Hotness puts e sufficiently far from the seed set, and the room radius keeps this entire row-root neighborhood clear, so every required row is selected in identity orientation and its motion norm is bounded by response. All required tangent values lie in current's radius-$3B$ neighborhood. Gauge covariance and norm preservation therefore give each relevant row obeys

$$\|\overline{\mathrm{Field}}_R(\mathrm{Ad}(z)\,\mathrm{lens}(u))\| \le E_o,$$

$$E_o = \mathrm{response}(o,u) + 2K_{\mathrm{st}}\,\mathrm{colorScale}(B+1)h\,\mathrm{current}(o,u).$$

Let K be the common local comparison constant for the actual predictor tags, supplied by the gauge-flattening/tree-slice construction. Its link estimate bounds $\|w_e\|$ by K times the macro-box sum of curl norms plus the predictor-bill norm. Apply it to $w = \mathrm{Ad}(z)\,\mathrm{lens}(u)$, zero on trunks

because trunks are not hot. Direct rows bound each curl by $E_o/\sqrt{\text{direct}}$ and noise bounds the predictor bill by $E_o/\sqrt{\text{penalty}}$. All macro roots occur in the selected neighborhood. Thus

$$\|\operatorname{lens}(u)_e\| \le ZE_o, \qquad Z = \frac{K|\operatorname{box}(\text{macroR})||\text{PlaqAxes}|}{\sqrt{\text{direct}}} + \frac{1}{\sqrt{\text{penalty}}}.$$

A clean-patch flattening gauge supplies rate-$-1/2$ h with its radius factors included, paying the squared-neighborhood absorption in the plate-floor proof. Tag uniformity comes from the actual tree-slice estimate.

12.18 Transported affine rows

Let $c_x(V;y)$ be transport along the coordinate-order route from x to its centered displacement toward y. It ends at y and has length at most $4\,\text{separation}(x,y)$ because each of the four centered coordinate lengths is bounded by that separation. Spin each tangent at parent y by $\operatorname{Ad}(c_x(V;y))$, then apply the cutoff flat row at x. The signed row first applies the actual replica tangent transformation, including the background adjoint and minus sign on reversed links. These transformations preserve each color norm and hence the K_{st} row bound; at $V = 1$ the straightened row is the flat row.

Carrier covariance is $c'_x = z(x)c_x z(y)^{-1}$, so input and carrier endpoint factors cancel and leave only $\operatorname{Ad}(z(x))$. The flat row commutes with this common action. The signed tangent transformation commutes with gauge change after pulling the gauge to the signed sites, giving the corresponding adjoint at the signed row root. A carrier to y lies in the queried radius-d hull when $4\,\text{separation}(x,y) \le d$. On that hull take $h \ge 0$. Then replacing exterior links by identity leaves the carrier unchanged. If its chords are at most h, carrier chord is at most dh. Truncating a tangent difference to the required radius-$R+\text{predictionReach}+4$ stencil similarly gives row difference at most $K_{\text{st}}M$ for local input error M.

For input bounded by M on the required stencil, every carrier changes it by at most $2\,\text{colorScale}\,dhM$. Thus if $4(R+\text{predictionReach}+4) \le d$,

$$\|\operatorname{straightened}_R(V,u) - \overline{\text{Field}}_R(u)\| \le 2K_{\text{st}}\,\text{colorScale}\,dhM.$$

The reversed-link tangent change at background V differs from its value at identity by at most $2\,\text{colorScale}\,hM$ and shifts its root by one. Radius-$d+1$ original data supply radius-d bounds after turning; summing reversal and carrier errors gives

$$\|\operatorname{orientedRow}_R(V,u) - \operatorname{orientedRow}_R(1,u)\| \le 2K_{\text{st}}\,\text{colorScale}(d+1)hM.$$

For a single-link input, signed transport and carrier spin preserve its single-link norm, while parent distance shifts by at most one. The impulse bound is therefore $2K_{\text{st}}e^{\text{width}_{\text{top}}(\text{predictionReach}+7)}\|a\|$ after multiplication by the original exponential distance weight, without smallness of V. The room condition gives locality at till using a half hull before turning. The source lift preserves these actual tangent transformations and route endpoints, not a bare signed permutation.

12.19 Affine accuracy on the protected stencil

Grafting on a radius-d hull means actual agreement $U = \exp(u)\,\text{replica}(V)$ there with $\|u_i\| \le H$. Gauge changes preserve it with adjoint-transformed u; signed changes retain it on radius $d-1$ because reversed stored roots shift by one. Smaller hulls and larger bounds preserve it.

At a row use $d = B/4$, $R = \mathrm{till}(x)$ and $V = \exp b$ on the hull, $\|b\| \le H$. Extend u by zero outside to v, and put $a = \mathrm{linearCopy}(b)$, both bounded by H. For eventually small H, the logarithmic product gives linkwise

$$\exp w = \exp v \exp a, \quad \|w - v - a\| \le JH^2, \quad \|w\| \le (1 + J)H.$$

Replica exponential equality is exact. Entry locality identifies actual C with $\mathrm{entry}(\exp w, \exp b)$ and background Y with $\mathrm{entry}(\exp a, \exp b)$. Let E_0 bound both logarithmic-entry errors against their full linear costs, and let M_0 bound the full linear cost of w, using the uniform entry-near estimates at enlarged envelope $(1 + J)H$. Put $z = \mathrm{linearCost}(w, b)$, $z_0 = \mathrm{linearCost}(a, b)$. Then $\|C\| \le M_0 + E_0$, and the exact five-term identity is

$$\begin{aligned} C - Y - \mathrm{straightened}(V, u) = {} & (C - z) + \mathrm{uncut}(w - v - a) + (z_0 - Y) \\ & + [\mathrm{uncut}(v) - \mathrm{truncated}(v)] \\ & + [\mathrm{truncated}(v) - \mathrm{straightened}(V, u)]. \end{aligned}$$

The respective bounds are E_0, q_1, E_0, q_2, q_3, with common stencil constant K and exponential-chart constant A,

$$q_1 = KJH^2, \qquad q_2 = KHe^{-\,\mathrm{width}\,R}, \qquad q_3 = 2K\,\mathrm{colorScale}\,BAH^2.$$

They use, in order, the full stencil bound, cutoff tail and transported-row comparison with chord at most AH. Locality removes exterior u before any estimate. The room inequality $4(R + \mathrm{predictionReach} + 4) \le d$ contains every input. Thus $E_{\mathrm{aff}} = 2E_0 + q_1 + q_2 + q_3$ also bounds $\|Y - \mathrm{linearCost}(a, b)\|$. For rate-$-1/2$ H, entry-near gives E_0 rate -1 and M_0 rate $-1/2$. The quadratic, cutoff and radius-weighted quadratic terms each have rate -1 under the stated radius/cutoff controls. Thus $M_0 + E_0$ has rate $-1/2$ and E_{aff} rate -1. The lift preserves all five terms and this local grafting contract; horizontal recovery does not license dropping a background transport error or reading unbounded exterior tangents.

12.20 Protected mean: stationary comparison field

Fix scale, geometry, selected rows and forest. In counting coordinates let A be the plate, D its selected linear form, b its bill and S the dormant diagonal. The exact pricing identity is $Av + b = D^\top \mathrm{priced}(v) + Sv$. Coordinates are expressed in the fixed orthonormal color basis. Field norms here are supremum norms; color norms are bounded by the sum of absolute color coordinates. A common Y bounds row sums of $D, D^\top$, with reach $B/4$: the matrix entries of D vanish beyond distance $B/4$.

A clean radius-$(d + 1)$ hull, $d \le 32B$, admits an axial gauge with radius-d link chord bound

$$u(x) = 16(2(32B(x) + 1) + 1)^5 |\mathrm{PlaqAxes}|\, 2\,\mathrm{fiery}(G, x, L).$$

The axial-patch bound uses the clean plaquette toll and room temperature comparison. The local logarithm extension then gives a bounded global a agreeing there with the gauged background and $\|a\| \le w = Cu$, of rate $-1/2$. For the actual chart product $h(p, q) = \log(\exp p \exp q)$, inverse-chart differentiation gives $Dh(0)(p, q) = p + q$. Analytic Taylor bounds on a fixed ball give exact exponential recovery, error $K \max(\|p\|, \|q\|)^2$ and size $K \max(\|p\|, \|q\|)$, linkwise. This is not an identification of group elements with tangent vectors. Applying this link by link proves the same uniform bounds for the logarithm of two successive increments.

The linear forest response spring has zero trunk values and coordinate bound on unit input

$$C_0 = \text{Response. total}(j, \eta, C) + e^{\eta}.$$

Sum the coordinate bound over colors on unit input, rescale by $\|a\|^{-1}$ for $a \neq 0$, and use linearity for $a = 0$. This gives $\|\operatorname{spring}(a)\| \leq C_s\|a\|$, where

$$C_s = \max(1, \max(0, C_0) \dim \text{Color}).$$

Signed changes preserve this bound and commute with linearCopy; the latter follows either from the copy formula or by differentiating the exact identity for its exponentiated replicas. For the same axial gauge z, let $v = \operatorname{Ad}(z^{-1}) \operatorname{spring}(a)$. It remains zero on trunks with size at most $C_s w$. Gauge covariance identifies the selected path, away from seeds, with request in the gauged background plus its actual oriented row on spring.

On the common clean hull the exact stationary cancellation is

$$\text{linearCost}(\text{linearCopy}(a), a) - \text{linearGuide}(a) + \text{uncutField}(\text{spring}(a)) = 0.$$

Indeed the stationary shifted field is linearCopy plus spring, and linearity gives its defining guide. Subtracting this identity leaves four errors: base-stroke linearization, guide linearization, background-oriented versus flat truncated row, and truncated versus uncut row. The first two envelopes A_1, A_2 have rate -1 from the earlier affine/guide estimates. The explicit remaining bounds and the padded-inverse mean calculation follow below. The source lift uses this same gauge, forest and hull for every row and column; the stationary cancellation is performed before norms or coordinate reduction. The last two stationary comparison errors are

$$U_1 = 2K_{\text{st}} \text{ colorScale}(B+1) C_{\text{ch}} C_s w^2, \qquad U_2 = K_{\text{st}} C_s w e^{-\text{ width till}(x)}.$$

Here C_{ch} bounds chord$(\exp a)$ by $C_{\text{ch}}\|a\|$. The first error has rate -1 because it is quadratic in w with only the radius factor; the second has rate -1 by cutoff decay. Thus $A_0 = A_1 + A_2 + U_1 + U_2$ has rate -1 and bounds each path of v outside the $20B$ seed neighborhood on its common radius-$2B$ agreement hull.

Let P be the nonsymmetric inverse-column pad, $E = AP - I$ and $\mu = -P^{\top} b$. Symmetry of A gives $E^{\top} = P^{\top} A - I$, and therefore proves exactly

$$\mu - v = E^{\top} v - P^{\top}(Av + b).$$

Only column window i of P is read at coordinate i. If the residual is bounded there by M and $\|v\| \leq W$, the proved window bounds give

$$|\mu_i - v_i| \leq \text{payE}\, M + \text{scraps}(B) W.$$

For any row, $|\sum_j M_{ij} t_j| \leq K \sum_j |M_{ij}|$ when $|t_j| \leq K$ on its nonzero entries. This uses no residual bound outside that support. Outside the $11B$ seed neighborhood $Sv = 0$: trunks have zero comparison field and every other coordinate is active. Pricing and the transpose row price then give $|(Av + b)_i| \leq Y \operatorname{scale} A_0$ when paths within distance B have the A_0 bound. Similarly local input error U gives path error at most $\dim(\text{Color}) \operatorname{scale}^{-1} YU$ on its radius-B coordinate neighborhood.

For a path root outside $30B$, flatten its clean radius-$(8B+1)$ hull on radius $8B$ and use one comparison v for all tested coordinates. Its paths within $6B$ obey A_0. All column windows needed within B lie within $3B$ and remain outside the $11B$ exclusion, so the residual bound above applies throughout each window. Therefore, with

$$W = C_s w, \quad M = Y \operatorname{scale} A_0, \quad U = \text{payE}\, M + \text{scraps}(B) W,$$

$$E_{\mathrm{mean}} = A_0 + \dim(\mathrm{Color})\,\mathrm{scale}^{-1}\,YU,$$

we have $\|\,\mathrm{realPath}_p(\mu)\| \le E_{\mathrm{mean}}$. The scraps estimate at any prescribed negative power gives rate -1 for U and then for E_{mean}. Arbitrarily negative-power scraps bounds pay the error, with eventual nonnegative envelopes. The same seed condition selects the root. This proves the protected-mean estimate used in the watch argument, with one common choice of gauge and comparison field for its entire finite stencil. The axial bound used here follows from the injective cover-box gauge, $16(2r+1)$ times face sum, bounded by $16(2r+1)^5|\mathrm{PlaqAxes}|q$ when $2r < n$ and $d+1 \le r$. The lift retains the single comparison field and the actual $P^\top$ mean; symmetrizing the covariance does not symmetrize this mean or enlarge its protected stencil.

12.21 Local normalization differences

The source condensation is $[\mathrm{joint}(A_{\mathrm{blank}}; e) - \mathrm{joint}(A; e)]/2$ on the same window. This subsection derives the remote normalization estimate used below from the local form comparison and the already established inverse-window bounds. Blank geometry retains forest, scale, windows and tags, with identity background and empty seeds; nontrunks are hot and all rows selected.

For positive plates on a common window U, put $f_A = A_U - I$, $g_A(s) = I + s(A_U - I)$ for $0 \le s \le 1$. The intensity is the distinguished diagonal of $f_A g_A^{-1}$. If absolute entries of $A - Z$ there are at most $e_0 \ge 0$, its census $|U| \le \Delta = \mathrm{ballCost}(B+2)m$ bounds the difference row sums by $d = \Delta e_0$. Suppose interpolation inverse row sums are at most $C_{\mathrm{inv}} = \mathrm{payE}$ and f_Z row price at most $Q + 1$. The exact resolvent identity

$$g_Z^{-1} - g_A^{-1} = g_Z^{-1}(g_A - g_Z)g_A^{-1}$$

gives price $C_{\mathrm{inv}}^2 d$. At the same interpolation value s, use the principal-window inverses throughout and decompose the intensity difference as

$$(f_A - f_Z)g_A^{-1} - f_Z(g_Z^{-1} - g_A^{-1}).$$

This gives diagonal bound

$$\mathrm{pulseHead}(e_0) = [C_{\mathrm{inv}} + (Q+1)C_{\mathrm{inv}}^2]\Delta e_0.$$

Positivity of the interpolated principal matrices gives continuity along s, justifying subtraction of the two integrals; so integration and color summation yield $|\,\mathrm{joint}(A; e) - \mathrm{joint}(Z; e)| \le \dim(\mathrm{Color})\,\mathrm{pulseHead}(e_0)$. Each individual diagnostic is bounded by $\dim(\mathrm{Color})(Q+1)C_{\mathrm{inv}}$.

For e outside the $30B$ seeds, flatten once on radius $6B$. Single-coordinate impulses compare the gauged and blank forms: trunk columns vanish in both, nearby nontrunk columns are hot in both, rows farther than $B/4$ vanish in both, and all remaining rows are selected by the seed margin. The transported-row bound gives entry error $\mathrm{swell} = 2\,\mathrm{scale}\,K_{\mathrm{st}}\,\mathrm{colorScale}(B+1)h_0$. Dormant diagonals agree. Expanding $ab - cd = (a-c)b + (b-d)c$ in $D^\top D$, with transpose row prices Y, gives plate entry error $\mathrm{repair} = 2\,\mathrm{swell}\,Y$ on every read window.

Block-diagonal orthogonal gauge action preserves whole-link windows, principal inverses, interpolation intensity and its color trace. Hence joint is gauge invariant. Flattening gives $h_0 = Z_{\mathrm{ch}} w$ with rate $-1/2$; radius and census factors have rate zero. Therefore $E_{\mathrm{cond}} = \dim(\mathrm{Color})\,\mathrm{pulseHead}(\mathrm{repair})$ has rate $-1/2$ and bounds remote condensation, enlarging its half factor harmlessly. The lift retains whole-color windows and their common forest/tags, so this determinant diagnostic transport is justified by the actual trace conjugation rather than entrywise gauge invariance.

12.22 Exact cut-gap locality and a reliable cutoff logarithm

For a projection in which neighboring sites have projected separation at most one, a walk of length at most R reads only its radius-R projected hull; the stored positive root of a negative link is paid by that step. The fine plotted inputs at relevant rows require at most R further travel plus their four plaquette steps; the noise input additionally queries its actual predictor through predictionReach. Induction proves equality of transport under hull agreement. A plotted slot needs $R + 4$, while cutoff F, I, N rows are banded at radius R and read their radius-R background hull. For the fine F row, lift its designated route in the replica: its endpoint projects to the coarse column and its cutoff bounds the projected separation. The lane rows use their coarse routes in the same way. Consequently a transpose reads input rows within R and its coefficients agree under radius-$2R$ background agreement. The same holds for $F^{\top} F w$ with arbitrary fixed w, since coefficients agree individually, without a locality assumption on w.

The plotted, noise-predictor and coarse terms therefore give exact lGap locality when

$$D \geq 3R + 4 + \text{predictionReach},$$

with identical actual predictor tags. The three row-square terms of dGap agree already under $2R$ background agreement, so this same D suffices for every fixed input array. These prove the local gap identities used in the chart transfer, without changing any clipping indicators in an unrelated data array.

Choose a smooth profile χ valued in $[0, 1]$, equal to one on the inner unit interval and zero beyond its prescribed outer interval. The inner-share operation evaluates it at $(\text{chord}(U)/d)^2$, a smooth squared matrix norm. Define

$$\text{cutLog}_d(U) = \chi_{\text{innerShare}}(d, \text{chord}(U)) \log(\text{mat}(U)).$$

The cutoff equals one on the inner shell and is zero beyond $2d$. For local constants C, s, analytic matrix radius t and inverse exponential color radius R, take

$$r = \tfrac{1}{2} \min(s, t/\,\text{colorScale}, R/C), \qquad \gamma = C.$$

For $0 < d \leq r$, every point with chord at most $2d$ obeys all three chart conditions and its logarithm has norm at most R. Nonzero cutoff values imply chord strictly below $2d$. The ordinary logarithm is only required to have its chart properties on this latter shell. Applying the inverse identity to $\text{Ad}(v) \log U$ and $-\log U$ proves conjugation covariance and inversion oddness there. The invariant cutoff extends both to all group arguments. Its global size is at most $2\gamma d$, and on chord at most d it is the genuine inverse chart. Smoothness follows from analytic composition on the outer shell and an identically zero neighborhood outside it; continuity implies Borel measurability. No regularity of an exterior logarithm choice is used. Readout preserves the hull, tags and squared-chord cutoff together, so these exact identities apply to the realized cut gaps and later signals on precisely the same domains.

12.23 Sector signals and a common grafted chart

Fix forest F, coarse V, tags and frozen values Y. Hot links are nontrunks whose coarse endpoints avoid the $10B$ seed dome. Input puts $\exp(u_e)\,\text{replica}(V)_e$ on hot links, frozen Y on other nontrunks and prescribed tree values on trunks. It remains on the same slice for all u. Define $\text{small}(e, U)$ as the cutoff logarithm of $U_e\,\text{replica}(V)_e^{-1}$ at radius $\text{width}\,t$. Saved is this logarithm on frozen nontrunks and zero elsewhere; lens is u on hot links and zero elsewhere. Signal is saved plus lens, hence zero on trunks.

Shell access at width equal to shellPrice makes the discrepancy chord at most width t under Interior, uniformly over forests and actual inputs. Taking x large puts width fiery in the reliable-log radius; the constants are chosen once after L. On hot links the grafting identity is definitional; on trunks it is the replica/tree identity; on frozen nontrunks the uncut logarithm recovers the discrepancy. Saved is globally bounded by 2γ width t. Thus Interior throughout a radius-B hull, hot tangent bound M and 2γ width $t \le M$ prove grafting there. Outside the $(11B+1)$ seed neighborhood, every nontrunk in that hull is hot: otherwise an endpoint touches the $10B$ dome and its parent is within one further site, contradicting the separation. Thus grafting follows directly without Interior for any nonnegative tangent bound. These are the required packet-annulus and optional remote-letter constructions, respectively.

For clean coarse background on radius $d+1$, flatten to $\exp b$ in one gauge, $\|b\| \le v(x)$ of rate $-1/2$. Grafting on radius d with rate-$-1/2$ tangent bound H transforms the fine field to $\exp(\mathrm{Ad}(g)u)$ times the replica of the gauged coarse field. Product-triangle bounds then give gauged fine chords by $D_{\mathrm{ch}}(H+v)$, because replica links are identity or coarse links. The single rate-$-1/2$ envelope $W = H + v + D_{\mathrm{ch}}H + D_{\mathrm{ch}}v$ bounds all chart data eventually, with no exterior condition. Restricting from $B/2$ to $B/4$ allows affine accuracy. The enlarged envelope W supplies both its background-log and tangent bounds. All entry, replica and straightened-row terms rotate by the same root adjoint, transporting size rate $-1/2$ and error rate -1 back to the original fields.

The identity cover field has frame, connectors and predictor all identity with strict minimum tests. Continuity on these branches, simultaneously over finite axes/tags, gives a product neighborhood for any chord tolerance. This proves the finite-stencil branch activation used in the local chart-mark argument without requiring any tags or defaults away from the finite hull. Choose the finite set I of restricted cover links and a common positive chord radius r meeting each coordinate neighborhood; chord-to-matrix conversion uses the fixed factor colorScale. Let k be the largest coordinate-order route length from zero to a root in I. Expanding any torus input at the embedded coarse root sends each required cover link to a fine link whose coarse-parent distance is at most that route length, hence at most k. Parent-distance chord bounds therefore place the expanded field in this same neighborhood, uniformly in period, with actual endpoint tags. The actual connector equals the cover test connector with its actual two endpoint tags, so its predictor meets the chosen tolerance. Every fallback tag has the same best value there, and normalized site/pin masses are one, their unnormalized values being reciprocal tag counts.

Choose eventually $k \le B/2$ and $W \le \min(r, \delta/4)$. Gauge invariance preserves masses; wire discrepancy is at most coarse chord W plus predictor chord $\delta/2$. A four-link plaquette in the unchanged hull has chord at most $4W$. Thus both actual wire and fine plaquette meet the desired small-curvature threshold. The uniform branch-cutoff and predictor estimates in this chapter give this finite-stencil estimate. The same chart proves that the actual local branch tags and holonomies lie in any prescribed small neighborhood eventually. The lift retains the same sector, frozen data and tag census; this local activation does not specify branch outcomes outside the finite hull.

12.24 Global and localized mismatch bounds

Let a cut transported row F and full flat F_0 have row/column prices at most C, difference prices Cq_0, and input comparisons $\|t_j\| \le Ch$, $\|f_j - t_j\| \le Chq_0$. The exact ordered decomposition

$$F^\top f - F_0^\top t = F^\top (f - t) + (F - F_0)^\top t$$

and the column-sum estimates give error $2C^2hq_0$ at every output coordinate. Applying the same estimate to $f = Fw$, $t = F_0 w$ gives $\|F^\top Fw - F_0^\top F_0 w\| \le 2C^2 \|w\|_\infty q_0$. Choose C to dominate the fine and coarse plotted bounds, predictor-noise bound and all three kernel bounds, intersect chart

radii and take $\tau = \min(\tau_{\mathrm{kernel}}, \mathrm{width}_{\mathrm{top}}, \mathrm{width}_{\mathrm{below}}/L) > 0$, $q_0 = \mathrm{widen}(R, h, \tau)$. Noise's quadratic remainder uses $h \le q_0$.

The uncut flat cross-roll identity cancels the lGap linear terms, leaving two transpose/input errors and one coarse plotted error; the full diagonal-roll identity cancels the three flat row squares against w. Hence

$$\| \mathrm{lGap} \| \le (4C^2 + C) h q_0, \qquad \| \mathrm{dGap}(w) \| \le 6C^2 \|w\|_\infty q_0.$$

The common $6C^2 + C$ pays both, including cutoff and transport errors for every $R \ge 0$, short periods and winding routes. The local transfer described next uses exactly these same gaps.

For local chord bounds h in one common fine/coarse gauge on radius d satisfying

$$d \ge 3R + 4 + \mathrm{predictionReach},$$

the bounded logarithm extension gives global a, b agreeing with the actual fine/coarse links there and bounded by $A_{\log} h$. Let $X = \max(1, A_{\log})$ and require $h \le \min(s_{\log}, r_{\mathrm{global}}/X)$. Exact gap locality identifies the global comparison with the actual tested gaps. Gauge covariance rotates lGap, and rotates both dGap and its arbitrary input w, preserving its independent bound q. Because $X \ge 1$ and both widening summands are nonnegative,

$$\mathrm{widen}(R, Xh, \tau) = (1 + R)Xh + e^{-\tau R} \le X \,\mathrm{widen}(R, h, \tau).$$

Thus, enlarging the global constant gives

$$\| \mathrm{lGap} \| \le C X^2 h \,\mathrm{widen}(R, h, \tau), \quad \| \mathrm{dGap}(w) \| \le C X^2 q \,\mathrm{widen}(R, h, \tau).$$

The source lift preserves this fixed arbitrary input during background replacement. In particular changing exterior links is not assumed to regenerate the same thinned data array: when w is the actual thinned data, changing the links outside the hull need not preserve that entire array. Only its separately retained supremum bound is used.

12.25 Quartic curvature deficit

For every $U \in G$, not only near identity, the Lie projection satisfies $P_{\mathfrak{g}}(\mathrm{mat}(U) - 1) = \mathrm{colorMat}(\mathrm{curvature}(U))$. Subtract the skew part $(\mathrm{mat}(U) - \mathrm{mat}(U)^*)/2$ from $\mathrm{mat}(U) - I$. The remainder is self-adjoint, orthogonal to every skew-adjoint Lie matrix by cyclicity of trace in the real Frobenius pairing. The surviving Lie projection of the skew part is the defining curvature. Pythagoras therefore gives exactly

$$\mathrm{colorScale}^2[\mathrm{chord}(U)^2 - \| \mathrm{curvature}(U) \|^2] = \| \mathrm{mat}(U) - 1 - P_{\mathfrak{g}}(\mathrm{mat}(U) - 1) \|^2.$$

Focus fixes nearby group matrices and has derivative $P_{\mathfrak{g}}$ at identity. Its ambient analytic remainder bounds the last norm by $C\| \mathrm{mat}(U) - 1 \|^2$ for that norm at most r. Squaring and restoring colorScale gives, with $K = C^2 \,\mathrm{colorScale}^2$,

$$\mathrm{chord}(U)^2 - K \,\mathrm{chord}(U)^4 \le \| \mathrm{curvature}(U) \|^2.$$

For $\delta = \min(r/\,\mathrm{colorScale}, \sqrt{1/(4K)})$, chord at most δ gives $K \,\mathrm{chord}(U)^2 \le 1/4$ and therefore curvature square at least $3/4$ chord square. Constants depend only on the group and precede block-length selection. This is a group-point restriction of an ambient analytic estimate, not a global configuration logarithm. The lift preserves both the global projection identity and the local

smallness domain under conjugation, with the same physical color normalization. For the source–readout correspondence, conjugation by a group matrix is an orthogonal action on the ambient real Frobenius space and preserves its Lie subspace. Therefore it commutes with $P_{\mathfrak{g}}$, and preserves the normal-projection norm in the exact deficit identity. The framed plaquette readout is conjugate to the physical plaquette by its base frame, so the deficit, quartic bound and chord threshold recover unchanged. Evaluation on the identity-frame section returns the physical equality. Thus the local estimate uses precisely the same group-only constants and smallness condition after realization.

12.26 Soft curvature and noise remainders

Define $\mathrm{flaw}(g) = [\mathrm{chord}(g)^2 - \|\,\mathrm{curvature}(g)\|^2]/2 \geq 0$. On a retained fine plaquette within its clipping threshold, expansion gives

$$\mathrm{crown} = \mathrm{direct\,flaw}(\mathrm{face}\,U) + \tfrac{1}{2}\sum_{i\in\mathrm{Bool}} \|\,\mathrm{entry}_{\mathrm{face},i}(U)\|^2.$$

The guide cross term and square complete both bill squares, bill being entry minus guide. Thus workF is direct times flaw plus those two half-squares. Without a retention premise, workI is penalty times $\mathrm{flaw}(VQ^{-1})$ plus its single bill half-square, with actual $Q = \mathrm{prediction}(U,k)$ and the same guide.

Affine accuracy bounds entry by M of rate $-1/2$ and bill-minus-reply by E of rate -1; subtracting the same guide leaves that affine error unchanged. The coarse maximum bounds guide by $A\,\mathrm{fiery}$, so bill has size $m = M + A\,\mathrm{fiery}$. Factoring squared norms gives

$$\left|\tfrac{1}{2}\|v\|^2 - \tfrac{1}{2}\|w\|^2\right| \leq E(2m+E)/2 =: Z$$

for $\|v\| \leq m$, $\|v-w\| \leq E$; Z has rate $-3/2$. Local branch activation puts the actual fine and noise holonomies in the quartic-deficit neighborhood. On this neighborhood the earlier bounds are $\mathrm{chord}(g)^2 - J\,\mathrm{chord}(g)^4 \leq \|\,\mathrm{curvature}(g)\|^2$ and $\frac{3}{4}\,\mathrm{chord}(g)^2 \leq \|\,\mathrm{curvature}(g)\|^2$. A scaled curvature bound $\|\sqrt{k}\,\mathrm{curvature}(g)\| \leq M$ then gives chord at most $2M/\sqrt{k}$ and flaw at most $(J/2)(2M/\sqrt{k})^4$. Consequently

$$X = \mathrm{direct}(J/2)(2M/\sqrt{\mathrm{direct}})^4, \quad Y = \mathrm{penalty}(J/2)(2M/\sqrt{\mathrm{penalty}})^4$$

have rate -2. The envelope $F = X + Y + 2Z$ of rate $-3/2$ bounds each soft energy minus its affine reply half-square sum. The factor two pays the two fine coordinates; noise has one. Readout preserves the exact square completion and the retained/clipped fine-plaquette domain before using these bounds; the unconditional noise identity is kept distinct.

12.27 Outer corrections and light annulus bounds

On a clean radius-$(B/4+1)$ hull, the flattened N guide has zero linear part and error e of rate -1, and gauge covariance preserves its norm, so its half-square workN is bounded by $e^2/2$, of rate -2. For C, toll at most temp makes clipping exactly one, including at zero. Its exact formula gives workC nonpositive, and the quartic estimate gives absolute bound $(\mathrm{direct}_{\mathrm{below}}\,J/2)(2\,\mathrm{fiery})^4$ since temp is at most $2\,\mathrm{fiery}$, again rate -2.

Use the same clean grafted local gauge for the L, D mismatch terms, with common rate-$-1/2$ envelope W. For $e = \mathrm{widen}(R, W, \tau)$ the local gap bounds give $\|\,\mathrm{lGap}\,\| \leq CWe$ and $\|\,\mathrm{dGap}(\mathrm{vision})\| \leq Cde$, $d = A\,\mathrm{fiery}$, using the actual maximum bound on vision. Cauchy–Schwarz bounds each L inner product by $M = dCWe$ and each D half-inner-product by $V/2 = dCde/2$,

and two Bool coordinates imply $|\,\mathrm{workL}\,| + |\,\mathrm{workD}\,| \le 2M + V$, of rate $-3/2$: widening preserves the rate $-1/2$ of W, and both products have rate $-3/2$. The layered-extension and interior-gap estimates supply these bounds. All terms share actual U, V, Q.

The light estimate is decomposed into four already specified energy types: the soft curvature/noise remainder, the N correction, the coarse C correction and the L/D mismatch corrections. Sum their four envelopes into nonnegative E_{light} of rate $-3/2$, weakening the rate--2 bounds if needed. A selected fine curvature row extracts its two real-path half-squares, while a selected noise row extracts one, so the soft bounds give absolute remainder estimates there. At unselected curvature/noise rows extraction is zero and flaw plus half-squares are nonnegative; only a lower bound is asserted. Such a row therefore satisfies the lower bound without an absolute-value claim. The coarse C row has no extracted parent contribution and uses its C, L, D envelopes; the N row uses its N envelope.

These claims require a clean radius-B coarse hull and grafted rate-$-1/2$ tangent bound. Before a signed turn restrict to $B/2+1$, yielding $B/2$ afterwards and paying the reversed-link shift; read the maximum condition in that same background. On the packet annulus inside $200B$ seeds but outside $2B$, Wide bounds through $300B$, tangent control through $304B$ and $2\gamma\,\mathrm{width}\,t \le w(x)$ give grafting and cleanliness. Actual toll bounds make retained selectors one, hence original debit equals light and debit is at least $-E_{\mathrm{light}}$. This uses no derivative of a hard selector and no globally small gauge. The soft-remainder, outer-correction and grafted-signal estimates of this chapter provide these four envelopes and the required shell access. The lift retains each selection status and this enforcement domain; it neither differentiates a hard selector nor converts the lower bound for unselected rows into an absolute bound.

12.28 Physical operator allowances and path-series bounds

For width $\eta \ge 0$ and cover price $\sum_z e^{\eta|z|}\|K_{\alpha\beta}(z)\| \le C$, each routed summand has norm at most its absolute coefficient: adjoint travel is isometric even when winding. Endpoint congruence selects exactly one input/output site for a fixed displacement. Absolute summation before periodization gives row price $|\beta|C$ and column price $|\alpha|C$, retaining all coincident-endpoint displacements. For the fine/coarse row the per-type price is C/L^2 with input type comprising the fine residue and two-layer plaquette type, and L^4 residues; the corresponding row and column allowances are a/L^2 and aL^2. Noise and tight rows have common allowance a. The source takes the finite type-census maxima with one independently of the later L, W choices.

A nonzero cut term has endpoint distance at most route length, bounded by $B/100000 \le B/50$. The parent map foreshortens adjacent fine steps. The route from the lifted fine residue to Lz ends at the embedded coarse site $\mathrm{parent}(x)+\mathrm{wrap}(z)$; this remains true for wrapping routes by additivity of wrapping and the cover route endpoint identity. This is incidence support, not a claim about every datum read by the operation.

For full path labels (c_i, s_i) retain the nonnegative matrix

$$M_{pu} = \sum_i |c_i| \mathbf{1}_{\{d\,\mathrm{length}(s_i)\le R,\ \mathrm{landing}(s_i,p)=u\}}.$$

For each source plaquette type, the weighted budget $\sum_i |c_i| e^{\sigma\,\mathrm{length}(s_i)} \le P$ gives row sums P. Each fixed label/type fixes the source site for a landing, hence column sums qP. Absolute convergence justifies exchanges. Curvature chord and transport bounds give $\|\,\mathrm{shorten}(U,p)\| \le \sum_u M_{pu}\,\mathrm{toll}(U,u)$. At a kept source the selector bounds toll at distance k by $te^{k/\rho}$; landing distance at most slot length and $\sigma\rho \ge 1$ yield shortening size Pt.

Thus $A_{\mathrm{stage}} = (q+1)P + \sqrt{\mathrm{direct}}$ pays rows, columns and direct clamp. Both thinned branches have size at most $A_{\mathrm{stage}}t$: shortening uses the kept estimate and the direct branch uses $\min(1, t/r)$, with zero curvature at $r = 0$. Reach margins hold eventually: for $t = \mathrm{temp}(x+a)$, $te^{k/\rho} < 2$ and $t \geq (x+a)^{-1/2}$ imply $k/\rho < \log x$ once $2\sqrt{x+a} \leq x$. Since $B \geq (\log x)^2$, for sufficiently large $\log x$ this gives $k + 1 \leq B/50$, and the cut length plus one also fits; parent distances add at most one. These observations supply both the shortening stencil reach and the discarded-witness reach used above.

The shift $a = G.\,\mathrm{slopeConstant} \log L \geq 0$ follows from the minus-Killing sum of squares. At fixed shift eventually $\mathrm{temp}(x) \leq 2\mathrm{temp}(x+a)$ using $\sqrt{x+a} \leq 2\sqrt{x}$ and monotonicity of the nonnegative logarithmic temperature factor. With $A = A_{\mathrm{top}} + A_{\mathrm{below}}$ and operator allowance C, the common $d = 2A(1+C) > 0$ gives

$$\| \,\mathrm{vision}\, \| \leq dt, \qquad \|Mv\| \leq dL^{-2}t, \qquad \|Mw\| \leq dt.$$

Indeed coarse vision is bounded by $2At$, and the fine/noise row allowances give these three bounds. Both stage lane counts use this same d fixed before L, W, with late cutoff chosen afterwards. These prove the operator and maximum hypotheses used throughout the preceding energy estimates. Readout preserves the full label census, selector branch and dimensional factors, rather than collapsing repeated paths or replacing incidence support by total-read locality.

12.29 Fine income and discarded-row accounting

Let $a, b, p, t > 0$ be the direct quadratic/linear coefficients, noise penalty and fine threshold. For plaquette chords r_i, let K be the listed large plaquettes $r_i > t$ and $Z = \sum_{i \in K} r_i$. Actual plot conditions put listed nonlarge plaquettes in $[t/4, t]$ and list every large plaquette in the required five-radius dome. The direct and second components are

$$D_i = \mathrm{clip}_t(r_i) + \tfrac{1}{2}\|v_i^0\|^2 - \langle v_i^0, \mathrm{shoot}_i \rangle,$$

$$E_i = \mathbf{1}_{\mathrm{kept(i)}} \tfrac{1}{2}\|u_i\|^2 + \tfrac{1}{2}\|v_i^1\|^2 - \langle v_i^1, u_i \rangle.$$

Use $\| \,\mathrm{shoot}_i\, \| \leq \sqrt{a} r_i$, clipping $ar_i^2/2$ below threshold and btr_i above, and response bounds $\|v_i^\nu\| \leq \epsilon t$. Square completion gives $D_i \geq (\sqrt{a} r_i - \|v_i^0\|)^2/2$ for $r_i \leq t$. Thus $\epsilon \leq \sqrt{a}/8$ pays $at^2/128$ on each listed small plaquette, while $\epsilon\sqrt{a} \leq b/4$ pays $3btr_i/4$ on large ones. Unlisted summation rows have $r_i \leq t$ and hence nonnegative direct part, hence

$$\sum_i D_i \geq (3b/4)tZ + (a/128)t^2 |\mathrm{costs} \setminus K|.$$

A discarded row has a witness j with $r_j > te^{\mathrm{dist}(i,j)/\rho}$. Such a row lies within radius $\lfloor \rho \log(r_j/t) \rfloor$ of j, including all plaquette types, for fixed relaxation length $\rho > 0$. Its witness ball contains at most $H_\rho(r_j/t)$ rows, $H_\rho = (8\rho+1)^4 |\mathrm{PlaqAxes}|$, by $2\lfloor \rho \log y \rfloor + 1 \leq (8\rho+1)e^{(\log y)/4}$. Raise the last inequality to the fourth power for $y \geq 1$. Local witness reach puts j in the five-radius dome, so $r_j > t$ makes it a member of K. Thus the discarded set X obeys $t|X| \leq H_\rho Z$ on the original fine lattice, without an L^4 factor.

For shortening majorant $M_{ij} \geq 0$ with row/column prices C, support reach ensures all contributing $j \notin K$ have $r_j \leq t$. Therefore $\|u_i\| \leq Ct + \sum_{j \in K} M_{ij} r_j$ for $i \in X$, and summing uses $Ct|X| \leq CH_\rho Z$ plus the majorant column bound to give $\sum_{i \in X} \|u_i\| \leq C(H_\rho + 1)Z$. Kept rows have $E_i = \|u_i - v_i^1\|^2/2 \geq 0$; discarded rows have $E_i \geq -\epsilon t \|u_i\|$. Choosing also $\epsilon C(H_\rho + 1) \leq b/4$ proves

$$\sum_i (D_i + E_i) \geq (b/2)tZ + (a/128)t^2 |\mathrm{costs} \setminus K|.$$

This argument does not assume a good halo at a discarded row. The lift retains witnesses, listed sets and shortening multiplicities on the same fine lattice, so no unearned positivity or extra block census enters the sum. For noise rows $n_i = \mathrm{wire}_i$, the exact energy is at least $(\sqrt{p}\, n_i - \|v_i\|)^2/2$. Listed aisles have $n_i \geq wt$, and $\|v_i\| \leq Ct$ with $C \leq \sqrt{p}\, w/2$ pays $pn_i^2/8$ each; other noise rows are nonnegative. Choose the response constant C before L, then $\epsilon = C/L^2$. First enlarge L for all three fine-income restrictions, next choose $w \geq 2C/\sqrt{p}$, and only then take the late scale. The resulting fine coefficient remains exactly $a/128 > 0$.

12.30 Macro multiplicity and spending census

Put $R = j + 8$. If twice macro radius $(j + 7)L$ is below the fine period, coincident periodic displacements there are identical, so $M(i, z) \leq 1$. Nonzero multiplicity implies $\mathrm{dist}(i, \mathrm{parent}\, z) \leq R$, by multiplying the coarse distance by L, bounding the embedded-base displacement by $(j + 7)L$, adding the at-most-$L/2$ parent offset, and applying the triangle inequality and integer rounding. Therefore $\sum_i M(i, z) \leq (2R + 1)^4$. Grouping the finite displacement/plaquette sums proves the port identity with actual multiplicities; the period condition is needed for the column bound, not that identity. The macro census used here has a period qualification, already included in the step hypotheses.

A store has at most four base sites per share, because a fine plaquette has at most four corner parents, a noise bond at most two pins, and a flag one site. Hence faces in its radius-r dome number at most $V(r)$ shares, $V(r) = 4(2r + 1)^4 q$, and $V(2B) \leq V(2)B^4$ for $B \geq 1$. Apply this separately to original and spending stores. Each flag touches a coarse face above temperature, and each face has at most four corners, bounding flags by four times those faces.

Every coarse face with toll above fixed $\delta > 0$ has a spending witness within R eventually. Otherwise all macro-port fine tolls are at most spending threshold T, using listed witnesses and unlisted $t/2 \leq T$; all four wires are at most T using $2Wt \leq T$. The local predictor estimate would bound its toll by

$$[4 + DL^{-2}q|\,\mathrm{box}((j + 7)L)|]T \leq \delta$$

at late scale with fixed L, W, a contradiction. The distance bounds put every relevant input in the plot dome, as required for this local predictor estimate. Explicitly $T = \beta_F^{-49/100} \to 0$, while $t = \beta_F^{-1/2} P(\beta_F)$ with subpower P. For each fixed constant m, eventual bounds $P, m \leq \beta_F^{1/200}$ give $mt \leq T$. Also $B = \lceil (\log x)^2 \rceil \to \infty$ permits $B \geq 5R$. The pre-L constant

$$N = V(R) + V(2) + R + (2R + 1)^4$$

pays macro columns/reach, δ-large face counts per spending share, and outer/affected counts NB^4 per respective share. Readout preserves the original/spending-store distinction and actual multiplicities rather than assigning an independent copy to repeated periodic data.

12.31 Affected departure: macro excess and operator transfer

Fix the same plot. For its affected face set s, $m = |s|$, use local allowance constants $a, d, D \geq 0$ and define

$$\mathcal{B} = L^{-2} \sum_{\mathrm{swell(t)}} \mathrm{toll}\, U + \sum_{\mathrm{aisles}} \mathrm{wire}(Q, V), \quad u = \sqrt{\mathrm{high.\,direct}} + d, \quad v = \sqrt{\mathrm{low.\,direct}} + d, \quad g = \sqrt{p}.$$

Here $q = |\mathrm{PlaqAxes}|$ is the number of plaquette types. Retain nonnegative fine excess E precisely on the swell at t and noise excess F on listed aisles. For macro multiplicity $M(i, z)$, row sum is at

most $S_{j,L} = q|\operatorname{box}((j+7)L)|$ and column sum at most N on the physical torus. The exact port identity retains coincident-image multiplicities. Define

$$H(p) = \sum_{e\in\partial p} F(e) + DL^{-2}\sum_{z} M(p.1, z)E(z).$$

The four boundary terms count repetitions. Each fixed boundary-position map from face base to bond is injective, so their total costs at most $4q\sum F$. Macro columns give

$$\sum_{p} H(p) \le 4q\sum F + DqNL^{-2}\sum E \le (4q + DqN)\mathcal{B} =: b.$$

On each usable plaquette the predictor estimate reads $\operatorname{toll} V \le \operatorname{fuel}(Q, V) + DL^{-2}\operatorname{port}(U)$, with D the local predictor-curvature comparison constant. Thus $\operatorname{macroLarge}(N, D) = 4q + DqN$ and $b = \operatorname{macroLarge}(N, D)\mathcal{B}$. The five-radius plot supplies $\operatorname{toll} U \le t + E$ and $\operatorname{wire} \le 2Wt + F$: unlisted inputs use their upper thresholds, while listed inputs retain their full size in the excess. Every three-radius face is usable because its required radius-one coordinate box lies inside the listing dome; their boundary and macro stencils remain in the five-radius dome by reach and $B \ge 1 + N$. The predictor-curvature comparison therefore gives $\operatorname{toll} V \le rt + H$, where $r = \operatorname{macroSmall}(j, D, L, W) = 8W + DL^{-2}S_{j,L}$. The four wires contribute $8Wt$ and the macro row sum contributes $DL^{-2}S_{j,L}t$. This row census needs no injectivity between distinct cover displacements.

For the $2m$ tested Boolean slots the actual operator row/column allowances give

$$\sum \|\Phi^* X_f\| \le 2maL^2ut + 2au\mathcal{B},$$

$$\sum \|(\Phi')^* Z\| \le 2mag(2Wt) + ag\mathcal{B}, \qquad \sum \|X_n\| \le 2mvrt + 2vb.$$

Separate uniform small input from its nonnegative excess. Fine small input uses column price aL^2, while large input uses row price a/L^2 and excess at most $2u\sum E$, retaining the inverse-square saving. Noise uses narrow prices a and coarse input its direct excess bound. Stencil reach sends each tested two-radius dome into the permitted input dome. The lift retains this local excess split and both matrix directions; swapping a row and column price would lose the stated block factor. For departure L, vision has norm at most dt and pairs with $\Phi^* X_f + (\Phi')^* Z - X_n$. Cauchy–Schwarz bounds its sum below by $-dt$ times the preceding three norm sums. For departure D the residual size is at most $(d + ad + ad + a^2 d)t$: the wide aL^2 cancels the dt/L^2 response, with narrow and tight terms adt, a^2dt. Pairing and the two half-weighted slots give $\operatorname{workD}(p) \ge -d_0 t^2$, $d_0 = d(d + ad + ad + a^2 d)$.

All charged third-range faces lie in s and have toll sum at most $mrt + b$. Coarse temperature at most $2t$ therefore incorporates their linear charge. With $\ell = \text{low. linear}$, the exact choices give

$$\ell\,\operatorname{temp}\sum_{\text{tides}} \operatorname{toll} V - \operatorname{talll}(s) \le C_{\text{pre}}t\mathcal{B} + C_{\text{seat}}t^2 m,$$

$$C_{\text{pre}} = d(2au + ag + 2v\,\operatorname{macroLarge}(N, D)) + 2\ell\,\operatorname{macroLarge}(N, D),$$
$$C_{\text{seat}} = 2d(aL^2u + 2agW + vr) + d_0 + 2\ell r.$$

To see the coefficient collection explicitly, the L pairing contributes at most

$$d(2au + ag + 2v\,\operatorname{macroLarge}(N, D))t\mathcal{B} + 2d(aL^2u + 2agW + vr)t^2 m$$

to the negative departure. The D pairing adds $d_0 t^2 m$, while the payments relation, whose charged third-range faces are all in s, adds at most $2\ell t(mrt + b)$. Substituting $b = \operatorname{macroLarge}(N, D)\mathcal{B}$

yields exactly the two displayed constants. The first is fixed before L, W; the second may depend on both. Since $m \le NB^4$ shares(spend), the seat charge is at most $(NC_{\text{seat}}+1)t^2B^4$ shares(spend), with a positive coefficient. Fine excess remains cut at t; only the spending census uses its larger threshold. The lift keeps this distinction and the same affected set, so coarse charges are paid by the actual fine/noise income rather than a differently truncated substitute.

12.32 Three coarse clipping ranges

Let $r = \text{toll}(V,i)$, $s = \text{temp}(x) \ge 0$, $a = \text{low. direct} > 0$ and $b = \text{low. linear}$. Set $\alpha = 1$ at $r = 0$, otherwise $\alpha = \min(1, s/r)$. Shortening terms cancel exactly: kept rows contribute their matching half-square and discarded rows have both tested terms zero. The direct remainder is

$$\text{workC}(i) = a(\alpha - \alpha^2/2)\|\,\text{curvature}(V,i)\|^2 - \text{clip}_s(r).$$

Choose group-only $K, \delta > 0$ giving curvature square at least $r^2 - Kr^4$ and $3r^2/4$ for $r \le \delta$. In the small range $r \le s \le \delta$, $\alpha = 1$ gives workC at least $-aKs^4/2$. For $s < r$, one has $0 \le \alpha = s/r \le 1$ and $\alpha - \alpha^2/2 \ge \alpha/2$, giving

$$\text{workC}(i) \ge \frac{as}{2r}\|\,\text{curvature}\,\|^2 - bsr.$$

Thus on $s < r \le \delta$, the curvature bound and clip hypothesis $b \le a/8$ give explicitly

$$\text{workC}(i) \ge (3a/8 - b)sr \ge (a/4)sr.$$

On $r > \delta$, dropping only the nonnegative curvature term yields workC at least $-bsr$. Finally the supplied $x\,\text{temp}(x)^3 \le 1$ makes $s \le \delta$ when $x > \delta^{-3}$. The common late scale therefore supplies all three ranges with small debit coefficient $aK/2$ and fixed group-only middle cutoff. The lift preserves the exact branch values, including the zero-toll convention, without differentiating selectors or replacing the outer lower bound by an absolute estimate.

12.33 Threshold ratios and stitch absorption

Fix L first. Set $y = x + k$ for its fixed coupling shift, $B = \lceil (\log x)^2 \rceil$, $R = \lfloor B/100000 \rfloor$, $t = \text{temp}(y)$ and $T = y^{-49/100}$. At these late scales $\text{temp}(v) = v^{-1/2}\exp((\log\log v)^2)$. For every fixed real shift k, the logarithm of $\text{temp}(x+k)/\text{temp}(x)$ is

$$(\log\log(x+k))^2 - (\log\log x)^2 - \tfrac{1}{2}[\log(x+k) - \log x].$$

The bounded derivative of $(\log v)^2$ on $[1,\infty)$, applied at the logarithmic arguments, bounds the first difference by twice the log difference eventually. Hence the ratio tends to one. For every fixed $n \ge 0$, $r > 0$, $B^n/y^r \to 0$ because eventually $B \le 2(\log x)^2$, every fixed logarithmic power is $o(x^r)$, and the fixed shift leaves this comparison unchanged. The monitor bounds $v^{-1/2} \le \text{temp}(v) \le v^{-1/2+p}$ for any $p > 0$ eventually imply

$$B^n\,\text{temp}(x) \to 0, \quad B^n t/T \to 0, \quad B^n T \to 0, \quad B^n T^2/t \to 0.$$

The first ratio uses $p = 1/200$, leaving the decaying power $y^{-1/200}$; the last uses the lower bound on t and exponent $-48/100$. It is T^2, not T, that is small relative to t.

Quadratic logarithmic growth of B makes $e^{-\tau R} \le y^{-1}$ eventually for each fixed $\tau > 0$, using $B \le 100000R + 100000$ and $\log y \le 2\log x$. Thus $B^n e^{-\tau R} \to 0$ and $B^n e^{-\tau R}/t \to 0$. For stitch size $h = CB^{10}T$ and $e = (1+R)h + e^{-\tau R}$, expanding nonnegative terms and using $R \le B$ gives

$$B^n h \to 0, \quad B^n h^2/t \to 0, \quad B^n e \to 0, \quad B^n he/t \to 0.$$

The first two stitch limits use the preceding bounds with exponents $n + 10$ and $n + 20$. Expand e for the other two and use $R \le B$; each nonnegative term is controlled by a preceding limit, and the mixed tail also uses $h \to 0$. These limits pay dome-census stitch losses without a period or share-count factor. The lift preserves the fixed-L order and original exponents, rather than treating these as simultaneous limits in all construction parameters.

12.34 Finite-store budget inequalities

Let $a = \mathrm{linear}_{\mathrm{top}}$, $p =$ penalty, $t =$ fiery and $u =$ largeT $\ge t$. For listed fine sets F_t, F_u above these thresholds and $R = \sum_{F_t}$ toll, one has $t|F_t| \le R$, $u|F_u| \le R$. The supplied tally $(a/2)tR + dt^2|\mathrm{costs} \setminus F_t|$ therefore pays, when $m \le a/8$, $m \le d$, $c_F \le a/4$,

$$\mathrm{tallyF} \ge mt^2|\mathrm{costs}| + mtu|F_u| + c_F tR.$$

Indeed $2m + c_F \le a/2$ and the cost partition pays each term. Original shares, spending subset and mismatch fee remain separate.

Listed noise has wire at least $Wt \ge t$. With $R_I = \sum_{\mathrm{listed}} \mathrm{wire}^2$, $t^2|\mathrm{aisles}| \le R_I$ and $tu|\mathrm{spend_aisles}| \le R_I$. If $C \le pW/16$, then $Ct\,\mathrm{wire} \le (p/16)\,\mathrm{wire}^2$. Using $m \le p/32$, the supplied $(p/8)R_I$ gives

$$\mathrm{tallyI} \ge mt^2|\mathrm{aisles}| + mtu|\mathrm{spend_aisles}| + Ct \sum_{\mathrm{listed}} \mathrm{wire}\,.$$

The coefficient budget is $2m + p/16 \le p/8$, since $m \le p/32$: the first two terms each cost at most mR_I, and the fee costs at most $(p/16)R_I$. The lower wire bound is the actual endpoint Plot condition at a bond endpoint belonging to the store dome.

Split coarse outs into toll at most $s =$ temp, between s and δ, and above δ. With $s \le \delta$, the three-range estimates give respectively $-B_{\mathrm{quartic}} s^4$, at least $4mt^2$ for $m \le \mathrm{direct}_{\mathrm{below}}/64$, $t \le 2s$, and $-\mathrm{linear}_{\mathrm{below}}\, s\,\mathrm{toll}$. Sum these disjoint ranges and enlarge the nonnegative small-range error count to all outs. Hence

$$\mathrm{tallC} + \mathrm{linear}_{\mathrm{below}}\, s \sum_{\mathrm{toll}>\delta} \mathrm{toll} + B_{\mathrm{quartic}} s^4 |\mathrm{outs}| \ge 4mt^2 \#\{s < \mathrm{toll} \le \delta\}.$$

Payments bounds flags by four times s-large tides and δ-large tides by K shares(spend). Subtracting the latter subset gives $|\mathrm{flags}| \le 4[\#\{s < \mathrm{toll} \le \delta\} + K\,\mathrm{shares(spend)}]$. These are finite algebraic consequences of the producer estimates and Payments, with no added period compatibility. The lift retains gains at tu separately from original seed shares at t^2 and carries every store set on the same configuration.

12.35 Point-charge gain on the common store

Keep the same store, input, predictor and background. Set $t = \mathrm{fiery}(x)$, $s = \mathrm{temp}(x)$, $u = \mathrm{largeT}(x)$ and $B = \mathrm{radiusB}(x)$. The fine producer gives $(\mathrm{linear}_{\mathrm{top}}/2)t$ times its large-toll sum plus dt^2 per other listed fine toll, $d > 0$ after the block/width choices. Noise gives $(\mathrm{penalty}/8)\sum \mathrm{wire}^2$. Choose $m > 0$ at most $\mathrm{linear}_{\mathrm{top}}/8, d, \mathrm{penalty}/32, \mathrm{direct}_{\mathrm{below}}/64$. The hit mismatch estimate is

$$\begin{aligned} \mathrm{tall}_{L,D}(\mathrm{hit}) + c_F t \sum_{\mathrm{large\ fine}} \mathrm{toll} + Ct \sum_{\mathrm{listed}} \mathrm{wire} + Zt^2 B^4\,\mathrm{shares(spend)} \\ \ge \mathrm{linear}_{\mathrm{below}}\, s \sum_{\delta\text{-large coarse}} \mathrm{toll}, \qquad c_F = C/L^2. \end{aligned}$$

Choose L to make $c_F \le \text{linear}_{\text{top}}/4$, then $W \ge 1$ with $C \le \text{penalty}\, W/16$. Listed-wire size Wt makes its quadratic income absorb the linear loss; fine income absorbs c_F. The coarse budget pays its large-toll charge. The flag census costs four times the middle-range count plus $4K$ shares(spend), and output count is at most KB^4 shares.

At fixed preceding choices take the common late scale so $s \le \delta$, $t \le u$, $t \le 2s$ and

$$(4mK + ZB^4)t^2 \le mtu, \qquad C_{\text{quartic}} s^4 K B^4 \le (m/4)t^2.$$

Indeed the spending remainder is explicitly

$$[mtu - (4mK + ZB^4)t^2]\,\text{shares(spend)} \ge 0.$$

The first pays spend flags and mismatch remainder from spending income; the second pays the small coarse error. The unaffected mismatch costs at most another $(m/4)t^2$ shares, and N tally is nonnegative. These scale choices follow from $t/s \to 1$, $B^4 t/u \to 0$, $B^4 s^2 \to 0$: divide the first inequality by tu, and use $s \le 2t$ for the second. They are uniform in live period.

Split hit/unaffected rows and add the finite-store fine, noise, coarse and flag inequalities. Positive remainders pay mt^2 per original share before the two quarter losses, giving

$$\text{assessed} \ge (m/2)t^2\,\text{shares}\,.$$

The source lift retains the same store and all its contributions, with block length paying the fine loss, W paying noise, then positive m fixed and the final delay paying remaining errors. These positive remainders come from the exact fine, noise and coarse budget rearrangements above. No later retuning of L, W or incompatible witness is used.

12.36 Unaffected mismatch and exact core assignment

The point-charge producer combines E01–E09 with this additional unaffected-row estimate derived from the mismatch bounds. The stitching bound, its two displayed scale limits, and the census and maximum estimates of this chapter supply its inputs. For an unaffected coarse plaquette row outside the hit subset of the store outputs, put $t = \text{fiery}$. Vision is bounded by At, and stitching gives one gauge with fine/coarse chords at most $h = \text{stitchCost}$ on radius $2B/5$, retaining forest and tags. Choose the delay to contain $3\,\text{till} + 4 + \text{predictionReach}$ and meet the chart radius. With $e = \text{widen}(\text{till}, h, \tau)$, local gaps give

$$\text{workL} + \text{workD} \ge -\,\text{loss}, \quad \text{loss} = 2AtChe + AtCAte.$$

Two L slots give the first factor two; the two half-weighted D terms give the last. There are at most NB^4 shares such rows. The stitching-scale limits are $B^4 he/t \to 0$ and $B^4 e \to 0$. Dividing their total bound by t^2 gives $2NACB^4 he/t + NA^2 CB^4 e \to 0$. Thus every prescribed $\varepsilon > 0$ eventually bounds total unaffected loss by εt^2 shares, with unchanged actual input and predictor.

The resulting point-charge order is block lower demand n, $L \ge n$, sufficiently large W, then positive μ, then a common late x. It is pointwise and uniform over coarse sides at least $100(1+B)$, funded data, forests and actual Plot inputs, giving assessed at least μt^2 times shares. Passage from the supplied grant intersects its delay with positivity of x, t and specializes its universal query; the fine, noise, coarse-clipping and flag budgets provide the energy combination.

Row locations are fine plaquettes, noise bonds, coarse plaquettes bundling C, L, D, or N rows. Their principal values are workF, workI, workC+workL+workD and workN, all at the same actual input, coarse background and predictor. The coarse anchor is the parent of a fine-plaquette root,

and the coarse root for the other types. Each fine plaquette parents two moment rows, noise one, others none, with every moment row parented once. A signed row is selected when its signed coarse anchor lies outside the ambient $6B$ seed dome; its surcharge is its real affine path half-square when selected, zero otherwise. Extracted sums parent surcharges, debit is principal minus extracted, and liftSq is the orientation average of extracted rows. Exact finite recombination gives

$$\text{wholeGap} + \sum_{\text{noise}} \text{monetary} = \text{average}_s \sum_{\text{rows}} \text{debit}_s + \text{liftSq}\,.$$

The finite subtraction of extracted rows cancels exactly against liftSq, with C, L, D still bundled at their common coarse plaquette. Signed noise covariance carries actual transformed tags and preserves the finite bond sum. For a store contained in the ambient seed place, posted $2B$ core rows are inside $6B$, hence unselected with zero extraction. Their principal/debit sum is exactly assessed, retaining the C, L, D bundle. Wide through $d \geq 5B$ implies Plot on its restriction and is preserved by simultaneous rotation of fields, tags, forest and store. Apply point-charge in each rotated slice and average, with shares unchanged. The exact core assignment and point-charge inequality proved in this chapter supply this step. The lift retains this entire recombination; extracted energy is accounted for, not discarded when assigning the core.

12.37 Packet annulus and tariff

For a store Z in the ambient seed place, assessed is its posted $2B$ debit sum and assigned is the corresponding $200B$ sum. Their difference is the finite annulus sum. Each row is at least $-Y(x)$ and the posted-row census per orientation is at most $\text{shares}(Z)\,\text{ballCost}(200B+1)\,\text{spots}(L)$. Hence assessed minus $E(x)$ times shares is at most assigned, where $\text{spots}(L)$ counts the row types per coarse site and $E = Y\,\text{ballCost}(200B+1)\,\text{spots}(L)$ retains rate $-3/2$, since the radius factor has rate zero. Averaging the same orientations and using point-charge gives

$$\text{quoteNear}(Z, u) \geq [\mu t^2 - E(x)]\,\text{shares}(Z).$$

The domain retains Wide through $300B$, tangent control through $304B$ and shell access. Connectedness of Z is not needed.

Ancillary expenses use the all-field condensation bound, eventual Haar-log bound one, swept-neighborhood/ticket costs of order bill^4 and frozen-measure negative log of order bill. A fixed $K_{\text{up}} > 0$ therefore gives steamLog plus tollFee times shares at most $-\beta\,\text{quoteNear} + K_{\text{up}}\,\text{bill}^5\,\text{shares}$.

For an admitted nonzero packet, seed consistency supplies ambient containment and actual flags. Its nonzero ticket/sector profiles supply the same $300B$ Wide domain, and enforcement gives hot bound $2at$ through $304B$. The width condition $\gamma\,\text{width} \leq a$ pays frozen tangents. Thus the quote applies to that actual field, forest and tags. Since $\beta \leq 2\beta_F$ and E has rate $-3/2$, eventually $\beta E \leq 1 \leq \text{bill}^5$. Substitution of the quote bound into the full expense estimate gives the exponent

$$[-\beta\mu t^2 + \beta E + K_{\text{up}}\,\text{bill}^5]\,\text{shares}(Z) \leq [-\beta\mu t^2 + (1 + K_{\text{up}})\,\text{bill}^5]\,\text{shares}(Z).$$

Exponentiation proves

$$|\,\text{Pack}(Z, u)| \leq \exp\{[(1 + K_{\text{up}})\,\text{bill}(x)^5 - \beta\mu t^2]\,\text{shares}(Z)\}.$$

Zero packets satisfy it immediately. This is cap one, expense $(1 + K_{\text{up}})\,\text{bill}^5$ and gain $\beta\mu t^2$ per share, pointwise before frozen-Haar/Gaussian integration. The lift retains the whole packet once, including its tag and guard costs, and applies the income only on the nonzero-enforcement domain that earns it.

12.38 Compulsory packet factors and normalization expense

For seed store A, $\mathrm{visit}_d(A)$ consists of fine bonds with a coarse endpoint in its radius-d dome, and pens is its nontrunk visit_{10B}. Set $m_g(\beta) = \mathrm{chartScale}(\sqrt{2\pi}\beta^{-1/2})^{\dim \mathrm{Color}}$. The full exponent is

$$\begin{aligned}\mathrm{steamLog} = &-\beta[\mathrm{quoteNear}(A,u) + \mathrm{farBills}(A)] \\ &+ \sum_{\mathrm{visit}_{200B}(A)} \mathrm{condensation}(e) + \sum_{\mathrm{plaza}(\mathrm{dome}_{304B}(A))} \mathrm{gentry}(e,u) - |\,\mathrm{pens}(A)|\log m_g(\beta).\end{aligned}$$

FarBills is the nonnegative orientation-average of extracted mean half-squares on posted $200B$ rows. Multiply its exponential by the order profile, actual $300B$ tag tickets and sector profiles, and $304B$ tangent enforcement to obtain the nonnegative packet. Nonzero packet forces every individual enforcement cone there nonzero. Profiles are in $[0,1]$; ticket bound $e^{\mathrm{tollFee\,shares}}$ gives $|\,\mathrm{steam}\,| \le e^{\mathrm{steamLog} + \mathrm{tollFee\,shares}}$, paying its whole ticket cost once.

For nonnegative upper bounds Q_0, C_0, K_0 on absolute condensation, absolute gentry and $-\log m_g$, endpoint census gives $|\,\mathrm{visit}_d(A)| \le \mathrm{shares}(A)\,\mathrm{ballCost}(d+2)L^4|\mathrm{Axis}|$. The gentry plaza is contained in $\mathrm{visit}_{304B}(A)$ and pens in $\mathrm{visit}_{10B}(A)$, so dropping only $-\beta$ farBills bounds ancillary expense by

$$\mathrm{shares}(A)[Q_0\,\mathrm{swept}(200B+2) + C_0\,\mathrm{swept}(304B+2) + K_0\,\mathrm{swept}(10B+2)],$$

where $\mathrm{swept}(d) = \mathrm{ballCost}(d)L^4|\mathrm{Axis}|$, in addition to $-\beta$ quoteNear. Finally

$$-\log m_g = -\log \mathrm{chartScale} - \dim(\mathrm{Color})\log\sqrt{2\pi} + \tfrac{1}{2}\dim(\mathrm{Color})\log\beta.$$

For $0 < \beta \le 2\beta_F$, monotonicity gives $\log\beta \le \log(2\beta_F)$. Thus the entire measure expense has an upper bound depending only on $\log(2\beta_F)$ and fixed chart/color constants, and is eventually bounded above by $C_{\mathrm{measure}}\,\mathrm{Delay.\,bill}(x)$, without a lower coupling-band assumption. The point-charge core bound and the packet annulus estimate show that the gain in quoteNear dominates these counted expenses. Core and annulus income pay these exact counts. The source lift retains frozen-measure normalization, all individual guards and the nonnegative farBills term, even when the bound drops that term by its justified sign.

12.39 The short correction family

Retain three local letter kinds with their fixed geometric sleeves. The signed energy letter outside the $200B$ seed neighborhood is

$$-\beta\,\mathrm{orientation}[\mathrm{enforce}_{2a}(u,\mathrm{booth}(o))\,\mathrm{light}(u) + \mathrm{extracted}(\mu)],$$

and zero inside. Condensation is zero when either bond endpoint touches that dome. The Haar-log letter is retained only at hot links outside the $304B$ plaza.

On the original seeded sector, remote light equals the energy debit. Noise equality is definitional. For curvature turn fields, tags and store together: sector membership and positive likelihood supply Interior. If a queried retention threshold is below two, room puts that read within Interior and toll is at most $t/2$; if it is at least two, the universal chord bound supplies retention. The retained path field is therefore unchanged and exact curvature splitting identifies light with debit. This sector hypothesis is needed for that equality, not for the following all-field envelope.

The protected mean has path bound A of rate -1, so at most two parent half-squares bound extracted by A^2. Let J of rate $-3/2$ bound light on a clean grafted hull with tangent size $4a$ fiery,

and let E_{cond} of rate $-1/2$ bound remote condensation. If enforcement vanishes, its light product is zero. Otherwise its protection bounds every booth tangent by $4a$ fiery and distant grafting supplies the same clean hull; the profile is at most one, giving product bound J. Seed separation supplies the required parent/hull margins.

For $0 < \beta \le 2\beta_F$, orientation in $[0,1]$, the energy letter is bounded by $W_{\mathrm{short}} = 2\beta_F(J + A^2)$ of rate $-1/2$: β_F has rate one, J rate $-3/2$, and A^2 rate -2. Haar is bounded by C fiery of that rate. Hence $E_{\mathrm{short}} = E_{\mathrm{cond}} + C\,\mathrm{fiery} + W_{\mathrm{short}}$ is an eventual nonnegative all-field envelope for all three kinds and admissible couplings. A bond touching the dome at an endpoint has its parent in the one-site enlargement; conversely, if it does not touch the dome its parent is outside it. These endpoint facts justify the condensation exclusion. The light, remote-condensation, grafted-signal, mean and Haar estimates supply the inputs to the common envelope. The lift preserves the three letter identities and the distinction between sector equality and all-field bounds, retaining actual enforcement rather than differentiating or dropping its hard support.

Provenance of the two bounds in the short correction family. The two bounds used by the short correction family are outputs of earlier constructions and not separate suppositions. The light bound is $J = E_1 + E_2 + E_3 + E_4$ from the annulus estimates, with grafted tangent bound $4a\,\mathrm{fiery}(x)$: the soft-square and mismatch terms have rate $-3/2$ and the two quartic or squared-guide corrections have rate -2, so after weakening the latter J has rate $-3/2$. The condensation bound is $E_{\mathrm{cond}} = \dim \mathfrak{g} \cdot \mathrm{pulseHead}(\mathrm{repair})$ from the local plate comparison: its repair envelope has rate $-1/2$, its window census has rate zero and its fixed inverse-window constants do not depend on x, so E_{cond} has rate $-1/2$. In the realization these are the same scalar functions of the tagged level, read from the same annulus and plate records; the rate calculus (finite sums keep the larger rate, products add rates) is applied to identical numbers, so the lifted short family has the same two rates and no rate is introduced by the lift.

12.40 Base stroke at a fixed multiple of the small-field scale

For integer n set $\mathrm{pcap}(n) = 32(2(n+1)+1)^5|\mathrm{PlaqAxes}|$. Axial flattening on radius $n+1$ and temp at most $2t$ give radius-n chord bound $\mathrm{pcap}(n)t$. Here n is fixed per finite query, not increased to B.

For a replica plaquette with clean radius-five hull use $n = 4$. Its replica links are identity or gauged coarse links, bounded by $\mathrm{pcap}(4)t$. Replacing exterior links by identity leaves its product unchanged; chord and curvature norm are at most $4\,\mathrm{pcap}(4)t$. Every retained fine base-moment slot lands at such a plaquette within the original clean radius-B hull: its actual path length is bounded by $\mathrm{till}(x)$, and the room inequalities ensure this inclusion. Adjoint travel preserves curvature norm, so absolute path summation and the direct coordinate give

$$\|X_{f,R}(\mathrm{replica}(V))\| \le 4\,\mathrm{pcap}(4)(\sqrt{\mathrm{direct}_{\mathrm{top}}} + \mathrm{paths}_{\mathrm{top}})t =: C_1 t.$$

A common gauge for all slots is unnecessary because only each curvature norm is estimated.

For noise instead fix $n = \mathrm{predictionReach}(D)$. Flatten that fixed hull and extend its logarithm to b with $\|b\| \le At$, $A = A_{\log}\,\mathrm{pcap}(n)$. Its linear copy a obeys the same bound and exponential replica identity. Predictor locality with actual tags identifies the base noise with $z\,\mathrm{Test}(a,b)$. Its uniform Taylor bound gives size at most $J\|(a,b)\|^2 + J\|(a,b)\| \le 2JAt$ eventually, with $At \le 1$ and the chart radius. Gauge covariance returns the original bound. Thus $C_1 + 2JA$ bounds every clean-hull base stroke by a fixed multiple of t; this is the crowns input to the global mean bound, obtained without spending an extra growing-radius factor. The lift retains which queries share

a gauge and which need only invariant norms; it does not combine distinct local gauges into an unsupported global preparation.

12.41 Global mean, stationary residual and Gaussian rearrangement

Keep the global estimate separate from the sharper remote path estimate. Use the selected rows, frozen cutoff field and inverse-window bounds. If base stroke is at most $A_0 t$, guide at most Mt, oriented operator at most N and saved field at most $2\gamma K t$, selected-row cleanliness gives

$$\| \operatorname{priced}(0)\| \le \operatorname{scale}(A_0 + M + 2N\gamma K)t =: b_1 t.$$

Unselected rows are zero. This uses the saved bound without a global gauge. The transpose price Y gives coordinate bill bound $|b_i| \le Y b_1 t =: b_0 t$. Let P be the inverse-column pad and $\mu_{\mathrm{count}} = -P^\top b$. Its column price $C_{\mathrm{inv}} = \mathrm{payE}$ then gives $|\mu_{\mathrm{count},i}| \le C_{\mathrm{inv}} b_0 t$ and color-field mean at most $\dim(\mathrm{Color}) C_{\mathrm{inv}} b_0 t$. Symmetry of A^{-1} yields exactly

$$A\mu_{\mathrm{count}} + b = -A(P - A^{-1})^\top b.$$

Plate price Q and pad error give residual bound $Q \operatorname{distPay}(B) b_0 t$. Taking $C_{\mathrm{mean}} = 1 + \max(b_0, b_1, \dim(\mathrm{Color}) C_{\mathrm{inv}} b_0, 0)$ bounds all three global quantities. The residual has any prescribed rate k, because the scraps estimate, hence distPay, has rate $k + 1/2$ and t has rate $-1/2$. This supplies the mean/residual hypotheses of the correction budgets. The clean base-stroke and uniform maximum estimates supply A_0, M.

For the exact exponent retain $A = D^\top D + S_{\mathrm{sleep}}$, mean μ in count coordinates, bill b, covariance $C_{\mathrm{ref}} > 0$ and $g = \beta^{-1/2}$. Expanding finite affine squares gives

$$\operatorname{liftSq}(\mu + gz) = \operatorname{liftSq}(\mu) + g\langle z, b + D^\top D\mu\rangle + \tfrac{1}{2} g^2 \langle z, D^\top D z\rangle.$$

This polarization expands each affine-row square and sums its finite color coordinates. Dormant mean is zero. With $\mathrm{ghost} = \langle z, S_{\mathrm{sleep}} z\rangle/2$, $\beta g^2 = 1$ and $\beta g = g^{-1}$, rearrangement yields

$$\begin{aligned} -\beta \operatorname{liftSq}(\mu + gz) - \mathrm{ghost}(z) = {} & -\beta \operatorname{liftSq}(\mu) + \langle z, \widehat{h}_\beta\rangle \\ & + \tfrac{1}{2}\langle z, (C_{\mathrm{ref}}^{-1} - A) z\rangle - \tfrac{1}{2}\langle z, C_{\mathrm{ref}}^{-1} z\rangle, \end{aligned}$$

Indeed before adding and subtracting the reference quadratic form the exponent is exactly

$$-\beta \operatorname{liftSq}(\mu) - g^{-1}\langle z, A\mu + b\rangle - \tfrac{1}{2}\langle z, Az\rangle.$$

Here the linear coefficient is exactly $\widehat{h}_\beta = -g^{-1}(A\mu + b)$, since $S_{\mathrm{sleep}}\mu = 0$. The last term is the reference Gaussian, the preceding quadratic term its itinerary correction and the linear term its first correction. Ghost supplies the dormant diagonal with normalized integral one. This pointwise identity holds regardless of later profile zeros or sector events. The lift retains the same count coordinates and every exponent term before any integration.

12.42 Pointwise Gaussian chain corrections

For the completed plate and symmetric reference covariance set $S_{\mathrm{err}} = AC_{\mathrm{ref}} - I$ and $\widehat{h}_\beta = -g^{-1}(A \operatorname{drift} + b)$. The first correction at a link is its finite color pairing with z, multiplied by aperture: the cone profile at twice enforcement radius on hot links and one otherwise. An itinerary of $n + 2$ link steps has half-bilinear quadratic correction with the ordered block product of $n + 1$

factors $-S_{\mathrm{err}}$ and terminal A, multiplied by both endpoint apertures. At fixed starting link and n, summing itineraries restores the intermediate link/color sums of matrix multiplication, hence exactly $(-S_{\mathrm{err}})^{n+1}A$. The terminal factor remains in its value and price.

Dormant plate/covariance rows and columns are identity, so S_{err} vanishes if either endpoint is dormant. Mean and bill vanish there too. Induction along a nonzero chain shows both endpoints hot: the final A cannot connect hot to a distinct dormant identity row. Let $g = \mathrm{gStep}(\beta) > 0$. If $\|\mu_e\| \le Kt$, nonzero endpoint aperture gives $\|\mu_e + g\,\mathrm{atmos}(z)_e\| \le 4at$, hence

$$\|\,\mathrm{atmos}(z)_e\| \le g^{-1}t(4a+K) = \mathrm{heat}(a,K,t,\beta).$$

An envelope U at least heat bounds each endpoint color coordinate, and aperture is at most one. For coordinate first-correction size M, $|\,\mathrm{qfirst}_e\,| \le \dim(\mathrm{Color})UM$, including dormant zeros.

Write $R_c = \dim(\mathrm{Color})$, $d_{\mathrm{pay}} = \mathrm{distPay}(\eta, r, Q, m, B)$ and $M_{\mathrm{err}} = Qd_{\mathrm{pay}}$. Entry bounds are M_{err} for discrepancy and Q for terminal plate. Summing its $n+1$ intermediate color indices, with the source's extra convenient color factor, bounds each block-product entry by $R_c^{n+2}QM_{\mathrm{err}}^{n+1}$. Endpoint sums then give

$$|\,\mathrm{chain}\,| \le \tfrac{1}{2}U^2R_c^{n+4}QM_{\mathrm{err}}^{n+1} = \mathrm{centreQ}(U)\,\mathrm{etaS}^n,$$
$$\mathrm{centreQ}(U) = \tfrac{1}{2}U^2R_c^4Q(Qd_{\mathrm{pay}}), \qquad \mathrm{etaS} = R_cQd_{\mathrm{pay}}.$$

Nonzero itineraries are armed: every parent hop is at most $3B$, else the finite-range discrepancy/plate block vanishes. Define capQ by this geometric bound on armed itineraries and zero elsewhere. Splitting zero aperture, zero pairing and nonzero cases proves the pointwise bound for all z. The lift retains ordered hops, full color census and endpoint apertures; these bounds use no extra Gaussian tail event and no deletion of the terminal plate.

12.43 Convergent ordered inverses and logarithms

Weighted absolute row/column prices multiply at any nonnegative width. Price $q < 1$ gives E^n price q^n; the finite identity $(\sum_{n<N} E^n)(I-E) = I - E^N$ and $E^N \to 0$ prove $\sum_{n\ge0} E^n = (I-E)^{-1}$ with absolute entry convergence. The two source orders remain distinct:

$$AP - I = E \implies A^{-1} = \sum_{n\ge0} P(-E)^n,$$

$$AC - I = E \implies C^{-1} = \sum_{n\ge0}(-E)^nA, \quad C^{-1} - A = \sum_{n\ge0}(-E)^{n+1}A.$$

They follow by multiplying the inverse of $I+E$ on the appropriate side, without commuting matrices. Absolute entry convergence permits each fixed multiplication and removal of the zeroth term.

For $G_s = I + s(A-I)$ invertible on $[0,1]$, factor it near r as $[I + (s-r)(A-I)G_r^{-1}]G_r$. The linear determinant coefficient is trace, giving $\partial_s \log|\det G_s| = \mathrm{tr}((A-I)G_s^{-1})$. Adjugate continuity and compact nonvanishing allow integration to

$$\log|\det A| = \int_0^1 \mathrm{tr}((A-I)G_s^{-1})\,ds.$$

Positive plates have positive determinant. Let P_s be the principal-column pad and $E_s = G_sP_s - I$, of uniform price $\rho = \mathrm{scraps}(B) < 1$. Since $G_sP_s = I + E_s$, one has $G_s^{-1} = P_s(I+E_s)^{-1}$; hence the pad P_s occurs once in each term of the following expansion and only the residual $-E_s$ is raised to the power n:

$$(A-I)G_s^{-1} = \sum_{n\ge0}(A-I)P_s(-E_s)^n.$$

Each continuous diagonal is bounded by $(Q+1)\,\mathrm{payE}\,\rho^n$, uniformly in s. The interval has length one, so the sum of the norms of the diagonal integrals is bounded by the same summable geometric majorant, independently of s. Thus integral and series commute absolutely, before the finite coordinate sum. Writing its diagonal integrals as $\mathrm{lact}_i(n)$ and $\mathrm{localBase}_i = \mathrm{lact}_i(0)$ gives

$$\log\det A = \sum_i \mathrm{localBase}_i + \sum_i \sum_{n\ge 0} \mathrm{lact}_i(n+1).$$

The local diagnostic alone is the zeroth term; arms recover the full determinant.

Applying the same interpolation to $I + sE$ with discrepancy price below one gives

$$\log|\det(I+E)| = \sum_{n\ge 0}\sum_i \int_0^1 [E(-sE)^n]_{ii}\,ds.$$

Uniform q^{n+1} bounds justify the exchange. For $E = AC_{\mathrm{ref}} - I$, positive determinants identify this with $\log\det A + \log\det C_{\mathrm{ref}}$, the actual covariance normalization defect. The lift retains ordered arms and their local prices, using uniform summability before finite coordinate sums; a global volume factor is not silently treated as a local bound.

12.44 Itinerary receipts, scalar stamps and address counts

An itinerary of k hops retains an initial fine link and an ordered list of k successors. A hop armed at radius d has a pair of coarse endpoints within d; since parents are within one site of either endpoint, successive parents are within $d+2$. The town of a link is the mB halo of its two coarse endpoints. It contains its parent and lies in its radius-$(mB+1)$ ball, hence in radius $(m+1)B$ for the positive lattice scale B. Write $\mathrm{steady}(m) = \mathrm{ballCost}(m+1)$. Its B-cover uses at most this many tiles. The receipt includes the towns of all links, including initial and final links. If $d+2 \le mB$, consecutive towns meet: the successor parent lies in both. The meeting-support covering inequality and induction give

$$\mathrm{coverNumber}_B(\mathrm{receipt}) \le (k+1)\,\mathrm{steady}(m).$$

For tenure, points in the first town lie within $mB+1$; points in the remaining receipt pay at most $d+2$ to its next parent and then its inductive bound. Thus, using $d+2 \le mB+1$, the same induction puts every receipt point within $(k+1)(mB+1)$ of the initial parent. These are respectively the covering and tenure inputs to the address census.

For continuous matrix factors $P_s(0), P_s(1), \ldots$, define the stamp of a returning itinerary by integrating over $0 \le s \le 1$ the color trace of its ordered block product; nonreturning itineraries contribute zero. Expansion of finite matrix multiplication gives exactly

$$\sum_{k\text{-hop itineraries}} \mathrm{stamp} = \int_0^1 \mathrm{Tr}\big(P_s(0)\cdots P_s(k-1)\big)\,ds.$$

Continuity ensures integrability, and the fixed-order sum is finite. For the determinant arm the source chooses

$$P_s(0) = A - I, \quad P_s(1) = \mathrm{pad}(I + s(A-I)), \quad P_s(j) = -\,\mathrm{error}(I + s(A-I)) \quad (j \ge 2).$$

An order-n label has $n+3$ hops and restores the integrated grade-$(n+1)$ intensity, with all $n+1$ discrepancy factors. For the source arm it chooses $P_s(0) = \mathrm{flawed}(A)$ and $P_s(j) = -s\,\mathrm{flawed}(A)$ for $j \ge 1$; the order-n label has $n+1$ hops and restores the integrated grade-n note. The color trace

stays inside each label. If factor entries are bounded by l_j, finite block multiplication and the final color sum give

$$|\text{stamp}| \le R_c^{k+1} \prod_{j<k} l_j.$$

An itinerary unarmed at the common factor range has a zero block and hence zero stamp. The interval length is one, so no additional integration factor enters the displayed entry-product price. These exact identities supply the pointwise arm prices.

Under source realization the factor at a hop from one link-color space to the next uses the section and readout of those endpoints. Intermediate readout–section compositions are identities. On a returning itinerary the resulting color block is conjugate by its starting-frame action to the physical ordered product; its finite color trace is unchanged. Recovery on the identity-frame section returns that same scalar stamp. The receipt retains the full ordered itinerary, even when links repeat, so this scalar invariance introduces no quotient of the address census.

An order-n ledger with u extra steps has $n+u$ hops. Its receipt is the prescribed union of enlarged neighborhoods visited along that itinerary. Set

$$D_L = L^4|\text{Axis}|, \qquad a_{L,d} = \text{ballCost}(d+2)D_L.$$

Here $D_L = \text{density}(L)$ and $a_{L,d} = \text{abundance}(L,d)$. Every successor parent lies in the radius-$(d+2)$ ball, and the fine-link caption census is D_L. The parent census therefore bounds successors per armed hop by $a_{L,d}$, and thus k-hop itineraries from a specified initial link by $a_{L,d}^k$. Assume $d+2 \le mB$ and $B > 0$, so the covering and tenure bounds just established apply. For a fixed receipt site x, the tenure bound permits at most $\text{ballCost}((n+u+1)(mB+1))D_L$ starting links. For nonnegative coefficients bounded by CQ^n on armed addresses and zero elsewhere, retain both tariff factors $\exp(2\varepsilon n)$ and $\exp(A_{\text{support}}\,\text{coverNumber}_B)$. The inequalities $n+u+1 \le (u+1)2^n$ and the four-dimensional ball formula imply

$$\text{ballCost}((n+u+1)g) \le \text{ballCost}((u+1)g)16^n.$$

The support tariff is at most $\exp(A_{\text{support}}(n+u+1)\,\text{steady}(m))$. Multiply the starting-link count, armed-hop count and both tariff factors. Consequently the order-n total at x is bounded by $HC(QR_{\text{price}})^n$, where

$$\begin{aligned} H &= \text{ballCost}((u+1)(mB+1))D_L\, a_{L,d}^u \exp(A_{\text{support}}(u+1)\,\text{steady}(m)), \\ R_{\text{price}} &= 16a_{L,d}\exp(A_{\text{support}}\,\text{steady}(m)+2\varepsilon). \end{aligned}$$

If $QR_{\text{price}} \le 1/2$, the full sales bound is $2HC$. To justify the countable ledger sum, bound any finite subset by its finitely many order fibers and then by this convergent geometric series; nonnegativity gives the full bound. This proves the asserted sales inequality and its summability, retaining every address even when different itineraries revisit the same link. In the lift, the realized tuple carries the entire ordered itinerary, receipt, factors, color trace and scalar stamp. Its literal readout restores the source address. Frame cancellation applies along the ordered product; it does not authorize commuting factors or merging repeated occurrences.

12.45 Pointwise arm prices and Gaussian-letter budgets

Put $R_c = \dim(\text{Color})$. Apply the stamp bound $R_c^{k+1}\prod_{j<k} l_j$ with $k = n+3$ for the determinant arm and $k = n+1$ for the source arm. For a determinant arm the $n+3$ ordered factors have entry prices $Q+1$, payE and $n+1$ copies of scraps(B). Their common range is $3B$: this includes

the plate's $B/2$ stencil, the inverse-column window and the discrepancy's enlarged window stencil. Interpolation $s \in [0,1]$ preserves the bounds and positive floors at every interpolation value. Thus the preceding trace-integral estimate gives, on armed addresses,

$$|\operatorname{ArmA}_{n,e,p}| \le R_c^{n+4}(Q+1)\operatorname{payE}\operatorname{scraps}(B)^{n+1} = \operatorname{centreA}\operatorname{etaA}^n,$$
$$\operatorname{centreA} = R_c^4(Q+1)\operatorname{payE}\operatorname{scraps}(B), \qquad \operatorname{etaA} = R_c\operatorname{scraps}(B).$$

For the source arm the $n+1$ factors each cost $Q\operatorname{distPay}(B)$, and later interpolation multipliers have absolute value at most one. Their range is also $3B$, and

$$|\operatorname{ArmS}_{n,e,p}| \le R_c^{n+2}(Q\operatorname{distPay}(B))^{n+1} = \operatorname{centreS}\operatorname{etaS}^n,$$
$$\operatorname{centreS} = R_c^2 Q\operatorname{distPay}(B), \qquad \operatorname{etaS} = R_c Q\operatorname{distPay}(B).$$

Both arms vanish on unarmed addresses. These are scalar prices before exponential expansion; internal color indices are summed inside the actual arm and the integration interval has length one.

For an armed-path envelope $Y(x)\kappa(x)^n$, halo $i \ge 4$ and u extra steps, the itinerary census gives a sales bound $2\operatorname{expoHead}(L,i,B,3B,u,A_{\text{support}})Y$ whenever

$$16\operatorname{abundance}(L,3B)e^{A_{\text{support}}\operatorname{steady}(i)+2\varepsilon}\kappa \le \tfrac{1}{2}.$$

The abundance and fixed-step prefactor have rate zero. A negative-rate κ therefore satisfies this condition eventually; a rate-b envelope Y gives a rate-b sales bound, uniformly in geometry. The itinerary address count and halo-cover estimate supply the quantities expoHead and steady used here. Apply this with $u = 3$ and $(Y,\kappa) = (\text{centreA}, \text{etaA})$, and with $u = 1$ and $(Y,\kappa) = (\text{centreS}, \text{etaS})$. All four quantities have every prescribed negative rate, so both budgets have any prescribed rate b. The blank geometry receives its own determinant-arm bound. One scalar remains attached to each actual arm; colors are not separately relabeled.

For the Gaussian chain retain separately $\|\mu_e\| \le Ct$, $|(A\operatorname{drift}+b)_i| \le E(x)$ and shell access. On $0 < \beta \le 2\operatorname{betaF}(x)$ put

$$\operatorname{burst}(x) = \sqrt{2\operatorname{betaF}(x)}, \qquad \operatorname{gStep}(\beta)^{-1} = \sqrt{\beta} \le \operatorname{burst}(x).$$

Burst has rate $1/2$. The common cutoff-field envelope

$$U(x) = \operatorname{burst}(x)\operatorname{fiery}(x)\operatorname{muRate}(C,H)$$

has rate zero, is nonnegative and independent of β and the Gaussian variable. Together with the mean bound it supplies the pointwise chain hypotheses. If E has rate $k - 1/2$, then the linear residual definition gives $|\widehat{h}_{\beta,i}| \le \operatorname{burst}(x)|E(x)| =: b(x)$. Thus b has rate k, and the first correction envelope $J(x) = \dim(\text{Color})U(x)b(x)$ has rate k. The caption is the parent of the fine bond. Its town lies in radius $(m+1)B$, giving sales price

$$\operatorname{ballCost}((m+1)B)L^4|\text{Axis}|J(x)e^{A_{\text{support}}\operatorname{ballCost}(m+1)},$$

also of rate k, since its geometric caption and covering factors have rate zero at fixed parameters. Higher quadratic-chain envelopes are the source's capQ evaluated at U. Their center is proportional to $\frac{1}{2}U^2\operatorname{flagP}^4 Q^2\operatorname{payE}\operatorname{scraps}(B)$, and their ratio is $\operatorname{flagP} Q\operatorname{payE}\operatorname{scraps}(B)$. The ratio has every prescribed negative rate, and so does the center because U has rate zero. Their pointwise bounds retain one scalar per actual arm, rather than splitting a trace into independently labeled colors. The itinerary estimate yields the rate-k chain sales bound for arbitrary prescribed k, retaining

series-index support charge and halo $m \geq 4$, together with coupling and residual factors. More generally a finite caption family of density n, radius-dB supports and constant envelope C_0 costs at most

$$\mathrm{ballCost}(dB)nC_0e^{A_{\mathrm{support}}\,\mathrm{ballCost}(d)}.$$

Indeed at most $\mathrm{ballCost}(dB)n$ supports contain a fixed site, and each has B-covering number at most $\mathrm{ballCost}(d)$. This is the fixed-envelope count for the first correction and local Short letters. In the source lift all envelopes, captions, receipts and rate hypotheses remain components of the realized record. Literal recovery therefore preserves each pointwise inequality and its eventual threshold; the geometric-series argument uses precisely the retained address census.

12.46 True walk supports, bundle budgets and connected tiling

The source's native support $\mathrm{native}(w)$ is the finite set of all visited sites, including start and endpoint. Induction on steps puts each visit within $\mathrm{length}(w)$ of the start. A walk of length at most M fits in one radius-M tile. Otherwise split off M steps; the two supports meet at the split endpoint. Induction and the meeting-support covering inequality give, for $M > 0$,

$$M\,\mathrm{coverNumber}_M(\mathrm{native}(w)) \leq M + \mathrm{length}(w).$$

Length counts every repetition and winding even if the visited set is smaller. A curvature slot's support is exactly the native support of its transport walk concatenated with its four-step plaquette: concatenate visit lists and wrap cover coordinates. Its slot length is that concatenated length. A d-slot bundle rooted at x is the union of slot supports with x inserted. These supports all meet at x, so

$$M\,\mathrm{coverNumber}_M(\mathrm{bundle}) \leq (1+d)M + \mathrm{totalLength}\,.$$

Every bundle site lies within totalLength of x, since each slot length is at most their nonnegative sum.

For $Y \subset \mathrm{halo}_{lB}(\mathrm{parent(bundle)})$, the parent-distance estimate bounds its extent from $\mathrm{parent}(x)$ by $(lB+1)(1+\mathrm{totalLength})$. Coarse footprint conversion at $M = B$ gives

$$\begin{aligned} \mathrm{baseFee} &= \mathrm{preCost}(l) + \mathrm{conversion}(L,l)(1+d), \\ \mathrm{legFee} &= \mathrm{conversion}(L,l)/B, \qquad \mathrm{originFee} = lB+1, \\ \mathrm{coverNumber}_B(Y) &\leq \mathrm{baseFee} + \mathrm{legFee}\,\mathrm{totalLength}\,. \end{aligned}$$

Apply the anchored-to-hit principle with degree equal to true totalLength, extent originFee and these covering coefficients. At a fixed fine root the input is exactly the original slot-family length budget. The root delta and that budget form a product budget; only the parent population L^4 supplies the root count, with no additional volume factor. This recovers the source's bundle_hit bound and its explicitly paid $e^{-\varepsilon m}$ tail for totalLength at least m. Atom footprints are precisely their slot-leg bundles; an ordered path pair is the two-slot bundle with the sum of both actual lengths.

For recovery of this support construction, retain the ordered visit list as part of the source path data, alongside its finite native set. Realization wraps each visit in the original period, and readout leaves that site address unchanged. Taking the image of the list, concatenating two lists, and then forming their finite union therefore commute with readout. Consequently the realized bundle has exactly the original native support, root and total length; repeated visits may collapse in the set while remaining in the length and label budget. The identity-frame section recovers each visit and

coefficient, so the covering and anchored-to-hit estimates use the same support currency and parent census.

The covering currency uses periodic maximum-coordinate separation. Its radius-M centers must be connected at range $2M+1$, so the currency includes connections between disconnected support pieces; it is not an unrestricted ball-counting number. A unit-speed line from x to y has duration separation(x,y): truncate each centered-lift displacement coordinate to $[-j,j]$ at time j, wrap and add x. Each coordinate changes by at most one and reaches its endpoint by the maximum absolute displacement.

For a finite center set D connected at range b, adjoin unvisited centers successively. A first exit along a connecting path supplies an edge from the visited set. Insert the line along this edge and its reverse at the visited endpoint in the current traversal. This adds at most $2b$ steps while retaining every earlier visit. Induction gives a unit-speed traversal visiting D of length at most $2b|D|$; the empty case uses the zero-length trivial traversal. This is the ring-path estimate used for extent and collapse.

A radius-kB halo has a cover whose centers are the image of $[-k,k]^4 \cap \mathbb{Z}^4$, scaled by B and translated to the halo center. Division of each centered coordinate by B leaves a remainder of absolute value at most B, proving coverage. Grid steps connect the centers; periodic identification only decreases their number. Thus $(2k+1)^4$ centers suffice. If a center set D gives a radius-mB cover of X, enlarge the balls to radius $(m+1)B$. Original connecting steps obey $2mB+1 \le 2(m+1)B$, so these enlarged halos meet along the connections. Tiling each halo and joining at these meetings yields

$$\operatorname{coverNumber}_B(X) \le |D|[2(m+1)+1]^4.$$

The same unit-speed line shows that halos of centers connected at range $2M+1$ are unit-range connected: the line lies in the two M halos, apart from the possible middle adjacent step. These arguments require no injectivity of periodic halos. In the lift, the walk record retains its ordered visit list and cover coordinates as well as its finite support. Readout therefore recovers both the true length and the connected covering currency, and the displayed induction and counting arguments commute with literal recovery even when wrapping identifies visits.

12.47 Coarse collapse and anchored-to-hit conversion

For block side L, a fine site is within $L/2$ of its embedded parent. Centered lifts show that embedded coarse separation is at least L times coarse separation. The triangle inequality yields

$$L\,\operatorname{separation}(\operatorname{parent}(x),\operatorname{parent}(y)) \le \operatorname{separation}(x,y) + L.$$

For a unit-step path $R_0,\dots,R_N$, sample parents every LB steps, $B>0$, retaining the final partial segment and allowing repeated samples. The resulting set D is connected at range $2B$, every path parent is within $2B$ of a sample, and $LB|D| \le N+LB$. Realize a scale-M cover of X by a connected center set A of cardinality $\operatorname{coverNumber}_M(X)$. The preceding traversal has length $N \le |A|(4M+2)$. Any two support sites are therefore separated by at most $|A|(4M+2)+2M$, by joining them to centers and traversing between centers.

Sample this traversal at scale LB. If $M \le 2B$, a support point's parent is within $3B$ of its center's parent, hence within $5B$ of D. The sB halo of parent(X) is thus covered by the $(s+5)B$ halo of D, retaining connectedness. Tiling costs at most $|D|\,\operatorname{ballCost}(s+6)$. For $B \ge 1$, the sample count and $M \le 2B$ give

$$L\,\operatorname{coverNumber}_B(\operatorname{halo}_{sB}(\operatorname{parent}(X))) \le [L + 10\,\operatorname{coverNumber}_M(X)]\,\operatorname{ballCost}(s+6).$$

The empty case is immediate. Otherwise at least one center pays the fixed endpoint term. Division by L recovers the preCost and conversion constants in the coarse hit budget. All block factors thus arise from parent separation, connected traversal and halo tiling.

Here is the full anchored-to-hit principle used in those budgets. Suppose nonnegative b_i satisfy, at every fine anchor y,

$$\sum_{o_i=y} b_i e^{Ad_i} \le \gamma,$$

the parent fibers have size at most P, and

$$x \in X_i \implies \operatorname{separation}(x, \operatorname{parent}(o_i)) \le W(1+d_i),$$
$$\operatorname{coverNumber}_B(X_i) \le v + v' d_i, \qquad 0 \le g_i \le C b_i (1+d_i)^k.$$

Assume $a \ge 0$, $\varepsilon > 0$ and $av' + 2\varepsilon \le A$. At degree $d_i = j$, the new weighted coefficient is bounded by $b_i e^{Aj} C(1+j)^k e^{av - 2\varepsilon j}$. Relevant anchors lie above the coarse radius-$W(1+j)$ ball about x, so there are at most $P \operatorname{ballCost}(W)(1+j)^4$ of them. Grouping a finite label set by anchor and applying the old budget bounds the degree-j total by

$$e^{av} \gamma C P \operatorname{ballCost}(W)(1+j)^{4+k} e^{-2\varepsilon j}.$$

The exponential series gives $(1+j)^{4+k} \le \varepsilon^{-(4+k)}(4+k)! e^{\varepsilon} e^{\varepsilon j}$. Summation over $j \ge m$ yields the tail factor $e^{-\varepsilon m}$ with payment

$$e^{av} \gamma C P \operatorname{ballCost}(W) \frac{\varepsilon^{-(4+k)}(4+k)! e^{\varepsilon}}{1 - e^{-\varepsilon}}.$$

Every finite subset obeys this bound, proving nonnegative countable summability as well.

For block parents, decomposition into parent and residue gives $P = L^4$. For old support Y with $M \le 2B$, put $\operatorname{preCost}(l) = \operatorname{ballCost}(l+6)$. Collapse gives

$$\operatorname{coverNumber}_B(\operatorname{halo}_{lB}(\operatorname{parent}(Y))) \le \operatorname{preCost}(l) + \frac{10 \operatorname{preCost}(l)}{L} \operatorname{coverNumber}_M(Y).$$

The extent bound from any chosen anchor is $(l+20)B(1 + \operatorname{coverNumber}_M(Y))$; these supply v, v', W. The connected traversal and coarse-collapse arguments above prove these covering and extent bounds; the displayed summation then supplies the hit budget. Inspection adds at most four fine-site steps and hence five parent steps, so increasing l by five includes its enlarged support. The quantifier order is retained: for $A > 2\varepsilon$, choose

$$L > \frac{10 \operatorname{preCost}(l) a}{A - 2\varepsilon}$$

before choosing x. Then $a\, 10 \operatorname{preCost}(l)/L + 2\varepsilon \le A$. For paths the coefficient is $10 \operatorname{preCost}(l)/(LB)$; with L fixed it tends to zero as $B \to \infty$, so any $\sigma > 2\varepsilon$ suffices eventually. The same delay ensures $\operatorname{radiusB}(\operatorname{betaF}) \le 2 \operatorname{radiusB}(x)$. Fixed powers of these radii and the monitor are subpower in betaF, while $e^{-c\,\mathrm{till}(x)}$ and $e^{-c\,\mathrm{monitor}^z}$, $c, z > 0$, beat every inverse power. These justify strict-exponent absorption without changing L after the delay. The lifted record retains parent fibers, connected covers, anchors, degrees and the order of parameter choices; literal readout preserves each finite-subset bound and its monotone passage to the countable budget.

Spanning-tree proof of the extent estimate. The centre-traversal construction used in the extent estimate is proved as follows. Realize the scale-M cover of a support X by a centre set A with $|A| = \mathrm{cov}_M(X)$ whose union of M-balls is connected by unit steps. Join two centres when their distance is at most $2M + 1$. This centre graph is connected, since a lattice path through the union passes from one ball to another only through overlapping or adjacent balls. A finite connected graph has a spanning tree (add an edge from the reached vertices to a new vertex until all are reached). Traverse the tree out and back: each of its $|A| - 1$ edges is replaced by a lattice path of length at most $2M + 1$ and is used twice, giving a unit-step path through every centre of length

$$N \le 2(|A| - 1)(2M + 1) \le |A|(4M + 2).$$

The empty cover has the constant path of length zero. Joining two sites of X to covering centres by paths of length at most M gives

$$\mathrm{diam}(X) \le |A|(4M + 2) + 2M,$$

on the periodic lattice as well as on the integer cover. The estimate is a statement about the support column of the realized tuple, which the readout does not alter; the spanning tree is a witness that may be chosen freely (its choice is a presentation ambiguity with no effect on the bound), and no frame data enters.

12.48 Support tariffs and path-tail conversion

For nonnegative envelopes w_i, supports X_i and integer extra charges e_i, the sales condition includes summability and, at every coarse site,

$$\sum_{i : x \in X_i} w_i e^{2\varepsilon e_i} e^{a\,\mathrm{coverNumber}_B(X_i)} \le \gamma.$$

Disjoint alphabets add their budgets. An additional finite orientation index costs the sum of orientation budgets. Rotation or translation preserves the bound because its site map is bijective and does not increase covering number. For nonempty X_i and $a, \varepsilon \ge 0$, evaluation at one of its sites gives $w_i \le \gamma$.

Reserve an old anchored support tariff before conversion by setting $v_i = w_i e^{2\varepsilon\,\mathrm{coverNumber}_M(Y_i)}$. Exactly,

$$v_i e^{(\delta - 2\varepsilon)\,\mathrm{coverNumber}_M(Y_i)} = w_i e^{\delta\,\mathrm{coverNumber}_M(Y_i)}.$$

Thus the original budget is unchanged with the reduced exponent. If X_i lies in the l-halo enlargement of the parent image of Y_i, $M \le 2B$, and the geometric reserve inequality holds, apply coarse conversion to v_i. This pays the new 2ε tariff for $z_i \le C w_i (1 + \mathrm{coverNumber}_M(Y_i))^k$. The payment is the preceding explicit anchored-to-hit payment with child factor L^4, $\mathrm{extentCost}(l, B)$, degree k, old budget γ, new support exponent a and geometric $\mathrm{preCost}(l)$. These retain the source's (NF9)–(NF10) dependencies.

For rooted bundles start instead from the true-length budget $\sum_i w_i e^{\delta \ell_i} \le \gamma$ and set $v_i = w_i e^{2\varepsilon \ell_i}$. The width left for conversion is $\delta - 2\varepsilon$. The bundle estimate supplies sales for $z_{\mathrm{root},i} \le C w_i (1 + \ell_i)^k$ on the allowed enlarged footprint. The root ranges over every fine site, accounted for by originFee and L^4. An inspected order-r atom footprint lies in the $(l + 5)$-halo of its actual path bundle; applying this transfer to its magnitude and exponential length budget gives the atom payment, retaining its degree-k polynomial mark. For paired tails, the parent-distance inequality puts the l enlargement of pairFoot in the $(l + 2)$ enlargement of the two-path bundle. This yields the tail payment with $r = 2$.

The omitted-pair small factor is computed before conversion. For actual slot lengths x, y, a nonzero omitted pair has $xd > R$ or $yd > R$, hence $(x+y)d \geq R$. The pair-price bound and half the length reserve give

$$\mathrm{pairPrice}\, e^{(\sigma/2)(x+y)} \leq 2e^{-\sigma R/(2d)}|u_x|e^{\sigma x}|u_y|e^{\sigma y}.$$

The product of the original budgets, each at most C_{paths}, bounds the half-width sum by $2e^{-\sigma R/(2d)}C_{\mathrm{paths}}^2$. At $R = \mathrm{till}(x)$ this has every prescribed negative rate. Under $8\varepsilon < \sigma$, the remaining width $\sigma/2 - 2\varepsilon$ is positive and supplies the geometric reserve and polynomial mark. Footprint payments multiply scalar envelopes and old budgets by fixed constants and a four-dimensional extent factor. If these have rates u, v and extent rate zero, the payment has rate $u+v$; any strictly larger exponent absorbs the fixed and subpower factors. This recovers the inherited six-family atom and tail rate deductions with the stated geometric conversion dependency. The lifted record retains extra charges, true lengths, masks, orientations and width reserves. Readout recovers each tariff identity and omitted-pair condition, so support conversion does not discard winding, multiplicity or polynomial marks.

The preservation of these sales conditions can be expressed without interchanging an uncontrolled series. For a fixed site x and finite label set F, define the nonnegative tariff reader

$$\mathcal{T}_{x,F}(w, X, e) = \sum_{\substack{i \in F \\ x \in X_i}} w_i \exp(2\varepsilon e_i + a\, \mathrm{coverNumber}_B(X_i))\,.$$

Its lifted version is this reader composed with π, including the recorded site and label addresses. Since readout recovers each support, charge and envelope, every finite sum is identical to its physical reader, and composition with the section recovers $\mathcal{T}_{x,F}$. Taking the supremum over finite F therefore preserves the full nonnegative sum, its bound and its summability. The reweighting identity above is an equality term by term before this supremum. Disjoint family tags preserve addition of budgets even when two letters have equal numerical coefficients or equal footprints. Thus neither a coincidence of values nor periodic identification merges separately charged labels.

12.49 Inherited envelopes and primitive dictionary

Fix halo l and $\varepsilon > 0$ with 4ε below the incoming support exponent and 8ε below both path widths. Choose one sufficiently large L for regular and forced support conversion, retaining the prescribed support price and extra tariff. Use the original envelopes before constructing any new marking reserve. A regular occurrence is bounded by its supplied b_i on its healthy support and is zero otherwise; its anchor is unchanged. A forced coefficient uses d_i without a health restriction. Choose a support inventory point as its anchor: the incoming hit bound implies an anchored bound there. The geometric factors have rate zero for fixed L. The regular rate $-11/10$ allows any eventual price betaF^u with $u > -11/10$; a forced incoming rate $v - 1$ allows betaF^v for any fixed v.

An order-k atom multiplied by $|\eta| \leq D(x)$ has healthy envelope

$$z_{\mathrm{site},j} = D(x)\, \mathrm{fiery}(x)^k\, \mathrm{magnitude}(\mathrm{atom}_j).$$

If D and the total atom budget have rates u, v, its preconversion rate is $u - k/2 + v$. When this is below a target r, choose an intermediate exponent, convert inspected supports and sum finite signed orientations; the strict margin absorbs that count. Retain true path length as extra charge. Omitted pairs pay every prescribed power after any fixed-rate scalar multiplier, using $8\varepsilon <$ width and the actual cutoff $\mathrm{till}(x)$. Sum plaquette types and orientations first at exponent $r - 1$, then

absorb their fixed number into exponent r. The detailed atom and tail support-conversion estimates are still the inputs to these two deductions.

The inherited logarithm has six disjoint families: regular occurrences, cubic atoms, quartic atoms, loop atoms, fine tails and coarse tails. Its phase is $\theta = -\text{orientation}\,\beta$, with $|\theta| \le 2\,\text{betaF}$; the coarse tail uses $-\theta$. Cubic and quartic multipliers have rate one and hence rates $-1/2$ and -1 after the field factors. The loop multiplier has rate zero and its own incoming budget; both tails allow arbitrary negative powers. For $r > -1/2$, choose $-1/2 < s < r$. All six prices are eventually at most betaF^s, weakening the regular bound to s. Their union costs at most $6\,\text{betaF}^s \le \text{betaF}^r$ eventually. Forced letters keep their independent arbitrary-exponent price. This proves the inherited dictionary bound with the actual fine and coarse beta factors already included. Every footprint is nonempty: regular and forced supports retain their anchor or inventory, and atoms and tails retain their root. These facts support the individual-letter smallness argument used in the optional dictionary assembly.

Before taking connected components the non-warning primitive dictionary has three disjoint kinds: inherited logarithms, local Short corrections and Gaussian chain letters. The last kind has five families: the negative actual log-determinant series, the positive blank-reference log-determinant series, the mean-source series, the first linear mean correction, and the remaining quadratic chain correction. Their exponent coefficients are, in order,

$$-\tfrac{1}{2}\,\text{ArmA}, \quad +\tfrac{1}{2}\,\text{ArmA}_{\text{blank}}, \quad +\tfrac{1}{2}\,\text{ArmS}, \quad q_{\text{first}}, \quad \text{chain}.$$

An inherited logarithm is its actual old exponent on the lifted fine field, multiplied by enforcement of its own full footprint at $2a$; inherited forced activities use their actual K coefficient with that enforcement. Profiles lie in $[0,1]$ and preserve bounds. The disjoint sum is farLog; each optional activity is $e^{\text{farLog}} - 1$. Supports are fixed before evaluation: enlarged old supports, Short sleeves, chain receipts and first-correction towns. Extra charges are respectively the old charges, zero for Short and first corrections, and series indices for other chains. Nonempty supports remain assigned even to zero coefficients. This keeps the support price independent of a field-dependent selection.

Let the chain families have common sales bound $N(x)$ of rate $-1/2$. Apply the estimate independently to blank geometry for its positive reference determinant. Short has envelope $Z(x)$ and sales bound $X(x)$ of rate $-1/2$. Its caption density is at most $|\text{Signed}|\,\text{spots}(L) + 2\,\text{itineraryDensity}(L)$. A halo-$l$ sleeve lies within radius $(l+1)B$ of its caption, so its price is

$$\text{ballCost}((l+1)B)\,\text{captionDensity}\,Z(x)e^{v\,\text{ballCost}(l+1)}.$$

The census has rate zero. Unite inherited, Short and chain log envelopes as b_0, adding sales prices. Choose $-1/2 < s < b < 0$ and put

$$T(x) = \text{betaF}(x)^s + X(x) + N(x).$$

It has rate s; uniformly before choosing geometry or fields, eventually $T \le 1$ and $2T \le \text{betaF}^b$. Nonempty supports imply $b_0(i) \le T$. Since $|t| \le b_0(i) \le 1$, $|e^t - 1| \le 2b_0(i)$. Thus $y_i = 2b_0(i)$ bounds both the raw logarithm and Mayer activity, with total sales price at most betaF^b. Forced envelopes keep their separately prescribed betaF^u price. Dummy geometry reads common numerical envelopes only; the bound applies to every admitted geometry, forest and field. This completes the dictionary assembly conditional on its inherited, Short and chain estimates given above. In the source lift, family tags, signs, masks, fixed supports and original envelopes remain in the realized tuple. The inherited log and forced activities have their own preassigned envelopes from the incoming representation. Recovery is componentwise, preserving the disjoint sum, exponential activity and every threshold's quantifier order.

12.50 Local driving and the Gaussian floor

Fix one real accepted sector and its forest; restrict u to active bonds as u_{hot}. Define

$$I_o(u)^2 = \sum_{\mathrm{dist}(o,\mathrm{addr}(e))\le 3B} \|u_{\mathrm{hot}}(e)\|^2,$$
$$R_o(u)^2 = \sum_{\mathrm{dist}(o,\mathrm{root}(p))\le R_{\mathrm{mac}}+2} \|\operatorname{motion}(u)_p\|^2.$$

Each summand norm is at most its corresponding square root, because all summands are nonnegative. Suppose a gauge z makes background links in the $2B$ box of o have chord at most $h \ge 0$. The accepted stencil estimate bounds the difference between flat and background-oriented rows on $\mathrm{Ad}_z\, u_{\mathrm{hot}}$ by $2Cc_G(B+1)hI_o(u)$ for rods rooted within $R_{\mathrm{mac}}+2$. The active-bond buffer places the whole selected rod stencil in the clean region. Adjoint action preserves color norm, and gauge covariance identifies the oriented row with its measured motion component. Thus the flat row is bounded by

$$E_o(u) = R_o(u) + 2Cc_G(B+1)hI_o(u). \tag{NF1}$$

For the logarithmic color variation w of $\mathrm{Ad}_z\, u_{\mathrm{hot}}$, the collected variation vanishes on every forest trunk. The forest link estimate gives, with $o = \mathrm{addr}(e)$,

$$\|u_{\mathrm{hot}}(e)\| \le K \sum_{x\in \mathrm{box}(R_{\mathrm{mac}})} \sum_{\alpha\in \mathrm{PlaqAxes}} \|\operatorname{curl} w(x,\alpha)\| + \|\operatorname{bill} w(o, e_{\mathrm{axis}})\|.$$

The low face row is $\sqrt{q_{\mathrm{dir}}}\operatorname{curl} w$ and the pin row is $-\sqrt{k}\operatorname{bill} w$, with $q_{\mathrm{dir}}, k > 0$. Their norms are bounded by $E_o(u)$, and parent distance puts every face root within $R_{\mathrm{mac}}+2$. Consequently

$$\|u_{\mathrm{hot}}(e)\| \le UE_o(u), \qquad U = K|\operatorname{box}(R_{\mathrm{mac}})||\operatorname{PlaqAxes}|q_{\mathrm{dir}}^{-1/2} + k^{-1/2}. \tag{NF2}$$

This uses no global pure-gauge representation. Clean-background flattening supplies $h \le Ys(x)$ on actual clean boxes, with s of rate $-1/2$. The chord chart writes each transformed link as an exponential of norm at most $s(x)$ and uses its uniform chord bound; its radius is fixed first. Set $M(x) = 2UCc_G(B(x)+1)Ys(x)$. Including the zero inactive case gives

$$\|u_{\mathrm{hot}}(e)\| \le UR_{\mathrm{addr}(e)}(u) + M(x)I_{\mathrm{addr}(e)}(u).$$

The neighborhood census is $\mathrm{swept}(L,d) = \mathrm{ballCost}(d)L^4|\operatorname{Axis}|$. Square with $(a+b)^2 \le 2a^2+2b^2$ and sum. When $2M(x)^2\,\mathrm{swept}(L, 3B(x)) \le 1/2$, absorption yields

$$\sum_e \|u_{\mathrm{hot}}(e)\|^2 \le 4U^2\,\mathrm{swept}(L, R_{\mathrm{mac}}+2)\sum_p \|\operatorname{motion}(u)_p\|^2. \tag{NF3}$$

The census has rate zero and M rate $-1/2$, giving this common tail. Constants are chosen after L, before sector and variation. The real plate is the motion Gram matrix plus identity on inactive coordinates, with exact normalization

$$\langle v, \operatorname{form}^T \operatorname{form} v\rangle = \mathrm{scale}^2 \sum_p \|\operatorname{motion}(u)_p\|^2.$$

Splitting active and inactive Euclidean squares, a common positive constant a in (NF3) gives

$$\langle v, \operatorname{plate} v\rangle \ge \min(\mathrm{scale}^2/a, 1)\|v\|^2.$$

The floor is therefore derived from the accepted branch's local forest inequality and flattening. The lift retains the branch, active mask, gauge, rod stencils, motion rows and metric normalization. Readout recovers their exact restrictions; adjoint isometry and literal recovery transport each local bound and the finite census sum. The resulting lower bound applies to the pulled-back form on the section metric, consistently with the operator-versus-form distinction below.

More explicitly, write H for the fixed active-coordinate projection of this accepted sector and Q = plate. For the tangent readout p and section j with $pj = I$, use jHp for the lifted active-coordinate operation. Then

$$p(jHp) = Hp, \qquad (jHp)j = jH.$$

Thus applying the active mask before or after physical readout gives the same u_{hot}; this statement uses the recorded sector mask. Put $c = \min(\mathrm{scale}^2/a, 1)$. The recovered floor is exactly

$$(p^*Qp)(\zeta,\zeta) = Q(p\zeta, p\zeta) \geq c\|p\zeta\|^2, \qquad (p^*Qp)(jv, jv) = Q(v,v) \geq c\|v\|^2.$$

Here p^* denotes pullback of the quadratic form and the norm on the section is the transported physical norm. The estimate consequently retains the source's Gaussian constant on recovered physical directions; the pullback vanishes on $\ker p$. The forest-coordinate gap derived next has its own specified inclusion and is recovered on that inclusion.

12.51 Forest differentiation and vertical Hessian gap

For a logarithmic direction w vanishing on forest trunks, put $U_t = \mathrm{represent}(tw)$. This stays on the same forest slice for every real t, and $U_0 = 1$. Predictor frame weights are usable near zero: their minimum condition holds in an identity neighborhood and there are only finitely many coarse roots. The nonlinear shell inequality with identity comparison gives

$$\mathrm{chord}(U_t(e)) \leq K\,\mathrm{port}_o(U_t) + \mathrm{chord}(\mathrm{prediction}(U_t)(o, e_{\mathrm{axis}})).$$

Bound each port by the full macro-box plaquette-chord sum. Periodic plaquette repetitions only enlarge this nonnegative sum. For an identity-based group curve V_t whose matrix derivative is $\mathrm{colorMat}(v)$, the chord normalization gives

$$|t|^{-1}\,\mathrm{chord}(V_t) \longrightarrow \|v\|.$$

Indeed take the norm of the matrix derivative quotient and divide by the fixed color scale. Edge, plaquette and actual predictor derivatives are the collected edge variation, linear curl and predictor bill. Divide the shell bound by $|t|$ and pass to zero through nonzero real t, including its finite port sums. This proves the forest inequality used in (NF2), with the pin term equal to the derivative of the chosen nonlinear predictor.

For $X = \mathrm{box}(R_{\mathrm{mac}})$, squaring that inequality and applying finite Cauchy–Schwarz gives

$$\sum_e \|w(e)\|^2 \leq \beta \sum_b \|\,\mathrm{curl}\, w(b)\|^2 + \gamma \sum_b \|\,\mathrm{bill}\, w(b)\|^2,$$

where

$$\beta = 2K^2L^4|\,\mathrm{Axis}\,||X|^2|\,\mathrm{PlaqAxes}\,|, \qquad \gamma = 2L^4.$$

For a fixed offset, the map from coarse root and plaquette type to fine plaquette is injective. Summation over roots therefore costs $|X|$, even when distinct offsets give the same periodic site. Parent fibers have exactly L^4 sites; these account for all factors without a period-dependent norm

equivalence. On a vertical forest coordinate v, the stretched collected field is zero on trunks and v_e on complementary bonds. The tested Hessian has lower bound $a\sum_b \|\operatorname{curl} w(b)\|^2 + k\sum_b \|\operatorname{bill} w(b)\|^2$, where $a > 0$ is the curvature-metric floor. The pin sign follows from vertical coarse mismatch equal to minus the bill. Hence

$$\operatorname{Hess}(v,v) \geq \frac{\min(a,k)}{1+\beta+\gamma} \sum_{e \notin \text{trunks}} \|v_e\|^2. \tag{NF4}$$

Restriction of the Hessian is composition of the joint analytic action with its linear vertical inclusion, so the second derivative is the joint second derivative on two included vertical vectors. These forest verticals remain distinct from the frame-fiber verticals of the source lift. The lifted realization retains the curve, predictor branch, trunk mask, linear inclusion and finite offset census. Readout recovers the curve before differentiation; its finite-dimensional linear differential therefore recovers curl, bill and the restricted Hessian, including the normalization and lower bound above.

12.52 Local stencil transport and constructed logarithmic backgrounds

The local stencil comparison in (NF1) has a separate elementary transport proof. A carrier from x to y has length at most $4\operatorname{dist}(x,y)$. On the declared surrounding radius-d box, with link chords at most h and the carrier path contained there, the source's product-chord estimate gives carrier chord at most dh. Replace exterior links by identity: locality preserves the carrier, so only local hypotheses enter. The adjoint inequality

$$\|\operatorname{Ad}_g b - b\| \leq 2c_G \operatorname{chord}(g)\|b\|$$

bounds straightening of a local variation of size M by $2c_G dhM$. A stencil of norm at most C transfers this to $2Cc_G dhM$. Explicitly, cut the variation difference to its lookup neighborhood; its sup norm obeys this local bound and the stencil reads the same difference. A signed coordinate change adds at most $2c_G hM$ for a reversed link and zero for a forward link. Its transformed lookup lies in the original box enlarged by one. Hence

$$\|\operatorname{orientedRow}(V,u) - \operatorname{orientedRow}(1,u)\| \leq 2Cc_G(d+1)hM.$$

The box and variation bounds preceding (NF1) give that estimate; adjoint action and signed displacement preserve each individual variation norm.

An impulse remains a single impulse after displacement and carrier conjugation, with unchanged color norm. Untransported impulse decay gives

$$e^{w\operatorname{dist}(e,s\operatorname{focus}(p))}\|\operatorname{orientedRow}_{s,p}(V,\mathbf{1}_e b)\| \leq 2Ce^{w(\operatorname{predReach}+7)}\|b\|.$$

Here $w > 0$ is the common stencil width; the extra one pays a possible signed-link endpoint shift. Thus the local comparison does not introduce a volume factor into the matrix estimate. The same transport calculation preserves the entrywise decay needed for the matrix upper price.

For a clean radius-$(d+1)$ box, $d \leq 32B(x)$, axial flattening constructs z with transformed chord on the radius-d box at most

$$u(x) = 16(2(32B(x)+1)+1)^5 |\operatorname{PlaqAxes}| \, 2\operatorname{fiery}(x).$$

The temperature comparison bounds the clean plaquette threshold by $2\operatorname{fiery}(x)$. Once u enters a fixed logarithm chart, a whole array b agrees with these links on the box and satisfies $\|b\| \leq C_{\log} u(x)$. Radius factors have rate zero and fiery rate $-1/2$, so $H = C_{\log} u$ has rate $-1/2$ and enters the chart eventually. This constructs the small-background function used in (NF3).

For completeness the analytic product chart $\mathcal{B}(a,b) = \log(\exp a \exp b)$ satisfies $\mathcal{B}(0,0) = 0$ and $D\mathcal{B}(0,0)(a,b) = a + b$ by matrix multiplication and the local inverse relation. A bounded second derivative on a small closed ball yields

$$\|\mathcal{B}(a,b) - (a+b)\| \le C\|(a,b)\|^2, \qquad \|\mathcal{B}(a,b)\| \le C\|(a,b)\|.$$

The exponential recovers the product there. Coordinatewise application gives uniform finite-lattice sup-norm constants and a genuine left chart with this quadratic error. The fixed chart has

$$\log 1 = 0, \qquad \exp(\log U) = U, \qquad \|\log U\| \le C\,\mathrm{chord}(U).$$

The last bound follows from chart Lipschitz continuity and $\|\mathrm{mat}(U) - 1\| = c_G\,\mathrm{chord}(U)$. Choose the radius within both the Lipschitz and right-inverse neighborhoods. Set $b_e = \log U_e$ on requested edges and $b_e = 0$ elsewhere. Then $\|b\|_\infty \le Ch$ and its exponential agrees on the requested set; no global logarithm or continuous cutoff is needed. Finally, for unitary U,

$$UXU^* - X = (U-1)XU^* + X(U^* - 1).$$

Frobenius unitary invariance and submultiplicativity bound this by $2\|U-1\|\|X\|$; division by the color scale proves the physical adjoint bound. Induction using product triangle inequality and inversion invariance proves the path-chord bound, including repeated edges. The realized record retains local lookup sets, carrier words, signed endpoint shifts and chart domains. Recovery therefore preserves each local comparison and impulse bound without extending local hypotheses to the whole lattice.

The local logarithm construction has an explicit recovery domain. For its recorded edge set D and fixed chart radius ρ, let

$$\mathcal{U}_D = \{U : \mathrm{chord}(U_e) < \rho \ (e \in D)\}, \qquad (L_D U)_e = \begin{cases} \log U_e, & e \in D, \\ 0, & e \notin D. \end{cases}$$

Choose ρ within both chart neighborhoods just specified. Its source domain is $\pi^{-1}(\mathcal{U}_D)$ and its source reader is $L_D\pi$. For the coordinate restriction R_D and coordinatewise exponential E,

$$R_D E L_D \pi = R_D \pi, \qquad L_D \pi \sigma = L_D.$$

These identities prove exact patch recovery and section recovery. The bound

$$\|L_D\pi(z)\|_\infty \le C \max_{e \in D} \mathrm{chord}(\pi(z)_e)$$

preserves the same constant. The construction is applied to the gauged physical links appearing in the local comparison. Their gauge and the set D remain recorded data. Consequently the recovered logarithm has precisely the stated local chart domain and the zero extension used by the source construction.

12.53 Curvature remainder and common non-packet envelopes

On a fixed sup-norm ball the local vertex estimate is

$$\|\,\mathrm{curvature}(\mathrm{face}(\exp b)) - \mathrm{curl}\, b\| \le C\|b\|_\infty^2,$$

with a lattice-independent constant. Indeed the analytic four-coordinate map

$$(a_0, a_1, a_2, a_3) \longmapsto \operatorname{curvature}(\exp a_0 \exp a_1 \exp(-a_2) \exp(-a_3))$$

vanishes at zero and has derivative $a_0 + a_1 - a_2 - a_3$. Bounded-second- derivative Taylor expansion gives the quadratic remainder and a uniform linear-part bound. Substitute the actual face coordinates, each bounded by $\|b\|_\infty$, including when periodicity identifies coordinates.

The regular-log and forced-letter assembly uses one envelope independent of sector, forest and Gaussian variable. Fix ε below one quarter of the support exponent and one eighth of each stencil width, footprint weight $v \geq 0$ and retained order $l \geq 4$. Choose the block side for inherited footprints, then fix the positive Gaussian floor, matrix upper prices, chart constants and remaining real-sector parameters. Assume covariance-window error of rate -1. For $-1/2 < b < 0$ and any forced exponent u, common nonnegative weights satisfy

$$|\operatorname{bag}_i(U)| \leq w_i, \qquad |\operatorname{farLog}_p(z)| \leq y_p, \qquad |e^{\operatorname{farLog}_p(z)} - 1| \leq y_p. \tag{NF5}$$

Their weighted footprint sums have budgets β_F^u and β_F^b, respectively. Specified footprint exponentials, overlap prices, radius charges and retained derivatives are paid before reducing to these powers.

To prove the bookkeeping, combine inherited physical, finite local and covariance-window letters into $b_p^{(0)}$. Its budget is $T(x) = \beta_F(x)^s + X(x) + N(x)$, with $-1/2 < s < b$ and X, N of rate $-1/2$. Thus T has rate s and eventually $T \leq 1$, $2T \leq \beta_F^b$. Evaluate uniform estimates on an empty-seed dummy sector at the same radius and cutoff and retain their background-independent weights. Its hypotheses hold and the estimates quantify over all real backgrounds; the weights are chosen before the integrated Gaussian value. Nonnegative weighted sales bounds imply $b_p^{(0)} \leq 1$ individually. The inequality $|e^q - 1| \leq 2a$ for $|q| \leq a \leq 1$ permits $y_p = 2b_p^{(0)}$; the arranged factor-two reserve pays its budget: multiplying the sales budget by two costs at most β_F^b. Forced letters retain their independent price. Real localization factors in $[0, 1]$ can only decrease these bounds, proving (NF5) uniformly for logarithms and Mayer factors.

The actual finite local types are row-energy remainders indexed by signed orientation and row, diagonal logarithmic corrections indexed by fine bond, and Haar-chart corrections indexed by active fine bond. Alongside inherited logarithms, the five chain types are real-plate principal log chains with coefficient $-1/2$, blank-plate chains with $+1/2$, covariance correction chains with $+1/2$, affine first corrections and residual quadratic chains. Chain length is the extra charge; the affine bond term has charge zero. Supports are full receipts enlarged by lB, including every lookup. The lifted record retains all these tags, supports and parameter choices; readout recovers the uniform weights before field evaluation. The local four-coordinate Taylor map is recovered before restricting periodic coordinates, so its bound and (NF5) transport without treating coincident coordinates as independent physical variables.

12.54 Explicit Short letters and their local census

The row Short exponent at center o vanishes within $200B$ of seeds and otherwise is

$$-\beta\vartheta\,[g_o(u)\,\operatorname{light}_o(u) + \operatorname{extracted}_o(\mu)], \qquad 0 \leq g_o(u) \leq 1, \quad 0 \leq \vartheta \leq 1. \tag{NF6}$$

The diagonal term vanishes if either bond endpoint meets the $200B$ seed dome and otherwise equals

$$\tfrac{1}{2}\big(\operatorname{joint}(\operatorname{plate}_{\text{blank}}) - \operatorname{joint}(\operatorname{plate})\big)_e.$$

The Haar term is the localized chart logarithm on active bonds outside the $304B$ seed dome, zero elsewhere. These are exponent signs; the activity is $e^{\mathrm{Short}} - 1$. On an intact real sector, light equals the actual debit. Outside the dome the local slice is interior and the predictor alive. Clean plaquette bounds supply the kept-path test locally; outside, its exponential threshold is at least two, so the universal chord bound supplies it. Thus the exact row-energy split applies with the configuration's existing choices.

A nonzero localization factor supplies the grafted-input variation bound, yielding $|g_o \operatorname{light}_o| \le J(x)$ of rate $-3/2$. Protected means bound each of at most two parent path values by $A(x)$ of rate -1. The extracted term consists of at most two nonnegative half-squares, hence is at most $A(x)^2$. Therefore

$$W(x) = 2\beta_F(x)(J(x) + A(x)^2)$$

bounds the row contribution and has rate $-1/2$. Local flattening and principal-inverse comparison for real and blank plates with the same windows give a diagonal bound E_{diag} of rate $-1/2$. The Haar bound is $C_{\mathrm{Haar}} \operatorname{fiery}(x)$ of rate $-1/2$. Their nonnegative sum gives a common short-letter bound E_{short}. The caption census and radius-$(l+1)B$ support containment give sales budget

$$\operatorname{ballCost}((l+1)B)(|\operatorname{Signed}|\operatorname{spots}(L) + 2\operatorname{density}(L))E_{\mathrm{short}}(x)e^{v \operatorname{ballCost}(l+1)}.$$

It retains rate $-1/2$ and is independent of total volume.

Before union, inherited regular activities use healthy-sector sup envelopes and vanish off the healthy set. A forced activity is anchored at a point of its nonempty inventory. Incoming support budgets then imply anchored budgets. Inspected-footprint conversion pays overlap and charge weights once 4ε is below the incoming exponent and L exceeds the fixed threshold. This yields regular price β_F^q for $q > -11/10$ and forced price β_F^u for every prescribed u. A degree-k translated atom with coefficient rate a and aggregate magnitude rate b has envelope equal to coefficient bound times fiery^k times fixed magnitude; its sales exponent can be any value strictly greater than $a - k/2 + b$. Finite signed orientations cost a fixed factor. Cubic and quartic coefficient rate one, and loop quadratic coefficient rate zero, place all three below every exponent greater than $-1/2$, leaving room for conversion and polynomial charges. True tails, including lower-scale tails, have arbitrary power saving at positive retained cutoff order, with finite orientation and plaquette sums. Choose an intermediate $s < b$ for the six inherited families; the factor six is absorbed by $\beta_F^b/\beta_F^s \to \infty$. This supplies the inherited part of (NF5). In the lift, seed domes, masks, chart domains, window geometries, exact exponent signs and inherited family tags are retained. Literal readout recovers the actual debit identity and each nonnegative census term before taking the uniform sales bound. The inherited letters satisfy their own support estimates before they are joined to these local types.

12.55 Ordered-chain census and exact chart localization

A route consists of a starting fine bond and an ordered list of successors. Allowed hops have endpoints within coarse distance d, so centers are within $d + 2$. The successor count is $P_d = \operatorname{ballCost}(d+2)\operatorname{density}(L)$, where $\operatorname{density}(L) = L^4|\operatorname{Axis}|$. Induction gives P_d^n routes of n hops, retaining repetitions. Attach the mB halo of both endpoints at every node. If $d+2 \le mB$, successive halos meet and their union has B-cover number at most $(n+1)\operatorname{steady}(m)$, with $\operatorname{steady}(m) = \operatorname{ballCost}(m+1)$. Each receipt point lies within $(n+1)(mB+1)$ of the start. For a letter with $n+u$ hops, charge n and magnitude Cq^n on allowed routes, the number of starting bonds for a receipt containing a fixed x is at most $\operatorname{ballCost}((n+u+1)(mB+1))\operatorname{density}(L)$. Using $n+u+1 \le (u+1)2^n$ gives the four-dimensional factor 16^n. Including $e^{2\varepsilon n}$ and the footprint price, the order-n sum is at

most $HC(q\mathcal{R})^n$, where

$$H = \mathrm{ballCost}((u+1)(mB+1))\,\mathrm{density}(L)P_d^u e^{v(u+1)\,\mathrm{steady}(m)},$$
$$\mathcal{R} = 16P_d e^{v\,\mathrm{steady}(m)+2\varepsilon}.$$

For $q\mathcal{R} \le 1/2$ the sales bound is $2HC$, uniformly in site and period. Every endpoint and full lookup halo is counted before shortening. Finite route sums followed by the nonnegative order sum justify the bound, as does bounding every finite subset. This independently recovers the ordered-chain support estimate used in the matrix assembly.

Record the exact localization identities

$$\gamma_a(u,e) = \chi_{\mathrm{inner}}(at, \|u_e\|), \qquad \Gamma_a(u,Y) = \prod_{e\in Y} \gamma_a(u,e).$$

Both are in $[0,1]$. For $at > 0$, $\|u_e\| \le at$ implies $\gamma_a = 1$; $\gamma_a \ne 0$ implies $\|u_e\| \le 2at$ and hence $\gamma_{2a} = 1$. Thus nonzero Γ_a bounds every selected variation and gives $\Gamma_{2a} = 1$ on the same set. For Haar density J, the localized logarithm is $\gamma_{2a}\log J$. If $\gamma_a \ne 0$ and $2at$ lies in the chart, exponentiating this logarithm gives J exactly. For its all-argument bound, if $\gamma_{2a} = 0$ the localized logarithm is zero; otherwise $\|u_e\| \le 4at$. Positivity, analyticity and $J(0) = 1$ give $|\log J(u_e)| \le C\|u_e\| \le 4Cat$ once $4at$ is in the fixed neighborhood. Multiplication by the cutoff preserves the bound, proving the uniform Haar contribution of (NF6) on one common tail. This size bound holds at all arguments, without assuming the argument is in the chart. The lift retains ordered route coordinates, complete receipts, selected edge sets and chart thresholds. Literal recovery preserves these exact cutoff implications and the density identity, as well as every term of the nonnegative census sum.

For a precise source domain, interpret the localized logarithm as the piecewise reader

$$\ell_a(u,e) = \begin{cases} \gamma_{2a}(u,e)\log J(u_e), & \gamma_{2a}(u,e) \ne 0, \\ 0, & \gamma_{2a}(u,e) = 0. \end{cases}$$

On the common tail $4at$ is within the positive chart, so the first branch is defined wherever selected. This defines the zero extension without requiring evaluation of the chart logarithm outside its domain. With $u = p\zeta$, put $\ell_a^Z(\zeta,e) = \ell_a(p\zeta,e)$ and define γ_a^Z, Γ_a^Z by the same composition. Then

$$\ell_a^Z(jv,e) = \ell_a(v,e), \qquad \gamma_a^Z(\zeta,e) \ne 0 \implies e^{\ell_a^Z(\zeta,e)} = J((p\zeta)_e).$$

Finite products on the recorded edge set recover Γ_a componentwise. The same implication and bound $|\ell_a^Z| \le 4Cat$ therefore hold on the whole realized tangent domain. These identities preserve the actual localized density factor used in the correction exponent.

12.56 Exact warning factors and quiet-box witnesses

The five detector addresses are a fine plaquette, two distinct coarse-bond types, a coarse site and a fine site. Their raw factors are respectively fine cutoff, predictor-noise cutoff, pin-frame mass, site-frame mass and contracted flag charge. A fixed district contains each focus and lies within one coarse step of it; warning support is its lB halo. A sixth type is the fine-bond logarithmic-cutoff complement with its two-endpoint support. A detector whose district meets the $300B$ seed dome is compulsory and its optional letter is zero. Otherwise the letter is exactly

$$\Gamma_{2a}(u, \mathrm{plaza}(\mathrm{halo}_B(\mathrm{district})))(\mathrm{rawStorm}(u) - 1). \tag{NF7}$$

Fine-bond warnings are $\gamma_a(u,e)-1$ on active bonds outside the $304B$ dome, zero otherwise. Put $C_{\mathrm{storm}} = 1 + |\,\mathrm{FrameTag}\,| + |\,\mathrm{Points}\,|$. Raw detectors lie between zero and this constant: cutoffs are in $[0,1]$, frame masses are bounded by their finite summand counts and contracted charges are zero or one. Hence all warnings have absolute value at most C_{storm}. For $u = \mu + \beta^{-1/2}z$, quietness of the radius-D box about o means

$$|z_{e,\alpha}| \le \xi t\sqrt{\beta} \quad \text{if } \operatorname{dist}(o,\operatorname{addr}(e)) \le D. \tag{NF8}$$

If $\|\mu_e\| \le Mt$, the Euclidean norm bounded by the coordinate absolute sum gives $\|u_e - \mu_e\| \le n_G\xi t$ and $\|u_e\| \le (M + n_G\xi)t$, without coordinate independence.

For background guide bound $A > 0$, choose first $\tau = \sqrt{q_{\mathrm{dir}}}/32$, then L so that $2(A/L^2 + \tau)/\sqrt{q_{\mathrm{dir}}} < 1/4$, then W so that $2(A+\tau)/\sqrt{k} < W$. For uniform row price Y, a concrete choice is

$$\xi = \min\left((n_G+1)^{-1}, \frac{\tau}{3n_G\,\mathrm{scale}^{-1}\,|Y|+1}\right) > 0.$$

This ensures $n_G\xi \le 1$ and noise row response at most $\tau t/3$. The other two row-bill contributions, nonlinear-to-linear error and protected row at μ, have rate -1 and are eventually at most $\tau t/3$ each since t has rate $-1/2$. Thus actual bills are at most τt. Adding the fine guide At/L^2 bounds weighted curvature by $(A/L^2+\tau)t$; curvature-to-chord conversion and the fine quota give chord below $t/4$. The predictor guide At and penalty k similarly give mismatch chord below Wt. Small-field tag identities make both frame masses exactly one. These statements hold outside the $30B$ dome on quiet radius-B boxes with the actual predictor choices.

If the $2B$ box about a detector focus is quiet, a compulsory letter is already zero. Otherwise the focus is outside the $300B$ dome. Every point within B is outside the $30B$ dome and has a quiet B box. Thus fine and noise cutoffs are on unit plateaus and frame masses equal one. A nontrivial contracted flag would give a touched plaquette of chord larger than t whose parent is in that same neighborhood, contradicting its chord below $t/4$. Its charge is consequently one too, proving (NF7) vanishes. The fine-bond complement vanishes when mean and noise put its variation on the unit plateau. Therefore any nonzero optional warning has a coordinate violating (NF8) in its fixed enlarged support. This holds for each processed warning regardless of other cutoff values; no product of independent warning probabilities is used. The source lift retains detector type, district, compulsory mask, actual predictor tags and the coordinate witness. Literal recovery preserves each raw factor and the quiet-box implication, supplying the same witness to the joint Gaussian catastrophe estimate.

The witness transport can be stated as a set inclusion. For a processed warning i, let $\mathcal{A}_i$ be its recorded finite set of eligible bond–color coordinates in the enlarged support, and write W_i for its actual optional coefficient, including its compulsory mask. With all the preceding mean and threshold hypotheses retained, define

$$E_{e,\alpha} = \{z : |z_{e,\alpha}| > \xi t\sqrt{\beta}\}.$$

The vanishing argument gives

$$\{z : W_i(\mu + \beta^{-1/2}z) \ne 0\} \subseteq \bigcup_{(e,\alpha)\in\mathcal{A}_i} E_{e,\alpha}.$$

Realize the full noise input and let p read its physical variation. Taking preimages under p preserves this inclusion and the finite union exactly; composing with j recovers it because $pj = I$. Thus the witness address is recovered with its actual coordinate threshold. This is an inclusion for each warning coefficient, available before any joint Gaussian integration or selection of several warnings.

12.57 Explicit inherited support-hit payment

For completeness retain the source's explicit normalization of the preceding anchor conversion. Let p have fibers of cardinality at most P, with anchors o_i, nonnegative weights b_i and integer sizes d_i satisfying $\sum_{o_i=y} b_i e^{Ad_i} \le \gamma$ at every fine site. Assume

$$x \in X_i \Rightarrow \operatorname{dist}(x, p(o_i)) \le W(1+d_i), \qquad \operatorname{cov}_B(X_i) \le v + v' d_i,$$

and $0 \le g_i \le C b_i (1+d_i)^k$, $a \ge 0$, $\epsilon > 0$, $av' + 2\epsilon \le A$. Then, with x fixed,

$$\sum_{\substack{i: x \in X_i \\ d_i \ge m}} g_i e^{a \operatorname{cov}_B(X_i)} \le \frac{e^{av} \gamma C P (2W+1)^4 \epsilon^{-(4+k)} (4+k)! e^{\epsilon}}{1 - e^{-\epsilon}} e^{-\epsilon m}. \tag{NF9}$$

At degree j, there are at most $P(2W+1)^4(1+j)^4$ possible anchors. Sum their anchored budgets and use

$$g_i e^{a \operatorname{cov}_B(X_i)} \le b_i e^{Ad_i} C (1+d_i)^k e^{av - 2\epsilon d_i}.$$

One term of the exponential series at $\epsilon(1+j)$ yields $(1+j)^{4+k} \le \epsilon^{-(4+k)} (4+k)! e^{\epsilon} e^{\epsilon j}$. The degree-$j$ bound is therefore the numerator in (NF9) times $e^{-\epsilon j}$. The resulting geometric tail proves (NF9). Every finite address subset obeys it, so infinitely many variants are allowed by nonnegativity.

For block ratio L, parent and four residue coordinates uniquely specify a fine site, giving $P = L^4$. For halo-lB enlargement of the parent image of Y_i, put $d_i = \operatorname{cov}_M(Y_i)$, $M \le 2B$. Then

$$v = (2(l+6)+1)^4, \qquad \operatorname{cov}_B(X_i) \le v + \frac{10v}{L} d_i, \qquad W = (l+20)B.$$

Each point is within $W(1+d_i)$ of the parent of any chosen anchor in Y_i. Absorb the extra tariff $e^{2\epsilon d_i}$ into the old weight first, replacing exponent δ by $\delta - 2\epsilon$. The exact reserve condition is consequently

$$a(10v/L) + 2\epsilon \le \delta - 2\epsilon.$$

This retains the L^4 child factor and polynomial-degree payment explicitly. For the direct geometric proof, with L the block ratio and the periodic maximum-coordinate distance used throughout, centered residues place a site within $L/2$ of its embedded parent, while embedded coarse distances scale by L. Thus

$$L \operatorname{dist}(p(x), p(y)) \le \operatorname{dist}(x, y) + L. \tag{NF10}$$

Centered representatives ensure this also holds on winding paths. The lift retains residues, anchors, degree and old exponent as separate data; recovery therefore preserves the two successive 2ϵ reserves and the explicit payment rather than replacing them by an unspecified constant.

12.58 Realizing the geometric input to (NF9)

Choose a realizing M-cover A of Y with $|A| = d = \operatorname{cov}_M(Y)$. Its centers are connected at range $2M+1$. Starting at one center, repeatedly attach an unvisited center along an edge from the visited connected set and insert an out-and-back detour of length at most $2(2M+1)$. This constructs a unit-step walk visiting every center with $N \le d(4M+2)$. Treat the empty support separately. Sample at times LBj, including zero. Parent samples form D with $LB|D| \le N + LB$. By (NF10), consecutive samples and any intervening walk-point parent versus its preceding sample are within $2B$. A point of Y is within M of a cover center, so its parent is within $3B$ of that center's parent for $M \le 2B$, $B \ge 1$. Thus $\operatorname{halo}_{lB} p(Y)$ lies within $(l+5)B$ of D with the required connected centers.

To replace this larger-radius cover by a B-cover, tile each radius-mB ball with the coordinate grid spaced by B in the four coordinate directions, with $|z_i| \le m+1$. Integer division puts every point within B of a grid point. Grids and their center connections give a connected union of size at most $|D|(2(m+1)+1)^4$; passage to torus images only reduces cardinality. For $m = l+5$,

$$\mathrm{cov}_B(\mathrm{halo}_{lB}\, p(Y)) \le |D|(2(l+6)+1)^4.$$

Since $4M+2 \le 10B$, multiplication by L and the sample count yield

$$L\,\mathrm{cov}_B(\mathrm{halo}_{lB}\, p(Y)) \le (L+10d)(2(l+6)+1)^4.$$

The traversal also gives $\mathrm{dist}(x,y) \le d(4M+2)+2M$ for $x, y \in Y$. Combine this with (NF10) and the halo radius to obtain the anchor-distance bound with $W = (l+20)B$, completing the geometric input to (NF9). The lifted record carries the realizing center set, traversal and sampling times, not merely the resulting number. Readout recovers their torus images and incidences, so connectedness and both cardinality bounds survive periodic identifications and repeated samples.

For clarity, the map p in (NF9)–(NF10) is the lattice parent map; write it as p_{par} in the recovery identities here. It is distinct from the tangent readout used for variations. The finite geometric record contains fine addresses x, y, their residues and parents, the centers A, the traversal and the sample times. Realization retains those discrete data, so its address readout R satisfies

$$R(p_{\mathrm{par}}(x)) = p_{\mathrm{par}}(R(x)), \qquad R(\mathrm{halo}_{lB}\, p_{\mathrm{par}}(Y)) = \mathrm{halo}_{lB}\, p_{\mathrm{par}}(R(Y)).$$

The identity-frame section recovers the original record. In particular $P = L^4$, $v = (2(l+6)+1)^4$ and $W = (l+20)B$ are identical on recovery. First reserving the charge and then applying (NF9) consequently transports precisely the condition

$$\delta \ge 4\epsilon + 10av/L,$$

which is the displayed two-reserve condition rewritten without changing its order of use. This also preserves the full factorial and geometric-tail payment in (NF9), including its degree parameter k.

13 Marked integration and coupling reset

What a marked integral records. Insertions must survive blocking as functions, including when a label occurs more than once. The marker algebra records occurrences separately; only the stated substitutions identify their final readings. This is why the numbered-tree estimates, repeated-occurrence factorials and distinct connected-set expansion all remain in the argument. The primitive integrand is the actual finite integrand with its chart Jacobian, Gaussian density, local interactions, insertion factors and masks. Its normalization is retained before any cluster regrouping.

The readout integrates this record and transports the prescribed insertion. Absolute weighted estimates justify the rearrangements and the passage from finite sums to a countable dictionary. Projection onto the coupling direction is followed by an exact reset identity; the unprojected part remains a residual activity, rather than disappearing. The loan cancellation restores the true kinetic base before the later bounds are used. At a group/logarithm interface, multiplication remains ordered: with $\exp \ell = UB^{-1}$, the neutral residual is $\exp \ell\, BU^{-1}$, not the reversed product. Its off-chart value stays in the record. This local correction preserves the supplied primitive integration and all subsequent paid restoration estimates.

13.1 Insertion algebra, restoration and local prices

Fix finite markers V and their allowed-mode census. A mode is an allowed subset $A \subseteq V$. Multiplication joins disjoint subsets and discards products outside the census. Write $[P]_A$ and e_A for coefficient and monomial; the empty mode is scalar. For positive w_v, define

$$t_w(A) = \prod_{v \in A} w_v, \qquad \|P\|_w = \sum_A t_w(A)|[P]_A|.$$

Triangle inequality and $t_w(A \sqcup B) = t_w(A)t_w(B)$ prove subadditivity and submultiplicativity by expanding disjoint coefficient pairs. On the zero-scalar nilpotent ideal, the finite exponential $\operatorname{Exp} P$ has norm at most $e^{\|P\|_w}$ and obeys exponential addition for commuting arguments. An insertion representation consists of density exponent E, primary functions Φ_v, forced profiles $\omega_v(q)$ and corrections $H(S), K(q)$, where $q = (S_q, I_q)$ is support plus flagged inventory. With scalar profile b_q and configuration flag set $F(U)$, restoration is exactly

$$X(U) = e^{E(U)} \operatorname{Exp}\left(\sum_v e_{\{v\}}\Phi_v(U) + \sum_S H(S,U)\right) \times \sum_{\mathcal{P}:\mathrm{Pattern}(F(U),\mathcal{P})} \prod_{q\in\mathcal{P}} \left(b_q(U) + \sum_v e_{\{v\}}\omega_v(q,U) + K(q,U)\right).$$

Pattern requires pairwise disjoint S_q and inventory union $F(U)$. All sums are finite at fixed torus and census. Corrections have only modes of size at least two, are measurable, gauge invariant, supported on their regions and scoped to markers pinned there. Exponential corrections vanish off the healthy region and have zero scalar part. Flagged corrections require $S_q \cap F(U) = I_q \neq \varnothing$ and $I_q \subseteq S_q$. Scalar multiplication preserves these permissions.

Correction prices use summable nonnegative envelopes a_i with $\|Y_i(U)\|_w \le a_i$ and

$$\sup_x \sum_{i:x\in S_i} a_i e^{ec_B(S_i)} \le f,$$

where c_B is the minimizing cover number. For $e \ge 0$, if each nonzero term has a marker pin in its support, charge it to all such pins. At least one exists by scope and positive degree, and the exponential weight is at least one. Summing the sitewise bound over pins proves $\sum_i \|Y_i(U)\|_w \le |V|f$. For the unmarked background profiles the same argument charges to all sites, giving $\sum_q \|b_q(U)\|_w \le |\Lambda|u$, where u is their sitewise activity budget. If $w_v|\Phi_v(U)| \le x$ and $\sum_q w_v|\omega_v(q,U)| \le y$, the exponential argument costs at most $|V|(x+f)$ and all field factors T_q together cost at most $|\Lambda|u+|V|(y+f)$. Dropping pattern restrictions gives

$$\sum_{\mathcal{P}} \prod_{q\in\mathcal{P}} \|T_q\|_w = \prod_q (1 + \|T_q\|_w) \le e^{\sum_q \|T_q\|_w}.$$

Consequently

$$\|X(U)\|_w \le \exp\{E(U) + |V|(x+y+2f) + |\Lambda|u\}.$$

The lawful-density estimate supplies the uniform majorant. For a normalized mode integral use $t_w(A)|[X(U)]_A| \le \|X(U)\|_w$, integrate against the probability prior and divide by the positive density integral. Its lower bound and the history treasury give the final bill $|\Lambda_k|\tau_k+|V|(x+y+2f)$. This separates polynomial algebra from the vacuum density and denominator estimates. The lift retains the mode census, weights, pins, inventories and permission predicates. Coefficientwise readout is

an algebra map on that same census, commuting with finite exponential, pattern restoration and probability integration; the retained positive denominator then gives exactly the same normalized mode bill.

Explicitly, for a mode-valued coefficient function P define its lift by

$$\mathscr{L}P = \sum_A e_A([P]_A \circ \pi).$$

The allowed-mode census is fixed in this definition. Since the coefficient of a product is the same finite sum over disjoint allowed pairs before and after composition, one has

$$\mathscr{L}(PQ) = (\mathscr{L}P)(\mathscr{L}Q), \qquad [\mathscr{L}P]_A = [P]_A \circ \pi, \qquad (\mathscr{L}P) \circ \sigma = P.$$

The same finite calculation gives $\mathscr{L}(\operatorname{Exp} P) = \operatorname{Exp}(\mathscr{L}P)$ on the zero-scalar ideal and $\|\mathscr{L}P(z)\|_w = \|P(\pi(z))\|_w$. Record the flag set and all support–inventory barcodes so that the pattern predicate is pulled back along with its coefficient. The restoration sum then is exactly $\mathscr{L}X$, including zero terms excluded by the predicate. Finally, for the realized probability prior ν_Z with $\pi_*\nu_Z = \nu$, coefficientwise integration gives

$$\int [\mathscr{L}X]_A \, d\nu_Z = \int [X]_A \, d\nu.$$

The displayed majorant ensures these integrals exist. Applying this also to the scalar density recovers the positive normalizing denominator, so the normalized mode integral and its stated bill are preserved.

13.2 Singleton restoration and response closure

For represented density R, primary function Φ and summable forced family g_i, define the singleton density

$$\operatorname{coin}_R(\Phi, g)(U) = R(U)\Phi(U) + e^{E_R(U)} \sum_i \operatorname{nickel}_{R,i}(U; g).$$

The second term is the first activity-gas variation in direction g. Absolutely convergent grouping by support-and-inventory barcode preserves that variation. Finite polynomial expansion shows the restored scalar mode is $R(U)$ and its $e_{\{v\}}$ coefficient is this coin: only one factor can supply a singleton, all others supply scalars; the exponential has scalar one, and higher corrections supply neither scalar nor singleton. The correction representation must preserve the actual singleton insertions after every step.

Conversely suppose the polynomial is $e^{E_R} \operatorname{Exp}(F) \operatorname{Field}(K, F_R)$, with zero scalar part of F, original background scalar profiles in K and prescribed forced singleton profiles g_v. We show that if the singleton coefficients of the whole expression are $\operatorname{coin}_R(\Phi_v, g_v)$, then the singleton coefficients of F are precisely Φ_v. If its singleton densities are $\operatorname{coin}_R(\Phi_v, g_v)$, subtraction cancels gas variations and gives $R(U)([F]_{\{v\}} - \Phi_v) = 0$. Nonvanishing of $R(U)$ permits cancellation. Projection onto modes of size at least two splits F into prescribed singleton plus higher exponential correction and each K_q into background, prescribed forced singleton and higher flagged correction. The original density is preserved pointwise.

For a misplaced forced singleton, take attached local responses $\psi_v(S, U)$ supported on regions containing pin o_v and vanishing when that region is healthy. Put $J = \sum_{v,S} e_{\{v\}} \psi_v(S, U)$. Closure joins overlapping background factors and responses into connected components, recording support and flagged inventory. Component expansion proves exactly

$$\operatorname{Field}(\operatorname{Closure}(K, \psi)) = \operatorname{Field}(K) \operatorname{Exp} J.$$

Replace K by its closure and F by $F - J$ to preserve the full polynomial. No-response terms reproduce K; terms with a response have zero scalar part; terms with at least two responses have zero singleton part. A sole response is multiplied by the background attached to its support, giving exactly its hook in the singleton profile. Thus no unspecified singleton change is introduced. The converse then identifies primary functions from known singleton densities. Higher-mode projection yields the two higher correction families, preserves locality and permissions, and decreases weighted prices. For field budget W, response budget c and pin occupancy M, the connected-component estimate bounds the higher closure budget by

$$\exp\{W + M\,\mathrm{siteFactor}(B)c\} - 1,$$

provided the support-exponent loss pays $(2B+1)^4\{W + M\,\mathrm{siteFactor}(B)c\}$. The forest-price argument below supplies this bound; marker-monomial reweighting transfers both higher families to the requested weights. These are the algebraic and numerical operations used in the one-step retained-insertion estimate. The source lift retains response attachments, barcode grouping and component partitions. Coefficientwise recovery commutes with finite projection and nilpotent exponential; absolute summability permits the same grouping. It therefore recovers both the exact closure identity and the nonvanishing cancellation establishing the prescribed singleton fields.

In the notation of the coefficientwise lift, let $P_{\geq 2}$ discard scalar and singleton modes. Its definition gives

$$P_{\geq 2}\mathscr{L} = \mathscr{L}P_{\geq 2}, \qquad \|P_{\geq 2}P\|_w \leq \|P\|_w.$$

Preserve each response label and its attachment, even when coefficients coincide. The overlap graph is determined by the recorded supports, so readout and the section recover its vertices, edges, connected partitions and inventory unions. Each component coefficient is a sum of products of these same labelled factors. The algebra-map identity therefore gives

$$\mathscr{L}\big(\mathrm{Field}(\mathrm{Closure}(K,\psi))\big) = \mathscr{L}(\mathrm{Field}(K))\ \mathrm{Exp}(\mathscr{L}J).$$

Absolute summability permits the stated barcode grouping before or after composition. Multiplication by $\mathrm{Exp}(\mathscr{L}(F-J))$ then recovers the unchanged full polynomial. For the converse, the nonvanishing domain is exactly $\pi^{-1}\{U : R(U) \neq 0\}$; there $R \circ \pi$ cancels in the lifted singleton identity. Section recovery returns the same Φ_v and the same two projected higher families, with their retained support permissions and weighted-price decrease.

13.3 Distinct-label component prices and exact closure bijection

Suppose supports X_i and nonnegative f_i obey

$$\sup_x \sum_{i:x\in X_i} f_i e^{(a+w)c_B(X_i)} \leq \gamma, \qquad a \geq 0, \quad (2B+1)^4\gamma \leq w.$$

For finite labels D, let $F_D(Y)$ sum $\prod_{i\in A} p_i$ over $A \subseteq D$ whose every overlap component meets Y, including the empty set, where $p_i = f_i e^{ac_B(X_i)}$. In each component choose a label meeting Y and remove it. Every remaining component meets the removed support. Distinct original components choose distinct labels. Chosen roots and their remainders reconstruct the subset, giving an injection. Allowing incompatible remainders enlarges the nonnegative sum, so

$$F_D(Y) \leq \prod_{i\in D: X_i\cap Y\neq\varnothing} \Big(1 + p_i F_{D\setminus\{i\}}(X_i)\Big).$$

Strong induction gives $F_D(Y) \le e^{wc_B(Y)}$: each factor is at most $\exp\{f_i e^{(a+w)c_B(X_i)}\}$, whose exponent sum is at most $|Y|\gamma \le (2B+1)^4 c_B(Y)\gamma \le wc_B(Y)$. The same estimate also gives the sharper $F_D(Y) \le e^{|Y|\gamma}$. A nonempty connected A whose union contains x is counted in $F_D(\{x\}) - 1$. Ordering its supports so each meets the preceding union and applying the meeting-support inequality proves $c_B(\bigcup_{i\in A} X_i) \le \sum_{i\in A} c_B(X_i)$. As $a \ge 0$,

$$\sum_{\substack{A\neq\varnothing \text{ connected} \\ x\in\bigcup_{i\in A}X_i}} e^{ac_B(\bigcup_{i\in A}X_i)} \prod_{i\in A} f_i \le e^{\gamma} - 1.$$

For countable labels, each finite subsum lies in a finite D and has the same bound. Nonnegativity proves summability without repeated-label or logarithmic expansion.

Apply this to the disjoint union of background and response labels with $\gamma = W + M\,\mathrm{siteFactor}(B)c$. Submultiplicativity prices each component by the product of label envelopes. Restricting to permitted components decreases the sum; grouping by output support and inventory preserves it. This proves the closure price and support-exponent loss above. For the exact identity, expand the field polynomial and $\prod_j(1+u_j)$ and join intersecting chosen labels. A nonzero response is off its healthy region, so its support meets a flag, which belongs to a chosen background inventory. Its component therefore contains a background factor. Original background supports are disjoint, and resulting nonempty inventories partition the original flagged set. Conversely, disjoint permitted components recover the unique background and response label sets by union, with unchanged coefficient products. This bijection proves the identity term by term. Singleton responses obey $u_j^2 = 0$, hence $\prod_j(1+u_j) = \mathrm{Exp}(\sum_j u_j)$. The lift retains distinct labels and component incidence rather than only unions of supports. Recovery preserves the decomposition bijection, polynomial products and finite-subset bound, and thus its countable summability conclusion.

To make the recursion's recovery map explicit, equip each finite D with its recorded label order. In a component meeting Y, select the least label whose support meets Y. Record that root together with the exact remaining label subset of its component. The inverse takes their disjoint unions and restores each root. Thus the injection used for the product bound is defined by labelled incidence alone. Realization preserves the order and incidence, and readout recovers this same root–remainder record. In particular, equal supports never identify different labels. For a finite D, every coefficient product and each cover number in $F_D(Y)$ is consequently recovered literally. Its two bounds

$$F_D(Y) \le e^{wc_B(Y)}, \qquad F_D(Y) \le e^{|Y|\gamma}$$

have the same constants after readout. Taking the supremum over finite label subsets recovers the nonnegative connected-component sum and its $e^{\gamma} - 1$ bound. The closure bijection uses precisely these retained label subsets and inventory unions, so its algebraic identity and its price estimate are recovered by the same discrete construction.

13.4 Finite transport inventory and exact ticket expansion

For a countable leaf family at one dictionary key with common support S, complex charges r_i and real-leaf bounds b_i, assume $\sum_i |r_i| b_i e^{sc_B(S)} < \infty$. Sum real and imaginary charge families separately. Each sum is a real leaf bounded by $\sum_i |r_i| b_i$; locality, gauge invariance, measurability and eligibility pass through absolute convergence. Sound and tuned transport identities pass the quote through the sum. Charges $1, i$ therefore reproduce both original and transported profiles with total marker price at most $2w_v \sum_i |r_i| b_i$. An insertion tale has finitely many keys, so grouping within each key preserves support weights and gives finite singleton leaves. A tale priced A_v costs at most $2w_v A_v$,

reproducing primary minus scalar displacement, original forced profiles, and coarse forced profiles minus response hooks.

For a higher coefficient h in mode A, split real and imaginary parts into two leaves with charges $1, i$. If $|h| \le b_A$, their combined price is $2t_w(A)b_A$. Coefficientwise scope, locality and permissions remain. Include only modes nonzero somewhere. A correction budget f becomes at most $2f$. Dormant regular slots of zero original coefficient exist for every nonempty support, and forced slots for every admissible nonempty inventory with exact eligibility. Place exponential and flagged corrections in these respective slots. Nonzero higher corrections have at least two scoped markers and hence nonempty support. Together both families cost at most $4f$, using this explicit history property: the existence of these dormant dictionary slots is part of the history construction. Their primary sum is $\sum_v e_{\{v\}}(\Phi_v - \nu_v) + \sum_S H_S$; the forced sum is $\sum_v e_{\{v\}}\omega_v(q) + K_q$. Higher leaves have no singleton quote. Occupancy converts singleton root budgets to $M\,\mathrm{siteFactor}(B)d$; with $4f$ this is the numerical budget T.

At a regular key its finite tagged monomials u_i have nonempty modes and $u_i^2 = 0$. Thus

$$\mathrm{Exp}\left(\sum_i u_i\right) = \prod_i (1+u_i) = 1 + \sum_i u_i + \sum_{\substack{J:|J|\ge 2 \\ \text{disjoint allowed modes}}} \prod_{i\in J} u_i.$$

Overlapping modes and products outside the census vanish. Each remaining product is a real product leaf at the same key, with product charge, union mode and product price. Composite leaves have degree at least two, leave singleton quotes unchanged and occur only at regular keys, so forced-slot contributions are untouched. This exactly replaces the exponential by one plus the enlarged key sum. The activity expansion now selects an old scalar or one marked leaf at each key; a ticket uses no key twice. Every term is the unselected old activity product times selected marked ingredients. Only finitely many keys are perturbed; lawful absolute summability permits barcode regrouping and interchange with the finite ticket sum. The result is the exact fine restored density, factoring out $\mathrm{Exp}(\sum_v e_{\{v\}}\nu_v)$. Expanding the represented activity gas now gives the finite transport expression. The exponential at a regular dictionary key must still be represented exactly.

For key k, let p_k sum its unweighted leaf prices. A marker pin in the common support implies $0 \le p_k \le T$. Composite prices are bounded by

$$\prod_{i \text{ at } k} (1+f_i) - 1 \le e^{p_k} - 1 \le e^{p_k} p_k \le e^T p_k.$$

The middle inequality integrates $e^t \le e^{p_k}$ on $[0, p_k]$. Composites share the support of their factors, and distinct keys use disjoint original labels. Weighted sitewise summation costs at most $e^T T$, giving total inventory price $T + e^T T$. Finally a normalized monomial has fee $c = t_w(A)|q|$ and value $e_A(q/c)$. For $c > 0$, multiplication recovers $e_A q$. If $c = 0$, positive marker weights imply $q = 0$; both sides vanish under the zero-inverse convention. The weighted norm is at most one, and its square is zero since $A \ne \varnothing$. The remaining input is the Mark-step reserve on this fully priced inventory. The lift retains key labels, charges, zero-fee convention and ticket exclusions. Coefficientwise recovery commutes with each finite expansion; absolute convergence justifies its countable leaf grouping. Thus the inventory reduction preserves the insertion itself and its transported quotes, not merely their bounds.

For the countable reduction, let f_i denote the real leaf functions at one fixed key, with $|f_i| \le b_i$. If I_N exhausts its labels, then

$$\sup_U \left| \sum_{i \notin I_N} r_i f_i(U) \right| \le \sum_{i \notin I_N} |r_i| b_i \longrightarrow 0.$$

Composition with π satisfies the same tail bound. Hence the regrouped coefficient obeys the literal identity

$$\left(\sum_i r_i f_i\right) \circ \pi = \sum_i r_i (f_i \circ \pi),$$

and the section recovers the original sum. At every transported profile the source's sound and tuned transport identities give the corresponding quote identity retained above. Finite key and mode labels are kept through the real/imaginary split and through each compatible composite subset. Thus every ticket has exactly the original distinct-key exclusion. The algebra lift carries its positive-fee normalization, including the zero-fee branch, term by term. Together with the lawful absolute summability of the remaining scalar activity series, this proves recovery of the exact ticket expansion as well as its $T + e^T T$ inventory price.

13.5 Repeated-occurrence rooted-tree estimate

Put $V_B = (2B+1)^4$, $t = V_B\gamma/\delta > 0$ and $F(Y) = \max\{\delta c_B(Y), 1\}$. For nonnegative vertex weights w_i with sitewise budget γ at exponent $a + \delta$, define

$$K(Y,i) = \frac{\mathbf{1}_{X_i \cap Y \neq \varnothing} w_i e^{(a+\delta)c_B(X_i)}}{tF(Y)}.$$

Every finite row sum is at most one: its numerator sum is bounded by $|Y|\gamma \le V_B c_B(Y)\gamma = t\delta c_B(Y) \le tF(Y)$. A forest on m numbered internal occurrences has parent map to $\{0,1,\dots,m\}$, with external root 0 supported on Y and ranks strictly decreasing toward each internal parent. Child and parent supports must meet. Distinct occurrences may carry the same activity label. For a fixed admissible parent map, the sum over label assignments of its kernel product is at most one. Sum a maximal-rank vertex first: it has no remaining children, so its label occurs only in its row factor. Remove it after applying the row bound and repeat. Nonnegative finite restrictions prove the assertion for arbitrary label sets.

If k_j counts children of vertex j, including the root, then $F(S)^{k_j} \le k_j! e^{\delta c_B(S)}$. For $\delta c_B(S) \ge 1$ use one exponential-series term; below one use $F = 1$, $k_j! \ge 1$. Multiplication over parents cancels each internal extra support weight in K, leaving only the root factor. Thus the original exponent-a overlap weight is at most

$$e^{\delta c_B(Y)} t^m \left(\prod_{j=0}^{m} k_j!\right) \prod_{i=1}^{m} K(S_{p(i)}, i).$$

Label summation and division by the occurrence factorial $m!$ give $e^{\delta c_B(Y)} t^m \prod_j k_j!/m!$. For all maps of m numbered objects into r bins, let $A_m(r)$ sum the products of bin-size factorials. Appending an object multiplies the total by $\sum_j (k_j + 1) = m + r$, so $A_0(r) = 1$ and $A_{m+1}(r) = (m+r)A_m(r)$. At $r = m+1$,

$$\sum_p \prod_{j=0}^{m} k_j! = \binom{2m}{m} m! \le 4^m m!.$$

This counts all maps and hence bounds admissible maps. The forest price is at most $e^{\delta c_B(Y)}(4t)^m \le e^{\delta c_B(Y)} q^m$. If an internal designated root has weights $u_i \ge 0$ with $\sum_i u_i e^{(a+\delta)c_B(X_i)} \le b$, multiply the external-root bound at X_i by $u_i e^{a c_B(X_i)}$ and sum. This gives bq^m with factorial $m!$ for the other occurrences. For $q < 1$, the size tail is $bq^m/(1-q)$. The argument includes $m = 0$, empty external root and arbitrary branching with unchanged normalization. All child factorials are paid together by an exact parent-map count. The lift retains numbered occurrences, activity assignments, parent

ranks and the occurrence factorial separately. Recovery thus preserves repeated labels and the kernel elimination order; the finite nonnegative bounds pass through readout before the countable summation.

The occurrence reader is explicit if $\lambda(r)$ denotes the activity label assigned to numbered vertex r. For a fixed admissible parent map h, set $S_0 = Y$ and $S_r = X_{\lambda(r)}$ for $r > 0$. Its kernel product is

$$\mathcal{K}_h(\lambda) = \prod_{r=1}^{m} K(S_{h(r)}, \lambda(r)).$$

The realization keeps the pair $(r, \lambda(r))$, so readout never replaces an occurrence assignment by its label range. Every finite assignment sum of this product is unchanged, including assignments with repeated labels. A maximal-rank occurrence can therefore be eliminated by exactly the same substochastic row bound. The section restores the assignment and parent map, hence also k_j, $m!$ and the bin-count payment. The source's final geometric bound uses a chosen q with $4t \le q$; its summed tail additionally uses $q < 1$. These conditions are retained with $t = V_B\gamma/\delta$, and the designated-root sum is recovered before passing to its nonnegative size tail. Thus the occurrence multiplicities and the convergence parameter are both preserved explicitly.

13.6 Graph intervals and distinct connected sets

Order occurrence vertices injectively and root at 0. For a connected graph H, let $d(v)$ be shortest-path depth and choose each nonroot parent as its least ordered neighbor at depth $d(v) - 1$. These edges form a ranked spanning tree p, whose parent map uniquely determines its depths. Let U_p consist of original graph edges changing depth by at most one and, for an edge from v to preceding-level u, satisfying $\mathrm{order}(u) \ge \mathrm{order}(p(v))$. Then graphs with breadth-first parent p are exactly

$$\mathrm{stem}(p) \subseteq H \subseteq U_p.$$

The forward implication follows from shortest depths and least-parent selection. Conversely the stem realizes assigned depths, the upper edge constraint precludes shorter paths, and the order constraint recovers the same least parent. These intervals partition connected subgraphs. If an optional edge lies in $U_p \setminus \mathrm{stem}(p)$, toggling it is a parity-reversing pairing and its alternating interval sum is zero. Otherwise the interval is a single tree with absolute sum one. Summing proves the rooted spanning-tree bound on the absolute connected coefficient.

For distinct labels D of size m, a parent map on D and a bijection $\{1, \dots, m\} \to D$ produce a numbered forest. This assignment is injective: its label range recovers D, its assignment recovers the bijection and transport recovers the original parent map. The $m!$ bijections have equal product weights divided by $m!$, so their sum is exactly the unnumbered weight. Restriction to this image proves the distinct-parent estimate and geometric size tail. A finite connected support union meeting x can be rooted at $\{x\}$, joining it to supports containing x and taking a spanning tree. The parent-map sum includes this tree and bounds its product. For the stronger sitewise prefactor choose a label containing x as internal root and charge it to the site's budget. Apply the internal-root estimate to other labels and sum possible roots, only enlarging the bound. Numbering distinct nonroot labels cancels their factorial; permitting the root label among nonroots in the majorant is harmless. Under readiness, the connected-set sum with actual union-support weight is therefore at most $\gamma/(1-q)$. This is the distinct-set bound for positive-multiplicity restoration. The lift retains occurrence ordering, parent map, optional edges and distinct-label bijections. Readout recovers the sign-reversing pairing before absolute values, and preserves the exact factorial cancellation in passage to sets.

13.7 Cluster logarithms with repeated occurrences

Let R be symmetric incompatibility with $R(i,i)$ for every label, and nonempty supports X_i such that incompatibility implies intersection. For real activities the hard-core gas is

$$Z(w) = \sum_{A \text{ compatible}} \prod_{i \in A} w_i,$$

including the empty subset. The logarithm instead uses ordered occurrence tuples $z = (i_1, \dots, i_n)$ with repetitions. On their incompatibility graph, define

$$\varphi(z) = \sum_{\substack{H \text{ connected} \\ H \subseteq \text{incompatibility graph}}} (-1)^{|E(H)|},$$

$$L(w) = \sum_{n \geq 1} \frac{1}{n!} \sum_{z \in I^n} \varphi(z) \prod_{j=1}^{n} w_{i_j}.$$

A singleton graph has coefficient one. Equal labels at different positions are incompatible, essential even though the gas uses distinct subsets. Breadth-first exploration partitions connected subgraphs into intervals with one fixed spanning tree as lower endpoint and an allowed edge set as upper endpoint. These intervals cancel by optional-edge toggling; each remaining interval is one tree. In each interval, if an optional edge exists, toggling that edge is a sign-reversing involution, so the alternating sum is zero. Thus $|\varphi(z)|$ is bounded by the number of compatible rooted spanning trees. Cancellation is performed inside the connected-graph sum before taking absolute values. Assume $a \geq 0$, $\delta, \gamma > 0$, $0 < q < 1$ and

$$\sup_x \sum_{i : x \in X_i} |w_i| e^{(a+\delta) c_B(X_i)} \leq \gamma, \qquad \frac{4(2B+1)^4 \gamma}{\delta} \leq q.$$

The designated-root estimate bounds size-n clusters with root-activity budget b by bq^{n-1}, normalized by $1/(n-1)!$. For an unrooted cluster containing a label with a chosen property, sum possible marked root positions and divide by n; its $1/n!$ normalization is bounded by that rooted expression. Tuple relabeling preserves graph coefficients and products, including repeated labels. If the union contains x, root at an occurrence containing x, with $b = \gamma$. A nonzero connected coefficient has overlap-connected union; meeting-support induction and $a \geq 0$ give

$$e^{a c_B(\cup_j X_{i_j})} \leq \prod_j e^{a c_B(X_{i_j})}.$$

Summing $n \geq m+1$ bounds the absolute logarithmic terms containing x, with actual union-support price, by $\gamma q^m/(1-q)$. Summing site root bounds proves absolute convergence at fixed torus with total at most $|\Lambda|\gamma/(1-q)$. The lift retains occurrence tuples, incompatibility edges, root positions and factorials. Recovery commutes with the finite graph cancellation and preserves the union-support inequality. Nonnegative absolute majorants then justify the same infinite occurrence sum, without identifying tuples that share the same label set.

The interval cancellation is itself a recoverable finite operation. Order potential edges by their recorded endpoint orders. For each tree interval with an optional edge, choose its least optional edge e_p and map H to $H \triangle \{e_p\}$. This operation stays in the same interval, squares to the identity and reverses $(-1)^{|E(H)|}$. Readout retains endpoint orders, depths and the least-parent rule, hence recovers both U_p and this involution before any absolute value is taken. For each finite occurrence

tuple the resulting φ is therefore identical on recovery. The section also recovers every tuple position, including positions carrying equal labels. The subsequent root-position sum keeps its $1/n!$ normalization, while distinct-set relabelling keeps its separate $m!$ cancellation. Finally the nonnegative bound $\gamma q^m/(1-q)$ controls the recovered size tail with its actual union support. This gives an absolute majorant for passage from finite tuples to the full lifted connected series.

13.8 The real logarithmic identity

Positivity of the gas is a conclusion of the identity. First restrict to a finite alphabet of s labels and put $P(t) = Z(tw)$, $L(t) = \sum_{n\geq 1} \ell_n t^n$. Connected-component decomposition, designating one occurrence and choosing its component and remaining graph, gives

$$n[P]_n = \sum_{k=1}^{n} k\ell_k [P]_{n-k}.$$

Position choices and component factorials give precisely this normalization. The cluster estimate gives $|\ell_n| \leq Cq^{n-1}$. Choose $u > 1$ with $qu < 1$. The series and differentiated series converge uniformly on compact subsets of $(-u, u)$, using the summable derivative majorant $Cn(qu)^{n-1}$. The recurrence yields $P' = L'P$, so $(Pe^{-L})' = 0$. Its value at zero is one, giving $P(1) = e^{L(1)}$. For arbitrary alphabets direct finite subsets by inclusion. Every fixed gas subset and occurrence tuple eventually appears unchanged. Dropping exclusions gives absolute gas majorant $\prod_i (1 + |w_i|) \leq e^{\sum_i |w_i|}$, finite by sitewise budgets at fixed torus. The connected-series absolute majorant was established above. Dominated convergence on both sides proves

$$Z(w) = e^{L(w)} > 0, \qquad \log Z(w) = L(w).$$

Removing all labels incompatible with a specified forced set only decreases absolute budgets, so the same identity applies. In the lift, coefficientwise recovery preserves the finite component recurrence. The retained absolute majorants justify passage through readout to the directed-alphabet limit; thus the recovered positive gas and its real logarithm obey this exact identity with the source's occurrence normalization.

Write Z_D and L_D for the gas and connected series restricted to a finite alphabet D. For a parameterized physical activity family $w(U)$, its lifted family is $w^Z(z) = w(\pi(z))$. The finite graph and occurrence identities established above give

$$Z_D(w^Z) = Z_D(w) \circ \pi, \qquad L_D(w^Z) = L_D(w) \circ \pi.$$

At each admitted input the same absolute majorants control the directed limits, yielding these identities for Z and L on the full alphabet. Since the real gas is positive there, scalar functional composition gives

$$\log Z(w^Z) = \big(\log Z(w)\big) \circ \pi = L(w) \circ \pi,$$

and composition with σ recovers the original real logarithm. If a forced set is specified, keep it and its incompatibility incidences in the record. Restricting the alphabet then commutes with readout, so the same recovery identities hold for the forced-compatible gas with its unchanged, or decreased, absolute budgets. The logarithm is consequently recovered on exactly the positive domain supplied by the source identity.

13.9 Primitive marked integrals and independent tree averaging

A marked team is a finite collection of real local leaves with distinct keys, weight equal to the product of bounds and ingredient equal to the product of functions. At fixed coarse configuration and local tree choice, choose a permitted flag sector, local layout and finite warning and regular-exponential selections. Together they form molecule address p. The selected integrand is the paid team ingredient times local vertex factors, zero unless every team key occurs in p. With local data X, sector k and selected integrand $I_{X,k,p}$, its coefficient is

$$a_{X,k}(p) = \mathbf{1}_{\mathrm{approved}(X,k)} \int I_{X,k,p}(h,z)\, d\nu_X(h)\, d\gamma_X(z).$$

Covariance and Gaussian density are those of this same tree choice; selectors and eligibility remain inside the integral. Average over finite layout boundary choices, assigning zero to disallowed layouts and retaining the original scalar overhead. The local vertex and integral estimates are developed in the packet, non-packet and warning arguments below.

Expand warnings by $\prod_j (1+d_j) = \sum_D \prod_{j\in D} d_j$ and regular exponentials by their absolutely convergent subset expansion. Marked regular keys are dormant: their original coefficient is zero and their unmarked increment $e^{-0}-1$ vanishes. Inserting and deleting these keys in selected lists is unique; the team ingredient supplies them. Forced marked keys must be present in the sector, otherwise both expansions vanish. Layout, sector and selection lists are recoverable from the molecule address, giving an injective assignment on permitted data, with explicit inverse obtained by reading these pieces from p. The activity majorant bounds integral norms by team weight times primitive tariffs. Summability, Fubini and integration of summable-norm series yield

$$\sum_p \mathrm{overhead}_X\, a_X(p) = \mathcal{P}(\text{team density})(U).$$

The Gaussian substitution is exact: $d\gamma_X(z) = \theta_X(z)\, dz$, with that same density and tribute prefactor in the transformed integrand. Thus each nonzero molecule occurs with exactly the specified coefficient, and disallowed molecule addresses contribute zero.

A molecule depends only on tree choices in its enlarged ground. Project fluctuations to the finite coordinates used there. Data agreeing on that ground have equal projected covariance and hence equal Gaussian marginal measures; integrands agree on projection fibers. Their integrals coincide. Local measure-preserving gauge coordinate maps give the analogous gauge statement. Compatible grounds are disjoint, so finite independent tree averaging factors:

$$\mathrm{Av} \prod_{p\in A} f_p = \prod_{p\in A} \mathrm{Av}\, f_p.$$

Factor the finite assignment sum over disjoint coordinate sets; unused coordinates contribute one. The overhead is tree invariant, preserving compatibility and inventory conditions after averaging, so averaging the team expansion yields the product of the averaged molecule coefficients. Every leaf key in a compatible molecule set belongs to exactly one molecule. Global distinct-key tickets therefore correspond bijectively to their local restrictions, with amplitudes factoring across molecules. Finite Mark-integral linearity gives exactly

$$\sum_{A\ \text{compatible,permitted}} \mathrm{overhead} \prod_{p\in A} \left(\sum_{D\ \text{ticket}} \prod_{i\in D} v_i\, a_D(p) \right)$$
$$= \mathcal{P}\left(\sum_D \prod_{i\in D} v_i\ \mathrm{teamDensity}_D \right)(U).$$

Primitive-integral and support-hit estimates supply the required summable norms and footprints. The lift retains local data, addresses, selected measures, ground projections and ticket partitions. Recovery commutes with these exact finite assignments and measure pushforwards; retained summability permits the same series integration. This reconstructs the literal Mark expansion used in the gas/log argument.

The measure part of this recovery can be written for each fixed local data choice X. Let R_X read the physical gauge and fluctuation coordinates from the realized integration domain, with realized measure $\widehat{\eta}_X$ satisfying

$$(R_X)_*\widehat{\eta}_X = \nu_X \otimes \gamma_X.$$

This is the same selected tree's measure, including its covariance. Lift the entire integrand, including selectors, by $\widehat{I}_{X,k,p} = I_{X,k,p} \circ R_X$. Then

$$\mathbf{1}_{\mathrm{approved}(X,k)} \int \widehat{I}_{X,k,p}\, d\widehat{\eta}_X = a_{X,k}(p).$$

The section recovers its coordinate formula. For a ground-coordinate projection Q_G, composition gives $(Q_G R_X)_*\widehat{\eta}_X = (Q_G)_*(\nu_X \otimes \gamma_X)$. Thus equal local marginals and fiberwise integrands give the same locality identity after realization. All address and tree-choice labels are retained, so the finite independent average and the global-to-local ticket bijection commute with these integral equalities. The summable integral-norm majorant then permits the same molecule series to be integrated on either side. No tree covariance is replaced during this recovery.

13.10 Forced-dictionary fusion and scalar normalization

Old forced labels $i \in \mathbb{N}$ have support S_i, nonempty inventory $I_i \subseteq S_i$ and coefficient $K_i(U)$. Attached logarithms ℓ_j have nonempty supports T_j and are nonzero only when T_j meets the flag set. Assume summable uniform envelopes for both. Use the disjoint atom alphabet with factors $h_{\mathrm{f},i} = K_i$ and $h_{\mathrm{a},j} = e^{\ell_j} - 1$. A new label is a finite overlap-connected atom set containing an old forced label, whose old forced supports are mutually disjoint. Its support and inventory are the respective unions, its coefficient the product restricted to exact inventory and original eligibility. These labels are countable and include every singleton old label; enumeration by $\mathbb{N}$ only names them. Expand the old gas times $\prod_j (1 + (e^{\ell_j} - 1))$, justified by absolute summability. In a nonzero term each attachment meets a flag supplied by an old inventory and hence an old support. Every component therefore contains an old forced label. Components give disjoint new supports with inventories covering all flags. Conversely a permitted new collection recovers the unique old label and attachment sets by union; nonempty atom supports prevent shared atoms. Exact inventory and disjointness recover local old eligibility. Products agree, proving

$$\mathrm{Gas}_{\mathrm{new}}(U) = \mathrm{Gas}_{\mathrm{old}}(U) \exp \sum_j \ell_j(U).$$

Regular labels and coefficients retain their regular role. Finite products and eligibility indicators preserve gauge invariance, measurability and union-support locality. Prior widening is an injective old-label placement in a universal dictionary with zero unused coefficients, preserving gas and logarithms by absolute reindexing. The regular dictionary and its original coefficients remain the regular part of the fused representation. Thus decomposition and union are inverse on all nonzero terms, with identical products. This choice only names the labels and does not change their functions or bounds. The dictionary fusion used twice in the coarse construction has an explicit formula.

For switched forced budget B_0 and attached-log budget w_0 at exponent $s+1$, inflate old envelopes to $e^{\kappa}b_i$, $\kappa \geq 0$, and assign attachments $2c_j$, where $|\ell_j| \leq c_j \leq 1$ in the admitted logarithm range. Use $|e^x - 1| \leq 2|x|$ there. The combined budget is $\gamma = e^{\kappa}B_0 + 2w_0$. Every new component contains an inflated forced atom, so undoing inflation pays at least $e^{-\kappa}$; other undone factors are at most one. Eligibility is switch independent. If $V_B\gamma \leq 1$, the connected-component bound spending one exponent unit gives output price $e^{-\kappa}(e^{\gamma}-1)$ at exponent s. The selected payment inequalities fit this within output allowance. Unmarked values follow the same fusion formula. Exterior-switch silence passes factorwise: omission of the required switch parent from the union means every factor and eligibility indicator is unchanged when erasing it. These supply value, price and scope after both coarse dictionary changes. The fusion estimate for switched coefficients follows from the distinct-component forest bound already proved.

For intermediate frozen exponent E_0, reset receipt $\mathcal{R}$ with coefficient α, and consumed attached-log sums L_1, L_2, retain the exact normalization equations

$$\begin{aligned} E_{\text{final}} + L_2 &= E_0 - \alpha\mathcal{R}, \\ \exp\{E_0 + L_1 - \alpha\mathcal{R}\} &= \text{overhead } Z_0, \\ e^{E_{\text{final}}}e^{L_1}e^{L_2} &= \text{overhead } Z_0. \end{aligned}$$

The last follows by addition of exponents. The scalar normalization needed after these fusions is an equality of exponents, not a comparison of partition functions. The lift retains atom types, exact inventories, switch scopes, both component decompositions and the reset receipt. Readout preserves their inverse union construction and these scalar equalities, with the paid-envelope bound transported through the same nonnegative connected-component sum.

Explicit fusion recovery uses the atom-set label itself, rather than its enumeration index. For a permitted connected atom set C, write S_C for its support union, I_C for its old-inventory union and $\mathbf{1}_{\text{elig},C}$ for its exact eligibility indicator. Then

$$K_C = \mathbf{1}_{\text{elig},C}\prod_{a\in C} h_a, \qquad K_C^Z = \mathbf{1}_{\text{elig},C}\circ\pi \prod_{a\in C}(h_a\circ\pi) = K_C\circ\pi.$$

The atom tags and incidence record recover S_C, I_C and the inverse component union; composition with σ recovers each coefficient. Absolute summability then transports the gas identity, including its attached exponential, with the same paid-envelope majorant. Apply this to both fusions, retaining their distinct attached sums. The recovered scalar multiplier is exactly

$$\left(e^{E_{\text{final}}}e^{L_1}e^{L_2}\right)\circ\pi = (\text{overhead } Z_0)\circ\pi.$$

Thus the switched expression uses the same recovered Z_0 as its unmarked normalization; subtracting $\log Z_0$ cancels that factor on its positive domain. No separate normalizing constant enters the lift.

13.11 The exact coupling-reset exponent

Both fusion-normalization sides use the same unmarked gas Z_0 as the switched formula; subtraction of its logarithm cancels that normalizer. Retain the stationary path and vacuum choices. For source v on listing X, the kinetic path family R and fixed threshold, support scale and Taylor order p, use the actual healthy masks. Set $\Delta = \delta(X.\,\text{read}(v))$ and retain the committed loop coefficient α. The reset is

$$\beta' = \beta + \alpha + \Delta, \qquad \tau' = \tau - |\text{Site}|^{-1}\sum_i v.\,\text{logRead}_i(1).$$

The receipt $\mathcal{R}(U)$ averages the masked plain low terms along the original paths. It is not obtained by changing paths to match a later choice. Regular-source decomposition into identity value, projected payout, fault and masked rebate consists of absolutely summable series, bounded by the original paid bill and fixed multiples. The calibrated projection, using the simple-group hypothesis and period $n > 20$, sums payouts at their anchors and gives

$$(X.\,\mathrm{read}(v)).\,\mathrm{additive}_p(U) = \sum_i v.\,\mathrm{logRead}_i(1) + \Delta \sum_o \mathrm{heads}_p(o,U) - \sum_i \mathrm{dust}_i(U) + \sum_i \mathrm{untagged}_i(U).$$

For averaged low and high corrections $F_\Delta, H_{\alpha,\Delta}$ and loan sum $L_{\alpha,\Delta}$, termwise identities give

$$(\alpha+\Delta)K_R + F_\Delta + H_{\alpha,\Delta} = \alpha\mathcal{R} + \Delta \sum_o \mathrm{heads}_p(o,\cdot) + L_{\alpha,\Delta}.$$

The kinetic base splits into live low terms and high summits; the high correction cancels $(\alpha+\Delta)$ times each summit. Each low term obeys

$$(\alpha+\Delta)\,\mathrm{live} + \mathrm{fade}_\Delta = \alpha\,\mathrm{maskedPlain} + \Delta\,\mathrm{sketch} + \mathrm{loan}_{\alpha,\Delta}\,.$$

Sketch sums equal head sums. Path bounds make live, masked plain, proxy, fade and summit sums absolute; the identity also proves loan summability. Finite seat averaging preserves this, justifying all cancellations. The reset additive sum is untagged plus $F_\Delta + H_{\alpha,\Delta}$; its attached log sum is dust plus $L_{\alpha,\Delta}$. Substitution into $\tau'|\mathrm{Site}| - \mathrm{inChannels}(\beta') - \mathrm{additive}$, using channel decomposition, yields

$$E_{\mathrm{reset}}(U) + \sum_i \log_i(U) = E_{\mathrm{frozen}}(U) - \alpha\mathcal{R}(U).$$

Exponentiation gives the unnormalized-evaluator identity with its exact constant and signs. It holds when the activity gas vanishes: that gas multiplies both sides and is never divided out. Fusion absorbs the attached log exponential. The lower projection, live-base and loan formulas retain the same threshold and paths. In the lift those paths, masks, projection hypothesis and paid series are realized data; readout recovers them before summation, so absolute convergence transports the exact reset and its vacuum scalar without a new path choice.

An explicit reset reader retains the pair of scalar outputs as well as the transformed dictionary. Put

$$c_0 = \sum_i v.\,\mathrm{logRead}_i(1), \qquad \mathfrak{r}(\beta,\tau,v) = (\beta+\alpha+\Delta,\ \tau - c_0/|\mathrm{Site}|)\,.$$

The listing, projection, paths, threshold masks and finite seat average are included in the same realized input tuple. Their scalar readers give exactly c_0 and Δ, so componentwise readout commutes with $\mathfrak{r}$, and the section recovers its original two outputs. Define the reset residual

$$\mathcal{C}(U) = E_{\mathrm{reset}}(U) + \sum_i \log_i(U) - E_{\mathrm{frozen}}(U) + \alpha\mathcal{R}(U).$$

The termwise low/high and projected-payout identities give $\mathcal{C} = 0$ with the original absolute sums. Composing each finite partial sum with π and taking the same convergent limits yields $\mathcal{C} \circ \pi = 0$. Exponentiation consequently preserves the unnormalized evaluator, and multiplication by the recovered activity gas preserves the identity at every input, including zeros of that gas. This recovers the vacuum shift and coupling shift together with the dictionary reset, rather than treating the shifts as independent parameters.

13.12 Gauge invariance and invariant color forms at the flat germ

Let $F(u) = f(\text{increment}(1, u))$ be the regular C^2 germ. Constant gauge transformations act linearly as $u_e \mapsto \mathrm{Ad}_g\, u_e$. The restriction of $DF(0)$ to each edge is an invariant linear form. Differentiation along $\exp(tx)$ shows it kills all brackets. Its kernel is an ideal, hence by simplicity is zero or the whole algebra. The zero case would be injective and force brackets to vanish, contradicting nonabelianness. Thus $DF(0) = 0$ after summing edge directions. Twice differentiating constant-gauge invariance gives adjoint invariance of $D^2F(0)$. For site field ϕ, set $(\nabla\phi)_{x,i} = \phi(x+e_i) - \phi(x)$. Gauge link exponentials by $\exp(-t\phi(x))$ and take local logarithms, giving a smooth chart C with

$$C(0,0) = 0, \qquad DC(0,0)[u,t] = u + t\nabla\phi.$$

The three link factors contribute $-t\phi(x), u_{x,i}, t\phi(x+e_i)$. The local logarithm recovers the gauged configuration near the identity, and gauge invariance gives $F(C(u,t)) = F(u)$. Its mixed second derivative has no right-side t term, and the D^2C term vanishes since $DF(0) = 0$. Hence $D^2F(0)[\nabla\phi, u] = 0$, and symmetry gives the reversed pairing. Locality transfers Hessians between fields agreeing on support, by comparison of diagonal exponential lines and polarization. These are smooth unmasked germs.

For adjoint-invariant color-bilinear B, differentiation gives $B([x,y],z) = -B(y,[x,z])$. Its antisymmetric part $A = B - B^{\mathsf{T}}$ obeys $A([x,y],z) = A([x,z],y)$. Cycling and bracket antisymmetry identify this value with its negative, so it vanishes. For fixed z, $A(\cdot, z)$ kills brackets and is zero by the preceding kernel argument. Thus B is symmetric. Write $B(x,y) = \langle Tx, y\rangle$ with real self-adjoint T. Invariance and adjoint skew-adjointness yield $T[x,y] = [x,Ty]$. The finite-dimensional spectral theorem supplies a nonzero eigenspace, which this identity makes an ideal. Simplicity makes it the whole color space, so $T = cI$ and $B = c\langle\cdot,\cdot\rangle$. Skew-adjointness follows from the real matrix-trace inner product and cyclicity of commutators. The proof uses no complex or infinite-dimensional color-representation assumption. In the lift these are physical gauge directions in the recovered link space. Frame-fiber directions remain the separate kernel of the frame readout. The recovered smooth chart and its differential transport the physical-gradient null pairing, while the section pulls back the Hessian as a form; neither operation identifies these two notions of verticality.

The Hessian recovery here includes the nonlinear readout chain rule. Choose compatible local charts at the identity and let Π be the readout germ, with $\Pi(0) = 0$, $D\Pi(0) = p$ and section differential j, so $pj = I$. For $\widehat{F} = F \circ \Pi$, the full second derivative is

$$D^2\widehat{F}(0)[\zeta,\eta] = D^2F(0)[p\zeta, p\eta] + DF(0)[D^2\Pi(0)[\zeta,\eta]].$$

The last term vanishes by the recovered critical-germ identity just proved. Consequently

$$D^2\widehat{F}(0)[ju, jv] = D^2F(0)[u,v], \qquad D^2\widehat{F}(0)[j\nabla\phi, \eta] = 0.$$

A frame-fiber vector $\zeta \in \ker p$ also gives a zero pairing, by the same pullback formula. These identities distinguish the lifted physical gradient from the frame-fiber kernel while recovering both null statements with their correct domains. Constant adjoint transformations and the color inner product are retained, so recovery of the invariant color forms preserves their scalar coefficient c as well.

13.13 Affine tests reduce to constant curl

Use centered lifts $z_i(x) = \text{lift}(x-o)_i$ and the linear jet

$$A_i(x) = a_i + \sum_j z_j(x) b_{ij}, \qquad F_{ij} = b_{ji} - b_{ij}.$$

On tested support with $10\,\mathrm{dist}(x,o) < n$, $n \geq 4$, centered coordinates increment ordinarily under either sign of a unit step: the old lift plus the step and the new lift wrap to the same point and remain in the centered range. Thus the discrete curl is F there. If $F = 0$, symmetry of b makes

$$\phi(x) = \sum_i z_i(x)(a_i - \tfrac{1}{2}b_{ii}) + \tfrac{1}{2}\sum_{i,j} z_i(x)z_j(x)b_{ij}$$

a primitive: $\phi(x + e_i) - \phi(x) = A_i(x)$. The diagonal subtraction removes the extra half-diagonal term, and symmetry combines cross terms. Hence curl-free affine fields agree locally with gauge gradients. Locality and the gradient-null Hessian identity show that equal-curl affine fields have the same Hessian pairings; the form depends only on both curls. For positive plane $a = (i,j)$, the wedge with sole component j equal to $z_i w_a$ has curl w_a in that plane and zero in all other positive planes. Summing therefore gives a primitive with exactly the prescribed flux. A threshold-scaled unit-color wedge is paid-admissible since $|z_i(x)| \leq \mathrm{dist}(x,o)$ and allowed growth dominates $1 + \mathrm{dist}(x,o)$; averages of two are admissible by triangle inequality.

For signs ε_i and permutation π, affine pullback has

$$b'_{ij} = \varepsilon_i\varepsilon_j b_{\pi i,\pi j}, \qquad a'_i = \varepsilon_i\left(a_{\pi i} - \mathbf{1}_{\{i \text{ reversed}\}} b_{\pi i,\pi i}\right).$$

The subtraction records the shifted base of a reversed positive link. Substitution of centered-coordinate increments proves the formula on the nonwrapping support, with curl equal to the signed-plane reframe. Pullback of exponential increments is linear on direction fields; the Hessian chain rule at the critical identity gives projected-form covariance. No global torus affine polynomial is identified. The source lift retains the center, tested support, lifted coordinates and reversal shift. Recovery therefore preserves the local primitive and signed pullback, transporting the curl-only Hessian without extending its nonwrapping domain.

This reduction defines a recoverable bilinear form on local flux data. Let $\mathcal{A}_o$ be the affine directions tested on the stated nonwrapping support and let $\mathcal{F}$ collect their constant positive-plane curls. The wedge construction gives a linear right inverse $W : \mathcal{F} \to \mathcal{A}_o$ of curl. Define

$$B_f(F,G) = D^2 f(1)[W(F), W(G)].$$

For any affine A, A' with curls F, G, the differences $A - W(F)$ and $A' - W(G)$ have zero curl. The local primitive formula and the recovered gradient-null identity annihilate every Hessian term containing either difference. Hence

$$D^2 f(1)[A, A'] = B_f(F,G).$$

In particular B_f is independent of the chosen affine primitives on this domain. Lifting its directions through j and using the preceding Hessian recovery gives the same B_f on readout. The recorded signed pullback, including the reversed-link base shift, recovers the same curl reframe and thus the same covariance of this flux form. Both the quotient construction and its section recovery retain the original local support restriction.

13.14 Affine-Hessian coefficients and paid bounds

Choose a unit color ξ in the nonzero simple algebra. The affine wedge $W_a(x)$ has curl x in positive plane a, is linear in x and commutes with color adjoints. Define

$$c_{ab}(f,o) = D^2 f(1)[W_a(\xi), W_b(\xi)].$$

Gauge invariance makes $B(x,y) = D^2 f(1)[W_a(x), W_b(y)]$ adjoint invariant. The proved simple-algebra result and evaluation at ξ, ξ imply

$$D^2 f(1)[W_a(x), W_b(y)] = c_{ab}(f,o)\langle x, y\rangle,$$
$$D^2 f(1)[\text{primitive}(w), \text{primitive}(z)] = \sum_{a,b} c_{ab}(f,o)\langle w_a, z_b\rangle.$$

The latter uses the wedge-sum primitive and bilinearity, with the curl-reduction and invariant-form hypotheses retained. The paid direction $tW_a(\xi)$ gives $t^2|c_{aa}| \le c_f$. Also admissible is $u = \frac{t}{2}(W_a(\xi) + W_b(\xi))$. At the critical identity repeated ordered left derivative equals the diagonal Hessian and

$$4D^2 f(1)[u,u] = t^2(c_{aa} + c_{bb} + 2c_{ab}).$$

The paid bound on this derivative and both diagonal bounds yield $t^2|c_{ab}| \le 3c_f$. These estimates use only the paid directions at the identity, rather than a limiting periodic Fourier test. These are actual normalized paid directions. The projected matrix is c_{ab} on nonwrapping enlarged anchored support, zero otherwise. Bounds hold in either case; the nonwrapping case has the exact flux formula. Signed rotations and translations preserve this cutoff and radius-two support, so the flux pullback gives signed-plane covariance, including the zero case.

Paid calculus controls all ordered left derivatives up to $p.\,\text{order}$ in the anchored direction class on the closed healthy set. Functions are smooth on full neighborhoods of tested points; these are derivatives of unmasked germs. Constant multiplication costs its absolute value, and sums or differences add bounds. Ordered Leibniz for an inner product has at most $2^{p.\,\text{order}}$ splittings, yielding

$$\text{pairingCost}(E,k) = 2^k \| \,\text{innerSL}_E\, \|.$$

Derivative and Hessian subtraction are linear on those neighborhoods; enlarging healthy support restricts tested configurations. These facts justify the residual and clover bills, including boundary points of the healthy sets. The lift retains the paid directions, smooth domains and nonwrapping cutoff. Recovery commutes with these finite-dimensional derivatives and wedge sums, so projected coefficients and their constants are recovered with the same scope rather than by differentiating masks.

Coefficient recovery can be given entry by entry. At the identity section use the lifted wedge $jW_a(\xi)$ and set

$$\widehat{c}_{ab} = D^2 \widehat{f}(1)[jW_a(\xi), jW_b(\xi)].$$

The critical-germ Hessian chain rule gives $\widehat{c}_{ab} = c_{ab}$. Retaining the nonwrapping indicator then recovers the actual projected matrix entry, including its zero branch. In particular the nonwrapping cutoff in each projected entry has already been transported before the label sum is taken. The transported direction norm on the section keeps $tW_a(\xi)$ and $\frac{t}{2}(W_a(\xi) + W_b(\xi))$ in precisely the original paid class. In particular the off-diagonal estimate recovers the explicit calculation

$$2t^2|c_{ab}| \le 4|D^2 f(1)[u,u]| + t^2|c_{aa}| + t^2|c_{bb}| \le 6c_f.$$

Thus the same matrix entries, flux pairing and factor $3c_f$ are recovered without introducing an ambient frame norm into the paid estimate. For higher ordered derivatives, the lifted readers are formed on the recorded smooth neighborhoods and restricted to the transported physical directions; section recovery applies before restriction to the healthy set. This preserves the stated Leibniz cost at boundary configurations as well.

13.15 Isotropy of the summed coupling projection

For the actual countable activity listing with regularity, paid bounds and signed-motion label permutations, define

$$T_o(a,b) = \sum_{i:\mathrm{anchor}(i)=o} \Pi_i(a,b).$$

The bound $t^2|\Pi_i(a,b)| \le 3c_i$ and summable bills prove absolute convergence and allow finite plane sums to interchange. Reindexing by the exact motion permutation, including anchor, support, value and nonwrapping cutoff transformations, gives

$$Q_{T_{\mathrm{affine}(s,b)o}}(w,z) = Q_{T_o}(\mathrm{reframe}_s\, w, \mathrm{reframe}_s\, z).$$

Test single-plane unit-color fluxes. Translations and plane permutations make every diagonal at every anchor equal to $\Delta = T_0((0,1),(0,1))$. For distinct planes, reflect an axis in one but not the other. This fixes plane labels and reverses their mixed pairing, so its coefficient equals its negative and vanishes. Hence

$$T_o(a,b) = \Delta \mathbf{1}_{a=b}, \qquad \sum_{\mathrm{anchor}(i)=o} Q_{\Pi_i}(w,w) = \Delta \sum_a \|w_a\|^2.$$

Use the actual clover flux and the same singleton healthy indicator at the anchor to recover the reset's projected payout. Individual support indicators cannot be substituted without the retained faults accounting for that change. The diagonal paid estimate gives the sharper $t^2|\Delta| \le c$, where c is the anchored listing budget: apply triangle inequality to the anchor-zero diagonal mass. No off-diagonal factor three is spent. Scalar and bound are obtained at the original finite period from the original label orbits. The lift retains those permutations, cutoff masks and summable coefficient families. Recovery therefore commutes with absolute reindexing and plane tests, preserving both the scalar payout and the distinct fault terms required for indicator changes.

For explicit recovery of the anchor sum, let D_N exhaust the labels at o and use the entrywise recovered matrices $\widehat{\Pi}_i = \Pi_i$. The tail estimate

$$t^2 \left| T_o(a,b) - \sum_{i\in D_N} \Pi_i(a,b) \right| \le 3 \sum_{\substack{i\notin D_N \\ \mathrm{anchor}(i)=o}} c_i \longrightarrow 0$$

shows that readout and section recovery preserve T_o entrywise. The recorded motion permutation reindexes this same absolutely convergent family. Thus the diagonal plane test and the mixed reflection test recover $\widehat{T}_o = \Delta I$ with the original Δ. Using the sharper diagonal estimate in that one entry gives

$$t^2|\Delta| \le \sum_{\mathrm{anchor}(i)=0} c_i \le c.$$

Finally the common singleton healthy indicator is part of the payout reader and is pulled back unchanged with the clover flux. Its recovery therefore preserves the scalar payout identity; individual support masks remain separate recorded readers whose differences supply the stated fault terms.

13.16 The local clover counterterm

Petal flux at o is the signed average of four oriented plaquette curvatures per plane, all based at o. Each square touches o and lies within radius two; singleton health bounds its chord by t. Gauge

transformations act by the adjoint at o, and signed motions by signed plane action. Curvature of a length-l word is its matrix word minus identity followed by the color projector, of norm at most inverse color scale. Its paid derivative bill is t colorBill, with

$$\mathrm{colorBill}(G,l,P,k) = G.\,\mathrm{colorScale}^{-1}\,\mathrm{feeWalk}(G,l,P,k).$$

Four coefficients of absolute value $1/4$ give that same bill at $l=4$ for each petal. Ordered Leibniz gives petal-pair bill $\mathrm{pairBill}(G,p)t^2$, where

$$\mathrm{pairBill}(G,p) = \mathrm{pairingCost}(G.\,\mathrm{Color}, p.\,\mathrm{order})\,\mathrm{colorBill}(G,4,p.\,\mathrm{power},p.\,\mathrm{order})^2.$$

For real plane matrix q, define

$$Q_q(U) = \tfrac{1}{2}\sum_{a,b} q_{ab}\langle \mathrm{petals}(U,o)_a, \mathrm{petals}(U,o)_b\rangle.$$

Its bill is at most $\frac{1}{2}\,\mathrm{pairBill}\; t^2 \sum_{a,b}|q_{ab}|$. It is smooth, gauge invariant, radius-two supported and linear in q.

On $U(t)=\exp(tv)$, differentiation of a based word at zero gives its signed link sum. For a square this is

$$[v_j(x+e_i)-v_j(x)]-[v_i(x+e_j)-v_i(x)].$$

For affine fields on the nonwrapping neighborhood it is their constant curl. Period $n>20$ ensures the requisite clover-coordinate increments; the signed average has derivative equal to positive-plane flux w. Consequently

$$Q_q(U(t))/t^2 \longrightarrow \tfrac{1}{2}\sum_{a,b} q_{ab}\langle w_a, w_b\rangle.$$

Simple-group gauge invariance makes the identity a critical point. Taylor's formula identifies the limit with half the diagonal logarithmic Hessian; thus the counterterm Hessian is exactly the q quadratic form on curls. Hessian symmetry and polarization recover mixed affine pairs, establishing residual-decomposition normalization. The lift retains all four based square words, their signs, anchor and period hypothesis. Ordered word recovery and its differential reproduce the clover derivative before the Taylor limit, preserving the counterterm's normalization and paid bill.

For each clover square ω based at o, the ordered frame cancellation gives its physical holonomy reader

$$H_\omega(\pi(r,A)) = r_o^{-1} H_\omega(A) r_o.$$

Apply the original color projector to this recovered matrix word and then take the same four signed averages. This defines $P_o^Z = \mathrm{petals}(\pi(r,A),o)$, with $P_o^Z\sigma = \mathrm{petals}(\cdot,o)$. The counterterm reader is therefore exactly

$$Q_q^Z = \tfrac{1}{2}\sum_{a,b} q_{ab}\langle P_{o,a}^Z, P_{o,b}^Z\rangle = Q_q\circ\pi.$$

The identity section recovers its word derivatives and radius-two support. At the critical germ the Hessian recovery formula gives $D^2\widehat{Q}_q(0)[jv,jv] = D^2Q_q(0)[v,v]$; the clover derivative then recovers the stated quadratic form on the curl of v. Polarization gives the corresponding mixed pairings. The norm and ordered derivative bounds are evaluated on these transported physical directions with the original color metric, preserving the colorBill and pairBill constants.

13.17 Masked residual decomposition

For regular f at anchor o with support D, set $D^+ = D \cup \mathrm{halo}_2(o)$ and let q_f be its projected-Hessian petal counterterm. Define

$$r_f = f - f(1) - q_f, \qquad d_f(U) = \begin{cases} 0, & D^+ \text{ healthy}, \\ f(1) + \mathbf{1}_{\{o\}\ \mathrm{healthy}} q_f(U) - \mathbf{1}_{D\ \mathrm{healthy}} f(U), & \text{otherwise}. \end{cases}$$

Then exactly

$$\mathbf{1}_{D\ \mathrm{healthy}} f = f(1) + \mathbf{1}_{\{o\}\ \mathrm{healthy}} q_f - d_f + \mathbf{1}_{D^+\ \mathrm{healthy}} r_f.$$

On healthy D^+ all smaller tests hold and this is the definition of r_f; on its complement the last term vanishes and this is the definition of d_f. In particular the constant remains outside the original healthy set. For paid bound c_f, projected entries satisfy $|\Pi_f(a,b)|t^2 \le 3c_f$. With q plane types, the counterterm bill is at most $M_p c_f$, where

$$M_p = \tfrac{1}{2} q^2\, 3\ \mathrm{pairBill}(G,p).$$

This counts all double-type entries and their petal pairing bills. Triangle inequality bounds the rebate's paid bill and fault's global absolute value by $(2+M_p)c_f$, while $f(1)$ costs at most c_f. The fault vanishes on healthy D^+ and has a genuine failure witness. The rebate is regular there, vanishes at identity and has zero affine Hessian under the retained nonwrapping and period hypotheses: its counterterm has precisely the original affine Hessian. Gauge invariance and measurability pass to all three functions. Faults read enlarged-support data and final-threshold flags as allowed for forced terms. Signed motions transport petals, projected matrices and actual healthy sets covariantly, commuting with decomposition. Projection and counterterms are linear, so differences of faults and rebates equal the fault and rebate of the difference. This permits the common parameterized source-series reset. The lift retains every distinct health predicate and the failure witness. Recovery preserves the two-case identity pointwise; derivative recovery acts on the regular unmasked rebate germ, preserving its zero affine Hessian and paid bound.

Write h_D, h_+ and h_o for these three distinct healthy-set indicators. Their lifted readers are $h_D \circ \pi$, $h_+ \circ \pi$ and $h_o \circ \pi$, respectively. With $r_f^Z = r_f \circ \pi$ and $d_f^Z = d_f \circ \pi$, the recovered pointwise decomposition is

$$(h_D f) \circ \pi = f(1) + (h_o q_f) \circ \pi - d_f^Z + (h_+ \circ \pi) r_f^Z.$$

It follows by composition of the two cases, with the constant $f(1)$ unchanged. The section recovers all four terms and their exact predicates. If $d_f^Z(z) \neq 0$, then $\pi(z)$ fails the enlarged-support healthy test, so its recorded failure witness is the original forced-term witness. For affine physical directions u, v, the smooth rebate germ instead satisfies

$$r_f^Z(\sigma(1)) = 0, \qquad D^2 \widehat{r}_f(0)[ju, jv] = D^2 f(1)[u,v] - D^2 q_f(1)[u,v] = 0.$$

Here the critical-germ chain rule is applied to the unmasked rebate before any healthy-set restriction. Thus the measurable fault identity and the smooth residual cancellation are recovered with their separate domains; the same transported paid class retains the $(2+M_p)c_f$ bill.

13.18 Low-path atoms and exact loan cancellation

A low index is a plaquette type and either a direct symbol or an ordered path-row pair (i,j). Direct indices have mark $m = c.\,\mathrm{direct}$, length zero and plain plaquette price. Pairs have

$$m = c_i c_j, \quad l = \mathrm{length}(s_i) + \mathrm{length}(s_j), \quad P(U) = \tfrac{1}{2}\langle s_i(U), s_j(U)\rangle.$$

Their flux analogues are respectively $\|w_a\|^2/2$ and $\langle w_{\mathrm{plaq}(s_i)}, w_{\mathrm{plaq}(s_j)}\rangle/2$. Evaluate on anchor petals to obtain proxy Q. Home H contains the anchor's B halo and both true path stencils for a pair, just the halo for a direct index. Let h_H, h_o be home and singleton health indicators. Live J is the clipped direct price or $\mathbf{1}_{\mathrm{kept}} mP$. Define

$$F_\Delta = \Delta m(Q-P), \qquad L_{\alpha,\Delta} = \begin{cases} 0, & h_H = 1, \\ (\alpha+\Delta)J - \Delta m h_o Q, & h_H = 0. \end{cases}$$

Healthy home implies kept selection, subthreshold direct plaquette, $J = mP$ and $h_o = 1$. Substitution, and the defining loan formula on its complement, prove in all cases

$$(\alpha+\Delta)J + h_H F_\Delta = \alpha h_H mP + \Delta m h_o Q + L_{\alpha,\Delta}.$$

This includes a healthy anchor inside an unhealthy home.

The selector is local: outside the B aperture its exponential threshold exceeds universal chord two by the decent-budget hypothesis; inside, plaquette corners lie in the aperture unless their test is automatic. Slots read true route and plaquette stencils, proxies the radius-two clover contained in H. Gauge conjugation preserves all chords, prices, predicates and transported inner products. All these functions are measurable; a loan is a finite combination of continuous functions and measurable healthy/kept indicators. Its extra flag read is precisely the final-threshold read allowed for forced labels. Put $t = p.\,\mathrm{threshold}$, $\rho = c.\,\mathrm{relaxation}$. Kept slots obey $\|s_i\| \le t e^{l_i/\rho}$. Chord two and the direct clip kink inequality give globally $|J| \le |m| 2t e^{l/\rho}$. Healthy-anchor proxy bill is $\frac{1}{2}\,\mathrm{pairBill}(G,p)t^2$, so

$$|L_{\alpha,\Delta}| \le |m| e^{l/\rho} \left(2t|\alpha+\Delta| + \tfrac{1}{2}|\Delta|\,\mathrm{pairBill}(G,p)t^2\right).$$

A nonzero loan witnesses unhealthy H, retaining length and relaxation for support charging. Loans are linear in (α,Δ) and freshness in Δ, so parameter differences use these same estimates. Along an affine exponential line, differentiating each finite slot word gives the constant plaquette curl. This follows by differentiating the finite link product; the two nonconstant affine edge terms cancel to the constant curl. On nonwrapping homes the plain and proxy values divided by the line parameter squared have the same flux limit. Their difference has zero quadratic limit. Smooth healthy freshness vanishes at identity; its affine Hessian is zero by Taylor's formula and polarization, under the retained period and nonwrapping hypotheses. The lift keeps ordered pairs, routes, masks and selector apertures. Literal recovery preserves the casewise cancellation and failure witness; differential recovery preserves the equal flux limits and zero affine Hessian.

For recovery of the parameterized loan family, define physical readers

$$A_H = (1-h_H)J, \qquad B_H = (1-h_H)(J - m h_o Q).$$

Then its piecewise definition is equivalently $L_{\alpha,\Delta} = \alpha A_H + \Delta B_H$. Lift A_H, B_H, P, Q and the two distinct masks by composition with π. This gives

$$L^Z_{\alpha,\Delta} = L_{\alpha,\Delta} \circ \pi, \qquad F^Z_\Delta = \Delta m(Q-P)\circ\pi.$$

Both are recovered by the section, and their parameter differences use exactly $L_{\alpha-\alpha',\Delta-\Delta'}$ and $F_{\Delta-\Delta'}$. The full cancellation therefore pulls back term by term with the original mark m and true length l. Nonzero $L^Z_{\alpha,\Delta}$ implies $(h_H \circ \pi) = 0$, recovering the original home failure witness, while its bound retains $e^{l/\rho}$ unchanged. The freshness derivative is taken on its smooth germ; section recovery of both plain and proxy flux limits gives its zero affine Hessian on the specified nonwrapping domain.

13.19 Recovering the actual kinetic base

Keep all ordered low pair labels. For $f_i = c_i s_i(U)$ with $\sum_i \|f_i\| < \infty$, absolute domination by $\frac{1}{2}(\sum_i \|f_i\|)^2$ and continuity give

$$\sum_{i,j} \tfrac{1}{2}\langle f_i, f_j\rangle = \tfrac{1}{2}\left\|\sum_i f_i\right\|^2 .$$

Thus live low sums are exactly the clipped direct price plus the kept shortening half-square. The selector is common to its entire row, so this holds on both branches. For flux analogues apply the same identity to zero-momentum symbol rows $S_{ab} = R.\,\mathrm{symbol0}(a,b)$. Their normalization

$$c.\,\mathrm{direct}\ \delta_{bi} + \sum_a S_{ab}S_{ai} = \delta_{bi}$$

gives

$$\sum_{\text{low } d} m_d\,\mathrm{analogue}_d(w) = \tfrac{1}{2}\sum_a \|w_a\|^2.$$

Weighted path bounds ensure absolute convergence, allowing the common singleton mask and petal substitution. The layer sums translated signed views divided by the number of signed orientations: translations are summed. Translation covariance moves the anchor-zero term to its translated anchor; signed-site reindexing proves that live low plus masked cubic and quartic layers equal the stipulated kinetic energy. The proxy layer equals $\sum_o \mathrm{heads}_p(o,U)$, since signed reframe preserves squared plane norms and the orientation count cancels its reciprocal. These are the reset's actual path identities. Only on everywhere healthy fields may all plain low masks be removed; the resulting layer is the original unclipped second term, the orientation average of $c.\,\mathrm{direct\,price} + \|\ell\|^2/2$.

High freshness retains the cubic and quartic atoms on their own footprints with coefficient $-(\alpha+\Delta)$. Arity-r paid bills are startup coefficients times t^r; $0 < t \le 1$ permits a common t^3 bound for $r = 3, 4$. Each slot is first order along exponential link lines, so division of an arity-r atom by the line parameter to power r has finite limit. Division by its square instead tends to zero. Taylor expansion at the critical identity and polarization give zero diagonal and mixed Hessians in all directions, retaining the existing masks and adding no boundary term. The lift retains ordered pair indices, common selectors, symbol normalization and the orientation-normalized layer. Recovery preserves absolute double sums and finite reindexing, so both kinetic and proxy identities, and the high-term Hessian vanishing, recover with these exact normalizations.

A direct tail estimate makes recovery of the ordered square explicit. For a finite label set D, put $M = \sum_i \|f_i\|$ and $\varepsilon_D = \sum_{i\notin D} \|f_i\|$. Then

$$\sum_{(i,j)\notin D\times D} \tfrac{1}{2}|\langle f_i, f_j\rangle| \le M\varepsilon_D \longrightarrow 0.$$

Each finite pair sum is recovered through the original ordered labels and color inner product, so the same limit recovers the full row half-square. Its common selector can be multiplied before or after this limit. The zero-momentum symbol identity is retained as a matrix identity, and the same reasoning recovers its flux analogue after petal substitution. For a local path term g, the kinetic layer has the precise reader

$$\mathrm{Layer}(g) = \frac{1}{|\,\mathrm{Signed}\,|}\sum_{s\in\mathrm{Signed}}\sum_b g_{s,b}.$$

Composition with readout commutes with this finite sum, preserving the translation census and the orientation reciprocal separately. Consequently the kinetic and head-sum identities recover with

their original volume factors. The cubic/quartic freshness terms use their recorded footprints and smooth germ derivatives; their order greater than two recovers the zero Hessian before the existing masks are evaluated.

13.20 The unmarked local scalar

Use the same positive local pool $C(V)$ as in the density expansion. Its paid, absolutely convergent trained connected-log series sums to $\log C(V)$. Incoming-to-current label equivalence preserves union support. Assign minus each stream as its fixed-support regular coefficient; its support-mask identity gives minus the trained log term. Absolute reindexing therefore gives regular sum $-\log C(V)$. Adjoin original-support path scores, whose masked sum is

$$-\gamma\mathcal{R}(V) - \beta(\mathrm{bill}_{\mathrm{cubic}}(V) + \mathrm{bill}_{\mathrm{quartic}}(V)) - \mathrm{bill}_{\mathrm{loop}}(V).$$

Thus the combined regular dictionary has additive sum equal to this expression minus $\log C(V)$. Anchors are chosen once from nonempty supports before field evaluation. Signed translated views preserve complete sums by finite path-seat and absolute connected-label reindexing, transporting supports and eligibility. Here the dictionary uses full-seat averaging, including translations, with pulled healthy masks; this preserves each complete sum exactly and retains its distinct normalization from kinetic orientation layers.

The positive scalar token is sector prefactor times incoming vacuum exponential times fixed deducted chart factor. Put $\tau = |\mathrm{Site}_c|^{-1}\log(\mathrm{token})$. Separating this from the same-threshold clipped-base exponential gives overhead

$$O(V) = \exp\left\{\tau|\mathrm{Site}_c| - \beta\,\mathrm{average}_{\mathrm{orientations}}\sum_i \mathrm{baseTerm}(V,i)\right\}.$$

Insert the regular sum into the frozen channel exponent. Cubic, quartic and loop bills cancel their path scores; the receipt correction cancels the regular receipt term, leaving $\log O + \log C$. Hence

$$\exp\{E_{\mathrm{frozen}}(V) - \gamma\mathcal{R}(V)\} = O(V)C(V).$$

A prior fusion adds its consumed attached-log sum on the left, recovering the earlier normalization formula. Positivity from the real log theorem justifies exponentiating $\log C$. The same forced canopy-pool gas obeys $O(V)C(V)\,\mathrm{gas}(V) = \mathrm{push}(R;V)$. Multiplying yields the early evaluator identity for every field, including zero gas, without division by gas or evaluation at a selected field. The lift retains the positive token, fixed anchors, exact pool and both averaging conventions. Recovery preserves their scalar equalities and the absolute stream sum, transporting the unnormalized push even at gas zeros.

Let $\mathcal{S}$ be the full finite seat set used for the dictionary, including translations. For its complete pulled sum G_s, invariance gives $G_s = G$, and therefore

$$\frac{1}{|\mathcal{S}|}\sum_{s\in\mathcal{S}} G_s = G.$$

The realized record retains $|\mathcal{S}|$ separately from $|\,\mathrm{Signed}\,|$, so readout preserves this normalization and the kinetic layer's distinct translation sum. The field-independent token $T > 0$ has the same scalar reader on the source object, giving $e^{\tau|\mathrm{Site}_c|} = T$ with the original site count. Pull back the exact pool, overhead, receipt and canopy gas together. Their recovery identities are

$$e^{(E_{\mathrm{frozen}}-\gamma\mathcal{R})\circ\pi} = (OC)\circ\pi, \qquad (OC\,\mathrm{gas})\circ\pi = \mathrm{push}(R;\cdot)\circ\pi.$$

Section recovery returns the original scalar and evaluator identities. These are multiplicative identities on every admitted field, so zeros of the canopy gas retain the same unnormalized push without any cancellation of that gas.

13.21 Pool averaging and literal score series

Probability averaging of single-forest activities preserves each common absolute tariff by triangle inequality. The thin and thick tariffs thus bound averaged entries and deposits. Renewed reserve supplies log-series readiness with unchanged supports and compatibility. Hence

$$\log C(V) = \sum_{q \text{ connected}} \text{Gas. term}(\text{entriesPool}(V), q),$$
$$C(V) = \exp\left(\sum_{q} \text{Gas. term}(\text{entriesPool}(V), q)\right) > 0.$$

The scalar proof uses this gas of averaged entries. Split each nonzero molecule family into nonempty-flag and empty-flag components, retaining nonempty ground. Their subtype union is injective and recovers every nonzero family, reindexing the integrated gas into thick and thin families. At fixed thick family A, excluding intersecting thin supports sets their entries to zero and decreases absolute budgets. The same log theorem and omitted-cluster restoration give excluded gas

$$C(V) \exp\left(-\sum_{q} \text{missing}_A(q, V)\right).$$

Multiply by the banked thick factor and sum to obtain the pool density identity. Throughout this step the forest averaging is performed at the level justified by disjoint-ground factorization, rather than assuming arbitrary products commute with a common forest average. Forest averages factor at disjoint grounds as proved earlier.

A low score label is a signed translated seat and low atom, with coefficient $-\gamma/|\text{Signed}|$ times its marked plain value and support equal to the viewed full home. Pullback of the actual healthy mask gives exactly a summand of $-\gamma\mathcal{R}$. Cubic, quartic and loop score labels carry their transformed footprints and coefficients respectively $-\beta/|\text{Signed}|$, $-\beta/|\text{Signed}|$ and $-1/|\text{Signed}|$ times the unmasked atom. Footprint masking gives the placed atom. Absolute atom budgets justify the sums. Reindexing seat translations by signed action identifies them with orientation averages of all anchors, preserving masks. This proves the exact score total used above, with no change of masks. Definition expansion gives exactly

$$\text{inChannels}(\beta) = \beta \,\text{average} \sum \text{baseTerm} + \beta(\text{bill}_{\text{cubic}} + \text{bill}_{\text{quartic}}) + \text{bill}_{\text{loop}}\,.$$

These are convergent equalities before smallness estimates or cubic coefficient cancellation. The lift retains pool entries, thin/thick subtype maps, omitted-cluster labels, seats and transformed masks. Recovery commutes with the justified probability averages and absolute series, preserving the exact score and channel identities in the reset.

For the pool reader, denote a single-choice integrated entry by $a_X(p)$ and its normalized finite averaging weights by λ_X. Then

$$\bar{a}_p = \sum_X \lambda_X a_X(p), \qquad \sum_X \lambda_X = 1, \qquad |a_X(p)| \le b_p \implies |\bar{a}_p| \le b_p.$$

The local integral recovery already proved gives the same $a_X(p)$ on readout, hence the same averaged entry and tariff. Form the gas and its connected logarithm from these recovered $\bar{a}_p$. At fixed

thick family A, retain the support exclusion indicator $\chi_A(p)$, so the restricted entries are $\chi_A(p)\bar{a}_p$. Readout preserves this indicator and the omitted connected labels, giving the same restoration factor

$$C \circ \pi \ \exp\left(-\sum_q \mathrm{missing}_A(q,\cdot) \circ \pi\right).$$

Absolute majorants justify the displayed sum. The thick/thin decomposition and its inverse union preserve every nonzero family's labels and grounds, so its density identity is recovered term by term. Finally each literal score coefficient is composed with readout together with its viewed full home and mask. Section recovery therefore returns the same three path bills and channel decomposition with their original seat coefficients.

13.22 Detector-to-warning implication with separate error bills

A radius-B quiet watch at o bounds fluctuation coordinates by $|z_i| \le \xi t/g$. For field $\mu + gz$, color dimension d and $\|\mu_e\| \le Mt$, coordinate absolute sums give

$$\|(\mu+gz)_e - \mu_e\| \le d\xi t, \qquad \|(\mu+gz)_e\| \le (M + d\xi)t.$$

A $2B$ watch supplies B watches at nearby sites. Choose the stationary guide bound A first, then $\tau = \sqrt{c_{\mathrm{direct}}}/32$, then L with $A/L^2 \le \tau$, then $W > 2(A+\tau)/\sqrt{c_{\mathrm{penalty}}}$. These ensure

$$\frac{2}{\sqrt{c_{\mathrm{direct}}}}(A/L^2 + \tau) < \tfrac{1}{4}, \qquad \frac{2}{\sqrt{c_{\mathrm{penalty}}}}(A+\tau) < W.$$

Keep three discrepancy contributions separate: fluctuation response, protected mean and nonlinear path-approximation remainder. Choose $\xi > 0$ with $d\xi \le 1$ and response at most $\tau t/3$, by the source's linear coefficient bound. Mean and approximation errors have reference-coupling order -1; each is eventually at most $\tau t/3$ since t is at least inverse square-root coupling. Total discrepancy is at most τt. Outside mandatory seeds, the same-link clean-patch and grafted-field estimates put site and pin tags on their plateaus and identify rows with actual fine and mismatch curvatures. Add guide bills At/L^2 and At. Curvature comparison gives fine chord below $t/4$ and mismatch below Wt by the displayed quotas. Fine and noise cutoffs and both tags equal one. A flag would require a nearby chord greater than t, contradicting $t/4$; the forced-charge detector equals one too. Thus each nonmandatory detector, a bounded auxiliary cutoff times a raw factor minus one, vanishes under the $2B$ watch. Mandatory detectors are zero by definition; active-link complements vanish on their plateau using the mean bound and $d\xi \le 1$. This is the required warning implication, dependent on protected means, physical path accuracy, clean-patch tags and curvature comparison. Those local geometric estimates remain dependencies of this paragraph; the Gaussian probability argument does not supply them. The lift retains those separate local error records and plateau predicates; readout preserves the implication with the same witness neighborhoods and ordered parameter choices.

13.23 Joint Gaussian warning estimate

Let centered finite-dimensional Gaussian z have covariance $A \ge 0$ with $v^T A v \le M\|v\|^2$. For coordinate set D and signs ϵ_i, choose $v_i = \epsilon_i$ on D, zero elsewhere. Then

$$\mathbb{E}e^{sv\cdot z} = e^{s^2 v^T A v/2} \le e^{Ms^2|D|/2}.$$

Markov's inequality for $s \geq 0$ on each signed orthant, followed by the $2^{|D|}$ sign union, gives

$$\mathbb{P}(|z_i| \geq t \ \forall i \in D) \leq 2^{|D|} e^{(-st+Ms^2/2)|D|}.$$

This includes degenerate covariance and empty D. Suppose at most m warning addresses and p coordinate addresses lie over one site. Each warning has a coordinate witness within R of magnitude at least t. For finite W there are at most $(V_R p)^{|W|}$ assignments, where $V_R = (2R+1)^4$. Their distinct-coordinate image D satisfies $|D| \leq |W| \leq V_R m |D|$, since one coordinate witnesses at most $V_R m$ addresses. Choose $H \geq 0$ with $M/2 + HV_R m \leq t$ and use $s = 1$. The exponential is at most $e^{-H|W|}$ and the sign factor at most $2^{|W|}$. Thus

$$\mathbb{P}(\text{every warning in } W \text{ has a witness}) \leq (2V_R p e^{-H})^{|W|}.$$

Shared witnesses and overlapping neighborhoods are counted explicitly.

For the actual family $R = 2B$, its census is bounded by

$$m = L^4 |\text{PlaquetteAxes}| + (|\text{Axis}| + |\text{Axis}| + 1 + L^4) + \text{itineraryDensity}(L).$$

Coordinate count p is the same block's tubular census, and M is its actual covariance bound. Its address census is the sum of the fine plaquette types, two coarse bond types, coarse and fine site types, and fine-link itinerary types. The warning threshold is $\xi t_{\text{field}}/g$, $g = \beta^{-1/2}$, with $t_{\text{field}}/g \geq P/2$ for fine monitor P. Put $\epsilon = \xi/4$ and $H = \epsilon\sqrt{P}$. Eventually $V_{2B} m \leq \sqrt{P}$ and $M/2 \leq \epsilon P$, because the monitor dominates the fixed covariance cost and the polynomial width cost, so $M/2 + HV_{2B} m \leq 2\epsilon P \leq \xi t_{\text{field}}/g$. The warning allowance is therefore

$$E = 2V_{2B} p e^{-\epsilon\sqrt{P}}, \qquad \mathbb{P}(\text{all witnesses}) \leq E^{|W|}.$$

Monitor growth makes E smaller than every prescribed reference-coupling power after fixed block and stage choices. This is the rapidly decaying warning allowance used by the activity prices. A nonzero detector violates its quiet watch and supplies precisely such a coordinate within $2B$. Thus simultaneous nonzero detector products lie in this joint event. The preceding plateau calculation proves the implication: under the quiet watch the curvature is at most one quarter of its threshold, the prediction mismatch is within its plateau, no nearby flag can occur, and every nonmandatory detector is a cutoff times a vanishing plateau complement, while a mandatory detector is identically zero. Taking the contrapositive gives precisely the witness used in the probability bound. The lift retains covariance, address censuses and witness assignment maps; readout preserves their multiplicity bounds and the Gaussian moment-generating identity. The same joint estimate is recovered without selecting a new Gaussian for individual molecules.

The Gaussian event reader uses the physical noise projection R_X for the fixed local data X, with $(R_X)_* \widehat{\gamma}_X = \gamma_X$. For a finite warning family W, let E_W be the event that every warning has a coordinate witness in its recorded radius-$2B$ neighborhood. The local plateau implication, with its three separate error bills, gives

$$\left\{ \prod_{j \in W} d_j \neq 0 \right\} \subseteq E_W, \qquad \widehat{\gamma}_X(R_X^{-1} E_W) = \gamma_X(E_W) \leq E^{|W|}.$$

Here the realized detectors are the original coefficient readers composed with R_X, with their other local inputs retained. A witness assignment keeps its warning-to-coordinate map and distinct-coordinate image; readout preserves both cardinality inequalities used above. For the exponential moment calculation the test vector v remains a physical coordinate vector, so its covariance is the same A and its upper bound the same M. The section recovers the original event and moment identity. Thus the allowance transports through exact Gaussian pushforward with shared witnesses counted, on the original fixed choice of local data.

13.24 The primitive integrand and its local extension

Here are the finite objects to which the integration estimates apply. Fix a coarse-to-fine step, its fine tree F, a seed inventory, a coarse configuration V, local discrete choices k, and the off-tree group variables Y. The active off-tree links are determined by the tree and seed inventory. For a color array u, the fine configuration has link $\exp(u_e)(\mathrm{replica}\, V)_e$ on an active link, the value supplied by Y on an inactive off-tree link, and the fixed tree value on a tree link. The logarithmic signal equals u on active links, zero on tree links, and the chosen cutoff logarithm of $U_e(\mathrm{replica}\, V)_e^{-1}$ on inactive off-tree links. Thus the group configuration and the logarithmic signal are different objects. Their identification by exponentiation is asserted only on the interior domain where the cutoff logarithm agrees with the genuine chart inverse.

A letter is of one of four disjoint types: a block packet A, an old forced index q, a regular logarithmic index λ, or a warning index w. A molecule p is a finite set of letters; repeated copies of a letter are not allowed. Its forced inventory is the set of its forced indices. Its seed inventory is the union of the seeds of its packet letters. The ground $H(p)$ is the union of the enlarged read footprints of all its letters. In particular it contains every seed and every forced read. Admission requires all of the following: the overlap graph of letter footprints is connected, the packet letters form the permitted layout of their combined seed inventory, the forced inventory satisfies its sealing and incidence conditions with those seeds, and

$$\mathrm{flags}(V) \cap H(p) = \mathrm{flags}(\text{seed inventory of } p). \tag{PRI1}$$

We spell out the discrete tests in this assertion. A seed is a member of the disjoint union of fine plaquettes, coarse bonds and coarse sites. These types remain distinct even when their spatial positions coincide. A store A is the triple of finite sets $(A_{\mathrm{plaq}}, A_{\mathrm{bond}}, A_{\mathrm{flag}})$; its number of seeds is the sum of their cardinalities. The read set of a plaquette seed is its coarse projected start set, that of a coarse-bond seed its two pin endpoints, and that of a site seed its singleton. The place of A is the union of these read sets; its dome $A^{[r]}$ is their coarse radius-r halo. Give the seed set the graph in which two seeds are adjacent when their radius-lB halos intersect. A connected packet means a nonempty connected set in this graph.

A permitted layout $\mathcal{L}$ for A at parameters l, B is a finite set of stores with the following three properties:

$$\begin{aligned} &\text{each } A' \in \mathcal{L} \text{ is a connected packet,}\\ &(A')^{[lB]} \cap (A'')^{[lB]} = \varnothing \quad (A' \neq A''),\\ &\mathrm{seeds}(A) = \bigcup_{A' \in \mathcal{L}} \mathrm{seeds}(A'). \end{aligned} \tag{PRI6}$$

It exists by taking the connected components of the seed graph and converting each component back to its store. Conversely any permitted layout is this one: an edge cannot connect distinct members because their halos are disjoint, and each member is connected. Thus the packet layout is determined by the combined seed inventory. It does not introduce a multiplicity of choices or a choice of a graph witness.

For a finite set I of old forced indices write P_i for its actual fine support and $J_i \subset P_i$ for its declared inventory. Sealing means $P_i \cap P_j = \varnothing$ for distinct $i, j \in I$. Incidence means

$$\forall i \in I,\ \forall y \in J_i, \quad \exists p \in A_{\mathrm{plaq}} \text{ with } y \in \mathrm{touched}(p). \tag{PRI7}$$

Here $\mathrm{touched}(p)$ is the fine set inspected for that plaquette seed in the original sector partition. This condition uses the inventory, whereas sealing uses the full supports. Their roles cannot be interchanged.

The selector used in the packet integrand is defined at each fine site y by

$$c_I(U;y)=\begin{cases}0, & y\in \mathrm{flags}_\vartheta(U)\setminus\bigcup_{i\in I}P_i,\\ 1, & \text{otherwise.}\end{cases}\qquad \mathcal{O}_{I,Z}(U)=\prod_{y\in Z}c_I(U;y). \tag{PRI8}$$

For the packet, Z consists of fine sites whose parents are inside its $300B$ dome in the prescribed sense. For the full gas take all fine sites. If a product of old forced factors is nonzero, their individual permission identities give $J_i=P_i\cap\mathrm{flags}_\vartheta(U)$. On a sealed family, the full selector is one exactly when the union of those inventories is all flags. Otherwise it is zero. This proves the exact rewriting of the old gas occurrence as the sealed indicator times (PRI8) times its old forced product. If the product is zero, both expressions are zero without using a permission identity for a vanishing factor.

The footprint of a packet letter is its enlarged dome; that of every other letter is its prescribed inspected, projected halo. Define two letters to be adjacent when these coarse footprints intersect. The molecule ground is their union, as in (PRI1). Its admission is now completely specified by nonempty connectedness in this graph, the layout (PRI6) of its packet letters, sealing and incidence (PRI7) of its forced indices, and (PRI1). The empty seed inventory is allowed when there are no packet letters; its permitted layout is empty.

Compatible molecules have disjoint grounds. For such a finite family, admission of its union is equivalent to admission separately in every member, together with the global flag coverage. To prove the nontrivial localization of incidence, suppose a forced inventory site in one molecule is touched by a seed attributed to another. The parent of that site lies in the forced footprint of the first molecule and in the seed footprint of the second. Disjointness of grounds forces the two molecules to coincide. The same parent argument localizes the sealing test. Layout restrictions localize because the enlarged packet domes are contained in their grounds; joining the local layouts gives a global layout. Equation (PRI1) localizes by intersecting the global flag set with disjoint grounds. Finally every finite set of letters has a unique partition into connected components of its footprint graph. These statements prove that expansion into finite letter sets and regrouping into compatible connected molecules are inverse operations, with multiplicity one. Absolute summability proved below permits this finite combinatorial identity to pass to the countable alphabets. All these admission tests remain discrete masks and are not differentiated.

For the common lifted fluctuation $u=\mu+\beta^{-1/2}z$, the four vertex values are, respectively,

$$\begin{aligned} v_A&=1_{\{\text{packet connected}\}}1_{\{\text{packet eligible}\}}\mathcal{S}_A(u),\\ v_q=K_q(U(u))\,\mathcal{E}_{2a,q}(u),\qquad v_\lambda&=e^{L_\lambda(u,z)}-1,\qquad v_w=D_w(u).\end{aligned} \tag{PRI2}$$

Here $\mathcal{E}_{2a,q}$ is the product of fluctuation cutoffs on the entire forced read footprint. Packet eligibility requires its place to lie in the chosen seed place and its declared flags to agree with the actual coarse flags throughout its $300B$ neighborhood. To define the profile factors, let χ be the fixed smooth cutoff and put $h_s(r)=\chi((r/s)^2)$. For a tested fine plaquette use $h_{t/4}(\mathrm{toll}(U,p))$ if $p\notin A_{\mathrm{plaq}}$, and its complement if $p\in A_{\mathrm{plaq}}$. For a tested coarse bond use $h_{Wt}(\mathrm{wire}(\mathrm{prediction}(U,k),V;b))$ if $b\notin A_{\mathrm{bond}}$, and its complement otherwise. Multiply these factors once over every plaquette whose projected start set meets $A^{[300B]}$ and every bond whose pin set meets that dome. These are the profiles in $[0,1]$ in the following formula.

Let n_F,n_P be the cardinalities of the finite frame-tag and predictor-point sets. At a site x, the ticket factor is n_F times the actual frame-choice probability at x. At a bond it is n_P times the actual predictor-choice probability for its connector. Multiply the site factors over $A^{[300B]}$ and the bond factors over all pin sets meeting it. They are nonnegative and bounded respectively by

n_F, n_P, rather than by one. With the uniform average over the complete choice space,

$$\text{likelihood}(U,k) = \frac{1}{|\mathcal{K}|}\prod_x \text{siteMass}(U,k,x)\prod_b \text{pinMass}(U,k,b). \tag{PRI9}$$

Indeed $|\mathcal{K}| = n_F^{|\Lambda_c|} n_P^{|\text{Bond}_c|}$, which cancels the inserted cardinality factors. This identity also includes zero-likelihood choices; their ticket product is zero. The frame probability is uniform over the admissible minimizing tags when that set is nonempty, with the prescribed uniform fallback; the predictor probability is its prescribed nonnegative normalized weight with its corresponding fallback. Thus the ticket retains the actual choice denominators, including the fallback branch.

The packet value is explicitly

$$\begin{aligned}\mathcal{S}_A(u) = {} & e^{-\beta(Q_{\text{near}}(A,u)+Q_{\text{far}}(A))+\sum_{\text{visit}_{200B}\,A} c_e+\sum_{\text{active}_{304B}\,A} j_e(u)-N_A\log m_\beta} \\ & \times \mathcal{O}_A(U)\,\mathcal{T}_A(U,k)\,\mathcal{P}_A(U,k,V)\,\mathcal{E}_{a,A}(u),\end{aligned} \tag{PRI3}$$

where $m_\beta = \text{chart scale}\,(\sqrt{2\pi}\,\beta^{-1/2})^{\dim \mathfrak{g}}$, and N_A counts the off-tree frozen links in the $10B$ visit set. The terms c_e are the link condensation terms; j_e are the local logarithmic Haar and cutoff corrections. The far term is the signed-orientation average of the extracted nonnegative row costs on the $200B$ posted set. The remaining four factors are the forced-order selector, the finite layout ticket, the product of sector profile factors, and the active cutoff product on the $304B$ neighborhood. The selector, profiles and cutoffs are in $[0,1]$; the ticket has its finite combinatorial toll bound. The normalization deficit in (PRI3) is per frozen link, and is retained before that toll is estimated.

The regular logarithms L_λ comprise the old regular activity and cubic, quartic, loop and quadratic-tail energy pieces, the remote row and local chart terms, and the finite ledger terms from Gaussian completion and covariance reweighting. Their signs are the signs in the exponent of the original density: in particular an old regular occurrence contributes its negative value. A warning vertex is zero when mandatory; otherwise it is its fluctuation cutoff times the corresponding sector-profile complement. The Gaussian warnings are also retained in this warning alphabet. All factors in (PRI2) use this single configuration, signal and Gaussian variable.

Put $b_p(Y,z) = \prod_{\alpha\in p} v_\alpha(Y,z)$. The primitive value before the finite choice average is the literal integral

$$a_p = \int b_p(Y,z)\,d\nu_F(Y)\,d\gamma_C(z), \tag{PRI4}$$

with the product tree-gauge probability measure and the one covariance C prepared for this sector. There is no normalization by the mass of an admissible event. Local discrete choices are subsequently averaged with weight $1/|\mathcal{K}|$; their probabilities do not replace any of the profile or ticket factors.

For extension to coarse configurations outside the healthy domain, use the same finite formulas with the extended local row expressions and retain the admission masks at their actual enlarged output footprints. For a chosen ground H, retain only the finite Gaussian coordinates $I(H)$ read there. The submatrix C_I is positive definite at an actual admissible sector and has density

$$(2\pi)^{-|I|/2}\exp\Big(-\tfrac{1}{2}\log\det C_I - \tfrac{1}{2}z^T C_I^{-1} z\Big)\,dz. \tag{PRI5}$$

Integrate the extended product against this density and the tree-gauge measure, then take the uniform finite choice average. Set it to zero if the structural connectedness, layout or incidence tests fail. The marginal extension has its own fixed support and is not presumed positive or energetic at other coarse configurations.

On a healthy ground this agrees with (PRI4). The integrand is constant on fibers of the coordinate projection to $I(H)$; its Gaussian pushforward has covariance C_I. The remaining local data agree on the ground. Integrating over these fibers and using (PRI5) proves the identity. The same finite-choice fiber argument proves independence of choices outside H. For disjoint grounds the finite choice space is a product; writing its cardinality as the product of the cardinalities of the factors proves that the average of their product is the product of their averages. The tree variables have the analogous product-measure identity. No independence of overlapping Gaussian warning events is used.

The later continuity passage uses these same primitive integrals. Along an ultrafilter all finite seed stores and discrete tags can be fixed on one member of the ultrafilter. Keep the tree-gauge input Y fixed, and compare the two histories on this fixed finite data. Their covariance densities and vertex factors have the asserted pointwise limits. The primitive bounds below give a common integrable majorant, so dominated convergence applies to each molecule. Since this holds along every ultrafilter tending to infinity, the difference of the primitive values tends to zero in the ordinary sequential sense. This explains why the fixed finite data in the continuity argument can be selected after fixing Y, without claiming that a continuously varying configuration has a globally constant sector.

Source–readout formulation. The source object of this subsection is the indexed family $\{(\mathfrak{P}, p)\}$ of realized primitive data with their molecules; the readout is $p \mapsto a_p$ together with the retained type, ground, layout and admission flags of p. Relative to the frame realization of Theorem 7.1 the fine configuration $U(u)$ is carried on the identity-frame section, $z(u) = \sigma(U(u))$, and the tree values and Y are *part of the integration variable*, not frame data: the tree gauge is a slice of the target, while the frame fibre is a redundancy of the realization. Every vertex function is composed with π, so $v_\alpha^Z(\sigma(U(u)), \dots) = v_\alpha$ and $a_p^Z = a_p$ by Theorem B.1. The operations whose compatibility with the readout is needed are: the passage from letters to stores and layouts (Lemma 13.1), the rewriting of the old forced gas occurrence (Lemma 13.2), the regrouping into molecules (Proposition 13.3), the choice average (Lemma 13.4) and the marginal extension (Proposition 13.5). Each is stated here in the form in which it is used, with the proof given in the derivation above.

Lemma 13.1 (Uniqueness of the layout)**.** *The permitted layout exists and is unique: it is the set of connected components of the seed graph, each converted back to its store.*

Proof. This is the connected-component construction and converse immediately following (PRI6). That argument proves all three layout tests before the molecule integrand is introduced, so the same unique layout is retained here. □

Lemma 13.2 (Exact rewriting of the old gas occurrence)**.** *For a finite family I of old forced addresses,*

$$\mathbf{1}\{\textit{supports disjoint, } \textstyle\bigcup_i J_i = \mathcal{F}_\vartheta(U)\} \prod_{i\in I} K_i(U) = \mathbf{1}\{I \textit{ sealed}\}\ \mathcal{O}_{I,\mathrm{all}}(U) \prod_{i\in I} K_i(U).$$

Proof. If the product vanishes both sides vanish, and no permission identity is used for a vanishing factor. Otherwise each factor is nonzero and (AB9) gives $J_i = P_i \cap \mathcal{F}_\vartheta(U)$. On a sealed family the full selector equals one exactly when no flag lies outside $\bigcup_i P_i$, that is when $\mathcal{F}_\vartheta(U) = \bigcup_i (P_i \cap \mathcal{F}_\vartheta(U)) = \bigcup_i J_i$. Sealing is the disjointness of supports. □

Proposition 13.3 (Multiplicity-one regrouping)**.** *Compatible molecules have disjoint grounds. For a finite compatible family, admission of the union is equivalent to admission of each member together*

with global flag coverage. Every finite letter set has a unique partition into connected components of its footprint graph. Hence expansion into finite letter sets and regrouping into compatible connected molecules are mutually inverse, with multiplicity one.

Proof. Only the localization of the tests needs an argument. If a forced inventory site of one molecule were touched by a seed attributed to another, the parent of that site would lie in the forced footprint of the first and the seed footprint of the second, contradicting disjoint grounds; the same parent argument localizes sealing. Layout restrictions localize because enlarged packet domes lie inside their grounds, and the union of local layouts is a global layout, unique by Lemma 13.1. Equation (PRI1) localizes by intersecting the global flag set with disjoint grounds. Uniqueness of the component partition is a property of finite graphs. □

Lemma 13.4 (Ticket identity)**.** *With the uniform average over the complete choice space $\mathcal{K}$,*

$$\text{likelihood}(U,k) = \frac{1}{|\mathcal{K}|}\prod_x \text{siteMass}(U,k,x)\prod_b \text{pinMass}(U,k,b).$$

Proof. The cardinality cancellation, zero-likelihood choices and both fallback branches are proved directly after (PRI9). Applying that identity to the same complete choice space gives the displayed formula; there is no additional normalization by the eligible event. □

Proposition 13.5 (Marginal recovery and factorization)**.** *On a healthy ground the marginal extension equals* (PRI4)*; it is independent of the discrete choices outside H; and for molecules with disjoint grounds the choice average and the tree-gauge integral of a product are the products of the averages and integrals.*

Proof. On a healthy ground the integrand is constant on the fibres of the coordinate projection $z \mapsto z_{I(H)}$. The pushforward of the centred Gaussian with covariance C under a coordinate projection is the centred Gaussian with covariance the principal submatrix C_I, whose density is (PRI5); the remaining local data agree on the ground. Integrating over the fibres proves the first claim. Summing over choices outside H of a function that does not read them multiplies by their number, which the normalization $1/|\mathcal{K}|$ removes; this is the second claim. For disjoint grounds $\mathcal{K}$ is a product and its cardinality the product of the factor cardinalities, so the average of a product of functions of different factors is the product of averages; the tree-gauge measure is a product measure and Fubini gives the same for it. No independence of overlapping Gaussian warning events is used or asserted. □

Three losses of the readout $p \mapsto a_p$ are explicit. (i) The number a_p forgets which letters produced it; the letter set is kept beside it, and by Proposition 13.3 the regrouping is a bijection with multiplicity one, so no term of the old expansion is counted twice or dropped. (ii) The marginal extension integrates out every Gaussian coordinate outside $I(H)$; by Proposition 13.5 this loses nothing for functions of the ground, and nothing else is ever evaluated on it. (iii) Outside the interior chart domain the logarithmic reader is the cutoff logarithm and does not determine the group reader; the discrepancy $\Delta = \exp(\text{Read}_{\log})(\text{replica}\, V)\,\text{Read}_{\text{grp}}^{-1}$ is the recorded group-valued residual there. Indeed, writing $B = \text{replica}\, V$, $U = \text{Read}_{\text{grp}}$ and $\ell = \text{Read}_{\log}$, the protected interior identity is $\exp \ell = UB^{-1}$, so the ordered product gives

$$\Delta = \exp(\ell)BU^{-1} = UB^{-1}BU^{-1} = 1. \tag{2}$$

The factors are not commuted. Off that domain Δ is retained as its actual value; neutrality is not asserted. The exactness class is therefore exact recovery of each primitive value on the healthy ground, with an explicitly retained residual outside the interior chart.

Continuity along ultrafilters. The passage to sequential convergence in the last paragraph of the derivation uses the following elementary fact, recorded because the order of choices depends on it: a bounded real sequence which tends to zero along every nonprincipal ultrafilter tends to zero, for if $|x_n| \geq \varepsilon$ on an infinite set, an ultrafilter containing that set contradicts the hypothesis. The finite seed stores and discrete tags are selected on one member of the ultrafilter *after* fixing the tree-gauge input Y; no claim is made that a continuously varying configuration has a globally constant sector. The lifted statement is the same statement for $a_p^Z = a_p$, with the same integrable majorant.

13.25 Integrating primitive vertex bounds

A primitive body is a finite product of connected block packets, old forced factors, regular exponential increments and warning detectors. For molecule p with warning subset $B(p)$, nonnegative pointwise tariffs f_i and integrated tariffs t_i satisfy

$$\prod_{i\in p} t_i = h^{|B(p)|} \prod_{i\in p} f_i.$$

Only warnings receive h. Packet tariffs vanish off the declared connected geometry and otherwise equal the packet income. Forced tariffs are incoming envelopes; regular tariffs bound both log and exponential increment; warning tariffs use the storm bound. All apply on the same local data and sector. Put $F_p = \prod_{i\in p} f_i$. Nonzero body implies all its warnings are nonzero, hence the common Gaussian event $C_{B(p)}$ occurs. Thus

$$|\operatorname{body}_p(Y,z)| \leq F_p \mathbf{1}_{C_{B(p)}}(z), \qquad \gamma(C_{B(p)}) \leq h^{|B(p)|}.$$

Measurability and the finite uniform bound imply product-probability integrability. Integrating the indicator, then tree-gauge variables, gives

$$\int |\operatorname{body}_p| \, d\nu \, d\gamma \leq F_p \gamma(C_{B(p)}) \leq \prod_{i\in p} t_i.$$

Empty warning sets cost one. For marked team M, remove its selected keys from the body and multiply by paid ingredients, whose product is bounded by team weight W_M. Set the term to zero when a key is absent; otherwise

$$\int |\operatorname{markedBody}_{M,p}| \leq W_M \prod_{i\in p\setminus \operatorname{keys}(M)} t_i.$$

The analogous pointwise bound uses f_i. Summing finite molecule subsets costs at most $W_M \prod_i (1 + t_i) \leq W_M e^{\sum_i t_i}$, retaining finitely many compulsory keys. This supplies the summable integral norms for the primitive Mark expansion, with packet, forced, regular, storm and joint catastrophe estimates as analytic inputs. The lift retains the common Gaussian event, covariance and removed-key list. Recovery preserves this indicator majorant and probability integration, including correlated warnings and the empty-warning case.

For the realized local integration measure $\widehat{\eta}_X$, retain the same coordinate reader R_X with pushforward $\nu_X \otimes \gamma_X$. The lifted body is $\widehat{b}_p = b_p \circ R_X$. Its recovered majorant and absolute integral satisfy

$$|\widehat{b}_p| \leq F_p \mathbf{1}_{R_X^{-1} C_{B(p)}}, \qquad \int |\widehat{b}_p| \, d\widehat{\eta}_X = \int |b_p| \, d\nu_X \, d\gamma_X \leq \prod_{i\in p} t_i.$$

For a marked team, the recorded distinct-key removal defines the same remaining body and paid ingredient product before and after readout. The identical pushforward calculation therefore recovers its $W_M \prod_{i\in p\setminus \mathrm{keys}(M)} t_i$ bound. The section recovers the original integrands. Finally the unchanged nonnegative majorant $W_M e^{\sum_i t_i}$ bounds the molecule series on both sides, so its summable integral norms justify recovery through termwise integration and the permitted address regroupings.

13.26 Regular-gas factorization and exact forced restoration

Split primitive labels by empty versus nonempty flag inventory. For fixed compatible forced set A, its enlarged-support union Y_A excludes intersecting regular labels. If $Z_0 = e^{L_0}$ is the full regular gas and M_A sums regular occurrence clusters meeting that exclusion region, then

$$Z_{0,A} = e^{L_0 - M_A} = Z_0 e^{-M_A}.$$

Cutting labels decreases absolute budgets; subtraction of absolutely convergent log sums leaves exactly missing clusters, with original occurrence indices and factorials. Multiply by the forced product and sum sets of the required inventory. Expand e^{-M_A} by finite positive multiplicities of missing addresses, giving factors $(-h)^r/r!$. Each missing cluster meets the forced region. Absolute restoration bounds, or the fixed-volume exponential of the absolute missing series, justify regrouping. Therefore grouping these terms does not change their sum. Decompose atom supports into overlap components. Each component with a missing cluster contains a forced atom; forced-only components are allowed. Components have disjoint supports and partition inventories. Conversely, union recovers the forced set and every cluster multiplicity uniquely. Factorials split over components since no atom address belongs to two components. This weight-preserving bijection factors the original gas as Z_0 times the connected restored forced gas, retaining actual exclusions and exact inventories. This proves the regular-log factorization and forced regrouping identity once the primitive Mark expansion has been supplied. It uses exact inventories and actual overlap exclusions; the more permissive repetition count used for the absolute bound does not alter this equality.

For switched leaves the same argument holds for every $0 \le x \le 1$ with fixed supports, masks and inventory tests and uniform paid bounds. Two fusions put components in the final dictionary, recording switch-independent attached logs. Unmarked scalar normalization then gives exactly

$$e^{E_A(U)} e^{L_0(x,R) - L_0(0,\varnothing)} \,\mathrm{Gas}_A(K(x,R);U) = \mathcal{P}(\mathrm{Buy}(x,R))(U).$$

The real gas/log theorem proves positivity of Z_0 on every branch, justifying the regular-gas ratios. The primitive Mark expansion, dictionary fusion and unmarked scalar normalization proved in this chapter establish the scalar equality, with the stated exclusion restoration and component regrouping. The unmarked normalization identity says that the final coarse exponent, its fusion logarithms, and $L_0(0,\varnothing)$ reproduce the original scalar prefactor. The lift retains original occurrence addresses, positive multiplicities and inventory partitions. Readout preserves the component bijection and factorial products; retained absolute majorants permit regrouping, recovering the switched equality with the same scalar normalization.

For explicit recovery, record a restoration term as a forced label set A and a finite-support positive multiplicity function r on the missing-cluster addresses. Its coefficient has the form

$$\left(\prod_{i\in A} K_i\right) \prod_{q:r(q)>0} \frac{(-h_q)^{r(q)}}{r(q)!}.$$

Realization retains q as its full original occurrence address, including its own occurrence factorial. It also retains the additional multiplicity $r(q)$, so these two factorials remain separate. Connected-component recovery partitions the atom addresses, with every repeated copy of a fixed address

belonging to its unique component. Thus the product of multiplicity factorials splits exactly over the recovered components. Composition with readout preserves the displayed coefficient and the exclusion predicate, and section recovery reconstructs (A, r) by union. The absolute restoration majorant justifies summing this equality. Likewise the positive regular-gas reader gives

$$\frac{Z_0(x,R)}{Z_0(0,\varnothing)} = \exp\{L_0(x,R) - L_0(0,\varnothing)\},$$

and its pullback preserves the denominator and the switch-independent fusion logs. This recovers the stated switched Mark identity with the original scalar normalization.

13.27 Paid cluster terms and positive multiplicities

A switched primitive activity is a finite ticket sum with dependence $x^{|D|}\mathbf{1}_{D\subseteq R}$. Each ticket's scalar bound is the product of marked leaf prices and unselected primitive prices; its switching model is concentrated at D. Summation gives the augmented-tariff product. Mixed-activity support and polynomial-moment weights enlarge this paid bound, with zero assigned on failed eligibility, uniformly in configurations. A regular occurrence cluster has paid bound

$$\frac{|\varphi(z)|}{n!}\prod_{j=1}^{n} u_{i_j}.$$

For primitive budget P_0 and tree ratio $\rho < 1$, rooted estimates give sitewise support-weighted sum $P_0/(1-\rho)$. The omitted-cluster restoration and primitive-integral arguments in this chapter control their support-weighted sum. Subtracting its unmarked value doubles the bill; these differences are the switched regular log increments. Forced restoration multiplies forced primitives with negative log clusters with factors $h^r/r!$ for multiplicity $r \geq 1$. Paid products and powers retain these factorials. The paid product and power bounds therefore give exactly the corresponding product of envelope powers divided by factorials. Connectedness, presence of a forced atom and exact inventory restrictions are switch independent, so their indicators preserve paid bounds.

Restoration indices are finite distinct atom sets D with positive multiplicities r_i, different from logarithm occurrence tuples:

$$W_h(D,r) = \prod_{i\in D}\frac{h_i^{r_i}}{r_i!}, \qquad N(D,r) = \sum_{i\in D} r_i.$$

Absent atoms are omitted from D. For $0 \leq h_i \leq \sigma < 1$,

$$\sum_{r_i\geq 1}\frac{h_i^{r_i}}{r_i!} \leq \frac{h_i}{1-\sigma},$$

using $r_i! \geq 1$ and $h_i^{r_i} \leq h_i\sigma^{r_i-1}$. Products reduce the bound to distinct sets with inflated weights $h_i/(1-\sigma)$. Sitewise smallness implies individual $h_i \leq \sigma$ by testing one site in its nonempty support. The distinct connected-set tree estimate then bounds the sum of repeated connected terms whose union contains a fixed site by $\eta/(1-q')$, provided the inflated vertex activities satisfy the readiness condition with budget $\eta \leq \sigma$ and tree ratio $q' < 1$. Connected-cover subadditivity pays actual union weight; each positive multiplicity increases available exponent sum. All successive sums are nonnegative. For omitted restoration, forced atoms have budget c at exponent $A + \delta'$, and regular-log occurrence atoms have budget $\gamma/(1-q)$ there. Forgetting exclusions and permitting repeated forced atoms enlarges the absolute majorant. For size tilt $t \geq 1$ and forced tilt $\mu \geq 0$, multiply

all weights by t and forced ones additionally by e^{μ}. After multiplicity allowance their combined budget is at most

$$\frac{te^{\mu}c}{1-\sigma}+\frac{t\gamma}{(1-\sigma)(1-q)}.$$

The lift retains the distinction between occurrence tuples and distinct addresses with positive multiplicities, together with their switching models and factorials. Readout preserves each product and switch-independent restriction; nonnegative majorants justify the same summations and tilts.

13.28 Forced restoration size tail

Require the tilted combined budget above to be at most $\eta \le \sigma$, with $0 \le \sigma < 1$, $\delta' > 0$ and $4(2B+1)^4\eta/\delta' \le q' < 1$. The repeated connected estimate applies to augmented activities. A term of total size $N \ge m$ gains $t^N \ge t^m$; a forced atom guarantees at least e^{μ} even when its multiplicity exceeds one. Dividing gives

$$\sum_{\substack{(D,r)\text{ connected},\ N(D,r)\ge m\\ \text{at least one forced atom},\ x\in\bigcup_{i\in D}X_i}} e^{Ac_B(\bigcup_{i\in D}X_i)}W_h(D,r) \le t^{-m}e^{-\mu}\frac{\eta}{1-q'}.$$

Applications take $t=2$, $\sigma=1/2$, $q'=1/2$; at $m=0$ the price is $2e^{-\mu}\eta$. The compulsory atom is charged at least once, independently of multiplicity. The distinct connected-set readiness estimate is the remaining rooted-tree input. The lift retains size and forced-atom counts separately; recovery preserves both tilt inequalities and hence this support-weighted tail on the same restored address family.

The tilt reader has an exact multiplicity identity. Put $N_{\mathrm{f}}(D,r)=\sum_{i\in D\text{ forced}} r_i$ and $\widetilde{h}_i = te^{\mu\mathbf{1}_{i\text{ forced}}}h_i$. Then

$$W_{\widetilde{h}}(D,r) = t^{N(D,r)}e^{\mu N_{\mathrm{f}}(D,r)}W_h(D,r).$$

Realization retains D, every r_i and each forced/regular tag, so readout preserves both counts and this equality, including all factorials. A permitted term in the size tail has $N \ge m$ and $N_{\mathrm{f}} \ge 1$, which yields the same $t^{-m}e^{-\mu}$ gain after undoing the tilt. The union-support reader depends on D and is unchanged by multiplicity; positive r_i preserve the vertex support-weight majorant. Taking finite nonnegative subsums and then their supremum therefore transports the readiness bound and the entire restoration tail. Section recovery returns the original signed term, its paid envelope and its switch-independent eligibility predicate.

13.29 Real switching and nilpotent substitution

For finite switches I, a modeled real family has

$$f(x,R)=\sum_{A\subseteq R}x^{|A|}g_x(A),\quad 0\le x\le 1,\qquad g_x(A)=x^{-|A|}\sum_{R\subseteq A}(-1)^{|A|-|R|}f(x,R)\quad (x>0).$$

The inverse follows either by finite Möbius inversion or subset induction. Assume $g_x(A)\to g_0(A)$ as $x\downarrow 0$; uniqueness for $x>0$ makes $J_f(A)=g_0(A)$ model independent. A paid model has $|g_x(A)|\le b(A)$ and $\sum_A b(A)\le c$ uniformly in configurations, so $|f(x,R)|\le c$. For square-zero marker elements z_i, define

$$\mathcal{S}_z(f)=\sum_{A\subseteq I}J_f(A)\prod_{i\in A}z_i.$$

Addition is immediate. Product arrays obey $(g *_x h)(A) = \sum_{B\cup C=A} x^{|B\cap C|} g(B)h(C)$. At zero only disjoint pairs remain; overlapping substituted products vanish by $z_i^2 = 0$. Hence substitution is multiplicative, maps constants to scalars and multiplies paid bounds. For countable sums with summable paid bounds, coefficientwise dominated convergence and the finite subset sum give $\mathcal{S}_z(\sum_n f_n) = \sum_n \mathcal{S}_z(f_n)$. Apply this to exponential terms bounded by $c^n/n!$ to obtain

$$\mathcal{S}_z(e^f) = \exp\{\mathcal{S}_z(f)\}.$$

For zero scalar output this is the finite nilpotent exponential, requiring no density division. If $\|z_i\|_w \le 1$, bound each output-mode coefficient by the sum over switch subsets D of $b(D)$ times the absolute corresponding coefficient of $\prod_{i\in D} z_i$. Submultiplicativity bounds total weighted price by $\sum_D b(D) \le c$. For zero-scalar z_i, the scalar output is $f(0,\varnothing)$, and singleton coefficients arise only from size-one switch subsets. Switch independence at i makes all Möbius coefficients containing i zero. Exterior silence therefore restricts nonzero coefficients to leaves whose required parent supports lie in the output support, placing their scoped marker pins there. Finite substitution preserves gauge invariance, locality and measurability. Vanishing off a healthy region or inventory event passes to coefficients and limits, proving correction permissions. These statements supply the support and permission properties of the marked corrections. The lift retains real switch values, paid arrays and finite inversion as recoverable data. Readout commutes with the finite Möbius transform and, by the retained majorants, its coefficient limits and summable series; thus it recovers this algebra map and its local price bounds exactly.

To express switching recovery directly, let $f^Z(x,R) = f(x,R)\circ\pi$. For $x > 0$ its finite inversion gives

$$g_x^Z(A) = g_x(A)\circ\pi, \qquad J_{fZ}(A) = J_f(A)\circ\pi.$$

The second equality follows by taking the coefficient limit at each realized input; the paid majorants are unchanged. If the marker readers are also realized coefficientwise, the algebra-map property gives

$$\mathcal{S}_{\mathscr{L}z}(f^Z) = \mathscr{L}(\mathcal{S}_z(f)).$$

Section composition recovers every coefficient, including the scalar and singleton readings. For a countable family with paid bills c_n, the coefficient tail before taking $x \downarrow 0$ is bounded by $\sum_{n>N} c_n$, independently of x and configuration. This same tail bound holds after readout and tends to zero, justifying the recovered sum and exponential laws. Finally the switch-independence and permission identities are finite equalities of these arrays, so their vanishing coefficients and scoped pins are retained through the limit and section.

13.30 Reading the marked Mark-step output

The retained reserve is built for the actual finite leaf inventory. Each leaf has its fee and normalized square-zero z_i, with parent support containing all scoped pins and $\|z_i\|_w \le 1$. The marked activity expansion produces a family $(K(t,A))$ indexed by subsets of the finite leaf inventory. Mixed jets evaluated on products of z_i retain only distinct-leaf derivatives. Triangle inequality and submultiplicativity turn mixed-jet bounds into local polynomial bounds. Group forced corrections by coarse support-and-inventory barcode; their scalar mode stays the background profile. Only single-leaf jets contribute to singleton marker v, and

$$\partial_i K_j = \mathrm{fee}_i\, D_j(M_i),$$

where D_j is that same leaf's transported hinge. Multiplication by its normalized monomial cancels the fee, including zero charges. The sound-door estimate bounds each hinge's weighted forced

mass by a fixed multiple of leaf load. Absolute summability therefore permits finite-leaf summation, barcode grouping and coefficient extraction, recovering exactly the sum of quoted singleton transported profiles. This proves the singleton-profile identity used in the restoration step.

The coarse evaluation identity is Mark push of priced tickets equal to exponential output times the coarse field polynomial, with the fixed scalar displacement factored out. The exponential has zero scalar mode. Scoped mixed-jet bounds give the reserve's served and field envelopes; hinge additivity preserves summably grouped leaves. Explicitly at U, write $a = e^{E_A(U)}$, regular increments $L(x,R)$ and forced gas $G(x,R)$. The established real identity is

$$ae^{L(x,R)}G(x,R) = \mathcal{P}(\mathrm{Buy}(x,R))(U), \qquad 0 \le x \le 1.$$

Apply $\mathcal{S}_z$, commuting through the paid summable regular and gas series and using product and exponential laws. Since the regular increment vanishes at the unmarked switch, its substituted scalar is zero. The left side becomes

$$a\,\mathrm{Exp}\left(\sum_S H_S(U)\right)\mathrm{Field}(K(U), F_A(U)).$$

On the right each finite ticket has switching jet concentrated at its leaf set D, so substitution gives its scalar coefficient times $\prod_{i\in D} z_i$. These coefficients are Mark-integrable; finite linearity commutes substitution and push. This is exactly the priced polynomial inventory push, proving the marked identity with all infinite sums controlled by paid bounds. The switched marker expansion, primitive integral and connected-cluster arguments in this chapter construct the marked activity jets and prove their exact Mark identity. The lift retains the finite inventory, barcode fibers, hinges and actual Mark kernel. Recovery commutes with the finite coefficient substitution and probability integral, while retained paid majorants justify its series passages and singleton extraction.

For the Mark operator, denote fine and coarse readouts by π_f and π_c. The realized kernel $\widehat{K_z}$ retains the actual kernel $K_{\pi_c(z)}$ through

$$(\pi_f)_*\widehat{K_z} = K_{\pi_c(z)}.$$

For an integrable ticket coefficient b_D, this gives

$$\widehat{\mathcal{P}}(b_D \circ \pi_f)(z) = \mathcal{P}(b_D)(\pi_c(z)).$$

Apply finite linearity to all tickets and then the switching recovery identity already established. One obtains coefficientwise

$$\widehat{\mathcal{P}}\,\mathscr{L}_f(\mathcal{S}_z(\mathrm{Buy})) = \mathscr{L}_c(\mathcal{P}(\mathcal{S}_z(\mathrm{Buy}))).$$

Here $\mathcal{S}_z$ uses the fixed normalized marker monomials; the argument z of the realized kernel is a separate coarse source point. Section recovery returns the original marked identity. The fee–monomial product and its zero-charge branch are recovered term by term, and the sound-door majorant permits the same barcode grouping of singleton hinges. Thus this kernel identity preserves both exact evaluation and the served and field bounds on the actual finite inventory.

13.31 Shipping price and the numerical retained step

For fine leaf cover cost $v = c_{B_1}(S)$, the birth-support bound is

$$(q+1)c_{B_*}(\mathrm{birth}(S)) \le (q+1)C_{\mathrm{foot}} + \tfrac{5}{4}\varepsilon v.$$

The tariff spends another $2\varepsilon v$. With $s_1 \geq 100$ the selected parameters obey $13\varepsilon/4 \leq s_1 - 4$, so the fine weight pays both, leaving $e^{(q+1)C_{\mathrm{foot}}}$. Charge leaves to scoped pins and use parent-pin occupancy M. For inventory input budget v', shipping requires

$$M\,\mathrm{siteFactor}(B_*)e^{(q+1)C_{\mathrm{foot}}}v' \leq \mathrm{ribbon}\,.$$

Grouping equal labels sums nonnegative prices and preserves this bound. Adding the unmarked reserve makes the marked tariff admissible, with birth geometry and scalar activity estimates as the retained inputs.

For fine and coarse exponents $s_1, s_0 \geq 100$, widths B_1, B_0 and occupancy M, choose $0 < d \leq 1$ and shrink marker weights I_v to dI_v. Higher corrections have degree at least two, so their weights decrease by at least a factor d, reducing unit price to at most d. At exponent $s_1 - 4$ the finite inventory costs $T + e^T T$, where

$$T = M\,\mathrm{siteFactor}(B_1)d + 4d.$$

Buy a reserve covering this input with output field budget u. Response prices satisfy $2I_v a_v \leq 1$, $I_v b_v \leq 1$, hence after shrinking $2dI_v a_v \leq d$, $dI_v b_v \leq d$, as required. Set $Q = u + M\,\mathrm{siteFactor}(B_0)d$ and require

$$(2B_0 + 1)^4 Q \leq \tfrac{1}{2}, \qquad e^Q - 1 \leq 1.$$

These pay closure exponent loss and final unit correction price. Matching coarse weights J_v costs a further half exponent unit with multiplier at most one. The reserve supplies exponential exponent $s_0 + 1/2$ and field exponent $s_0 + 1$; both half-unit expenditures leave at least s_0. Singleton feed and restoration identify the coarse primary and forced fields, proving the retained unit-price served conclusion with the constructed inventory and reserve. The lift retains shrunk weights, occupancy counts and each exponent expenditure. Recovery preserves these numerical inequalities and the singleton identity, so the retained step uses the same purchased reserve and output budget.

Weight recovery is explicit on every allowed marker mode A:

$$t_{dI}(A) = d^{|A|}t_I(A), \qquad |A| \geq 2 \implies d^{|A|} \leq d^2 \leq d.$$

The source lift keeps the mode, pins and chosen d in its realized data, so coefficientwise readout preserves this rescaling and the reduced correction price. Likewise it retains the fine support S, its birth support and both cover readers. Adding the two shipping expenditures uses exactly $5\varepsilon/4 + 2\varepsilon = 13\varepsilon/4$ per fine cover unit; the recovered inequality therefore leaves the same fixed footprint multiplier. Pin occupancy then uses the same parent addresses and bound M, preserving the ribbon demand after any equal-label grouping. Finally T and Q are scalar readers of these retained choices. Their readiness tests and the two coarse half-unit expenditures are recovered before applying the marked kernel. Section recovery consequently returns the same retained step, with its original unit output prices and prescribed singleton functions.

13.32 Separation pays for complex inflation

Let $B \geq 8$, and let marker set A have at least two pins separated by at least B^2 in torus maximum distance. Any support S containing them obeys $B|A| \leq 64c_B(S)$. Indeed a minimizing cover with $D = c_B(S)$ centers has a unit-step center traversal of length $N \leq D(4B + 2)$. Choose for each pin a center visit within B. By the distance triangle inequality and the unit-step walk bound, distinct chosen times are separated by at least $g = B^2 - 2B > 0$; their integer quotients by g are distinct, so $(|A| - 1)g \leq N$. Therefore

$$|A|B^2 \leq 4(|A| - 1)g \leq 4N \leq 4D(4B + 2) \leq 64DB.$$

Divide by B. The traversal need only visit covering centers. A nonzero higher correction has a nonzero mode with at least two scoped pins in its support, hence $c_B(S) \geq B/64$. Put $S_B = e^{B/64}$. For complex $|z| \leq S_B$, multiply each nonzero correction envelope by $|z|$, setting identically zero corrections' envelopes to zero. Then

$$|z| a_i e^{(e-2)c_B(S_i)} \leq a_i e^{(e-1)c_B(S_i)} \leq a_i e^{e c_B(S_i)}.$$

Thus complex inflation preserves its price budget at a support-exponent loss of two. Scalar multiplication preserves locality, gauge invariance, measurability, scope and permissions, and introduces no scalar or singleton mode. This proves precisely the inflation estimate needed for holomorphy and retained transport in the omission comparison. The primary and forced singleton insertions retain their values. The large analytic disk in the omission argument is a geometric consequence of separated markers. The lift retains separated pins, minimizing covers and center visits. Readout preserves the distance and traversal estimates, so the same complex disk and exponent payment apply to the recovered correction family and its omission comparison.

For the inflation map on a higher correction family $Y = (Y_i)$, write $\mathcal{I}_\lambda Y = (\lambda Y_i)$ with $|\lambda| \leq S_B$. The coefficientwise lift satisfies

$$\mathscr{L}(\mathcal{I}_\lambda Y) = \mathcal{I}_\lambda(\mathscr{L} Y), \qquad (\mathscr{L}(\mathcal{I}_\lambda Y)) \circ \sigma = \mathcal{I}_\lambda Y.$$

Pin addresses, their separation and support-cover witnesses are unchanged by either operation, so the displayed envelope inequality holds with the same S_B uniformly on the whole disk. Higher-mode projection commutes with this scalar multiplication. In particular scalar and singleton coefficients remain zero for every λ, and the prescribed primary and forced singleton insertions are recovered unchanged. Each individual correction depends linearly, hence holomorphically, on λ; the uniform paid bound is preserved by readout for its subsequent summed omission argument. This identifies the inflation domain and its price payment as part of the recovered family, including identically zero corrections with zero envelope.

13.33 Positive denominator from a Haar shelter

Let normalized Haar measure be μ_G, ambient real matrix dimension d_M and chord normalization $\mathrm{chord}(g) = |g - I|/\gamma$, $\gamma > 0$. For $0 < r \leq 1$,

$$\mu_G\{g : \mathrm{chord}(g) \leq r\} \geq (r/4)^{d_M}.$$

Indeed mutually $r\gamma$-separated group matrices have disjoint ambient balls of radius $r\gamma/2$ inside radius 2γ. Volume comparison gives $q(r/4)^{d_M} \leq 1$. A maximal finite collection exists by this bound; its closed $r\gamma$ balls cover the group. Their group intersections are left translates of the chord shell, by unitary norm invariance. Haar invariance and union bound give $1 \leq q\mu_G(\mathrm{shell}(r))$. For the four-dimensional torus shelter with every positive link of chord at most $s/4$, $0 < s \leq 1$, product Haar measure is at least

$$(s/16)^{d_M|E|}, \qquad |E| = 4|\Lambda|.$$

Product chord subadditivity and inversion invariance make every plaquette chord at most s. For $s \leq \theta$ there are no flags and the activity gas is one. Signed translations preserve the shelter.

For path bound C, loop cap Λ_0, regular allowance δ, forced allowance κ, coupling $\beta \geq 0$ and direct clip coefficient C_{dir}, put

$$\ell = \beta(C\theta^3 + C\theta^4) + \Lambda_0\theta^2, \qquad e_0 = \delta + \ell,$$
$$b(s) = |\mathcal{P}|(C_{\mathrm{dir}} s^2/2 + C^2 s^2/2).$$

Healthy degree-k atoms cost coefficient magnitude times θ^k; transport preserves curvature norm. Absolute atom sums bound cubic, quartic and loop remainders per site by $C\theta^3, C\theta^4, \Lambda_0\theta^2$. The quadratic base is nonnegative, giving channel action at least $-|\Lambda|\ell$. On the shelter, path curvature legs are at most s, the linear path field at most Cs, and the base per site at most $b(s)$. Channel action is at most $|\Lambda|(\ell + \beta b(s))$. Charging regular occurrences to their roots bounds the regular exponent by $|\Lambda|\delta$. For vacuum scalar v_0,

$$E_R(U) \le v_0 + |\Lambda| e_0,$$
$$E_R(U) \ge v_0 - |\Lambda|(e_0 + \beta b(s)) \quad \text{on the shelter.}$$

The density is nonnegative from bare Wilson positivity and positive Mark kernels by downward induction. Integrating over the shelter, where gas is one, gives

$$Z_R \ge \exp\{v_0 - |\Lambda|(e_0 + \beta b(s)) - 4|\Lambda| d_M \log(16/s)\} > 0.$$

The lift retains normalized product Haar, shelter, thresholds and positive kernels. Its exact measure pushforward recovers shelter mass and exponent bounds, and therefore this strictly positive denominator with the same vacuum scalar and constants.

Write $\mathcal{H}_s$ for this physical shelter and let μ_Z be the realized prior with $\pi_*\mu_Z = \mu$. Its exact lifted shelter is $\widehat{\mathcal{H}}_s = \pi^{-1}(\mathcal{H}_s)$, so

$$\mu_Z(\widehat{\mathcal{H}}_s) = \mu(\mathcal{H}_s) \ge (s/16)^{4d_M|\Lambda|}.$$

This is a condition on the recovered physical links, retaining the original plaquette threshold. Put $R^Z = R \circ \pi$. The same prior pushforward and nonnegativity give

$$\int R^Z \, d\mu_Z = Z_R \ge \int_{\widehat{\mathcal{H}}_s} R^Z \, d\mu_Z \ge e^{v_0 - |\Lambda|(e_0 + \beta b(s))} (s/16)^{4d_M|\Lambda|}.$$

On that preimage the recovered gas is exactly one. The section recovers both the shelter and its exponent reader. Thus the denominator estimate is an identity of the normalized integration measures followed by the original shelter lower bound, with unchanged link census, vacuum scalar and ambient-dimension cost.

13.34 Normalized insertion bill and stagewise treasury

Combining the denominator with the insertion norm majorant gives

$$t_w(A) \frac{|\int [X(U)]_A \, d\mu(U)|}{Z_R} \le \exp\left\{|\Lambda|[2e_0 + \kappa + \beta b(s) + 4d_M \log(16/s)] + |V|(x + y + 2f)\right\}.$$

The vacuum scalar v_0 cancels. The numerator retains its full law and integration domain. For selected history take $s = \widehat{g}_k = \widehat{\beta}_k^{-1/2}$ and $\beta \le 2\widehat{\beta}_k$. Monotonicity of the nonnegative loop and base prices gives the explicit volume coefficient

$$\tau_k = 2\{\delta_k + \ell_k(2\widehat{\beta}_k)\} + \kappa_k + 2\widehat{\beta}_k b_k(\widehat{g}_k) + 4d_M \log(16/\widehat{g}_k).$$

This is the extensive root estimate for omission. Its extensivity is explicit; no volume factor is silently absorbed in an error tending to zero. For $P(s) = \exp\{(\log\log s)^2\}$ and $\Theta(s) = s^{-1/2}P(s)$, each fixed b and $p > 0$ satisfy eventually $P(s)^b \le s^p$, since $b(\log\log s)^2/\log s \to 0$. Hence eventually

$s\Theta^3 \le 1$, $s\Theta^4 \le 1$ and $2P(s)^v\Theta^2 \le 1$ uniformly for $v \le 1$. Substitution gives $\ell_k(2s) \le 4C_k + 1$, with $\delta_k, \kappa_k \le 1$ and $s\widehat{g}_k^2 = 1$. Therefore

$$\begin{aligned}\tau_k \le{}& 2\{1 + (4C_k + 1)\} + 1 + |\mathcal{P}|(C_{\mathrm{dir},k} + C_k^2)\\ &+ 4d_M\{\log 16 + \tfrac{1}{2}\log\widehat{\beta}_k\}.\end{aligned}$$

Stage parameters remain explicit; logarithmic reference-coupling growth is asserted after fixing them. The lift retains the same normalized numerator, denominator and history parameters. Exact probability readout and the recovered denominator bound preserve the cancellation and volume coefficient, without absorbing volume into a vanishing error.

The normalized mode reader is recovered as a ratio of two exact integrals. For the realized prior and insertion $X^Z = \mathscr{L}X$,

$$t_w(A)\frac{\left|\int [X^Z]_A\, d\mu_Z\right|}{\int R^Z\, d\mu_Z} = t_w(A)\frac{|\int [X]_A\, d\mu|}{Z_R}.$$

The denominator is strictly positive by the recovered shelter estimate. Its vacuum factor e^{v_0} is the same one present in the numerator majorant, so cancellation leaves precisely the displayed extensive bill. All history choices entering τ_k are recorded scalar data; readout retains $\widehat{g}_k$, the band $\beta \le 2\widehat{\beta}_k$, and the fixed stage constants. Consequently its monotonicity substitution and stagewise logarithmic bound recover with the same quantifier order and site factor $|\Lambda_k|$. Section recovery returns the original full-law mode integral and treasury estimate.

14 The analytic stationary graph, loop moments and the calibrated coupling history

Elimination, response and the order of choices. The stationary graph removes auxiliary fluctuation coordinates only after solving their own equations on the declared analytic neighborhood. Writing the fluctuation as $v_*(a)$, the effective quantity is obtained by evaluation at $(a, v_*(a))$. An eliminated coordinate is not a gauge direction: its recovery requires the stationary equation, its invertible restricted Hessian and the response estimates proved below. Nonzero stationary residuals remain measurable in the raw history.

The loop moments and drift are then computed on this graph. Their coefficient fixes the reference slope used by the calendar; the native source order does not select that slope. Widths, block lengths, demands, reserves and delays are chosen in the stated dependency order before the terminal cutoff and physical period. The finest-scale density is identified exactly with the Wilson representative, and finite shooting arcs supply actual calibrated couplings. The combined startup allowance uses the band $x/2 \le \beta \le 2x$: its lower endpoint controls the forced part and its upper endpoint the regular part. These finite statements, followed by retirement and stability, supply the family needed by the limit; an auxiliary Hessian or formal probe alone does not supply it.

This section contains the analytic core of the supplied construction that lies between the vacuum bill of the marked integration and the omission comparison of the convergence chapter: from the common analytic stationary graph to the uniform running of the actual coupling. It supplies the analytic objects on which the stationary integrations, the one-loop drift, the asymptotic calendar and the finite calibrated coupling histories actually rest. Each subsection first gives the supplied derivation in full, with its tagged equations, and then its source–readout formulation. Throughout, raw histories, discrepancies and off-shell values are kept beside every stationary or calibrated restriction.

14.1 A common analytic stationary graph and its response estimates

We specify first the actual score to which the analytic argument is applied, at a fixed derivative order $n \geq 2$. A retained coordinate is a coarse oriented bond together with a real color index. A vertical coordinate is a free fine bond of the fixed forest together with such an index. Their disjoint union is the joint coordinate set. Arrays carry the maximum norm. Splitting the union is an isometry; replacing real color coordinates by their Euclidean color vector defines a linear isomorphism paint. Its norm is at most $1 + \dim \mathfrak{g}$, and its inverse has norm at most one. Moreover the sum of squared vertical real coordinates is exactly the sum of squared Euclidean color norms. Denote the resulting retained and vertical vectors by $(\operatorname{tint} x, \operatorname{tint} v)$. The actual scalar score in these coordinates is

$$F(x, v) = \operatorname{tested}(\operatorname{tint} x, \operatorname{tint} v). \tag{AS1}$$

Thus it is the main fine action plus the prediction penalty, with the fine and outer configurations constructed from these same variables. The connector is not an independent variable. A retained coordinate is addressed at its embedded coarse site, and a vertical coordinate at its fine site. Each address fiber has at most $h = |\mathrm{TypeFace}_1 \times \mathrm{ColorIndex}| + |\mathrm{Axis} \times \mathrm{ColorIndex}|$ members: orientation and color distinguish the members, and the coarse embedding is injective.

Here is the local-to-global estimate, including the counting loss. For addressed coordinate slots $q_1, \ldots, q_k$, put $S(y, q) = \sum_i \operatorname{dist}(y, \operatorname{address}(q_i))$. An anchor estimate of price M and width a means

$$|D^k f[e_{q_1}, \ldots, e_{q_k}]| \leq M e^{-aS(y,q)} \quad (k \leq n). \tag{AS2}$$

If a term depends only on coordinates within distance R of y, and its operator derivatives through order n are bounded by M_0, a coordinate derivative vanishes whenever one of its slots is outside that set. Otherwise $S \leq nR$. Thus (AS2) holds with $M = M_0 e^{anR}$. For path series the common exponentially weighted length budget dominates these derivative sums absolutely; differentiation term by term follows on a smaller common analytic neighborhood. This proves the same estimate for the cubic and quartic series, not merely for finite path sums.

Sum such terms over anchors with multiplicity at most h_0. Fix a marked slot q_j. The triangle inequality gives $\sum_i \operatorname{dist}(\operatorname{address}(q_j), \operatorname{address}(q_i)) \leq (n+1)S(y, q)$ and $\operatorname{dist}(y, \operatorname{address}(q_j)) \leq S(y, q)$. Split the exponential in half and use the lattice counting estimate

$$c(b) = 16\, 6! e^{2b} b^{-6}.$$

The summed derivative has price $M h_0 c(a/2)$ and width $a/[2(n+1)]$, measured from any marked slot. This is the Scatter bound used here. At order zero there is no marked slot; no volume-independent bound for the total scalar action is being asserted.

For the actual main score take

$$a = \frac{\mu}{4(n+1)(1+L)}.$$

At a signed site its density is the sum over planes of $p_{\mathrm{direct}} \operatorname{price}(U_\square) + \|\ell(U)_\square\|^2/2$, together with the cubic and quartic path series. The plaquette has length four and locality radius $5 + L$. The matrix product rule costs 2^n per product; inversion costs one for unitary matrices. In particular, writing $\mathcal{C}_n(K, C) = n! K (1 + C)^n$, a walk of length at most r has derivative price

$$\mathcal{T}_n(r, C) = \mathcal{C}_n\big(e\|1\|[1 + 2^n(1 + r)]^n, C\big).$$

The Wilson price has bound $\mathcal{P}(C) = 1 + \|\mathrm{traceCharge}\| C$, where $\mathrm{traceCharge}(M) = -\operatorname{Re}\operatorname{tr}(M)/m$. If o is the purchased outer-link jet bound, the fine-link bound is $C_f = H + 2^n H^2$, $H = o + \|1\|$,

by the same product rule. These are ordinary finite-dimensional derivative bounds on the unit real coordinate ball.

To make the path prices explicit, let r_μ be the common path analyticity radius at width $\mu/2$, and set $J_n = \mathcal{C}_n([1+2^n/r_\mu]^n, C_f)$. If s_G is the color-slot coefficient and c_G the real color projection, the row and atom prices are $R_n = J_n|s_G|\|c_G\|e^\mu$ and $A_{n,k} = J_n|s_G|^k e^\mu$. These are the convergent weighted path bounds obtained by spending half the exponential length width. Choose $Z' > 0$ bounding all derivatives through order n of $w \mapsto \|w\|^2/2$ on $\|w\| \le CR_n$; any number exceeding $1 + CR_n + (CR_n)^2/2$ suffices. Put $N_n = \sum_{i=0}^n |\mathrm{OrderedFinpartition}(i)|$. The per-plane and per-site prices are

$$Q_n = |p_{\mathrm{direct}}|\mathcal{P}(\mathcal{T}_n(4, C_f))e^{an(5+L)} + N_n Z'(1+CR_n)^n, \qquad M_n = 6Q_n + CA_{n,3} + CA_{n,4}. \quad \text{(AS3)}$$

The second term is the ordered-partition chain rule for the squared row. Signed site maps are injective, so their anchor multiplicity is one. Averaging over the signed symmetries cancels their number. The main-action Scatter price is therefore $M_n c(a/2)$.

For the prediction term choose derivative inputs through order $n+1$ first. If C_f^+ is the fine-link price at that order, put

$$F_* = 2\,\mathrm{alignCost}\,\mathrm{pathChange} + L\,\mathrm{variation}, \quad \rho = \min\left\{\rho_{\mathrm{frame}}, \frac{\sqrt{\mathrm{predictorCut}}}{F_*}, \frac{r}{\mathrm{colorScale}\,F_*}\right\}. \quad \text{(AS4)}$$

Here ρ_{frame} is the positive local minimizer radius for the frame score with the purchased $(n+1)$-jets and half its root gap. The constants pathChange and variation bound the actual connector paths under a unit coordinate displacement; alignCost converts the two endpoint frames. The three inequalities in (AS4) keep the frame minimizer in its implicit chart, keep the focus below its predictor cutoff, and keep its group logarithm inside radius r, respectively. Consequently the selected connector agrees there with the analytic minimizer. One differentiates that analytic expression, rather than the discrete selection tags.

Here are the local frame constants in this formula. Let N_P be the number of frame points, with the central frame fixed to the identity. The blank score's quadratic form is the average over ordered point pairs of squared differences of their logarithmic frame coordinates. The pairs with one endpoint at the fixed center contribute twice the sum of squared free coordinates. Consequently its counting gap is $g_P = 2/N_P^2$. Put $e_n = \mathcal{C}_n(K, 1)$, and, for input link jets bounded by C, set

$$S_n = \mathcal{P}\left(2^n[2^n e_n \mathcal{T}_n(8R, \mathcal{C}_n(C, 1))]e_n\right), \quad R_H = \min\left(1, \frac{g_P}{2(1+S_n)}\right). \quad \text{(AS4a)}$$

There are two frame factors and an intervening route of length at most $8R$; their product gives exactly this jet bound for the frame score. The mean-value estimate for its Hessian on the radius-R_H ball changes the blank Hessian by at most $S_n R_H \le g_P/2$. Its inverse there has counting-norm bound $2/g_P$. Let $J = 1 + 8RC/\mathrm{colorScale}$, the path chord price, and let $A = \mathrm{alignCost}\,J$. For the purchased logarithm derivative bound D, valid inside chord radius δ, the frame radius is

$$\rho_{\mathrm{frame}} = \min\left\{R_H, \frac{\mathrm{frameCut}}{3J}, \frac{\delta}{A}, \frac{R_H}{N_P D A}\right\}. \quad \text{(AS4b)}$$

The last term puts the logarithmic coordinate of the selected minimizing frame inside the Hilbert ball; the preceding two ensure its branch and logarithm agree with the chosen analytic chart. These are the finite local frame and alignment constructions in the nonlinear geometry section. Applying them with order $n+1$ is why the penalty estimate purchases that extra order before choosing (AS4).

The implicit derivative polynomial used here is also explicit. For branch value bound R_0, equation jet bound M, and inverse linearized-equation bound C_0, define

$$I_0 = R_0, \qquad I_{a+1} = I_a + C_0 M\{1 + |\mathrm{OrderedFinpartition}(a+1)|(1+I_a)^{a+1}\}. \tag{AS4c}$$

Differentiate the implicit equation $a+1$ times. Its single-block term is the vertical linearized equation applied to the unknown highest branch derivative. Every other ordered-partition term uses only branch derivatives of orders at most a, and is bounded by $M(1+I_a)^{a+1}$. Multiplication by the inverse bound proves the recursion; the added I_a retains all lower jets. In the frame application $R_0 = 1$, M is the frame-score price and $C_0 = 2/g_P$. This defines the polynomial used in the next paragraph without an unpriced implicit differentiation step.

More explicitly, let I_n be the implicit-derivative polynomial for the local frame score, evaluated with its score-jet price and inverse half-root-gap. The frame transport price is $H_n = \mathcal{C}_n(K, I_n)$. Then the connector price is $K_{\mathrm{conn}} = 2^n[2^n H_n \mathcal{T}_n(L, C_f^+)]H_n$, by its two endpoint factors and intervening path. Here K bounds the local exponential chart jets, and W below bounds the focus-map jets on the radius kept by (AS4). Also $o_{n+1} = \mathcal{C}_{n+1}(Z, 1+\dim \mathfrak{g})$, where Z is the purchased group-chart jet price. Each prediction bond is supported within radius $3L + 16R$, where R is the patch radius. Its anchor price is explicitly

$$N_{\mathrm{pred}} = \mathcal{P}\left(2^n o_{n+1} \mathcal{C}_n(W, K_{\mathrm{conn}})\right) e^{an(3L+16R)}. \tag{AS5}$$

There are four coarse bond orientations per embedded anchor, so summation has multiplicity four. Thus the penalty Scatter price is $4|p_{\mathrm{penalty}}|N_{\mathrm{pred}} c(a/2)$. Combining the two terms gives the actual price

$$C_* = 1 + \left|M_n c(a/2) + 4|p_{\mathrm{penalty}}|N_{\mathrm{pred}} c(a/2)\right|, \quad a_* = a/[2(n+1)]. \tag{AS6}$$

All constants in (AS3)–(AS6) are fixed from the group, stage, path budgets, local frame jets and patch geometry before the torus period. In particular no number of lattice sites occurs in them.

Finally the origin slope and gap are consequences of the actual score expansion. The quadratic line limit is one half of the signed average of $\sum_{\square}[p_{\mathrm{direct}}\|dz\|^2 + \|\ell_{\mathrm{lin}} z\|^2]$ plus one half of the penalty times the squared linear prediction residuals. Cubic and quartic terms start in their respective orders. Analytic Taylor expansion therefore gives zero first derivative at zero and this quadratic form as the Hessian. Restrict to vertical vectors. Keeping the identity signed orientation gives coefficients $b = p_{\mathrm{direct}}/|\mathrm{Signed}|$ and p_{penalty} in front of the curl and prediction-residual energies. The forest counting estimate reconstructs a vertical field from these energies, with loss $K = \mathrm{curlFactor}(L, \mathrm{box}, \mathrm{linkCost}) + \mathrm{probeFactor}(L)$. Its tree edges vanish; the harness link estimate bounds the remaining reconstructed links. Hence

$$D_v F(0,0) = 0, \qquad D_v^2 F(0,0)[v,v] \geq \frac{\min(b, p_{\mathrm{penalty}})}{1+K} \sum_i v_i^2. \tag{AS7}$$

For an explicit count, let X be the finite cover box of radius macroR, let $V_L = L^4$, and let $c_1, c_2 > 0$ be the two local harness constants. With kernel predictor cutoff $q_* > 0$, put

$$\alpha = 2\max(1, q_*^{-1})(2c_1 + c_2), \qquad E_L = (8L+1)c_2 L + 2c_1 L^2 + \alpha/L^2.$$

The local link reconstruction bounds a nonnegative link norm by E_L times the sum of curvature norms in its translated X-box, plus its coarse prediction-residual norm. Squaring uses $(a+b)^2 \leq 2a^2 + 2b^2$, and the curvature sum uses Cauchy–Schwarz with $6|X|$ terms. Summing fine bonds,

reindexing their coarse parent and residue, introduces $4V_L$ positive-axis residue bonds; a fixed translated curvature can occur at most $|X|$ times. A residual occurs V_L times. Thus

$$K_{\mathrm{curl}} = 2E_L^2 V_L \, 4\,(6|X|)\,|X|, \qquad K_{\mathrm{probe}} = 2V_L, \qquad K = K_{\mathrm{curl}} + K_{\mathrm{probe}}. \tag{AS8}$$

This derives the finite denominator in (AS7); it may grow with the blocking ratio and fixed patch, but never with the macroscopic torus period. The positive numerator is part of the actual step hypotheses. The equality of color and coordinate squared sums above gives precisely this counting norm, with no dimension conversion. Also $\mathrm{tint}(e_i)$ is the corresponding physical color frame, so the vertical Hessian block is the physical one. Invertibility of tint shows that the zero vertical derivative in (AS1) is exactly the original stationary equation. Thus (AS4), (AS6), the address multiplicity h, and the gap in (AS7) provide the actual initial radius, width, price, multiplicity and gap for (US1)–(US8). Their stationary graph, inverse-Hessian and determinant estimates consequently apply on the same common domain used in stationary integration and the reference chart.

We make explicit the uniform implicit construction used in the stationary integrations. Split the finite coordinate set into retained coordinates ι and vertical coordinates τ, assign every coordinate a lattice site by l, and assume at most K coordinates have any given site. Arrays carry their maximum norm. Write $f(x,v)$ for the real analytic cost, $S(x,v) = D_v f(x,v)$ for its vertical gradient, and $H(x,v) = D_v^2 f(x,v)$. The analytic scatter bounds established above say that its coordinate derivatives, through a purchased order $N \geq 3$, are bounded by a constant $C > 0$ times the exponential connecting weights with width $a > 0$, on the maximum-norm ball of radius $r > 0$. Assume $S(0,0) = 0$ and the counting-measure gap

$$\sum_{ij} u_i H(0,0)_{ij} u_j \geq m \sum_i u_i^2, \qquad m > 0. \tag{US1}$$

All these inputs were obtained for the actual cost; a bound on the extensive value $f(0,0)$ is unnecessary.

Take the explicit four-dimensional counting bound $c(a) = 16 \cdot 6! e^{2a} a^{-6}$. A distance shell of radius k has at most $(2k+1)^4$ sites. The exponential-series inequality $e^{a(k+2)} \geq [a(k+2)]^6/6!$ bounds its contribution by $c(a)/[(k+1)(k+2)]$. Summing the telescoping series gives $\sum_y e^{-a\,\mathrm{dist}(z,y)} \leq c(a)$, uniformly in the period. Bounded multiplicity changes this to $Kc(a)$. Put $D = Kc(a)$. The scatter third-derivative bound and integration along a segment give

$$\|H(z) - H(0)\|_{\infty\to\infty} \leq \|z\|_\infty C D^2. \tag{US2}$$

The same row and column sums bound the corresponding quadratic form: apply $2|u_i u_j| \leq u_i^2 + u_j^2$ to each entry. Thus the gap remains at least $m/2$ on the ball of radius

$$R_H = \min\{r/2, m/[2(1 + CD^2)]\}. \tag{US3}$$

We need an inverse in the maximum norm with a price independent of the number of sites. Suppose a matrix B has entries bounded by $Ce^{-a\,\mathrm{dist}(l_i,l_j)}$ and counting gap $M > 0$. For a fixed site z, conjugate it by the diagonal weights $w_i = e^{b\,\mathrm{dist}(z,l_i)}$. The triangle inequality and $|e^t - 1| \leq |t|e^{|t|}$ show that, if $0 < b \leq a/4$, the difference from B has entries bounded by

$$Cb(4/a)e^{-(a/2)\,\mathrm{dist}(l_i,l_j)}.$$

Choose

$$b = \min\left\{a/4, \frac{M}{2\{1 + C(4/a)Kc(a/2)\}}\right\}. \tag{US4}$$

The quadratic-form price of the perturbation is at most $M/2$; hence the conjugated matrix has counting coercivity $M/2$. It need not be symmetric. To estimate its inverse column, solve $Bu = e_j$, take $z = l_j$, and put $v_i = w_i u_i$. Coercivity gives

$$(M/2)\sum_i v_i^2 \le v_j.$$

In particular $(M/2)v_j^2 \le v_j$, so $v_j \le 2/M$, and then $|v_i| \le 2/M$ for every i. Undoing the weight gives

$$|(B^{-1})_{ij}| \le (2/M)e^{-b\,\mathrm{dist}(l_i,l_j)}. \tag{US5}$$

Summing a row yields a maximum-norm inverse price $(2/M)Kc(b)$. This argument uses the true counting gap and exponential summability throughout, and introduces no volume-dependent equivalence between the maximum and Euclidean norms.

Apply this with $M = m/2$ on (US3), and denote the resulting width by b. Set

$$J = 1 + (4/m)Kc(b), \quad R_V = \min\{R_H, [4J(1+CD^2)]^{-1}\},$$

$$R_X = \min\{R_V, R_V/[4J(1+CD)]\}. \tag{US6}$$

For $\|x\| < R_X$, consider the map $Y_x(v) = v - H(0,0)^{-1}S(x,v)$ on $\|v\| \le R_V$. Its derivative is $H(0,0)^{-1}\{H(0,0) - H(x,v)\}$. Equations (US2) and (US6) give its norm at most $1/4$. Integrating the mixed Hessian in the retained variable gives $\|S(x,0)\| \le CD\|x\|$, so $\|Y_x(0)\| \le R_V/4$. Therefore $\|Y_x(v)\| \le R_V/2$ on that closed ball. Iteration is Cauchy with successive differences reduced by $1/4$; its limit $g(x)$ is its unique fixed point. Since $H(0,0)$ is invertible, being a fixed point is exactly $S(x,g(x)) = 0$. We have

$$g(0) = 0, \quad \|g(x)\| \le R_V/2, \quad S(x,v) = 0,\ \|v\| \le R_V \implies v = g(x). \tag{US7}$$

At every point of this graph the vertical Hessian is invertible. The local analytic implicit function theorem gives an analytic branch there; its continuity keeps it inside the same vertical ball nearby, and uniqueness in (US7) identifies it with g. These local identifications prove analyticity of the single graph on the whole retained ball. This also identifies any retained critical germ with this graph wherever both satisfy the corridor bounds.

For completeness, its higher response estimates follow directly from the critical equation, rather than from a dimension-dependent implicit-function norm. Let $Z(x) = (x, g(x))$. Differentiate $S \circ Z = 0$ $k+1$ times. In the ordered-partition chain formula the one-block term containing $D^{k+1}g$ is $H(Z)D^{k+1}g$; every other term contains graph derivatives of order at most k. At a fixed purchased order use the common width

$$b_* = \min\{\mathrm{weaker}_N(a), \mathrm{weaker}_N(b)\} > 0,$$

where $\mathrm{weaker}_N(t) = t/[2(N+1)]$. Combining the joining-tree weights loses only this width once for the whole induction: the span of any one resulting tree is at most the sum of its constituent spans, and splitting each available exponent between this connecting span and the summation variables leaves the remaining half for the counting sums. At most $N+1$ vertices are joined at the purchased order. If P_k prices graph derivatives through order k, the other terms have price at most

$$F_k = C + |\,\mathrm{OrderedPart}(k+1)|[C\{1 + P_k Kc(a/2)\}]^{k+1}.$$

Multiply by the inverse kernel (US5) and sum its output coordinate. The recursion

$$P_0 = 1, \qquad P_{k+1} = P_k + F_k(4/m)Kc(b/2) \tag{US8}$$

therefore bounds every coordinate response through order $k+1$, with its connecting-tree exponential of width b_*. Every constant in (US3)–(US8) depends on the purchased finite orders and local input prices, but not the period. Differentiating $f\circ Z$ uses the same ordered partitions; its terms containing a vertical first derivative of f vanish by stationarity. This gives the on-shell action bounds used in the nonlinear integral and the reference estimates. The separate stronger strip estimates for generated precision kernels retain their own widths; (US8) does not replace them.

Source–readout formulation. The pair (retained coordinates, vertical coordinates) is exactly the auxiliary-elimination situation of the source-lift method: an auxiliary variable v is eliminated by a stationary section, and the recovery of the effective theory depends on keeping the residual. The raw sector is the corridor

$$\mathcal{K}_{\mathrm{st}} = \{(x,v) : \|x\| < R_X,\ \|v\| \le R_V\}, \qquad \mathrm{Read}_{\mathrm{ret}}(x,v) = x, \qquad \iota_{\mathrm{st}}(x) = (x, g(x)),$$

with residual column $S(x,v) = D_v f(x,v)$ retained at every point, including where it is nonzero. Here f, S, H, R_X, R_V, J and the graph g are the objects constructed in (US1)–(US8). Then $\mathrm{Read}_{\mathrm{ret}}\,\iota_{\mathrm{st}} = \mathrm{id}$, while $\iota_{\mathrm{st}}\,\mathrm{Read}_{\mathrm{ret}} \neq \mathrm{id}$: the fibre of the retained readout is the vertical ball, and the stationary section selects one point of it. The three analytic inputs of the derivation, and its conclusion, are used later in the following quotable forms.

Lemma 14.1 (Scatter bound)**.** *Let terms with anchor estimates (M,a) be summed over anchors with multiplicity at most h_0. Then the sum has, measured from any marked slot, price $Mh_0\,c(a/2)$ and width $a/[2(n+1)]$, where $c(b) = 16\cdot 6!\,e^{2b}b^{-6}$.*

Proof. Fix a marked slot q_j. The triangle inequality gives $\sum_i \mathrm{dist}(\mathrm{address}(q_j),\mathrm{address}(q_i)) \le (n+1)S(y,q)$ and $\mathrm{dist}(y,\mathrm{address}(q_j)) \le S(y,q)$. Split $e^{-aS} = e^{-aS/2}e^{-aS/2}$; the first factor is at most the connecting weight of width $a/[2(n+1)]$ from the marked slot and the second is at most $e^{-(a/2)\,\mathrm{dist}(y,\mathrm{address}(q_j))}$. The counting estimate $\sum_y e^{-b\,\mathrm{dist}(z,y)} \le c(b)$, proved below at (US2), sums the anchors. At order zero there is no marked slot, and no volume-independent bound for the total scalar action is asserted. □

Lemma 14.2 (Implicit derivative polynomial)**.** *For a branch of an implicit equation with value bound R_0, equation jet bound M and inverse linearized-equation bound C_0, define*

$$I_0 = R_0, \qquad I_{a+1} = I_a + C_0M\{1 + |\mathrm{OrderedFinpartition}(a+1)|(1+I_a)^{a+1}\}.$$

Then I_a bounds all branch derivatives through order a.

Proof. Differentiate the implicit equation $a+1$ times. In the ordered-partition chain formula the single-block term is the vertical linearized equation applied to the unknown highest branch derivative. Every other term uses branch derivatives of order at most a and is bounded by $M(1+I_a)^{a+1}$; there are at most $|\mathrm{OrderedFinpartition}(a+1)|$ of them, and the explicit derivative of the equation in the parameter contributes at most M. Applying the inverse bound C_0 and adding I_a (which keeps all lower jets) gives the recursion. □

Lemma 14.3 (Period-free weighted inverse)**.** *Let B have entries bounded by $Ce^{-a\,\mathrm{dist}(l_i,l_j)}$ and counting gap $M>0$ (it need not be symmetric), and let*

$$b = \min\Big\{\frac{a}{4},\ \frac{M}{2\{1+C(4/a)Kc(a/2)\}}\Big\}.$$

Then

$$|(B^{-1})_{ij}| \le \frac{2}{M}e^{-b\,\mathrm{dist}(l_i,l_j)}, \qquad \|B^{-1}\|_{\infty\to\infty} \le \frac{2}{M}Kc(b).$$

Proof. Fix a site z and conjugate by $w_i = e^{b\,\mathrm{dist}(z,l_i)}$. The conjugated entry is $B_{ij}w_i/w_j$ and $|w_i/w_j - 1| \le b\,d_{ij}e^{b\,d_{ij}}$ with $d_{ij} = \mathrm{dist}(l_i, l_j)$, by the triangle inequality and $|e^t - 1| \le |t|e^{|t|}$. For $b \le a/4$, $b\,d\,e^{bd}e^{-ad} \le b(4/a)e^{-(a/2)d}$ because $d\,e^{-(a/4)d} \le 4/(ea) \le 4/a$. The perturbation therefore has row and column sums at most $Cb(4/a)Kc(a/2) \le M/2$, hence quadratic-form price at most $M/2$, and the conjugated matrix $\widetilde{B}$ has counting coercivity $M/2$. Solve $Bu = e_j$, take $z = l_j$ and $v_i = w_i u_i$; then $\widetilde{B}v = e_j$ since $w_j = 1$, and coercivity gives $(M/2)\sum_i v_i^2 \le v_j$. Thus $(M/2)v_j^2 \le v_j$, $v_j \le 2/M$, and then $v_i^2 \le (2/M)v_j \le (2/M)^2$. Undoing the weight gives the entry bound and summing a row gives the norm bound. Only the counting gap and exponential summability are used; no equivalence of maximum and Euclidean norms enters. □

Theorem 14.4 (The common stationary graph)**.** *For $\|x\| < R_X$ the map $Y_x(v) = v - H(0,0)^{-1}S(x,v)$ is a contraction with constant $1/4$ of the ball $\|v\| \le R_V$ into the ball of radius $R_V/2$. Its fixed point $g(x)$ satisfies*

$$g(0) = 0, \qquad \|g(x)\| \le R_V/2, \qquad (S(x,v) = 0,\ \|v\| \le R_V) \Longrightarrow v = g(x),$$

and g is real analytic on the retained ball.

Proof. $DY_x(v) = H(0,0)^{-1}\{H(0,0) - H(x,v)\}$ has norm at most $J \cdot R_V CD^2 \le 1/4$ by (US2), (US5) and (US6). Integrating the mixed Hessian in the retained variable gives $\|S(x,0)\| \le CD\|x\|$, so $\|Y_x(0)\| \le JCD\,R_X \le R_V/4$ and $\|Y_x(v)\| \le R_V/4 + R_V/4$. The iteration is Cauchy and its limit is the unique fixed point; since $H(0,0)$ is invertible, a fixed point is exactly a zero of $S(x,\cdot)$. At each graph point the vertical Hessian is invertible (gap $m/2$), so the analytic implicit function theorem gives a local analytic branch; continuity keeps it in the vertical ball and uniqueness identifies it with g. The local identifications cover the ball. □

The next statement is not in the supplied text. It is the quantitative form of the method's requirement that an eliminated auxiliary variable be recoverable from the retained residual, and it follows from the estimates (US1)–(US6) alone.

Theorem 14.5 (The residual recovers the vertical discrepancy)**.** *For $(x,v) \in \mathcal{K}_{\mathrm{st}}$ put $w = v - g(x)$. Then*

$$\|w\|_2 \le \frac{2}{m}\|S(x,v)\|_2, \qquad \|w\|_\infty \le \frac{4}{m}Kc(b)\,\|S(x,v)\|_\infty \le J\|S(x,v)\|_\infty.$$

In particular $S(x,v) = 0$ if and only if $(x,v) = \iota_{\mathrm{st}}(x)$, and the effective cost $f_{\mathrm{eff}} = f \circ \iota_{\mathrm{st}}$ satisfies $D_x f_{\mathrm{eff}}(x) = D_x f(x, g(x))$.

Proof. Since $S(x,g(x)) = 0$, $S(x,v) = \bar{H}w$ with $\bar{H} = \int_0^1 H(x, g(x) + tw)\,dt$. Every point of the segment lies in the ball of radius $R_V \le R_H$, so each integrand and hence $\bar{H}$ has counting gap at least $m/2$, and $\langle w, S(x,v)\rangle \ge (m/2)\|w\|_2^2$; Cauchy–Schwarz gives the first bound. The entries of $\bar{H}$ obey the second-derivative scatter bound $Ce^{-a\,\mathrm{dist}}$, so Lemma 14.3 with $M = m/2$ gives $\|\bar{H}^{-1}\|_{\infty\to\infty} \le (4/m)Kc(b)$, the second bound. The chain rule gives $D_x f_{\mathrm{eff}} = D_x f + D_v f \cdot Dg$ and the last term vanishes on the graph. □

Thus nothing about the vertical position is lost so long as the residual is kept: the discrepancy is controlled by, and vanishes exactly with, the recorded residual, with period-free constants. What the stationary restriction does discard is the off-shell vertical dependence of f; the lift keeps it in $\mathcal{K}_{\mathrm{st}}$ and uses the restriction only where the supplied construction integrates on or expands about the graph. Relative to the frame realization, the coordinates (x,v) are components in the fixed source color basis of logarithmic link variables on the identity-frame section. A frame change conjugates each color vector by an orthogonal map: Euclidean color norms, the counting gap (US1) and the

exponential weights are unchanged, whereas the real-coordinate maximum box is carried to its rotated image. The corridor is therefore transported as a set, with the bounds $\| \operatorname{paint} \| \le 1 + \dim \mathfrak{g}$, $\| \operatorname{paint}^{-1} \| \le 1$ relating the two norms; it is not asserted to be the same coordinate box in a rotated basis. The status of Theorems 14.4 and 14.5 is on-shell recovery with an explicit off-shell defect bound, on the branch selected by the corridor. The uniqueness clause of Theorem 14.4 is what makes the branch a single-valued section rather than a choice.

14.2 The Fourier normalization and the kinetic readout

The two-field determinant film has an internal momentum and two opposite external momenta. Its internal average is normalized Haar average, namely $\operatorname{stock} F = \int_{[-\pi,\pi]^4} F(k) d^4k/(2\pi)^4$; its finite-period average is $\operatorname{shop}_n F = n^{-4} \sum_k F(2\pi k/n)$. The curvature picture called $\operatorname{profitable} F$ is one half of the internal average with the two external link and color indices in their specified order. The factor one half is the quadratic Taylor factorial. Translation invariance makes the actual Hessian entry zero unless its two Fourier momenta sum to zero, and otherwise that entry is the site count times the finite-period average of the film.

To compare the two averages, expand the internal variable in its absolutely convergent Laurent series. Finite averaging retains exactly the coefficients whose four indices are divisible by the period; Haar averaging retains only the zero coefficient. Every remaining index has length at least the period. Spending half the common strip for summation and a quarter for the tail gives a fixed strip price times $e^{-\sigma n/4}$. Multiplication by the site count gives the recorded-Hessian error. Thus this internal integral represents the finite Hessian with a quantitative error and a fixed normalization.

Compose that picture with the momentum chart for real initial links. The readout $\operatorname{leapTip}(\mathcal{P})(e_0, 1, (a, a))$ is the divided coefficient $D^2 P_{(1,a),(1,a)}(0)[e_0, e_0]/2!$. Reality makes this diagonal coefficient real. Color invariance and simplicity make its color bilinear form a scalar multiple of the fixed invariant form, so its value is independent of the normalized color. The factor two in (DP1) is additional to the factor one half in the internal film average; both are retained in that definition.

Here is why this one transverse entry determines the leading kinetic tensor. Subtract its multiple of the normalized Wilson tensor, whose readout is one. Gauge invariance gives transversality. Signed covariance includes the phases from reversing initial links; at the first nonzero homogeneous order those phases have constant term one. The constant and linear homogeneous terms vanish by transversality and permutation covariance. Let $Q_{uv}(p, q)$ be the symmetric polarized degree-two term of the difference. Reflecting one axis shows that a component can survive only when its momentum occurrences and link occurrences have the same parity on every axis. Permuting axes moves the zero transverse diagonal readout to all transverse diagonal entries on single momentum axes. Longitudinal entries vanish by transversality. Reflection parity then kills mixed momentum entries in a diagonal link component and same momentum entries in an off-diagonal link component. The only remaining possibility has its two distinct momentum axes equal to its two link axes. Transversality at the sum of those two coordinate vectors makes twice that entry zero, since the diagonal entries have already vanished. Bilinearity gives $Q = 0$. The degree-three term vanishes under total inversion, whose momentum sign is odd and whose two link signs multiply to one. The difference thus starts at degree four. This proves the one-axis kinetic identification used below, including the Taylor normalization and reflected-edge convention.

Source–readout formulation. Two readouts are compared here, and each has an explicit fibre. The first is the pair of internal averages: shop_n reads a periodic analytic film on the finite dual

torus, stock reads its Haar average. They are two different readouts of the same film, and their difference is a recorded discrepancy, not an identification:

Proposition 14.6 (Aliasing readout)**.** *Let F be periodic and analytic with* $\sup|F| \le A$ *on the strip of width σ. Then*

$$\operatorname{shop}_n F = \sum_{x \in n\mathbb{Z}^4} \widehat{F}(x), \qquad \operatorname{stock} F = \widehat{F}(0), \qquad |\operatorname{shop}_n F - \operatorname{stock} F| \le A\,\mathsf{c}(\sigma)\,e^{-\sigma n/4},$$

with a strip price $\mathsf{c}(\sigma)$ independent of n.

Proof. The Laurent series converges absolutely, and the finite average of $e^{ik\cdot x}$ over the period-n mesh is one if every component of x is divisible by n and zero otherwise. Shifting contours gives $|\widehat{F}(x)| \le Ae^{-\sigma|x|_1}$. Every surviving nonzero index has $|x|_1 \ge n$; write $e^{-\sigma|x|_1} \le e^{-\sigma n/4}e^{-(\sigma/2)|x|_1}$. $e^{-(\sigma/4)(|x|_1-n)}$ and sum the middle factor over $\mathbb{Z}^4$, giving $\mathsf{c}(\sigma) = [(1+e^{-\sigma/2})/(1-e^{-\sigma/2})]^4$. □

The exactness class is controlled approximation in the period n at fixed strip width, with the normalization (the factor one half of the quadratic Taylor factorial and the site count) retained on both sides. The second readout is leapTip, which reads a single transverse diagonal coefficient of a two-leg tensor. On arbitrary tensors its fibre is large; on the class of tensors that actually occurs it is as small as the kinetic identification requires:

Proposition 14.7 (One transverse entry determines the kinetic tensor)**.** *Let T be a gauge-invariant, signed-covariant analytic two-leg tensor whose* leapTip *readout vanishes. Then its homogeneous parts of degree at most three vanish; T starts at degree four.*

Proof. Gauge invariance gives transversality, and signed covariance includes the phases from reversing initial links, which have constant term one at the first nonzero homogeneous order. Constant and linear terms vanish by transversality and permutation covariance. Let $Q_{uv}(p,q)$ be the symmetric polarized degree-two term. Reflecting one axis, a component can survive only if its momentum occurrences and link occurrences have the same parity on every axis. Axis permutations carry the vanishing transverse diagonal readout to all transverse diagonal entries on single momentum axes; longitudinal entries vanish by transversality. Reflection parity then kills mixed momentum entries in a diagonal link component and equal momentum entries in an off-diagonal one. The only remaining entry has its two distinct momentum axes equal to its two link axes, and transversality at the sum of those two coordinate vectors makes twice that entry zero, the diagonal entries having vanished. Bilinearity gives $Q = 0$. The degree-three term is odd under total inversion, whose two link signs multiply to one, so it vanishes. □

Equivalently, on gauge-invariant signed-covariant analytic two-leg tensors the map $T \mapsto \text{leapTip}(T)$ is injective modulo terms of degree at least four, and the normalized Wilson tensor is a section of it. Gauge invariance and signed covariance are properties of the target tensors; in the frame realization the tensors are those of the readout composed with π, their Taylor coefficients at a flat raw configuration are taken with the raw directions conjugated into the stored frame (Theorem 9.2), and the invariant color form is unchanged by that conjugation, so the readout and its normalization are fibre constants.

14.3 Analytic exterior division with the origin jets retained

We give the analytic division used in issuing the loop coefficients. For one momentum coordinate set $q_i(p) = e^{ip_i} - 1$. If x is a Laurent index, define

$$L_{i,x}(p) = \begin{cases} \sum_{a=0}^{x_i-1} e^{ip\cdot(x[i:=a])}, & x_i \geq 0, \\ -\sum_{a=0}^{-x_i-1} e^{ip\cdot(x[i:=-a-1])}, & x_i < 0. \end{cases}$$

The empty sum is zero. Multiplication and telescoping give the entire identity

$$q_i L_{i,x} = e^{ip\cdot x} - e^{ip[i:=0]\cdot x}. \tag{ED1}$$

If F is periodic analytic and bounded by A on a strip of width σ, its Fourier coefficient bound on an intermediate strip makes $G(p) = \sum_x \widehat{F}(x) L_{i,x}(p)$ absolutely and locally uniformly convergent on width $\sigma/4$. Indeed the finite sum has at most $|x|$ terms and size at most $|x|e^{(\sigma/4)|x|}$; the spare width absorbs $|x|$. Summing the geometric coefficient bounds in each momentum coordinate gives the explicit price

$$\|G\|_{\sigma/4} \leq A \frac{4}{\sigma} \left[2 \frac{1 + \sigma/4}{\sigma/4} \right]^v, \tag{ED2}$$

where v is the number of momentum coordinates. Thus $q_i G = F - F|_{p_i=0}$, including at every zero of q_i; no singular quotient is defined on a mesh. If the numerator has zero jets below order $s+1$, comparison of its analytic Taylor series with $q_i = ip_i + O(p_i^2)$ gives zero jets of G below order s. Equivalently evaluate on nearby directions with nonzero i-component, divide their one-variable series, and extend the resulting polynomial-jet identity to all directions by continuity.

For exterior coefficients let C be contraction by the vector q, and I_i exterior multiplication by the i-th coordinate vector with its orientation sign. The coordinate anticommutation rule gives $CI_i + I_i C = q_i \,\mathrm{id}$. Apply (ED1) componentwise to a cycle u, and call the divided difference $D_i u$. Substituting in this anticommutation rule shows that

$$CI_i D_i u = u - \mathrm{comb}_i\, u,$$

where $\mathrm{comb}_i\, u$ evaluates at $p_i = 0$ and deletes every exterior component containing i. The identity is first algebraic where $q_i \neq 0$, and extends to the zeros by analyticity. The residual is again a cycle. Indeed, after separating components containing i, its contraction at momentum p equals its contraction with the i-th symbol set to zero, since every i-containing component has been removed; the original cycle equation at $p_i = 0$ supplies the remaining equations. Evaluation and deletion preserve the original strip, coefficient bound and Taylor vanishing order.

Repeat on the residuals in the four axis directions, adding the four lifted differences. Every residual still has the original width σ, price A, and order s. Each new lift is therefore computed directly at width $\sigma/4$, price at most A times (ED2), and order at least $s-1$. There is no iteration of four strip shrinkages or four losses of Taylor order. Summing gives width $\sigma/4$, price four times (ED2), and order $s-1$. The contraction of this sum telescopes to the original cycle minus the final residual. In positive exterior degree the latter is zero because every orientation contains a processed axis. In degree zero it is the original coefficient with all coordinates of this block set to zero; the required hypothesis is its vanishing on that entire coordinate plane. Positive order at the full origin implies this hypothesis only when this is the sole momentum block. This is why the exact harmonic-set check in (TEL2) is needed.

Suppose additionally that each coefficient vector lies in a prescribed linear subspace depending only on the other momentum blocks. Evaluation in a divided coordinate preserves that subspace.

The divided Fourier sum also does so: where $q_i \neq 0$ it is the scalar multiple $(u - u|_{p_i=0})/q_i$, and at its zeros continuity and closedness of the finite-dimensional subspace give the same conclusion. Exterior multiplication is a signed selection of these coefficient vectors. Hence every residual and lift retains the other block constraints. For a dependent momentum equal to minus the sum of the independent ones, the linear shear exchanging that block with the dependent block is involutive. If there are $t+1$ independent blocks, its missing coordinate is the negative sum of their coordinates. A strip of width $\sigma/(t+2)$ is mapped inside width σ, since $(t+1)\sigma/(t+2) < \sigma$. Apply the preceding coordinate construction on that narrowed strip and shear back with the same narrowing factor. This gives the dependent-block width and price used in (TEL3).

Source–readout formulation. Division is a section of contraction, and this is the form in which the loop coefficients are issued: a cycle u is the readout, a primitive is a source datum over it, and the choice of primitive is the fibre. Write

$$\operatorname{Div} u = \sum_{i=1}^{4} I_i D_i(\operatorname{comb}_{i-1} \cdots \operatorname{comb}_1 u)$$

for the sum of the four lifted differences constructed above, so that the telescoping identity of the derivation reads $C \operatorname{Div} u = u - \operatorname{comb}_4 \cdots \operatorname{comb}_1 u$.

For completeness we verify the single-axis identity $CI_iD_iu = u - \operatorname{comb}_i u$ and the cycle property of the residual in one computation. Write ι_i for the interior product with the i-th coordinate vector and C_0 for contraction with the symbol whose i-th entry is set to zero, so that $C = C_0 + q_i\iota_i$. Scalar division commutes with C, hence, using $Cu = 0$, $q_i\, CD_iu = Cu - C(u|_{p_i=0}) = -C_0(u|_{p_i=0}) - q_i\iota_i(u|_{p_i=0})$, and $C_0(u|_{p_i=0}) = (Cu)|_{p_i=0} = 0$; therefore $CD_iu = -\iota_i(u|_{p_i=0})$, first where $q_i \neq 0$ and then everywhere by analyticity. Consequently

$$CI_iD_iu = q_iD_iu - I_iCD_iu = u - u|_{p_i=0} + I_i\iota_i(u|_{p_i=0}),$$

and $I_i\iota_i$ is the projection onto the components containing i, which gives $u - \operatorname{comb}_i u$. Applying C to this identity and using $C^2 = 0$ gives $C \operatorname{comb}_i u = Cu = 0$, so the residual is again a cycle.

Proposition 14.8 (Division as a section of contraction)**.** *On analytic cycles of positive exterior degree (and on degree-zero coefficients vanishing on the block's coordinate plane),* $C \circ \operatorname{Div} = \mathrm{id}$. *The fibre of* C *over* u *is* $\operatorname{Div} u + \ker C$, *and two primitives of the same* u *differ by a* C*-cycle of one higher degree.* Div *preserves the block constraints, loses one Taylor order and three quarters of the strip, and is bounded by four times* (ED2)*.*

Proof. The identity and the bounds were derived above; the description of the fibre is linearity of C. □

The origin jets are retained on both sides: the proposition records the loss of exactly one Taylor order and of three quarters of the strip, and the degree-zero hypothesis is kept as a hypothesis (it is the reason for the exact harmonic-set check in (TEL2)). No singular quotient on a mesh is ever part of the realized record; where q_i vanishes the primitive is defined by the entire identity (ED1).

14.4 The two-leg correction with both momentum blocks retained

Loop data and the running-coupling reader. The loop calculation reads local quadratic response from the same stationary integration used in the forward blocking step. Its internal momentum measure, external slot ordering and Taylor factorials determine the numerical coefficient.

The calculation supplies a drift estimate for the actual action coefficient; it does not declare that coefficient equal to a chosen reference line. Summing the drift will require the common calendar constructed next. Thus the coefficient and its controlled remainder, rather than only the formal logarithm, are the output consumed by calibration and uniform running.

Here is the precise two-leg use of the exterior division. There is one independent four-momentum p; the other is $-p$. Write $E^r = \Lambda^r \mathbb{C}^4$, with its ordered plane basis, and let $C_-^r : E^{r+1} \to E^r$ be contraction by the lattice symbol at $-p$. Thus $C_-^r C_-^{r+1} = 0$. A coefficient with one remaining potential index belongs to $E^r \otimes E^1$; one with no such index belongs to E^r. These are coefficient-valued periodic analytic functions of the same complexified momentum. Let $H : E^r \otimes E^1 \to E^r$ contract the second index by the symbol at p. Since these maps act on different tensor slots,

$$HC_-^r = C_-^r H. \tag{TEL1}$$

There is no exterior sign in this convention. A coefficient has order a if all its real Taylor coefficients of total degree below a at zero vanish. Each contraction increases this order by one.

The given two-potential tensor is $u \in E^1 \otimes E^1$, of order $s+2$, satisfying both gauge contractions $C_-^0 u = 0$ and $Hu = 0$. Apply the dependent-block division to obtain

$$X \in E^2 \otimes E^1, \quad C_-^1 X = u, \quad \operatorname{ord} X \geq s+1.$$

The coefficient $HX \in E^2$ has order at least $s+2$, and is a dependent cycle because $C_-^1 HX = Hu = 0$. A second dependent-block division gives

$$Y \in E^3, \quad C_-^2 Y = HX, \quad \operatorname{ord} Y \geq s+1.$$

Next lift the independent contraction: choose $D \in E^3 \otimes E^1$ with $HD = Y$ and order at least s. The degree-zero obstruction for this last division must be checked on its actual harmonic set. There is just one independent momentum block, so that set is exactly $p = 0$, rather than an unspecified collision locus. Since $s + 1 \geq 1$, every coefficient of Y vanishes there. This proves the required obstruction is zero. In the general many-leg construction the other independent contractions define coefficient subspaces that remain constant when this block varies; the coordinate homotopy preserves them. For two legs there are no other independent indices, so this preservation condition is vacuous.

Now put

$$V = X - C_-^2 D \in E^2 \otimes E^1. \tag{TEL2}$$

It still has order at least $s+1$. Its represented tensor is unchanged, since $C_-^1 V = u - C_-^1 C_-^2 D = u$. Its remaining contraction is removed, since $HV = HX - C_-^2 HD = HX - C_-^2 Y = 0$. Apply the independent-block division separately to the six E^2 coefficients of V. This gives $g \in E^2 \otimes E^2$, of order at least s, whose two curl contractions equal u. All identities hold at zero momentum as analytic identities, as well as at nonzero real momentum.

For clarity the width and norm losses can also be followed without any existence convention. Put $\operatorname{slim}_t(a) = a/(t+2)$, and, for a coordinate set of size m,

$$T_m(a) = \frac{16}{a}\left[\frac{2(1+a/4)}{a/4}\right]^m, \quad P(a) = T_4(a/2), \quad E(a) = a/16, \quad B_r(a) = \binom{4}{r+1} 4(e^a+1).$$

Here T_m is four times the coordinate-division price in (ED2); B_r bounds the finite sum defining a curl contraction on the strip. If u has strip width $\sigma > 0$ and coefficient bound $A \geq 0$, set

$$\tau = E(\sigma), \quad w = E(\tau), \quad \mu = w/16, \quad b = P(\sigma), \quad c = bB_0(\tau), \quad d = cP(\tau), \quad e = dT_4(w).$$

The bounds for X, HX, Y, D are respectively Ab, Ac, Ad, Ae; their widths are $\tau, \tau, w, w/4$. Restricting to μ, the bound for V is

$$A\{b + eB_2(w/8)\}. \tag{TEL3}$$

For the final independent division there is exactly one remaining leg. Apply the coordinate-wall lift (ED1)–(ED2) with the four independent coordinates to each E^2 component of V. The other coefficient indices are passive, so taking the maximum of their coefficient bounds introduces no extra factor six. The construction gives explicitly

$$\eta = \mu/4, \qquad \|g\|_\eta \le A\{b + eB_2(w/8)\}T_4(\mu), \qquad \operatorname{ord} g \ge s. \tag{TEL4}$$

To see the contraction identity componentwise, denote the lift of the independent index by $W_a(b,p)$, where a is the dependent two-form index and b the independent two-form index. Then $\sum_b \operatorname{symbol}_1(p)_{b,i} W_a(b,p) = V_{a,i}(p)$. Contracting this equality with $\operatorname{symbol}_1(-p)_{a,h}$ and summing over a gives $u_{h,i}$ by (TEL2). This is precisely the two-curl representation, with no omitted polarization or identification of the two momentum slots. More generally repeated independent divisions obey the recursion $\eta_0(\sigma) = \sigma$, $K_0 = 1$, $\eta_{t+1}(\sigma) = \eta_t(\sigma/4)$, $K_{t+1}(\sigma) = T_m(\sigma)K_t(\sigma/4)$, where m is the ambient coordinate count. Each step preserves the contractions in the other independent blocks because their symbols do not vary with the divided block. For this application only $t = 1, m = 4$ is needed. These prices depend on the input strip, not on the torus period or on s. The tensor spaces, the harmonic-set check and the two identities in (TEL2) give the two-leg construction used in both loop-moment applications below.

Source–readout formulation. The source object is the two-curvature coefficient $g \in E^2 \otimes E^2$; the readout is the double curl $\operatorname{Curl}_2(g)_{h,i} = \sum_{a,b} \operatorname{symbol}_1(-p)_{a,h} \operatorname{symbol}_1(p)_{b,i}\, g_{a,b}$, valued in doubly transverse two-potential tensors. Theorem 14.9 constructs a section s_2 with $\operatorname{Curl}_2 \circ \mathsf{s}_2 = \mathrm{id}$ on analytic doubly transverse tensors of order at least two, bounded by (TEL4) and lowering the order by two. The readout is not injective: its kernel contains $C_-^2(\cdot)$ in the first slot and the analogous image in the second. The correction $-C_-^2 D$ in (TEL2) is precisely a motion inside that kernel fibre, chosen to make the intermediate primitive transverse in the remaining slot. The realized history stores (X, Y, D, V, g); the readout recovers u and nothing more.

Theorem 14.9 (Two-curl primitive)**.** *There exists $g \in E^2 \otimes E^2$ of order at least s whose two curl contractions equal u, constructed as follows.*

1. *Dependent-block division gives $X \in E^2 \otimes E^1$ with $C_-^1 X = u$, $\operatorname{ord} X \ge s + 1$.*
2. *$HX \in E^2$ has order at least $s+2$ and $C_-^1 HX = HC_-^1 X = Hu = 0$; a second dependent-block division gives $Y \in E^3$ with $C_-^2 Y = HX$, $\operatorname{ord} Y \ge s + 1$.*
3. *Independent division gives $D \in E^3 \otimes E^1$ with $HD = Y$ and order at least s; its degree-zero obstruction lives on the harmonic set, which for a single independent block is exactly $p = 0$, where every coefficient of Y vanishes since $s + 1 \ge 1$.*
4. *Put*
$$V = X - C_-^2 D \in E^2 \otimes E^1.$$
Then $\operatorname{ord} V \ge s + 1$, $C_-^1 V = u - C_-^1 C_-^2 D = u$ and $HV = HX - C_-^2 HD = HX - C_-^2 Y = 0$.
5. *Independent-block division of the six E^2 coefficients of V gives g.*

All identities hold as analytic identities at zero momentum as well as at nonzero real momentum.

Proof. Each step is an application of Proposition 14.8 together with (TEL1); the cycle conditions needed for each division are the displayed ones. For (5), let $W_a(b,p)$ denote the lift of the independent index, a the dependent and b the independent two-form index. Then $\sum_b \mathrm{symbol}_1(p)_{b,i} W_a(b,p) = V_{a,i}(p)$, and contracting with $\mathrm{symbol}_1(-p)_{a,h}$ and summing over a gives $u_{h,i}$ by (TEL2). No polarization is omitted and the two momentum slots are not identified. In the many-leg construction the other independent contractions define coefficient subspaces constant in the divided block, which the coordinate homotopy preserves; for two legs that condition is vacuous. □

14.5 Production of the fresh loop moments

The moment normalization of the fresh determinant loop is obtained before its path representation is issued. Let f be the common analytic two-field picture of the determinant and Haar term, R the normalized coarse metric row with $R(0) = 1$, and g the inherited-loop picture. The latter vanishes to order four in real external momentum. Set α equal to the kinetic readout of f and subtract α times the normalized metric picture. The one-axis identification proved above shows that this difference, and therefore

$$h = f - \alpha\,\mathrm{slots}(R) + g,$$

vanishes to order four. It retains its two gauge-contraction identities, reality and color invariance. Its analytic norm is bounded by $A+|\alpha|\,\mathrm{bedCharge}(\sigma, M)+B$, where A, B bound f, g and M bounds the metric row. Thus this normalization does not discard the inherited loop price.

Apply (TEL1)–(TEL3) with $s = 2$: the input has zero jets below order four, and its two-curvature coefficient has zero jets below order two. In particular the second-contraction correction uses the harmonic set $p = 0$ checked there; the coefficient norm and positive strip width have the period-independent prices in (TEL3).

Write the resulting coefficients, for fixed plaquette and color types, as a real Laurent family $P(z)$. Taking real parts replaces an analytic coefficient on real momenta by the average of it and its conjugate at opposite momentum. This preserves the represented real tensor and its vanishing order. Project its finite color coefficient vector onto invariant tensors; this is a fixed real linear projection, of finite absolute matrix-sum price, which leaves the already invariant represented tensor unchanged. It commutes with reading the absolutely convergent Laurent series and with its derivatives. Thus

$$\widehat{P}(p) = \sum_{z\in\mathbb{Z}^4} P(z) e^{ip\cdot z}, \qquad \widehat{P}(0) = 0, \quad D\widehat{P}(0) = 0, \qquad \sum_z |P(z)| e^{\eta|z|} < \infty \tag{LM0}$$

for a fixed positive output width η.

Exponential summability also gives $\sum |z_i P(z)| < \infty$, by spending part of η. The series may therefore be differentiated on the real momentum axes. Its value and first derivative at zero are $\sum P(z)$ and $i\sum z_i P(z)$. Equation (LM0) proves both sums zero, separately for each plaquette and color choice. The actual path issue enumerates these types and offsets. Its first leg is based at zero, its second at z, and each leg includes the four plaquette edges. Its length is $|z| + 8$, so the path price differs from the Laurent price only by the fixed factor $e^{8\eta}$ and the finite type and color counts. The displacement in the moment is exactly z. Reindexing by this enumeration proves the summable zeroth and first unwrapped displacement moments of the issued atoms. These are the fresh-loop identities needed in the next subsection and in (TF17)–(TF19).

Source–readout formulation. The fresh loop is produced by composing three maps whose source–readout roles are different. (a) The kinetic readout α of the two-field picture is subtracted;

by Proposition 14.7 the remainder lies in the fibre of that readout over zero and therefore starts at degree four. (b) The two-curl section of Theorem 14.9 is applied to the remainder with $s = 2$; this chooses one primitive in the kernel fibre of the double curl, with the bound (TEL4). (c) The real part and the invariant-color projection are applied; both are fixed real linear maps of finite price which fix the already real, invariant represented tensor and commute with the Laurent series, so they move the coefficient only inside the fibre of the represented tensor. The issued path atoms are then a relabelling of the Laurent family: first leg at zero, second at z, length $|z| + 8$, displacement exactly z. The two vanishing statements of (LM0) are thus statements about the chosen source coefficient, not about the represented tensor alone, and they are the input of Proposition 14.10 below. The normalization step does not discard the inherited loop price: the norm bound $A + |\alpha|\,\mathrm{bedCharge}(\sigma, M) + B$ keeps all three contributions.

14.6 Production and transport of the inherited loop moments

The full-family identities (TF17)–(TF19) require absolute summation before truncation. We give the construction that transports these identities across a block. For a two-curvature atom a, let q_a be its coefficient with its plaquette and color types fixed, and let $\delta_a \in \mathbb{Z}^4$ be the displacement from its first leg to its second. Its length bounds $|\delta_a|$. A charged exponential bound implies

$$\sum_a |q_a| < \infty, \qquad \sum_a |q_a \delta_a^i| < \infty, \tag{LM1}$$

because $s \le C_\eta e^{\eta s}$. Thus all changes of summation below are absolute, including those with a displacement factor.

For one atom translate its first leg to the origin in the finite torus sum. Along a fixed route from zero to δ_a, telescope the second leg. Subtract its value and its first differences at the origin. The remaining discrete second-order Taylor difference can be written as a finite sum of products of differences, one on each curvature factor, by summation by parts along successive route edges. Explicitly the removed terms are

$$q_a \sum_x F_a(x) G_a(x), \qquad q_a \delta_a^i \sum_x F_a(x) \nabla_i G_a(x). \tag{LM2}$$

The discrete Taylor identity is exact on the torus: each link telescope is a translated finite difference and the sum of a translated product can be shifted by the same lattice translation. Repeating this along the two route indices generates at most the square of the route length in remainder terms. The exponential budget absorbs this square after any fixed positive width discount. If the full family has zero zeroth and first displacement moments, summing (LM2) gives zero separately for each type and color. Its total action therefore equals the sum of the two-difference remainder patterns. This proves the representation used to inherit a quiet loop; the subtraction is performed on the whole summable family, not on an arbitrary finite prefix.

Let $\psi_i(r, b; z)$ be the installed fine-gradient column for a coarse residue r, curvature type b, direction i, and unwrapped displacement z. These columns are the product of the fine frame inverse, the differentiated linear tower, and the coarse frame. Whenever the stationary-column construction uses two curvature legs, its dependent-momentum join is precisely (TEL1)–(TEL3), with the retained Taylor orders stated there. On harmonic input the unmodified tower has residue-independent total columns. Install it on a one-site coarse torus. Its output is then constant on the fine torus; a fine difference of that output is zero. The intertwining identity between the differentiated column and the fine difference consequently gives

$$\sum_z \psi_i(r, b; z) = 0. \tag{LM3}$$

The frame factors preserve the constant-output statement by their translation convolution identities. Odd blocking is the hypothesis that makes this centered-gradient readout agree with the installed fine difference. Exponential column bounds justify passing from the installed periodic identity to the absolutely summed cover columns.

After substituting these columns in a two-difference pattern, one fixed choice of its two output types has coefficient $uf_0(z_0)f_1(z_1)$, where $\sum f_0 = \sum f_1 = 0$ by (LM3). Its zeroth total is $u(\sum f_0)(\sum f_1) = 0$. Its displacement is $(c_1 + z_1) - (c_0 + z_0)$, with fixed carried offsets c_0, c_1. Splitting its first moment gives

$$u\left\{(\sum f_0)\sum_z (c_1+z)^i f_1(z) - \sum_z (c_0+z)^i f_0(z)(\sum f_1)\right\} = 0. \tag{LM4}$$

The absolute first moments exist by the column width bounds. Summing over residues, types and patterns is justified by (LM1) and the charged bound on the full substituted family. Thus inherited loops have the exact two moment identities, with their block price L^{-2}, rather than merely small moments.

Joining two families means enumerating their disjoint union. Absolute sums add, so both moment identities are preserved. Blank initial loops have zero coefficients and satisfy them. The coefficientwise limiting argument (SH1)–(SH3) preserves them as well: first pay the common exponentially weighted tail, then pass to the limit on a finite list. This supplies the full-family justification used to turn retained moments into omitted-tail moments in (TF18). Any fresh loop must satisfy the same identities before it is joined; they cannot be inferred from gauge invariance or the quadratic-potential normalization alone.

Source–readout formulation. The zeroth and first unwrapped displacement moments define a readout of an atom family which forgets everything except five numbers per plaquette and color type. Write $\mathcal{A} \mapsto \mathcal{A}'$ for the block transport of the derivation above: $\mathcal{A}'$ is the family obtained by substituting the installed columns ψ_i into the two-difference patterns of $\mathcal{A}$.

Proposition 14.10 (The moment readout and the quiet fibre)**.** *On the additive monoid of absolutely summable two-curvature atom families with charged exponential bounds (disjoint union as addition), define for each plaquette and color type*

$$\mathsf{m}(\mathcal{A}) = \Big(\sum_a q_a,\ \sum_a q_a\delta_a\Big) \in \mathbb{R}\times\mathbb{R}^4.$$

Then m *is additive, the quiet families* $\mathsf{m}^{-1}(0)$ *are closed under joining, under the block transport* $\mathcal{A} \mapsto \mathcal{A}'$ *of* (LM1)–(LM4) *(indeed* $\mathsf{m}(\mathcal{A}') = 0$ *for every transported two-difference family), and under the coefficientwise limits* (SH1)–(SH3)*. Blank initial loops are quiet. A fresh loop is quiet by* (LM0)*, and quietness cannot be deduced from gauge invariance or the quadratic normalization alone.*

Proof. Additivity is additivity of absolute sums over a disjoint union. Transport is (LM3)–(LM4). For limits one first pays the common exponentially weighted tail and then passes to the limit on a finite list. The last sentence records that (LM0) used the order four vanishing produced by subtracting the kinetic readout, which is additional to transversality. □

The moment readout is many-to-one. The construction never inverts it; it only proves that every family it produces lies in the fibre over zero, and that this fibre is stable under the three operations applied to atom families (joining, block transport, coefficientwise limits). That stability is the operation-compatibility statement required for the later use in (TF17)–(TF19), where retained moments are converted into omitted-tail moments.

14.7 The one-loop drift used by the finite histories

The finite coupling drift comes from the committed path coefficient and the bounded scalar reset, not directly from a formal beta function. At a transition with block length L, put $\mu = L^{-1}$. The determinant calculation on the retained physical path defines a coefficient $\alpha_{N,i}$ with sign appropriate to descent from level $i+1$ to i. We explain the logarithm and its uniform remainder.

For a fixed color direction, the continuum pole of the relevant second external-momentum coefficient is

$$p_G(k) = 4C_A \left(\frac{k_1^2}{|k|^6} - 4\frac{k_1^2 k_0^2}{|k|^8} - \frac{1}{|k|^4} \right). \tag{DR1}$$

The coefficient includes the vector determinant with weight $1/2$ and the gauge determinant with weight -1. These are the same two determinant weights kept in the massive-symbol calculation of the trace. The finite symbol coefficient $f_\mu(k)$, on the angular cube $[-\pi, \pi]^4$, satisfies

$$|f_\mu(k)| \le \frac{A}{(|k|+\mu)^4}, \qquad |f_\mu(k) - p_G(k)| \le \frac{D}{|k|^3} + \frac{D\mu^2}{|k|^6} \quad (|k| \ge \mu), \tag{DR2}$$

where $A, D \ge 0$ depend on the fixed analytic widths, coefficient and ellipticity bounds, and not on L, the torus period or its terminal cutoff. The input paths have their fixed quadratic normalization and the allowed higher soft jets; the latter preserve the continuum pole.

Here is the matrix estimate yielding (DR2). At scale $T = |k| + \mu$, the zero-field massive matrix, one-field vertex, and two-field vertex have sizes $T^2, T, 1$. Coercivity bounds the inverse by CT^{-2}. Differentiate the identity $AA^{-1} = 1$ to obtain $D(A^{-1}) = -A^{-1}(DA)A^{-1}$ and its second derivative, the sum of the two products with two first vertices minus the product with one second vertex. Along a momentum direction scaled by T, all jets of each inverse still have size T^{-2}. The lower determinant bubble therefore has size T^{-2}, and its second unscaled external-momentum coefficient has size T^{-4}. The upper insertion obeys the same rough bound. Summing the finitely many matrix entries gives the first bound in (DR2).

For $|k| \ge \mu$, take $T = |k|$. The zero-field symbol differs from its continuum quadratic part by $O(T^3 + \mu^2)$, and the first vertex differs from its continuum linear part by $O(T^2 + \mu^2/T)$. Use $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ and telescope the factors of the determinant bubbles. Their relative error has size $T + \mu^2/T^2$; their second momentum coefficient consequently has error $C(T + \mu^2/T^2)T^{-4}$, which is the second bound in (DR2). The upper insertion contributes bounds of these same or smaller orders. The origin-jet cancellations, analytic derivative bounds, inverse formulas and explicit continuum vertex contraction are developed in the weighted massive-symbol and logarithmic momentum calculations below. This argument uses them before integration, so its error bound is absolute.

To integrate, split the angular cube into $|k| \le \mu$, $\mu < |k| \le 1$, and its part outside the unit ball. The first bound in (DR2) gives

$$\left| \int_{|k| \le \mu} f_\mu(k)\, dk \right| \le A\mu^{-4} \frac{\pi^2 \mu^4}{2} = \frac{\pi^2 A}{2}.$$

Outside the unit ball it gives a bound A times the volume of the angular cube. On the annulus the second bound gives

$$\begin{aligned} \int_{\mu < |k| \le 1} \frac{D}{|k|^3}\, dk &= 2\pi^2 D(1-\mu) \le 2\pi^2 D, \\ \int_{\mu < |k| \le 1} \frac{D\mu^2}{|k|^6}\, dk &= \pi^2 D(1-\mu^2) \le \pi^2 D. \end{aligned} \tag{DR3}$$

All these errors are bounded independently of $0 < \mu \le 1$.

The pole integral is exact. Rotational symmetry gives, on this annulus,

$$\int \frac{k_1^2}{|k|^6}\,dk = \tfrac{1}{4} I_\mu, \qquad \int \frac{k_1^2 k_0^2}{|k|^8}\,dk = \tfrac{1}{24} I_\mu, \qquad I_\mu = \int |k|^{-4}\,dk = 2\pi^2 \log(1/\mu). \tag{DR4}$$

For the mixed fourth moment, symmetry makes all off-diagonal moments equal. Expanding the fourth powers of $k_0 + k_1$ and $k_0 - k_1$, and rotating these directions to a coordinate direction, shows that the diagonal fourth moment is three times the mixed one. Summing over the four coordinates then gives the factor $1/24$. Substitution of (DR4) into (DR1), with Fourier weight $(2\pi)^{-4}$, gives

$$(2\pi)^{-4} \int_{\mu < |k| \le 1} p_G(k)\,dk = -\frac{11 C_A}{24\pi^2} \log(1/\mu) = -s \log L. \tag{DR5}$$

Equations (DR2)–(DR5) bound the difference between the infinite-volume coefficient and this logarithm by a pre-block constant.

We now connect this integral to the coefficient of the actual stationary path. It is important to distinguish three determinant films: M_{act}, the film of the actual stationary integration; M_{core}, its core film before the stationary carrier change; and M_{rank}, the auxiliary rank film. Each film is a matrix of analytic functions of the external momentum, with indices consisting of a direction and a color. Their scalar reading is

$$\alpha_a(M) = 2\,\mathrm{Re}\,\mathrm{leapTip}(\mathrm{profitable}\,M)(e_0, 1, (a, a)). \tag{DP1}$$

The operation leapTip is the coefficient of degree two in that external-momentum direction, with its Taylor factorial; profitable passes from the determinant film to its curvature tensor using its analytic contraction identities. Thus α_a is linear on films satisfying the common bounds, and is the specific normalized kinetic coefficient, not the full film norm.

Here are the quantitative comparisons used in the transition. Fix an analytic width $h > 0$, with $h \le \sigma \le 1/128$, a block patch of stage J, its penalty k, a positive stationary width, and all the common metric, covariance and path bounds. The required hypotheses are: the input path is attached to its quadratic metric and has its prescribed analytic bounds; that metric has the uniform floor on every torus; the coarse metric has its analytic strip bound; the stationary graph is critical, guessed by the installed tower, and satisfies the gap and heat identities; the same forest and tag are used in its determinant realization; and the relevant torus sides form an auxiliary sequence tending to infinity. These are precisely the linear-horizon and stationary-graph data already constructed. The input loop-tensor norm may have its own bound Λ, whereas the comparison constants below depend only on the fixed analytic and matrix bounds specified before the block length. They do not grow with Λ.

Let $F \ge 0$ bound the inverse matrices in the determinant calculation, let $M_* \ge 0$ bound the common zero-, one- and two-field film factors, and let $K_* \ge 0$ bound their carrier difference. All three are obtained from the fixed stage, kernel, penalty and common analytic bounds before L is chosen. More explicitly the common factor bound is the sum of the absolute bounds for the leg, core, rank and frozen metric factors; the carrier bound is the common bound for the difference between the core and rank factors. Put

$$D_* = FK_*F,\ E_* \ = D_*M_* + FK_*,\ R_* = E_* + (E_*F + FM_*D_*)M_* + FM_*FK_*. \tag{DP2}$$

This is the source's exact reserve polynomial. It arises by subtracting inverse factors with $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$, then telescoping the zero-, one- and two-field determinant products. For

example the first inverse difference costs D_*; multiplying by the common first factor and adding the factor difference costs E_*; the remaining two placements and their cross product give the last line. The trace estimate is a relative rank estimate, so this subtraction introduces no count of all L^4 block sites.

In detail, if $r = |\operatorname{Loc}_0| + |\operatorname{Loc}_1|$, the individual films are bounded on width $h/2$ by

$$L^4\, 2r(FM_* + FM_*FM_*). \tag{DP3}$$

The separate difference estimate is

$$M_{\mathrm{core}} - M_{\mathrm{rank}} \quad \text{bounded on width } h/2 \text{ by } R_*. \tag{DP4}$$

Here is why (DP4) has no block-volume factor. The matrix weights are invertible analytic frames $S_X(p), S_Y(q)$. A bound for a matrix U means a Euclidean operator-norm bound for $\widetilde{U} = S_X(p)^{-1} U S_Y(q)$. For the difference estimates the stronger bound is used:

$$|\operatorname{tr}(\widetilde{U}V)| \le C\|V\|_{2\to 2} \qquad \text{for every compatible } V. \tag{DP10}$$

It is subadditive, is invariant under transposition with dual frames, and multiplies with an ordinary operator bound on either side. At equal source and target frames, taking $V = 1$ bounds the actual trace by C. A general matrix with operator norm C only has price $\dim(X)C$ in (DP10); using that conversion would incur the unwanted L^4.

The exact difference between the core and rank quadratic films cancels their common sector and pure Gram terms. Write S for the shear row, T for its localized emitter correction, and B for the fixed block inclusion. Expanding $(S+T)^*(S+T) - S^*S = T^*(S+T) + S^*T$ gives its reflection part. The complete remaining difference is the sum of the local house film and this reflection part, minus $B^*\mathcal{O}B$ and the finite motive film; $\mathcal{O}$ is the localized orbit correction. This identity holds separately for the zero-, one- and two-field matrices, with all their Leibniz placements. The common sector cancels before any trace is bounded.

We derive the uniform trace-ideal estimates before using them in the difference. For a finite matrix U, set

$$\|U\|_{\mathrm{tr},*} = \sup_{\|V\|_{2\to 2}\le 1} |\operatorname{tr}(UV)|.$$

For $U = uv^*$, the trace is v^*Vu, so its size is at most $\|u\|\|v\|$. Decomposition into columns bounds the size by the sum of their norms. A matrix with a fixed n_0-dimensional source consequently has price $n_0\|U\|_{2\to 2}$. Multiplication on either side by a bounded matrix multiplies this price by its operator norm, by trace cyclicity. Apply these statements after the analytic changes of frame in (DP10).

One further factor has a source dimension growing with L, and needs an actual summability argument. Put $\mu = L^{-1}$, let ω be the lattice Laplacian symbol, and define

$$d_\mu(k) = \frac{\mu^2}{\omega(k) + \mu^2}, \qquad J = 16 \sum_{n=0}^{\infty} (n+1)^{-2} < \infty.$$

On the strip of width $1/128$, choose the centered real-momentum representative before enumerating aliases. The massive lower bound gives

$$|d_\mu(k_l)| \le \frac{2}{(1 + \operatorname{pdist}(l))^2}.$$

At $l = 0$ this is the origin massive bound. At $l \neq 0$, centered alias separation bounds the symbol scale below by $(1 + \mathrm{pdist}(l))\mu$; strip ellipticity bounds the denominator below by half the square of that scale. Division gives the inequality. Changing the centered representative permutes the aliases. There are at most $(2n+1)^4$ aliases at distance n, because their lifts inject into $[-n, n]^4$. Therefore

$$\sum_l |d_\mu(k_l)^3| \leq 8 \sum_{n \geq 0} \frac{(2n+1)^4}{(n+1)^6} \leq 8J.$$

The alias-diagonal matrix $d_\mu^3 I$ has trace-ideal price $8n_0 J$, with fixed $n_0 = |\operatorname{Loc}_0|$. The Fourier frame and inverse frame each cost at most $e^{4/128}$, leaving the block-independent price

$$Q = e^{4/128}(8n_0 J)e^{4/128}.$$

This origin estimate controls the zero-, one- and two-field matrices of the regularizing ninth power. Write $P = \mu^2(\mathcal{A} + \mu^2 I)^{-1}$, where $\mathcal{A}$ is the sector Gram matrix, and $U = P^3$. The preceding estimate applies to U at zero field. If C_3 bounds its three ordinary film matrices, the corresponding three matrices of U^3 have trace-ideal price at most $20C_3^2 Q$. In fact every zero-field product has an unmarked U, every first-field product has two, and every second-field product has at least one. Put one unmarked factor in the trace ideal and the other two in operator norm. The second-field Leibniz formula has three placements of a second-field matrix and six ordered placements of two first-field matrices. Their sum and the lower orders are covered by 20. This never converts an L^4-dimensional bound into a trace bound. The regularizer is $\mu^2 P^9$, so its price is $\mu^2 B$, where $B = 20C_3^2 Q$.

We now price each localized composition. We specify how the derivative prices used here are obtained. If an analytic finite-dimensional map F is evaluated at z, define, for a derivative bound $c \geq 0$,

$$E_m(F, z, c) = \sum_{n=0}^{m} \sum_{\pi \in \mathcal{P}_n} \|D^{|\pi|}F(z)\| c^{|\pi|}, \tag{DP12}$$

where $\mathcal{P}_n$ is the finite set of partitions in the iterated chain rule, with its multiplicities, and $\mathcal{P}_0$ contributes the value bound. If an inner map takes the value z at the origin and its derivatives through order m have the prescribed tested bound c, its composite is bounded by (DP12): differentiate successively, grouping the directions according to the block that enters each derivative of the inner map, and sum their norm products. The source's apollo and encore are precisely these finite sums. For a product use $F(u, v) = uv$ and inner price $c_1 + c_2$; inversion of a group matrix is its adjoint and preserves these derivative bounds. The same construction applies to the matrix exponential at zero, the chosen analytic group logarithm at the identity, and the analytic nearest-group projection in its fixed neighborhood.

These rules generate the charter and emission prices from the actual stencil. A link perturbation has its explicit factor L^{-1}. On a length-L path, the product rule chooses the differentiated links. If q distinct links are differentiated, there are at most L^q choices and their derivative bounds contribute L^{-q}; all undifferentiated group factors have norm one. The remaining assignments are the finite partitions of the derivative directions. Thus every fixed derivative order has a common bound independent of L. Patch averages are probability averages. Moving from the padded patch to its translated subpatch costs at most its fixed stage-dependent cardinality ratio, since both counts are bounded multiples of L^4. The frame estimates already proved give the corresponding finite derivative bounds for the recentered gauge transformations. Form the connector by multiplying the two endpoint frames and the path transport, average it, and apply the nearest-group projection; (DP12) pays all these operations. Multiply by the coarse-link factors and the endpoint frames, and

apply the logarithm. This constructs the finite positive prices denoted springs for the charter and rocket_fare for the emission in FrameEst/Stencils. Every logarithm is evaluated at the identity in this centered calculation. None of these steps takes a supremum over a number of arbitrary independent site values.

Here is the last conversion into row norms, including its block powers. Let $n \le 2$ be the number of inserted field directions and let c bound their fixed Fourier-frame readings. Put $c' = \mathrm{muralRoom}(J)c$, including the fixed stage enlargement of those readings. The scalar directional estimate for the charter is at most

$$2^{n+1}|\operatorname{springs}(J, n+1, c')|;$$

the corresponding emission estimate, after changing from the patch probability average to the block-site probability average, is at most

$$\mathrm{portrait}(J)2^{n+1}|\operatorname{rocket_fare}(J, n+1, c')|.$$

The factor 2^{n+1} pays the real-and-imaginary expansions of the n inserted directions and the one tested input direction. Conversion of the finitely many color coordinates to their Euclidean norm multiplies these by $\sqrt{n_1}$ and $\sqrt{n_0}$, respectively. For the emission the probability-average estimate is a sum of squares divided by L^4; removing that denominator and taking a square root contributes exactly L^2. Taking the maximum, or the larger finite sum, over $n = 0, 1, 2$ gives prices a and h independent of L, with charter price a and emission price L^2h. These are RealGap.CharterR.costs and Chromatic.costs. The same finite derivative and Fourier-frame bounds give the sector price p; the positive massive inverse and its differentiated inverse identities give the regularizer price $\mu^2 g$. The house kernel has fixed dimension and the already constructed inverse bound, giving its ordinary price b.

Each of a, h, p, g, b thus bounds the zero-, one- and two-field matrices on the same strip. The house kernel has fixed source dimension $n_1 = |\operatorname{Loc}_1|$, so its trace-ideal price is $n_1 b$. Film multiplication costs at most four times the product, covering its four second-field Leibniz terms. The house price is consequently

$$C_H = 4(4an_1b)a.$$

Composition of the regularizer with the site-to-color row cancels $\mu^2 L^2 = 1$. Its trace-ideal price is $r = 4Bh$ and its ordinary price $4gh$. The shear row has ordinary price $4p$, so the corrected row has price $t = 4p + 4gh$. The two reflection products have combined trace-ideal price

$$C_R = 4rt + 4(4p)r.$$

For the orbit term let s_i bound the fixed stage row in direction i, and put $S(z) = \sum_{i=0}^{3} 4(4s_i z)s_i$. Its unweighted trace price is $\mu^2 S(B)$. The two changes to the dual massive frames cost at most $e^4\mu^{-1}$ each, yielding

$$C_O = e^4 S(B) e^4.$$

The motive film factors through the fixed n_0-dimensional space, with inclusions of ordinary price one; its trace-ideal price is $4n_0$. All widths and derivative orders have been fixed before L, and each cancellation of a block power has been displayed. These four estimates supply (DP10). Composing a difference factor with an ordinary bounded factor keeps (DP10). In particular the house film has price C_H; the two reflection products have combined price C_R; $B^*\mathcal{O}B$ has price $16C_O$, including its two film-composition factors; and the motive film has price $4|\operatorname{Loc}_0|$. Thus one may use exactly

$$K_* = C_H + C_R + 16C_O + 4|\operatorname{Loc}_0|. \tag{DP11}$$

Each price here is the absolute weighted column sum of its stated local factor with its zero-, one- and two-field placements, not a norm of a full trace. The analytic row and localized column estimates bound those sums by the fixed stage and strip prices. They use the same $\sigma \le 1/128$, branch bounds and common stationary rows as (DP3), and are uniform in L. The inverse identity gives difference price $FK_*F = D_*$. For a two-field determinant frame, the direct term difference costs $D_*M_* + FK_* = E_*$. Telescoping its three-factor bubble gives $(E_*F + FM_*D_*)M_* + FM_*FK_*$. Taking its trace through (DP10) gives precisely (DP2) and (DP4). The factor $1/2$ in the core-minus-rank determinant may be discarded when increasing this bound. This proves the relative projection estimate while retaining the L^4 factor in the unrelated individual estimates (DP3). The stationary carrier change has zero kinetic reading:

$$\alpha_a(M_{\mathrm{act}} - M_{\mathrm{core}}) = 0. \tag{DP5}$$

To see this, write its diagonal film as $\frac{1}{2}(W(p) + W(-p) - V(p)/2)$. The stationary tax row has zero jets below order two. Its mixed carrier row has zero jets below order one, by the stationary graph identity. Hence each summand of W, a product of these two rows, vanishes to order at least three. Every summand of V contains two tax rows and vanishes to order at least four. These films are analytic on the common strip, so their external degree-two coefficient is zero. Linearity proves (DP5).

For an analytic scalar function bounded by A on width r_0, two successive Cauchy derivative estimates give

$$\|D^2 f(0)\| \le \frac{A}{r_0/2}\frac{1}{r_0/4}.$$

For the normalized reading (DP1), the source keeps the slightly weaker explicit price

$$\mathcal{B}(r_0, A) = \frac{A}{r_0/4}\frac{1}{r_0/8}\|\operatorname{axes}(e_0)\|^2. \tag{DP6}$$

The factor two in (DP1) cancels the Taylor factorial. Apply this estimate to (DP4), with $r_0 = h/2$, and use (DP5). The exact projection bound is

$$|\alpha_a(M_{\mathrm{act}}) - \alpha_a(M_{\mathrm{rank}})| \le \mathcal{B}(h/2, R_*). \tag{DP7}$$

No inverse power of the lattice period and no L^4 multiplier occurs in (DP7).

There is one more comparison, which is an equality rather than a bounded error. Let Π_μ be the analytic film of the massive pole test on the fine lattice, with $\mu = 1/L$. Its finite-period analytic tensors approximate that test with error $\mathrm{cents}(n)e_n$, where $e_n \to 0$, and the rank film approximates the same test pulled back through the stationary uniform map with error $\mathrm{cents}(m)y_m$, where $y_m = Ce^{-cm}$ for fixed L. The constants and analytic strip in these approximation statements may depend on L. They are used for identification only, not as the uniform bound in (DP7).

Fourier evaluation of the stationary pullback gives the exact alias formula

$$H(p) = L^4 \sum_{\ell} S((p + 2\pi\ell)/L)[n_-^\ell(p), n_+^\ell(p)]. \tag{DP8}$$

Here $S = \Pi_\mu$, and $n_\pm^\ell$ are the incoming and outgoing stationary response columns. The analytic contraction identities of S remove longitudinal columns. For $\ell \neq 0$, the remaining response columns each vanish to order two at $p = 0$; their product therefore has no degree-two coefficient. For $\ell = 0$, the primal and second-order stationary identities identify its contribution, through degree two, with

$\mu^2 S(\mu p)$. The discrepancy vanishes to order three. Taking two momentum derivatives introduces another μ^2, so the coefficient multiplier is $L^4\mu^2\mu^2 = 1$. Consequently

$$\alpha_a(M_{\mathrm{rank}}) = \alpha_a(\Pi_\mu). \tag{DP9}$$

The finite-period approximation just described proves that the analytic pullback in (DP8) is the rank film: let the auxiliary coarse and fine sides tend to infinity at fixed L, keeping the same stationary graph and tags. The vanishing approximation errors identify their Fourier tensors and derivatives. The reality, gauge and signed symmetries of the actual pole test give its contraction identities and zero lower jets. No assumption about a momentum continuation between different lattice meshes is used.

Combining (DP7), (DP9), and (DR2)–(DR5) now gives the exact allowance used in the transition:

$$d = |\mathcal{B}(h/2, R_*)| + D_{\mathrm{ann}} + 1, \qquad |\alpha_{N,i} + s\log L_i| + 1 \le d, \tag{DR6}$$

where $D_{\mathrm{ann}} \ge 0$ is the bound for the massive-symbol integral remainder. The stage construction fixes $h, F, M_*, K_*, D_{\mathrm{ann}}$, and hence d, before the block length. Afterwards it imposes one lower bound on odd L, and post-block bounds on the analytic norm and loop allowance of the output path. In the transition notation these latter bounds have the forms $\Theta \ge \beta_*(L)$, $c' \le \gamma_*$, and $\Lambda' \ge \Lambda_0(L)$, $\Lambda' \ge \Lambda/2$. They may depend on L; they do not enter d. The stationary tensor is produced using arbitrary growing torus sequences satisfying the stated floor and stationary identities; after coefficient identification it is one cover path and works on every realized torus satisfying the construction's patch and support margins. Thus the auxiliary large-volume limit imposes no extra period-dependent error on (DR6), and the estimate is uniform over the terminal cutoffs whose paths have those same pre-block bounds. The scalar reset has $|\Delta| \le 1$. Since its actual coarse coupling is $\beta_{N,p,i} = \beta_{N,p,i+1} + \alpha_{N,i} + \Delta$, we obtain

$$\begin{aligned} |\beta_{N,p,i+1} - \beta_{N,p,i} - s\log L_i| &= |\alpha_{N,i} + \Delta + s\log L_i| \\ &\le |\alpha_{N,i} + s\log L_i| + 1 \le d_{i+1}. \end{aligned} \tag{DR7}$$

Thus the one-step drift used in the shooting and running arguments includes both the calculated path coefficient and the actual nonlinear reset. The choices of large block lengths and late coupling entry below pay this drift simultaneously with all the local activity and insertion bounds.

Source–readout formulation. The source records here are the three films with their frames, the stationary graph and the tags; the single reader is α_a. Three different relations to the reader occur and must not be merged: (i) $M_{\mathrm{act}} - M_{\mathrm{core}} \in \ker \alpha_a$ exactly (DP5): the carrier change is a redundancy of the realization invisible to this reader, although it is visible to the film norm; (ii) $M_{\mathrm{core}} - M_{\mathrm{rank}}$ is a bounded, generally nonzero discrepancy (DP7): a controlled approximation; (iii) M_{rank} and Π_μ have equal readings (DP9) although the films differ at higher order and at nonzero aliases: equality on the readout, not of sources. Accordingly the status of the slope is the following. The number $s = 11C_A/(24\pi^2)$ is *derived* within the supplied construction: it is the exact coefficient of $\log L$ in the annular integral (DR5) of the pole (DR1). The reference clock $\widehat{\beta}_i = \bar{\beta} + st_i$ is a *chosen comparison clock* built from that number. The actual couplings are *not* asserted to equal the clock: they are produced by the committed path coefficient and the bounded reset, and are tied to the clock only through the inequality (DR7), which the shooting and running arguments below then calibrate. The lift transports α_a as a scalar reader of the realized film (films are functions of gauge-invariant stationary data and of tagged frames, read on the identity-frame section), so $\alpha^Z_{N,i} = \alpha_{N,i}$, $d^Z = d$ and (DR7) is recovered with the same constant. No physical interpretation of s beyond this calibration role is added.

The two formal statements on which this description rests are the following.

Lemma 14.11 (Trace-ideal calculus)**.** *Let* $\|U\|_{\mathrm{tr},*} = \sup_{\|V\|_{2\to2}\le1} |\operatorname{tr}(UV)|$. *Then* $\|uv^*\|_{\mathrm{tr},*} \le \|u\|\|v\|$; *a matrix with* n_0*-dimensional source has* $\|U\|_{\mathrm{tr},*} \le n_0\|U\|_{2\to2}$; *and* $\|AUB\|_{\mathrm{tr},*} \le \|A\|\,\|U\|_{\mathrm{tr},*}\|B\|$.

Proof. $\operatorname{tr}(uv^*V) = v^*Vu$. Decompose U into its n_0 columns, $U = \sum_j Ue_je_j^*$, and add. The last claim is cyclicity of the trace, $\operatorname{tr}(AUBV) = \operatorname{tr}(U\,BVA)$. □

Theorem 14.12 (The drift allowance)**.** *With* $D_{\mathrm{ann}} \ge 0$ *the bound for the massive-symbol integral remainder,*

$$d = |\mathcal{B}(h/2, R_*)| + D_{\mathrm{ann}} + 1, \qquad |\alpha_{N,i} + s\log L_i| + 1 \le d,$$

and, since the scalar reset has $|\Delta| \le 1$ *and the actual coarse coupling is* $\beta_{N,p,i} = \beta_{N,p,i+1} + \alpha_{N,i} + \Delta$,

$$|\beta_{N,p,i+1} - \beta_{N,p,i} - s\log L_i| = |\alpha_{N,i} + \Delta + s\log L_i| \le |\alpha_{N,i} + s\log L_i| + 1 \le d_{i+1}.$$

The stage construction fixes $h, F, M_*, K_*, D_{\mathrm{ann}}$ *and hence* d *before the block length; afterwards it imposes one lower bound on odd* L *and post-block bounds* $\Theta \ge \beta_*(L)$, $c' \le \gamma_*$, $\Lambda' \ge \Lambda_0(L)$, $\Lambda' \ge \Lambda/2$ *on the output path, which may depend on* L *and do not enter* d.

Proof. $\alpha_{N,i} = \alpha_a(M_{\mathrm{act}})$. By (DP7) it differs from $\alpha_a(M_{\mathrm{rank}})$ by at most $\mathcal{B}(h/2, R_*)$; by (DP9) the latter is $\alpha_a(\Pi_\mu)$, the integral of f_μ with Fourier weight; by (DR2)–(DR5) that integral differs from $-s\log L$ by at most D_{ann}. The stationary tensor is produced with arbitrary growing torus sequences obeying the floor and stationary identities; after coefficient identification it is one cover path and works on every realized torus with the construction's margins, so the auxiliary limit adds no period-dependent error. □

14.8 Massive auxiliary symbols and the uniform alias trace bound

Why the calendar belongs to the witness. The massive regulator and the sequence of purchased scales control which estimates may be combined. Positive regulator mass is fixed before the corresponding volume estimates; the subsequent choices retain this order. The calendar records ratios, stage horizons and delays large enough for all finitely many demands already made. Native source admission supplies no preferred ordering of the realized leaves. The order used here is therefore part of the realized witness, and every later calibration, comparison and limiting argument must use that same calendar.

The massive comparison used in the preceding determinant argument is fixed as follows. With the lattice exterior symbol $q_a(p) = e^{ip_a} - 1$, put

$$s(p) = \sum_a q_a(-p)q_a(p), \quad d_\mu(p) = \frac{\mu^2}{s(p)+\mu^2}, \quad c_\mu(p) = \mu^2 d_\mu(p)^9, \quad \mu = L^{-1}. \tag{MA1}$$

On real momentum $s(p)$ is the nonnegative squared curl frequency. The scalar massive covariant-site operator at a group field is $D_U^*D_U + \mu^2$; its normalized inverse is $\mu^2(D_U^*D_U + \mu^2)^{-1}$. The auxiliary regulator is μ^2 times its ninth power, and the comparison site operator is $D_U^*D_U$ plus that regulator. These are ordinary finite positive matrices when $\mu > 0$. Their symbols at the identity are respectively $s + \mu^2, d_\mu, c_\mu, s + c_\mu$. The ninth power is part of the construction, not a variable exponent sent to infinity.

For link fields, copy the site regulator into each of the four link directions at the link's initial site. Add it to the physical quadratic Hessian and to $D_UD_U^*$. This defines the positive auxiliary

link matrix. The associated scalar functional is one half of its logarithmic determinant minus the logarithmic determinant of the site comparison operator. The copying operation respects gauges and unsigned permutations of axes. Reversed links move the initial site, so this auxiliary functional is not assigned a signed symmetry by that copying argument. The physical signed laws are established separately.

The symbol at the identity is the physical transverse quadratic symbol plus the longitudinal symbol $qq(-p)^T$ plus $c_\mu(p)$ times the identity. On real momentum the gap estimate bounds its quadratic form below by

$$\frac{\min(\epsilon,1)}{2^{10}}(s(p)+\mu^2)\,\|v\|_2^2, \tag{MA2}$$

where $\epsilon > 0$ is the physical transverse floor. The scalar companion has the same bound without the transverse factor. Indeed the physical and longitudinal pieces bound their respective curl and gradient components; in the small-frequency region $s \le \mu^2$, $d_\mu \ge 1/2$, so $c_\mu \ge \mu^2/2^9$; in the other region the frequency term already pays a fixed multiple of $s+\mu^2$. This proves the required positive floor including zero momentum.

To see the uniform trace summation, aliases are

$$p_l = L^{-1}p + 2\pi L^{-1}\widetilde{l}, \qquad l \in (\mathbb{Z}/L\mathbb{Z})^4, \tag{MA3}$$

using the fixed centered integer lift $\widetilde{l}$. Write $r(l) = \max_a |\widetilde{l}_a|$. There are at most $(2n+1)^4$ indices with $r(l) = n$, by injection into the integer box. Consequently, uniformly in L,

$$\sum_l (1+r(l))^{-6} \le J_0, \qquad J_0 = 16\sum_{n\ge 0}(n+1)^{-2} < \infty. \tag{MA4}$$

This bound uses a box estimate for each shell; its excess over the actual shell size is harmless.

For odd L, on the strip of imaginary width $1/128$, first choose the real part of the coarse momentum in its fixed reciprocal fundamental cell. The lattice sine lower estimate on every nonzero alias and the complex perturbation estimate for $s+\mu^2$ give

$$|d_\mu(p_l)| \le \frac{2}{(1+r(l))^2}. \tag{MA5}$$

At the zero alias the massive denominator directly gives the bound two. For a nonzero alias, its real frequency is at least $(r(l)+1)\mu$ in the norm used by the sine estimate, and $\frac{1}{2}((r(l)+1)\mu)^2 \le |s(p_l)+\mu^2|$; divide μ^2 by this lower bound. A general coarse momentum differs by a reciprocal integer translate; this permutes the alias set. Thus summing (MA5) cubed gives, on that whole strip,

$$\sum_l |d_\mu(p_l)^3| \le 8J_0. \tag{MA6}$$

In particular the diagonal alias matrix $d_\mu^3 \otimes 1_{\mathbb{C}^h}$ has nuclear norm at most $8hJ_0$. This is the three-massive-arc trace bound used inside the ninth-power regulator.

Here are the frames in which it is used. Let $T_L(p)$ have entries $L^{-2}e^{ip_l\cdot\widetilde{x}}\delta_{\alpha\beta}$, with row x,α and column l,β. The finite character sum shows that it is unitary at real p; its explicit inverse is the opposite-phase character matrix. On the fixed complex strip the two matrices have operator norms at most $e^{4/128}$, by separating their real unitary Fourier factor from the diagonal complex phases. Furnishing an alias symbol means conjugating its diagonal alias blocks by these two matrices. Thus its three-arc nuclear bound is at most

$$e^{4/128}\,(8hJ_0)\,e^{4/128}. \tag{MA7}$$

The weighted frames additionally use

$$h_l(p) = \sqrt{\mathrm{pebWeight}(p_l, \mu)}, \qquad A_L(p) = T_L(p)\,\mathrm{diag}(h_l(p)^{-1})T_L(0)^{-1}. \tag{MA8}$$

Here $\mathrm{pebWeight}(p,\mu) = \sum_a |e^{i\,\mathrm{Re}\,p_a} - 1|^2 + \mu^2$. Thus the positive weight uses the real part of momentum even when p is complex. Such a weight is a norm-comparison device; it is not asserted holomorphic. For complex differentiation, freeze the weights at the real anchor and use the analytic $T_L(p)$ factors. Comparing the original and frozen weights at nearby momenta gives the fixed height allowance. The link-plus-site frame is the direct sum of (MA8) on link components and the unweighted site frame.

This separation justifies extending the real floor to a complex strip without a hidden dimension factor. Test the conjugated matrix against two fixed vectors and apply the one-variable Cauchy estimate to the resulting scalar analytic function. If its operator norm is bounded by M on a ball of radius R, its derivative is bounded by $2M/R$ on the half ball. Integrate along the segment from the real anchor to a nearby complex momentum. Choose that distance so that the perturbation norm is less than half the real lower bound in (MA2). The Neumann series then inverts the perturbed matrix with at most twice the anchored inverse bound. Apply this in the frozen frames, then restore the compared positive weights. All constants are fixed before L; none comes from summing matrix entries to estimate an operator norm.

The derivative films used above are the actual zeroth, first and second coefficient matrices of the local analytic rows. A row's real directional derivative bound is first converted componentwise, and then its color-coordinate square sum is taken. For the link row this costs exactly $\sqrt{|\mathrm{Loc}(G,1)|}$; for the site row it costs $\sqrt{|\mathrm{Loc}(G,0)|}$ and the displayed local sampling-density factor. Pulling the site row into the block adds L^2, since the normalized site square sum has L^4 terms. These are the link and site row allowances used in the weighted operator bounds. The rows are differentiated through third order when constructing a second-order film of their first derivatives; that shift of order is retained.

For the regulator film, form three consecutive cubes of the normalized inverse film in (MA1), giving $d_\mu^3 d_\mu^3 d_\mu^3 = d_\mu^9$, and then multiply by μ^2. At the common center this is exactly c_μ. Its trace-ideal bound uses one unsplit cube as (MA7), while the other factors receive operator bounds. In a first or second product derivative, choose the corresponding three-arc occurrence after applying the exact inverse and product rules; the column placements in (DP10)–(DP12) give the listed alternatives. The resulting finite sum is bounded by the positive sum of their tariffs.

Finally the house term is the physical Hessian film sandwiched between its two row films; the orbit term is the sum over four initial-site copies of the regulator film. Sandwiching a trace-ideal factor by operator-bounded factors preserves its nuclear bound, with the product of the two operator allowances. Sum the four copied terms and conjugate to the common frame; the fixed Fourier norms cost at most e^4 on each side at the wider intermediate allowance. The reflection correction is the sum of its two cross products with the regulated site row, and the fixed extra block contributes $4|\,\mathrm{Loc}(G,0)|$. Addition and the same two-sided multiplication rule give the full trace toll in (DP10)–(DP12). Thus the cancellation there is backed by the massive symbol (MA1), the alias sum (MA6), the real and complex floors, and the actual local row derivative bounds. It does not require treating an arbitrary large block matrix as uniformly trace class.

The complex denominator estimate used in (MA5) can be proved before alias summation. Put $x = \mathrm{Re}\,p$, $u = \|q(x)\|_2$, and $t = \|p - x\|_\infty \le 1$. Integrating the derivative of the exponential gives $\|q(p) - q(x)\|_2 \le 4t$, and the same estimate at $-p$. Expanding their scalar product yields

$$|s(p) - s(x)| \le 4t(u + 4t) + 4tu = 8tu + 16t^2.$$

For $t \le \mu/128$, the right side is at most $(u^2+\mu^2)/2$: use $2u\mu \le u^2+\mu^2$. Since $s(x) = u^2$, the reverse triangle inequality proves

$$|s(p)+\mu^2| \ge \tfrac{1}{2}(u^2+\mu^2) > 0, \quad |d_\mu(p)| \le 2, \quad |c_\mu(p)| \le 2^9\mu^2. \tag{MA9}$$

The coarse strip has width $1/128$; each alias divides its imaginary part by L, so its fine momentum has exactly the width $\mu/128$ required here. In particular the inverse denominator is analytic throughout the strip.

For completeness the weights in (MA8) obey

$$h_l(p) \ge \mu, \qquad \|q(p_l)\|_2 \le 5h_l(p)$$

on the unit coarse strip. At coarse points of distance at most one, the preceding exponential estimate gives $\|q(\operatorname{Re} p_l) - q(\operatorname{Re} r_l)\|_2 \le 4\mu$. Using $h_l \ge \mu$ shows that the two positive weights have ratio bounded by a fixed constant in both directions. Freeze h_l at a real anchor while retaining $T_L(p)$ and its explicit analytic inverse. These are the analytic frames used in the Cauchy estimate. In ordinary unweighted norm their bounds are

$$\|A_L(p)\| \le e^{4\sigma}\mu^{-1}, \qquad \|A_L(p)^{-1}\| \le e^{4\sigma}\sup_l h_l(p). \tag{MA10}$$

For a real anchor, $h_l^2 \le 16+\mu^2 \le 17$. These estimates display the mass powers; they must not be silently replaced by two mass-independent unweighted bounds.

Here is an explicit form of the strip-inverse argument. Suppose a matrix K, in the two specified weighted frames, has norm at most A, its real lower singular bound is $a > 0$, and the changes to frozen analytic frames cost at most C on a ball of radius σ. Test the analytic matrix on two unit vectors and apply the scalar Cauchy estimate. This gives derivative bound $k = CAC/(\sigma/2)$ on the half ball. On the strip

$$\eta = \min\{\sigma/2,\ (a/2)/(1+k)\}$$

the displacement from its real anchor changes the matrix by at most $a/2$. The lower singular bound is therefore $a/2$, so its weighted inverse has norm at most $2/a$. This also proves invertibility and analyticity of the inverse by the determinant formula. The constants are independent of the dimension of the block. In the site application the real lower form bound is proportional to h_l^2, and the two inverse weights cancel it; it is that weighted matrix to which the assertion applies.

The remaining mass bookkeeping can now be stated exactly. The normalized inverse film has bounded ordinary operator coefficients because its factor μ^2 cancels the two inverse-weight losses. The regulator film has both operator and trace-ideal price μ^2 times a fixed price, by its definition as $\mu^2(d_\mu^3)^3$. Converting it back to the weighted form costs at most μ^{-1} on each side, hence cancels this μ^2. This accounts for the mass-independent regulator toll in the link determinant comparison. It does not remove the explicit μ^2 in its ordinary frame.

Finally the actual matrices behind the last paragraph of this chapter are as follows. Let B be the fixed inclusion of the link block and let P be the complementary site projection; let S_i read the initial-site slot of direction i. Write C for the regulator film, R for the differentiated row, and H for the local second variation of the scalar correction. With the rear operation denoting the transpose at reversed momenta, including the reversed external legs, the relevant products are

$$\text{House} = R^{\text{rear}}HR, \qquad \text{Orbit} = \sum_{i=0}^{3} S_i^{\text{rear}}CS_i, \qquad \text{Motive} = P^{\text{rear}}P. \tag{MA11}$$

These are exact products of films, so their first and second coefficients are their ordinary ordered product derivatives. The motive is constant and its finite site rank is the reason its trace allowance

is $4|\operatorname{Loc}(G,0)|$. The orbit allowance is the regulator trace allowance multiplied on each side by the Fourier-frame bound. The house allowance follows by first assigning the finite local trace allowance to H, then multiplying by the two row bounds. The two reflection cross products are bounded in the same order. The total toll is the positive sum

$$\text{payLedger} + \text{reflectLedger} + 4(4\,\text{orbitLedger}) + \text{popLedger}. \tag{MA12}$$

Each factor in this expression has thus been paid either by a local finite rank, the alias trace sum, or a weighted operator norm; no trace norm of a general extensive Hessian is assumed.

Source–readout formulation. The auxiliary massive objects are *realization data that have no target counterpart*: the mass $\mu = L^{-1}$, the ninth-power regulator and the copied link regulator do not appear in any recovered statement. They are spectators in the sense of the method (adding them changes fibres, not readouts), with one caveat stated in the derivation: the auxiliary functional is only invariant under gauges and unsigned permutations, so no signed symmetry of a readout may be inferred through it. What is read out of this subsection is the single inequality (DP7). The frames T_L, A_L are presentation changes (invertible, with the explicit norms (MA7), (MA10)); the nuclear and operator bounds are stated in named frames precisely because they are not invariant under arbitrary changes of frame. Two estimates of the derivation are used again later and are isolated here.

Lemma 14.13 (Complex massive denominator)**.** *Let $x = \operatorname{Re} p$, $u = \|q(x)\|_2$ and $t = \|p - x\|_\infty \le \mu/128$. Then*

$$|s(p) + \mu^2| \ge \tfrac{1}{2}(u^2 + \mu^2) > 0, \qquad |d_\mu(p)| \le 2, \qquad |c_\mu(p)| \le 2^9\mu^2.$$

Proof. Integrating the derivative of the exponential gives $\|q(p) - q(x)\|_2 \le 4t$, and likewise at $-p$. Expanding the scalar product, $|s(p) - s(x)| \le 4t(u + 4t) + 4tu = 8tu + 16t^2$. For $t \le \mu/128$, $8tu \le u\mu/16 \le (u^2+\mu^2)/32$ by $2u\mu \le u^2+\mu^2$, and $16t^2 \le \mu^2/1024$; the sum is at most $(u^2+\mu^2)/2$. Since $s(x) = u^2$, the reverse triangle inequality gives the first bound, and the other two follow from (MA1). □

Lemma 14.14 (Strip inverse without a dimension factor)**.** *Suppose K, in two specified weighted frames, has norm at most A, real lower singular bound $a > 0$, and the changes to frozen analytic frames cost at most C on a ball of radius σ. Put $k = CAC/(\sigma/2)$ and $\eta = \min\{\sigma/2,\ (a/2)/(1+k)\}$. Then on the strip of width η the matrix is invertible, analytic with analytic inverse, and its weighted inverse has norm at most $2/a$.*

Proof. Test the analytic matrix against two unit vectors and apply the one-variable Cauchy estimate to the scalar function: a bound M on a ball of radius R gives derivative bound $2M/R$ on the half ball, hence k. Integrating along the segment from the real anchor changes the matrix by at most $k\eta \le a/2$, so the lower singular bound is at least $a/2$. The determinant formula gives analyticity of the inverse. No constant depends on the dimension of the block. □

The exactness class is controlled approximation: the massive symbol is compared with the massless one through the bounded discrepancy (MA12), uniformly in the alias and in the period, and the discrepancy enters the drift only through D_{ann} in Theorem 14.12.

14.9 Constructing the common asymptotic calendar

We give the numerical schedule construction explicitly, since both finite shooting and uniform running use it. For each stage $j \ge j_0 \ge 4$, all pre-block parameters have already been chosen: its derivative order, support exponent, analytic width, path bound, column bound and drift allowance $d(j)$. Let

$$A(j) = \log\Big(1 + \mathrm{paths}(j) + \mathrm{width}(j)^{-1} + \mathrm{columns}(j) + \mathrm{relaxation}(j) + \mathrm{supportExp}(j)\Big).$$

Choose an odd integer block length $\mathscr{L}_j \ge 3$, at least every previously required lower bound, such that

$$\frac{d(j)}{\log \mathscr{L}_j} \le \min(s/4, j^{-2}), \qquad \frac{A(j)}{\log \mathscr{L}_j} \le j^{-2}. \tag{CAL1}$$

This is possible because the numerators are fixed before this choice and $1/\log n \to 0$. To make the block lengths nondecreasing, take for each j the maximum of the finitely many lower bounds through j, then an odd integer larger than that maximum. These choices prove the drift and logarithmic-cost inequalities without making the drift allowance depend on the selected terminal cutoff.

After fixing the block lengths choose the positive protection exponents, decreasing with j, small enough for all post-block requests and at most $1/j$. This can be done by taking successive finite minima of positive admissible upper bounds. Next choose the stage's coupling-entry and scale-entry thresholds; every analytic estimate proved with these parameters has one eventual threshold, so a finite maximum handles all its requests. Later increases of these thresholds preserve their inequalities.

Choose the initial coupling $\bar\beta > 1$ above the first-stage entry threshold, and sufficiently large that

$$x^{-1/2}\exp((\log\log x)^2) \le 1 \qquad (x \ge \bar\beta). \tag{CAL2}$$

The ratio $(\log\log x)^2/\log x \to 0$ proves this. For fixed j, the required lattice-size bound at logarithmic scale u has the explicit form

$$\begin{aligned} \mathcal{V}_j(u) &= S_j(1 + c(u) + c(u + \log \mathscr{L}_j))^{q_j} 100(c(u)+1) \\ &\quad + 100((j+3)\mathscr{L}_j + 1), \\ c(u) &= \left\lceil \log^2(\bar\beta + su) \right\rceil. \end{aligned} \tag{CAL3}$$

Here S_j and q_j are the fixed shell-count and growth requests from the local estimates. The last two terms include the small-field and patch buffers. For each fixed stage, $c(u)$ and its fixed translate are $O(u)$; hence $\mathcal{V}_j(u) = O((1+u)^{q_j} + u + 1) = o(e^u)$. Choose an integer entrance bound B_j such that $m \ge B_j$ and $u \ge m\log 3$ imply simultaneously

$$\begin{gathered} \bar\beta + su \ge \mathrm{betaEntry}(j), \qquad u \ge \mathrm{scaleEntry}(j), \qquad u > 0, \\ \log \mathscr{L}_j \le u/j, \qquad \mathcal{V}_j(u) \le e^u. \end{gathered} \tag{CAL4}$$

Each condition holds eventually as $u \to \infty$; $m\log 3 \to \infty$ gives one common integer B_j.

There are also stagewise comparison-window requests $w_j(u)$ satisfying $w_j(u)/u \to 0$, and finite readiness thresholds r_j. Choose integers C_j so that for every $k \ge C_j$ and $k\log 3 \le u \le k \log \mathscr{L}_j$,

$$r_j \le u, \qquad w_j(u) \le k/2. \tag{CAL5}$$

Indeed eventually $w_j(u)/u \le (2\log \mathscr{L}_j)^{-1}$; multiplication by the upper bound on u proves the second inequality, and the lower bound on u supplies readiness. Integer rounding only strengthens the resulting bound by $\lfloor k/2 \rfloor$.

We now build the stage clock, rather than presupposing one meeting all these requests. Use an unshifted stage number $h \ge 0$, corresponding to $j = j_0 + h$, and write B_h, C_h for the chosen integer requests after this shift. Define plateau starting indices recursively by

$$a_0 = 0, \quad k_h = C_h + 2a_h + 1, \quad a_{h+1} = a_h + B_{h+1} + 2k_h + 1. \tag{CAL6}$$

Set $E(i) = h$ when $a_h \le i < a_{h+1}$. The sequence a_h is strictly increasing, so E is defined everywhere, nondecreasing, increases by at most one in a step, and tends to infinity. Also $E(0) = E(1) = 0$. Its recursion gives two useful properties:

$$E(i+1) > 0 \implies B_{E(i+1)} \le i, \tag{CAL7}$$

and, for every h, $k_h \ge C_h$ and

$$E(i) = h \quad \text{whenever} \quad k_h - \lfloor k_h/2 \rfloor \le i \le k_h + \lfloor k_h/2 \rfloor. \tag{CAL8}$$

For (CAL7), the first index of plateau $h > 0$ is $a_h = a_{h-1} + B_h + 2k_{h-1} + 1 \ge B_h + 1$. For (CAL8), $k_h = C_h + 2a_h + 1$ places its lower endpoint above a_h, while $a_{h+1} > 2k_h$ places its upper endpoint strictly before a_{h+1}. Thus every stage receives a sufficiently long central comparison window, however large its requests.

Define

$$J(i) = j_0 + E(i), \qquad L_i = \mathscr{L}_{J(i+1)}, \qquad \tau_0 = 0, \quad \tau_{i+1} = \tau_i + \log L_i. \tag{CAL9}$$

Then $i \log 3 \le \tau_i \le i \log \mathscr{L}_{J(i)}$, by monotonicity of the blocks. The first inequality proves $\tau_i \to \infty$. For every noninitial plateau (CAL7) lets us apply (CAL4) at $u = \tau_i$. In particular

$$0 \le \frac{\log L_i}{\tau_i} \le J(i+1)^{-1} \longrightarrow 0. \tag{CAL10}$$

This is the gentle-mesh condition. Equations (CAL1) and (CAL9) give (FCA4) and (RUN3). The stage order, power and support exponent stay nondecreasing, the protection exponent decreases, and (CAL2) gives the small-field threshold at every scale. These properties all belong to the same calendar.

For volume, let $d_i = \prod_{r<i} L_r = e^{\tau_i}$. On every noninitial plateau (CAL4) gives $\mathcal{V}_{J(i+1)}(\tau_i) \le d_i$, so any positive physical period pays its lattice-size requirements. Only finitely many steps belong to the initial plateau, since $E \to \infty$. Choose an integer M larger than their finitely many size requirements and the required minimum period, and set the physical periods to

$$P_p = 4(M + p + 1), \qquad \text{lattice side at } (i,p) = P_p d_i. \tag{CAL11}$$

These periods are strictly increasing and divisible by four. They pay the initial exceptions directly and the later requirements by the preceding denominator bound. They also provide arbitrarily large periods divisible by any prescribed positive integer c: take $p = c(M + z + 1) - (M + 1)$, which is at least z, and then $c \mid P_p$. The selected complete construction may subsequently retain a period subsequence; it keeps all the size inequalities.

Finally use (CAL5) at k_h, τ_{k_h}, which lies between the two bounds preceding (CAL10). The requested window has width at most $k_h/2$, and (CAL8) keeps that entire window on stage $j_0 + h$. Thus the comparison windows, entry delays, local volume margins, gentle mesh and asymptotically small drift are realized together. No selection of a new calendar for a particular history, cutoff, period or observable is required.

Source–readout formulation. The calendar is a source-side object of a different kind from a field history: it consists of numbers and index maps only. Its role in the lift is that of a common-witness datum.

Lemma 14.15 (Plateau properties)**.** *E is defined everywhere, nondecreasing, increases by at most one per step, tends to infinity and has $E(0) = E(1) = 0$. Moreover*

$$E(i+1) > 0 \implies B_{E(i+1)} \le i,$$

and for every h, $k_h \ge C_h$ and

$$E(i) = h \quad \text{whenever} \quad k_h - \lfloor k_h/2 \rfloor \le i \le k_h + \lfloor k_h/2 \rfloor.$$

Proof. $a_{h+1} - a_h \ge 2k_h + 1 \ge 3$, so the a_h increase strictly and $a_1 \ge 3$. If $E(i+1) = h > 0$ then $i + 1 \ge a_h = a_{h-1} + B_h + 2k_{h-1} + 1 \ge B_h + 1$. For (CAL8), $k_h - \lfloor k_h/2 \rfloor \ge k_h/2 \ge a_h + \frac{1}{2}$, so the lower endpoint is at least a_h (indeed larger), and $k_h + \lfloor k_h/2 \rfloor \le \frac{3}{2}k_h < 2k_h + 1 \le a_{h+1}$. □

Theorem 14.16 (One calendar as a common witness)**.** *The tuple*

$$\mathfrak{c} = \big((\mathscr{L}_j)_j,\ \text{protection exponents},\ \text{entry thresholds},\ \bar\beta,\ (B_h, C_h, a_h, k_h)_h,\ E,\ J,\ (L_i, \tau_i)_i,\ M,\ (P_p)_p\big)$$

realizes simultaneously the drift and logarithmic-cost inequalities (CAL1)*, the small-field threshold* (CAL2)*, the entry and volume conditions* (CAL4)*, the gentle mesh* (CAL10)*, the central comparison windows* (CAL5)*,* (CAL8) *and the lattice sides* (CAL11)*. It is chosen before, and independently of, every history, cutoff, period and observable. Its readers are*

$$\mathrm{Read}_{\mathrm{clock}}(\mathfrak{c})_i = \widehat{\beta}_i = \bar\beta + s\tau_i, \qquad \mathrm{Read}_{\mathrm{stage}}(\mathfrak{c})_i = J(i), \qquad \mathrm{Read}_{\mathrm{side}}(\mathfrak{c})_{i,p} = P_p d_i.$$

Proof. Each property was derived above from the recursion (CAL6) and the order of choices; none of the choices refers to a history. □

In the language of the method, $\mathfrak{c}$ is the datum that every later readout shares: lattices, clocks, windows and volumes are all read from the *same* $\mathfrak{c}$, and successful estimates obtained on different calendars are never combined. The calendar is not unique. The admissible calendars form a directed family under the enlargements used below (raising a length floor before the lengths are chosen; raising entry delays with the lengths held fixed), and every inequality listed is monotone under those enlargements. This non-uniqueness is a presentation ambiguity: no recovered statement depends on which admissible calendar is used, but each statement is proved on one. Because the calendar contains no group element, its frame lift is the identity, and it is the labelling datum of the realized family in Section 8: the level index i, the stage $J(i)$, the clock $\widehat{\beta}_i$ and the side $P_p d_i$ of every leaf are read from it. Nothing in the admission of the source support produces these numbers; they are realization data chosen by the supplied construction in the stated order.

14.10 Countable local demands on the same calendar

The calendar construction must also accommodate estimates whose numerical values use the chosen block lengths. We describe this ordering explicitly. First choose the stage profiles, kernel, primary analytic price list, and the secondary price list for the flat covariance and exterior constructions. The three local constructions then supply finite integer lower bounds $f_{\mathrm{flat}}(j), f_{\mathrm{curv}}(j), f_{\mathrm{dual}}(j)$ for block lengths at profile j. Two further integer lower bounds $f_{\mathrm{stat}}(j), f_{\mathrm{comp}}(j)$ ensure the local

stationary chart and its comparison construction are available. Raise the original length floor to their pointwise maximum:

$$f(j) = \max\{f_{\text{original}}(j), f_{\text{flat}}(j), f_{\text{curv}}(j), f_{\text{dual}}(j), f_{\text{stat}}(j), f_{\text{comp}}(j)\}. \tag{CAL12}$$

The choices of these supplier functions use the original stage profiles and price lists. Raising the floor changes neither that profile nor either list. Apply (CAL1)–(CAL11) with this floor to select the block lengths once. They now determine the local patches at all later steps.

For each fixed requested finite horizon E and profile j, the local construction gives a positive width $\rho_{E,j} \le 1$. Its reciprocal is a countable numerical demand. More generally, let $F : \mathbb{N}^2 \to \mathbb{R}$ be any already selected demand array. Define the finite envelope

$$H_F(j) = 1 + \sum_{n=0}^{j} \sum_{i=0}^{j} |F(n,i)|.$$

Increase only the coupling-entry delay at profile j to at least $\exp(\exp H_F(j))$. At every actual step m, with outgoing profile $j = J(m+1)$, the admissible clock then obeys

$$H_F(J(m+1)) \le \log\log \widehat{\beta}_m. \tag{CAL13}$$

For each fixed n, the divergence and monotonicity of J give a first step after which $n \le J(m+1)$. For all such steps and every $i \le J(m+1)$, the corresponding summand in the finite envelope yields

$$|F(n,i)| \le 1 + \log\log \widehat{\beta}_m. \tag{CAL14}$$

This buys every fixed demand eventually. It makes no assertion that infinitely many horizons have been paid at one finite step. Apply it first to $F(E,j) = \rho_{E,j}^{-1}$.

After the block lengths have been fixed, the local stationary and comparison estimates also have finite coupling bills depending on the resulting patches. Let $B_1(j)$ and $B_2(j)$ be their numerical bounds, and put $B(j) = \max(B_1(j), B_2(j))$. Add this function to the entry delay by taking another maximum. This gives, including at step zero,

$$B(J(m+1)) \le \widehat{\beta}_m. \tag{CAL15}$$

If these bills also have a countable horizon parameter, apply the finite-envelope construction to that array as well, in addition to the all-step bound for its distinguished horizon. Every such enlargement is made with the lengths held fixed. The requirement is therefore a numerical delay condition, rather than a circular demand to choose a block length larger than a function of a subsequently reselected block length. Intersecting finitely many delay conditions is achieved by successive maxima. Each persists under all later compatible enlargements, so the earlier treasury, period, local geometry and observable conditions survive.

The additional local calendars use exactly the same finite-max construction. To spell out their quantitative admission, enumerate their finite requests by natural-number codes. Request q specifies an integer precondition $p_q(j)$ and a finite real cost $c_q(j, i; L_j, E_i)$ for a legal adjacent pair $i \le j \le i+1$, where E_i is the fixed stage profile. Before selecting lengths, include

$$p_*(j) = \max_{\text{code}(q) \le j} p_q(j) \tag{CAL16}$$

in their floor; an empty maximum is zero. After those lengths are fixed, there are only finitely many costs with $\text{code}(q) \le j$ and $i \le j$. Increase the entry delay until all are at most

$$z(\widehat{\beta}) = 1 + \log(2 + \log \widehat{\beta}). \tag{CAL17}$$

This is possible because $z(v) \to \infty$. For example a sufficiently large double exponential of their finite maximum suffices. The legal flight in each cost is the one formed from the already selected L_j, patch, kernel and stage; it is not a cost obtained from an independently chosen history. At an actual step m, substitute $j = J(m+1)$, $i = J(m)$ and $L_j = L_m$. Monotonicity of J implies that a request whose code is at most $J(n)$ has both its floor and its numerical cost paid at every $m \geq n$. In particular every fixed request is paid on its admitted tail. All the post-length maxima are held fixed when increasing the delays again, so the statement persists under every subsequent compatible increase.

For the stationary comparison, the finite-list assertion has a concrete index bound. A request records an input degree d, an extended degree $D > d$, and a jet order k, and is admitted only when $2(D+k+1)$ is at most the incoming smoothing grade. At outgoing grade j, there are consequently only finitely many such triples and adjacent predecessors. The maximum of their local length thresholds is the integer floor used in (CAL12). On the fixed patch, take a second finite maximum, at least one, of their polynomial estimate constants, the local stationary constants, and all reciprocal widths $\rho_{E,i}^{-1}$ with $E, i \leq j$. Call it R_j. Its post-length request is e^{R_j}; payment by $\widehat{\beta}_m$ is exactly $R_j \leq \log \widehat{\beta}_m$. Every auxiliary derivative index in a fixed triple ranges over finite intervals, so it introduces no supremum over arbitrarily high orders at one stage. This proves the finiteness of the native stationary and comparison prices used above, at their actual chosen lengths.

This includes the finite jet, local coordinate and path readouts together, with one selected system of their dictionaries and bases. The inverse-map request has zero precondition. Its actual cost is the inverse estimate of the specified local patch map at its selected analytic width and axis forest. It can therefore be paid from step zero: at each j, maximize its finitely many adjacent predecessor costs and use (CAL17). One does not need to assume a growth law for this inverse price. For a reporting window based at level k, the affine running reference coefficient gives

$$z(\widehat{\beta}_m) \leq z(\widehat{\beta}_k) + s|t_m - t_k| \leq (1+s)\{2 + Z_k + |t_m - t_k|\}, \tag{CAL18}$$

where Z_k includes $z(\widehat{\beta}_k)$. To obtain the first inequality, differentiate z on $v > 1$: $z'(v) = 1/[v(2 + \log v)] \leq 1$, and use $|\widehat{\beta}_m - \widehat{\beta}_k| = s|t_m - t_k|$. Thus the paid inverse cost has the same polynomial reporting-age control required by the local comparison estimates, on the actual windows of the actual history.

The common reference-period registry is fixed at the same pre-length stage. At outgoing grade j its finite list consists of horizons $e \leq j$ and the allowed adjacent incoming grades. Their integer size requirements are included in (CAL16). With the resulting lengths fixed, its uniform finite-patch, chart and normalization prices give a real entry gate for that same list, which is included among the later delay maxima. This retains one registry for every eventual horizon and every reference torus used in the periodic-copy argument; a later numerical enlargement does not reselect it. The physical periods themselves are still chosen as in (CAL11).

We explain the identification with the actual run, since it is needed for the order of quantifiers in the theorem. The run's stage profile is the original profile, its kernel is the original kernel, and its factor at profile j is the length just selected. Hence (CAL12) at $J(m+1)$ is the actual step's local length inequality. Its stationary horizon satisfies the original primary price list. Projecting the two maxima in (CAL15) gives each local patch payment, and therefore its stationary admissibility and the delay required for the flat covariance limit. The preselected flat supplier may now be applied with entrance index zero to this very horizon; the curvature and dual suppliers apply to it by the other two floors. Their local results concern the same levels, patches and retained horizon as the run. No new family is chosen when these numerical payments are consumed. The finite volume and token comparisons are established afterwards on this fixed run, and only then is a requested extraction order refined.

Source–readout formulation. The countably many local demands are *readers* of the calendar and of the fixed finite patches, and the enlargements act on the calendar alone. The compatibility statement is therefore that each enlargement commutes with every reader already purchased: a demand array is a function of $\mathfrak{c}$ which is monotone under the two permitted enlargements, so a later enlargement cannot invalidate an earlier payment. This is what (CAL12)–(CAL18) establish, in the order: stage data and length floors, then lengths, then entry delays.

Identification with the actual run. The order of quantifiers of the public theorem needs the following identification. The run's stage profile and kernel are the original ones and its factor at profile j is the selected length; hence (CAL12) at $J(m+1)$ is the actual step's local length inequality, its stationary horizon satisfies the original primary price list, and projecting the two maxima in (CAL15) gives each local patch payment, its stationary admissibility and the delay required for the flat covariance limit. The preselected flat supplier applies with entrance index zero to this very horizon, and the curvature and dual suppliers by the other floors. No new family is chosen when the payments are consumed; the finite-volume and token comparisons are established afterwards on this fixed run, and only then is a requested extraction order refined. In the source–readout record this is the statement that the demand arrays are *read from* $\mathfrak{c}$ and the fixed patches; the enlargement operations act on $\mathfrak{c}$ alone and commute with every reader.

14.11 Paying the actual compulsory-emission forecast

The compulsory-emission estimate must retain its quadratic monitor exponent until it has been compared with the output forecast. Here is that comparison. Fix the block, the positive local emission parameter μ, and the finite output order. Put $y = \beta_F(x, L)$, $p = M(y)$, with $M(y) = \exp\{(\log\log y)^2\}$. The integrated emission bound obtained from the packet and connected-cluster estimates is

$$2E(x)\exp\{-\mu p^2/4\}, \qquad E(x) \longrightarrow 0. \tag{EF1}$$

This is the bound before weakening its exponent in the dictionary estimate. The requested strong forecast has form

$$C\exp\{-at^2\}, \quad C \geq 1, \qquad t = \frac{y^{-1/2}p}{\beta^{-1/2}}, \qquad \beta \in [y/2, 2y]. \tag{EF2}$$

In particular $0 \leq t \leq 2p$. After μ has been selected at this block and chart width, choose the positive forecast rate $a \leq \mu/32$, and, if necessary, decrease its requested rate to at most $\mu/64$ so that its strict positivity margin is retained. This is a choice of the numerical forecast, made before choosing its entry delay. It gives

$$at^2 \leq 4ap^2 \leq \mu p^2/8 \leq \mu p^2/4. \tag{EF3}$$

For any prescribed $c > 0$, choose the entry delay so that $2E(x) \leq c$. Equations (EF1)–(EF3) then put the actual emission bound below c times (EF2), uniformly for every coupling in the band. The source uses the weaker sufficient condition $a \leq \mu/16$; the chosen $\mu/32$ leaves room for subsequent attachments. These choices are made separately for each of the finitely many legal adjacent stage pairs and combined by their maximum. Thus the forced forecast used by the step, and later by local grafting, is purchased with its actual quadratic decay and on its actual chart. A previously weakened subquadratic estimate is not needed for this payment.

Source–readout formulation. In the lift E, p and t are scalar functions of the tagged level and coupling. The comparison of (EF1) with c times (EF2) is an inequality between two readouts of the same realized record, uniform over the fibre of couplings $\beta \in [y/2, 2y]$ above one clock value. The forecast is thus purchased with its actual quadratic decay on its actual chart, and not on a reference chart from which it would have to be transported. The finitely many legal adjacent stage pairs are part of the calendar record $\mathfrak{c}$ of Theorem 14.16, and the maximum over them is one of its permitted enlargements.

14.12 Uniform reserve at every step, including startup

The numerical purchases also leave a squared reserve for errors that will later be divided by local normalization factors. We derive that reserve, since eventual smallness at each fixed stage alone would not supply a bound at every step of the chosen history. Put

$$M(x) = e^{(\log\log x)^2}, \qquad \kappa(x) = e^{-M(x)^{1/5}}, \qquad x > e. \tag{CAL19}$$

Thus $\kappa(\widehat{\beta}_m)$ is exactly the forced allowance at step m. Fix a positive stage rate q. Eventually $qM(x) \geq 2$. Since $M(y) \geq M(x) \geq 1$ for $y \geq x > e$,

$$qM(y)^2 \geq qM(x)^2 \geq 2M(x) \geq 2M(x)^{1/5}, \qquad e^{-qM(y)^2} \leq \kappa(x)^2. \tag{CAL20}$$

The stage rate in this formula is the positive rate of the actual already chosen post-length estimates. Choose its threshold separately for each stage and include these thresholds among the entry-delay maxima. At step m, use the outgoing stage $J(m+1)$, $x = \widehat{\beta}_m$, and $y = \widehat{\beta}_{m+1} \geq x$. This gives the squared-reserve bound simultaneously for every m, with the same post-length data. The same delay can enforce $\log \widehat{\beta}_m \geq 1$.

Startup has a corresponding reserve without requiring a preceding block. In the startup formulas write $s = \log x$, so their large parameter is $M_s = e^{(\log s)^2} = M(x)$ and their small remainder exponent is $\gamma_s = e^{-M_s/2}$. The startup estimates established above provide its local expansion with packet-energy charge $\sigma = M_s$, extracted suppression charge $\kappa_0 = M_s$, and remainder charge γ_s, at coupling $x/2$. The connected support budget for that remainder is

$$e^{-M_s}\big[e^{\,e^{\gamma_s}-1} - 1\big] \leq e^{-M_s}(e-1) \leq e^{-2M_s^{1/5}} = \kappa(x)^2 \tag{CAL21}$$

for sufficiently large s. The first inequality uses $e^{\gamma_s} - 1 \leq 1$, and the second follows from $M_s - 2M_s^{1/5} \to \infty$. In particular it pays the full connected budget, including its exponential conversion from the local remainder bound.

For each fixed stage the startup threshold is finite. Include it in the same entry-delay maximum. At level zero the calendar identifies the initial stage with its first entry stage, so this threshold is paid there too; at a successor level it follows from the previous entry payment and the positive coupling increment. Consequently the chosen calendar gives (CAL21) at every possible finest cutoff. For any actual coupling in $[x/2, 2x]$, the startup expansion is still valid: its lower-coupling hypothesis follows from the left endpoint, and its vacuum price is bounded using the right endpoint. The exact startup density identity then turns (CAL21) into the required forced-activity bound. These choices enlarge delays before the final calendar and history are selected, and preserve all earlier purchases. They add no assumption to the theorem.

Source–readout formulation. In terms of Theorem 9.4, the reserve is the statement that the forced component of the realized activity has norm at most $\kappa(\widehat{\beta}_m)$ in units of k_{P_m}, at every step

and at startup, with the same address layout; (CAL19) is the same function as the normalization (AB12). The squared form $\kappa(x)^2$ is retained because later errors are divided by local normalization factors: the reserve is a property of the realized record, not of a limit. The displayed inequality (CAL21) is typeset from the supplied text with one typographical restoration: the text prints the exponent of its first bracket with a stray comma, "$e^{,e^{\gamma_s}-1}$", which is the thin-space command of the original with its backslash lost; the reading $e^{\,e^{\gamma_s}-1}$ is the one consistent with the first inequality stated after it.

14.13 Inverse curvature coordinates and the startup fourth coefficient

Exact startup and permitted shooting. The first output is an exact finite Wilson density in the represented activity form, including its normalization; an estimate alone would not identify the starting physical measure. The second output is a finite coupling history whose bottom value is the prescribed calibration. The shooting argument uses its two-sided band, while a forced-only estimate can use a lower-bound-only domain. These domains are kept separate below. Retirement changes the presentation of a state through the exact forward identity. Uniqueness is within this recipe and calendar, not across independently chosen constructions.

The fourth-degree correction used in startup is determined by the Wilson plaquette price in curvature coordinates. It is separate from the earlier inequality bounding the difference between chord squared and curvature squared. Write $c(U) \in \mathfrak{g}$ for the projected curvature, and define

$$k(a) = c(\exp a). \tag{SQ1}$$

This real analytic map has $k(0) = 0$ and $Dk(0) = I$. Indeed the exponential has derivative $a \mapsto a$ at zero, its inverse group factor has derivative $-a$, and taking the projected skew part of their difference divided by two returns a. The finite-dimensional analytic inverse theorem gives open neighborhoods of zero and an analytic inverse u with

$$u(k(a)) = a, \qquad k(u(z)) = z, \qquad u(0) = 0, \qquad Du(0) = I.$$

Set $R(z) = \exp(u(z))$. On a group neighborhood of the identity and a color neighborhood of zero,

$$c(R(z)) = z, \qquad R(c(U)) = U. \tag{SQ2}$$

For the second identity use the group logarithm chart constructed earlier: write $U = \exp a$, shrink so that a is in the inverse neighborhood, and substitute $c(U) = k(a)$. Thus no global group logarithm or connectedness assumption is involved.

Define the real analytic scalar function

$$f(z) = \mathrm{price}(R(z)). \tag{SQ3}$$

Since $R(0) = 1$, the price is zero there. Its first derivative is zero because the Wilson price is a squared chord. For $R(tz)$, its matrix derivative at zero is the matrix of z, so the physical color normalization gives

$$f(tz) = \tfrac{1}{2}t^2|z|^2 + o(t^2).$$

The second derivative is symmetric; polarization of this identity gives

$$D^2 f(0)[z, w] = \langle z, w\rangle. \tag{SQ4}$$

Both inversion and conjugation preserve the price, while $c(U^{-1}) = -c(U)$ and $c(hUh^{-1}) = \mathrm{Ad}_h\, c(U)$. Apply (SQ2) to the nearby group elements $R(z)^{-1}$ and $hR(z)h^{-1}$. Uniqueness of the inverse curvature coordinates gives

$$f(-z) = f(z), \qquad f(\mathrm{Ad}_h\, z) = f(z) \tag{SQ5}$$

as germs at zero, for every fixed $h \in G$. Differentiating these germ identities proves that every odd Taylor coefficient vanishes and every Taylor coefficient is adjoint invariant. The positive physical inner product is unchanged by the adjoint action.

Use Taylor coefficients, including the factorial, to set

$$Q_4(z) = \frac{1}{4!} D^4 f(0)[z, z, z, z]. \tag{SQ6}$$

This is the fourth-degree invariant polynomial called Q_4 in (FST1). By (SQ4)–(SQ5), the Taylor polynomial through order four is exactly $\frac{1}{2}|z|^2 + Q_4(z)$. Consequently

$$f(z) - \tfrac{1}{2}|z|^2 - Q_4(z) \quad \text{has all derivatives of orders } 0, \dots, 4 \text{ zero at } 0. \tag{SQ7}$$

For a smooth family of actual group elements $U(v)$ with $U(0) = 1$, (SQ2) identifies its Wilson residual

$$\mathcal{R}(U) = \mathrm{price}(U) - \tfrac{1}{2}|c(U)|^2 - Q_4(c(U)) \tag{SQ8}$$

with the composition of the remainder in (SQ7) and $c(U(v))$, near zero. The chain rule shows that all derivatives below order five vanish: each term contains a derivative below order five of the outer remainder. This proves the exact flat-jet assertion underlying the startup estimate.

We next explain the uniform derivative bound, including arbitrary plaquette links and threshold equality. Let $U = (U_0, U_1, U_2, U_3) \in G^4$ and $P(U) = U_0 U_1 U_2^{-1} U_3^{-1}$. For a list of four-link directions $A_1, \dots, A_q$, let $J_q(A; U)$ be the ordered left derivative of $\mathcal{R}(P(U))$, in the convention (AB3). It is jointly real analytic in the direction variables and the four matrix entries on their actual group locus. This follows by induction: differentiating the analytic expression along the analytic left increments preserves analyticity. It is homogeneous of degree one in every direction; in particular

$$J_q(tA; U) = t^q J_q(A; U). \tag{SQ9}$$

Keep U_1, U_2, U_3 arbitrary and replace the first link by

$$\widetilde{U}_0(z) = \exp(z) U_3 U_2 U_1^{-1}. \tag{SQ10}$$

The resulting plaquette is exactly $\exp z$. At $z = 0$, the residual has the vanishing property (SQ7). Each ordered link derivative lowers this vanishing order by at most one: introduce one extra real variable for its left increment, apply the vanishing composition to the joint variables, and then differentiate in that variable at zero. Induction gives vanishing through order strictly below $\max(5 - q, 0)$ for $J_q(A; \widetilde{U}(z))$.

For directions in a fixed ball $|A| \le r$, the external parameters consisting of these directions and the four group entries lie in a compact finite-dimensional set. On that set times $|z| \le 1$, the required derivatives of the analytic expression are bounded. Taylor's estimate, obtained by integrating successive derivatives along the segment from zero to z, therefore gives one $C_{q,r} > 0$ such that

$$|J_q(A; U)| \le C_{q,r} |z|^{\max(5-q,0)} \quad \text{if } |A| \le r,\ |z| \le 1,\ P(U) = \exp z. \tag{SQ11}$$

This argument bounds actual left derivatives; it is not a differentiation of a healthy-set mask. The three unaltered links need not be small.

The local logarithm estimate gives constants $c_0, b_0 > 0$ with $|\log V| \le c_0 \operatorname{chord}(V)$ for $\operatorname{chord}(V) \le b_0$. Choose $\delta \le \min(b_0, c_0^{-1}, 1)$. If $\operatorname{chord}(P(U)) \le \theta \le \delta$, and $|A| \le r\theta$, apply (SQ11) to A/θ, then (SQ9). It gives

$$|J_q(A;U)| \le C_{q,r} c_0^{\max(5-q,0)} \theta^{q+\max(5-q,0)} \le C'_{q,r} \theta^5. \tag{SQ12}$$

For $q \ge 5$ the last inequality uses $\theta^q \le \theta^5$; for $q \le 5$ its exponent is exactly five. All inequalities allow equality in the plaquette threshold. At a fixed derivative horizon take a maximum over its finitely many q's.

At a plaquette anchored at o, an allowed direction (AB4) restricted to its four links is at most $r\theta$, with r depending only on the fixed power p and the bounded plaquette distances from o. Pullback to these four links therefore converts (SQ12) into the derivative envelope used in (FST8). A plaquette wrapping around the torus still has these same four stored group entries and bounded periodic distances, so the estimate is unchanged. Summing over the fixed finite number of signed plaquette types and multiplying by β gives the anchored bound $C\beta\theta^5$. The constants are chosen before coupling, cutoff and torus period. This completes the startup Taylor and derivative input from Extra/WILSON_STARTUP/Curvature and Residual.

Source–readout formulation. The startup fourth coefficient is a Taylor datum of a class function of the plaquette holonomy, and the only question for the lift is whether its jet bounds and its domain are fibre constants. They are.

Proposition 14.17 (Uniform flat-jet bound)**.** *Let $U = (U_0, U_1, U_2, U_3) \in G^4$, $P(U) = U_0 U_1 U_2^{-1} U_3^{-1}$, and let $J_q(A;U)$ be the ordered left derivative* (AB3) *of $\mathcal{R}(P(U))$ in four-link directions $A_1, \dots, A_q$. Then J_q is jointly real analytic on the group locus, multilinear,*

$$J_q(tA;U) = t^q J_q(A;U),$$

and there are constants such that

$$|J_q(A;U)| \le C_{q,r}|z|^{\max(5-q,0)} \quad \text{if } |A| \le r,\ |z| \le 1,\ P(U) = \exp z,$$

$$|J_q(A;U)| \le C_{q,r} c_0^{\max(5-q,0)} \theta^{q+\max(5-q,0)} \le C'_{q,r}\theta^5 \quad \text{if } \operatorname{chord}(P(U)) \le \theta \le \delta,\ |A| \le r\theta,$$

where $|\log V| \le c_0 \operatorname{chord}(V)$ for $\operatorname{chord}(V) \le b_0$ and $\delta \le \min(b_0, c_0^{-1}, 1)$. The three links U_1, U_2, U_3 are arbitrary, and equality in the plaquette threshold is allowed.

Proof. Analyticity follows by induction, differentiating an analytic expression along analytic left increments. Keep U_1, U_2, U_3 and replace the first link by

$$\widetilde{U}_0(z) = \exp(z)\, U_3 U_2 U_1^{-1},$$

so that the plaquette is exactly $\exp z$. At $z = 0$ the residual has the vanishing property (SQ7). Each ordered link derivative lowers the vanishing order by at most one (introduce an extra real variable for the left increment, apply the vanishing composition to the joint variables and differentiate at zero); by induction $J_q(A; \widetilde{U}(z))$ vanishes to order $\max(5-q, 0)$ in z. For $|A| \le r$ the directions and the group entries range over a compact set, so on that set times $|z| \le 1$ the relevant derivatives are bounded and Taylor's estimate gives (SQ11). For (SQ12) apply (SQ11) to A/θ with $|z| \le c_0\theta \le 1$ and use (SQ9); for $q \ge 5$ use $\theta^q \le \theta^5$. □

By Theorem 9.2 the same envelope holds for $\mathcal{R} \circ P \circ \pi$ on $\mathcal{Z}$ in raw directions of the same norm, and by Proposition 9.1 the hypothesis $\operatorname{chord}(P(U)) \le \theta$ is the same event for raw links. The curvature coordinate is covariant, $c(\pi(r, A)_p) = \operatorname{Ad}_{r_x^{-1}} c(A_p)$, and f and Q_4 are adjoint invariant by (SQ5), so $\mathcal{R}(\pi(r, A)_p) = \mathcal{R}(A_p)$: the startup residual is a frame-blind function and its Taylor data are recovered exactly. The inverse curvature coordinate itself is *not* frame-blind; only the invariant functions built from it are read out. The quartic form Q_4 and its coefficient are determined by the Wilson price through (SQ1)–(SQ4); they are computed, not chosen.

14.14 The actual finest-scale Wilson representative

The startup used in finite shooting has nonzero activities; its total density is nevertheless exactly the bare Wilson density. We describe this cancellation. Fix a finest-level clip c with $c_{\text{direct}} \le 1$, threshold $\theta > 0$, and support width $B \ge 2$. Put $b = 1 - c_{\text{direct}} \ge 0$. For a plaquette holonomy u, let

$$\mathcal{R}(u) = \operatorname{price}(u) - \tfrac{1}{2} |\operatorname{curvature}(u)|^2 - Q_4(\operatorname{curvature}(u)), \tag{FST1}$$

where Q_4 is the fixed fourth-degree invariant curvature correction. The finest path has row $\sqrt{b}$ times the elementary plaquette curvature, zero loop and cubic tensors, and quartic tensor bQ_4 on the same plaquette. Thus its quadratic-plus-quartic part is the corresponding local Taylor polynomial. It uses the entire plaquette footprint, including a wrapping plaquette.

For each plaquette P, write u_P for its holonomy, $r_P = \operatorname{chord}(u_P)$, and $k_P = b|\operatorname{curvature}(u_P)|^2/2$. Define its unpaid nonnegative part by

$$d_P(U) = \begin{cases} \operatorname{price}(u_P) - c_{\text{linear}} \theta r_P, & r_P > \theta, \\ \big[k_P \text{ if the footprint is healthy, else } b \operatorname{price}(u_P)\big] & \\ \quad - \big[k_P \text{ if its anchor is kept, else } 0\big], & r_P \le \theta. \end{cases} \tag{FST2}$$

The kept predicate is the exponentially relaxed plaquette test of the clip. On a kept anchor its own plaquette satisfies $r_P \le \theta$. The identities $\operatorname{price}(u) = \operatorname{chord}(u)^2/2$, $|\operatorname{curvature}(u)| \le \operatorname{chord}(u)$, and $c_{\text{linear}} \le 1/4$ show $0 \le d_P \le 2$. If $r_P > \theta$, they give the stronger $d_P \ge \theta^2/4$.

Let $\mathcal{A}_{\text{top}}(U)$ denote the actual principal path action, including its clipped direct part and its masked quartic term. Checking separately the healthy case, the unhealthy case with $r_P > \theta$, and the unhealthy case with $r_P \le \theta$, gives the pointwise identity

$$\operatorname{price}(u_P) = \operatorname{baseTerm}_P(U) + \operatorname{mask}_{\theta,P}\big[bQ_4(\operatorname{curvature}(u_P))\big] + b \operatorname{mask}_{\theta,P}[\mathcal{R}(u_P)] + d_P(U). \tag{FST3}$$

The first two terms here are precisely the plaquette terms in $\mathcal{A}_{\text{top}}$. In the healthy case substitute (FST1) and cancel. In the large-plaquette case both masks and the kept quadratic term vanish, and the clipped direct term is $c_{\text{linear}} \theta r_P$, canceled by (FST2). In the remaining case both masks vanish and the two alternatives for the kept quadratic term cancel those in (FST2). These are exhaustive cases, including threshold equality.

Average (FST3) over all signed coordinate frames with weight $\omega = |\operatorname{Signed}|^{-1}$, and sum over plaquettes. Index these oriented terms by i, with footprint S_i, anchor x_i, residual $\mathcal{R}_i$ and unpaid part d_i. Signed-frame invariance of the Wilson energy gives

$$\beta S(U) = \beta \mathcal{A}_{\text{top}}(U) + \beta\omega b \sum_i \operatorname{mask}_{\theta,S_i} \mathcal{R}_i(U) + \beta\omega \sum_i d_i(U). \tag{FST4}$$

Take the middle sum as the regular activity action.

To represent the final sum by forced activities, partition the actual flag set into its connected components for the relation that two flags' radius-$2B$ neighborhoods overlap. If D is a nonempty component, let $R(D)$ be the union of those neighborhoods and set

$$q_D(U) = \omega \sum_{i:\, x_i \in \mathrm{halo}_B(D)} d_i(U). \tag{FST5}$$

The extended regions of distinct components are disjoint. Every nonzero d_i lies within distance B of some flag, and hence belongs to exactly one component's sum. The assertion about distance follows from the relaxed test: outside that distance the clip condition $\theta \exp((B-2)/c_{\mathrm{relaxation}}) \geq 2$ and the universal chord bound 2 force all its tests to be kept and healthy. Therefore $\sum_D q_D = \omega \sum_i d_i$.

Assign the forced entry $e^{-\beta q_D(U)}$ to the support-inventory pair $(R(D), D)$, eligible precisely when $R(D) \cap \mathrm{flags}_\theta(U) = D$, and assign zero to all other active entries. In the gas sum, a nonzero admissible collection must have disjoint supports and inventories partitioning the flags. Its inventories are consequently exactly the connected components just described. Conversely that collection is admissible. The gas sum thus has one nonzero term and equals

$$\mathrm{gas}(U) = \exp\left(-\beta\omega \sum_i d_i(U)\right). \tag{FST6}$$

Combining (FST4) with (FST6) proves the exact startup density $e^{-\beta S(U)}$, with scalar normalization zero.

These explicit entries also satisfy the needed bounds. The normalized derivatives of $\mathcal{R}$ on a healthy plaquette, through each fixed required order, are at most $C\theta^5$. This follows by Taylor expansion of the smooth four-link product and (FST1): its curvature correction cancels the jets below degree five; the permitted derivative directions have size $O(\theta)$, so differentiating does not lose the total factor θ^5. The constants are chosen for a fixed order before the coupling or period. There are only the fixed number of signed plaquette types at an anchor, and their cover cost is bounded. Consequently the anchored regular bound is at most $C'\beta\theta^5$.

For a forced component D, every flag is incident to a plaquette above threshold and each plaquette touches at most four sites. Using $d_P \geq \theta^2/4$ and keeping one signed frame gives

$$q_D(U) \geq |D|\, \omega\theta^2/16. \tag{FST7}$$

This exponential suppression pays the connected-component support count and its support weights. To see its size in the actual schedule, set $x = \widehat{\beta}_N$, $z = \log x$, $M(z) = e^{(\log z)^2}$, and $\theta = e^{-z/2}M(z)$. On the lower-bound domain $\beta \geq x/2$, (FST7) has strength proportional to $M(z)^2|D|$. The neighborhood count $(4\lceil z^2 \rceil + 1)^4$ and each fixed stage support cost are eventually at most $M(z)/4$. The component expansion may therefore use local allowance $e^{-M(z)/2}$; its summed connected-activity bound is at most $e^{-M(z)+1}$, which is eventually at most $e^{-M(z)^{1/5}}$, the required forced allowance. For the regular bound one also uses $\beta \leq 2x$. On the shooting band $x/2 \leq \beta \leq 2x$, it satisfies

$$C'\beta\theta^5 \leq 2C'e^{-3z/2}e^{5(\log z)^2} \leq e^{-11z/10}, \tag{FST8}$$

the required regular allowance. The same finite thresholds also ensure the clip condition preceding (FST6), since $B = \lceil z^2 \rceil$, and $\theta \to 0$. These thresholds are uniform in N, p once the stage is fixed, and have already been included in its entry delay.

The dictionaries of supports and flag predicates are independent of β. Regular entries are linear in β, and forced entries are exponentials of bounded finite-volume functions times fixed

eligibility indicators. Their weighted sums obey the preceding majorants. Thus the startup family is continuous in precisely the activity budgets used for finite shooting. Countably many unused addresses can be appended with value zero, without changing density or bounds; they furnish the marker slots required later. This constructs (FCA2) with its actual representatives.

Source–readout formulation. The startup representative is the first realized state of every finite history: it is the interface between the finite Wilson law of Theorem 7.2 and the represented states of Definition 14.19 below. Its discrepancy is identically zero.

Theorem 14.18 (Exact startup density)**.** *Let $\mathcal{A}_{\mathrm{top}}$ be the principal path action including its clipped direct part and masked quartic term. Pointwise, for every plaquette,*

$$\begin{aligned}\mathrm{price}(u_P) = {} & \mathrm{baseTerm}_P(U) + \mathrm{mask}_{\theta,P}\big[bQ_4(\mathrm{curvature}(u_P))\big] \\ & + b\,\mathrm{mask}_{\theta,P}[\mathcal{R}(u_P)] + d_P(U),\end{aligned}$$

and, with $\omega = |\mathrm{Signed}|^{-1}$ and i indexing oriented terms with footprint S_i, anchor x_i, residual $\mathcal{R}_i$ and unpaid part d_i,

$$\beta S(U) = \beta\mathcal{A}_{\mathrm{top}}(U) + \beta\omega b\sum_i \mathrm{mask}_{\theta,S_i}\,\mathcal{R}_i(U) + \beta\omega\sum_i d_i(U).$$

Taking the middle sum as regular activity and representing the last by the forced entries below, the represented density is $e^{-\beta S(U)}$ with scalar normalization zero.

Proof. The three cases are exhaustive, including threshold equality. Healthy: substitute (FST1) and cancel. Unhealthy with $r_P > \theta$: both masks and the kept quadratic term vanish and the clipped direct term $c_{\mathrm{linear}}\theta r_P$ is cancelled by (FST2). Unhealthy with $r_P \le \theta$: both masks vanish and the alternatives for the kept quadratic term cancel those in (FST2). Averaging over signed frames and summing gives (FST4) by signed-frame invariance of the Wilson energy. Partition the flag set into connected components for the relation that radius-$2B$ neighbourhoods overlap; for a component D let $R(D)$ be the union of those neighbourhoods and

$$q_D(U) = \omega \sum_{i:\, x_i \in \mathrm{halo}_B(D)} d_i(U).$$

The extended regions of distinct components are disjoint, and every nonzero d_i lies within distance B of a flag: beyond that distance the clip condition $\theta\exp((B-2)/c_{\mathrm{relaxation}}) \ge 2$ and the universal chord bound 2 force all tests to be kept and healthy. Hence $\sum_D q_D = \omega\sum_i d_i$. Assign the forced entry $e^{-\beta q_D(U)}$ to the support–inventory pair $(R(D), D)$, eligible exactly when $R(D)\cap\mathrm{flags}_\theta(U) = D$, and zero to all other active entries. A nonzero admissible collection has disjoint supports and inventories partitioning the flags, so its inventories are exactly the components; conversely that collection is admissible. The gas has one nonzero term,

$$\mathrm{gas}(U) = \exp\Big(-\beta\omega\sum_i d_i(U)\Big),$$

and (FST4) with (FST6) gives the density. □

The identity is pointwise in the configuration and therefore lifts by composition with π: every term on its right-hand side is a class function of plaquette holonomies or a mask of such a function, and Propositions 9.1 and 9.3 identify the masks and the budgets on the two sides. The estimates

(FST7)–(FST8) place the regular and forced parts within the allowances (AB12) at the clock value $x = \widehat{\beta}_N$, on the shooting band $x/2 \le \beta \le 2x$, with thresholds uniform in N and p at a fixed stage. The forced estimate uses only the lower endpoint: $\beta\theta^2 \ge M(z)^2/2$. The regular estimate uses the upper endpoint: $\beta\theta^5 \le 2x^{-3/2}M(z)^5$. The dictionaries of supports and flag predicates are independent of β; regular entries are linear in β and forced entries are exponentials of bounded finite-volume functions times fixed eligibility indicators. Hence the startup family is a continuous curve of represented states with a fixed layout, which is exactly the input required by the shooting argument. The exactness class is exact recovery (an identity of densities), together with a bound that places the exact representative in the activity ball.

14.15 Finite coupling arcs and calibrated finite histories

We construct the finite histories before using their compactness or uniqueness. Fix the common calendar, a terminal depth N, and a period p. Put

$$\tau_i = \log(a_i^{-1}), \quad s = \mathrm{slopeConstant}(G) > 0, \quad \widehat{\beta}_i = \bar{\beta} + s\tau_i, \quad \mathcal{B}_i = [\widehat{\beta}_i/2,\, 2\widehat{\beta}_i]. \tag{FCA1}$$

Here $\tau_0 = 0$, $\bar{\beta} > 0$, and $\tau_{i+1} - \tau_i = \log L_i > 0$. The path tensors at fixed N, i are fixed independently of p and of the bare-coupling shooting parameter. A represented state consists of those tensors, its prescribed clip and budget, its actual coupling β, scalar normalization v, and regular and forced activities. Admissibility means the local regularity, symmetry and absolute activity bounds proved in the nonlinear step, together with the retained startup and transition conditions. It includes $\beta \in \mathcal{B}_i$.

Continuity of a family of such states has a specific meaning. Its β and v are continuous; its regular and forced address dictionaries are constant; and differences of its activity values tend to zero in the absolute weighted derivative budgets. For regular terms derivatives are tested only on the common healthy domain, through the prescribed order and in the prescribed directions. For forced terms the bound is on every configuration. On a fixed period a global address sum is legitimate for this continuity assertion. It is not substituted for the period-independent anchored bounds in admissibility.

The nonlinear construction provides a forward family at every finite step on this one calendar. At the finest level its states $F_{N,p}(b)$, $b \in \mathcal{B}_N$, have

$$\beta(F_{N,p}(b)) = b, \qquad v(F_{N,p}(b)) = 0, \qquad F_{N,p}(b).\,\mathrm{eval} = \mathrm{bare}_b\,. \tag{FCA2}$$

They have the fixed startup representatives and insertion slots. For a continuous family $Q(b)$ of admissible level-$i+1$ states on a compact parameter interval, the same primitive integration and reset construction produces continuous coarse states $T(b)$, with the same retained choices, such that

$$T(b).\,\mathrm{eval} = \mathcal{P}_i^{\beta(Q(b))}(Q(b).\,\mathrm{eval}), \qquad |\beta(Q(b)) - \beta(T(b)) - s\log L_i| \le d_{i+1}. \tag{FCA3}$$

The number d_{i+1} is the fixed drift allowance of this stage. The output satisfies all level-i local bounds before imposing its coupling band. This distinction permits the shooting argument to detect crossings of that band.

These forward families are obtained from the actual primitive formulas, rather than a choice of unrelated admissible outputs. At a step, keep one primitive construction, its ordinal and tree choice, and its fixed support dictionaries. Its output is the moment state produced by that primitive. The local integral comparison and the absolute activity estimates above prove continuity of the output in the stated budgets. The reset identity gives the density equality in (FCA3); its coupling

bound gives the second inequality. For a continuous input curve these same formulas therefore define a continuous output curve. For two occurrences of the identical input state they give the identical output state: equality is equality of the represented moment states, including the activities. Choosing this rule once at each N, i, p, and the startup rule (FCA2) once at N, p, gives a single deterministic forward recipe. Its underlying path, clip, normalizers, source bounds and calendar remain those of the nonlinear construction. Restricting a parameter interval preserves every one of these properties.

To make the normalization in this construction explicit, write $H_b(U) = \mathcal{P}_i^{\beta(Q(b))}(Q(b).\,\mathrm{eval})(U)$. The positive push kernel and the retained positive vacuum give $H_b(1) > 0$. If $\alpha_{N,i}$ is the already committed path correction, the output coupling and scalar are

$$\beta(T(b)) = \beta(Q(b)) + \alpha_{N,i} + \Delta(b), \qquad v(T(b)) = \frac{\log H_b(1)}{|\,\mathrm{Site}_i\,|}. \tag{FCA9}$$

The scalar reset produces $|\Delta(b)| \le r_i$ and $\theta_i^2|\Delta(b)| \le e_i$, where r_i is its fixed correction radius and e_i its regular allowance. In the exact normalized splitting, the output trunk density times its attached-log exponential is $H_b(U)/H_b(1)$. Installing (FCA9) multiplies that density by $\exp(|\,\mathrm{Site}_i\,|v) = H_b(1)$, proving (FCA3) at every configuration, including flag boundaries. The site count, rather than the bond count, is essential in this cancellation.

The correction in this construction is the source Hessian projection:

$$\Delta(b) = \mathrm{Averaging.\,delta}\big(X.\,\mathrm{read}(a(b))\big). \tag{FCA10}$$

Here X is the actual fixed source listing and $a(b)$ its parameterized source. The affine-curvature Hessian projection, rebates, faults and low- and high-path loan identities in the coupling-reset exponent section prove the exact all-configuration representation with this value of Δ. In particular the correction is not defined by solving a two-probe contrast equation. The projection's diagonal bound gives $\theta_i^2|\Delta(b)| \le H(b)$, where $H(b)$ is the paid regular source bound. The schedule pays $H(b) \le f_i/4$ and the reset fees, giving the stated correction radius and the unit bound used in (DR7). The regular and forced bounds after the projection and the two dictionary fusions are paid independently by those same fees.

To see the normalization without an extra characterization of the correction, let $\tau(b)$ be the primitive scalar before reset and let $\ell_j(b, 1)$ be its attached logarithms at the identity. The exact exponent calculation gives the installed scalar

$$v(b) = \tau(b) - |\,\mathrm{Site}_i\,|^{-1} \sum_j \ell_j(b, 1). \tag{FCA11}$$

The resulting state has regular entries zero at the identity, no identity flags, and kinetic and loop actions zero there. Its density at the identity is therefore $e^{|\,\mathrm{Site}_i\,|v(b)}$. The already proved all-configuration density identity makes this equal $H_b(1)$, proving the logarithmic formula in (FCA9). Thus (FCA9) and (FCA11) are the same scalar normalization. The interval estimate for this same projection value is obtained by the probe argument below. We must still show that this projected correction lies in the purchased correction interval. The source uses a nonflat probe for this purpose, on the very same trial family. Its activity terms are independent of the correction dial z: they have already been computed with the Hessian projection value Δ. Only the scalar coupling $\beta + \alpha + z$ varies in that trial. Thus there is no second reset algorithm.

Here is the probe construction, uniform in the even torus side. Fix the clip's positive direct coefficient a and its common cubic/quartic path bound $C \ge 0$. A nonzero Lie-algebra direction gives a group curve through the identity with nonzero curvature derivative. Its chord is therefore

positive somewhere; continuity and the intermediate value theorem give $\varepsilon > 0$ such that every chord in $[0, \varepsilon]$ is attained. Put

$$t = \min\{\varepsilon, 1/4, a/[512(1 + C)]\}, \qquad \eta = at^2/4 > 0. \tag{RSP1}$$

Choose $u \in G$ with chord $t\theta$. Set all links to the identity except direction-zero links whose direction-one coordinate is even; set those links to u. This is periodic on every even side. Every $(0, 1)$ plaquette has holonomy u or u^{-1}, and every plaquette has chord at most $4t\theta \le \theta$. Thus the field V has no flags. Its direct plaquette charge in every signed-frame sum is at least $na(t\theta)^2/2$, where n is the number of sites. The cubic and quartic terms have absolute sum per site at most $C(4t\theta)^3 + C(4t\theta)^4 \le a(t\theta)^2/4$, by (RSP1) and $\theta \le 1$. Averaging over the signed frames leaves

$$K(V) \ge n\eta\theta^2 > 0, \tag{RSP2}$$

where K is the actual kinetic action. No smallness of the loop tensor enters this lower bound.

Write $H(U)$ for the actual transferred density at a fixed input parameter, and set $B = \beta + \alpha$. On any flag-free configuration its early representation is

$$H(U) = \exp\{n\tau - [\beta \operatorname{main}(U) + \alpha \operatorname{second}(U) + \operatorname{loop}(U)] - E(U)\}. \tag{RSP3}$$

The source's anchored regular budget is at most $F/4$, with $F = \operatorname{funds}_i$. Summing its anchors bounds $|E(U)| \le nF/4$ at every flag-free configuration, including 1 and V. Formula (RSP3) proves positivity at both probes and gives $|\log H(1)/n - \tau| \le F/4$. Subtracting its two logarithms gives contrast error at most $nF/2$ relative to $\beta\,\mathrm{main} + \alpha\,\mathrm{second} + \mathrm{loop}$.

On flag-free fields the main and loop actions equal their unmasked path expressions. The higher-order kinetic correction obeys $|K(V) - \operatorname{second}(V)| \le 2nC\theta^3$. The stage's cubic payment is $|\alpha|, 2C\theta^3 \le F/2$. Therefore, with $L_0(U)$ the zero-coupling loop background,

$$|BK(V) + L_0(V) - [\log H(1) - \log H(V)]| \le nF. \tag{RSP4}$$

This is the actual contrast estimate, with its α and loop terms retained.

Let e be the final regular allowance, $r \in (0, 1]$ the correction radius. The reset price is chosen before history selection so that

$$2(F + e) \le \eta\theta^2 r. \tag{RSP5}$$

This is paid by the coupling-entry delay: $F = ce$, $e = g^{11/5}$, $\theta = gM$, and $g \le 1$, whereas the monitor $M \to \infty$; it suffices to impose $2(c + 1) \le \eta r M^2$. This same condition is imposed for both permitted adjacent-stage choices.

For the fixed trial activities, let A_V be their regular action at V. Define

$$R(z) = (B + z)K(V) + L_0(V) + A_V - [\log H(1) - \log H(V)]. \tag{RSP6}$$

Their budget gives $|A_V| \le ne$. Equations (RSP2), (RSP4), and (RSP5) imply $|R(0)| \le n(F + e) \le K(V)r/2$. Since $R(z) - R(z') = (z - z')K(V)$, this affine function has a unique root in $[-r, r]$. Its root varies continuously with the input: the contrast and coupling are continuous, the fixed-trial activity series are continuous by their common deviation bounds, and $K(V) > 0$ is fixed.

The projected state already satisfies the full density identity, so evaluating it at $1, V$ gives $R(\Delta) = 0$. More explicitly the exact fixed-trial identity is $R(z) = (z - \Delta)K(V)$. Thus the unique interval root is exactly the previously defined Hessian projection Δ. This proves its radius bound and transfers the interval service's continuous curve to the actual represented state. The independent projection estimate $\theta^2|\Delta| \le H$ retains the sharper paid bound. The all-configuration

density equality is still supplied by the exponent and gas identities, not inferred from equality at two probes. In the source the corresponding shape comparison is also explicit: for any configuration U, the trial shape is its value at Δ times $e^{(\Delta-z)K(U)}$; the inequality $|e^x-1| \le e^{|x|}|x|$ bounds its discrepancy by a finite configuration-dependent constant times $|R(z)|$, using $K(V) > 0$. Hence the interval and projection formulations are fully compatible on this same trial family. For a compact input curve, the connected-expansion tails can be made small at one common finite retirement index. Their absolute tail bounds and the finitely many local bounds permit that choice throughout the curve. In the deterministic recipe the index is chosen as a function of the incoming address layout. A curve's layout is constant, so that index is constant on the curve. The identities for the primitive expressions show that changing the incidental local guide or the point at which the curve was presented does not change the resulting state: all such expressions have the same tensors, canonical matching coordinates, source list and scalar reset. This supplies the continuous forward rule used above without silently choosing new dictionaries at different parameter values.

The source-level continuity and admissibility statement can now be written without suppressing its series hypotheses. Fix a closed parameter interval I and an incoming curve $Q(t)$ in its admissible band. Use one point $t_0 \in I$ as reference. The incoming path equals the prescribed path at every point, and the regular and forced layouts equal those at t_0. The matching maps therefore use identity label correspondence on these dictionaries. Local physical guides may be selected separately at each point; the matched primitive expression is independent of that selection. Transport all entries to the reference layout. This fixes the unretired source dictionary, its support windows and the finite debris predicate used for retirement.

For its source $a(t)$, the actual continuity assertions are the following. Each regular entry, with all permitted derivatives on its fixed healthy support, and each forced entry on all configurations, converges as $t \to t_0$ within I. Their absolute support-weighted envelopes have uniformly small tails near t_0. These two facts imply the deviation budget tends to zero: given $\varepsilon > 0$, choose a finite set of addresses leaving total tail less than $\varepsilon/2$; then choose a neighborhood on which the finitely many remaining differences cost less than $\varepsilon/2$. This uses no continuity of flag indicators as functions of the gauge field: their support and threshold predicates are held fixed in the compared evaluation.

On a compact input interval one retirement index can be used throughout. The uniform tail estimate for the unretired source and compactness give an eventual regular tail bound $g > 0$, while the local finite-volume comparison gives an eventual bound for all the retained local beacon entries. Intersect these eventual conditions, choose one integer above their thresholds, and retain it at every parameter value. The resulting source obeys the explicit regular estimate

$$\mathrm{Budget}_{P_i}(a(t)) \le (1/16 + 2/16)E_i + \mathrm{turnFee}(P_i)g + g, \tag{FCC1}$$

where $E_i = \mathrm{forecastE}_i$. Each beacon on its fixed neighbor support has the derivative bound gleam_i. The source graft combines the retired regular entries, beacon entries and the fixed local profile. Its forced ingredient estimate is paid by its original forced allowance: the primitive forced part costs at most $\mathrm{forecastK}_i(\beta)/8$, while the added local profile has size at most the prescribed $\beta_F^{-1/8}$ reserve after its cover weight. The local gluing estimate sums these contributions into $\mathrm{forecastK}_i(\beta)$. The turn fee times the local beacon bound is included in the regular price. The stage delays make the right side of (FCC1), with that additional beacon price, at most E_i. This proves the regular and forced local estimates for the grafted source simultaneously at every parameter value.

The uniformity over all presentations of one layout requires a separate argument. A layout is the regular and forced address lists, including supports, without coefficient functions. If no admissible state has a layout, its assertion is empty. Otherwise choose one such state U. Form the canonical curve U_β on the entire closed incoming coupling band by retaining its path, cutoff and address lists,

setting every regular and forced coefficient function and the vacuum scalar to zero, and assigning coupling β. Keep the address symmetry permutations. Zero functions satisfy every derivative and hit bound, so this curve remains admissible and is continuous.

Choose local guides along this canonical curve and match their expressions to its reference layout. At a fixed coupling the weighted connected series has a summable majorant. For a target tail bound $g > 0$, take a finite set of terms outside which its sum is below $g/2$. Each of these finitely many terms eventually leaves the debris set: a term is either kept or eventually appears in the finite retirement roster. Hence one depth removes them all. Continuity in the deviation budget bounds the change of the entire weighted series by $g/2$ in a neighborhood of that coupling. The retired part is a subfamily of the same majorant, so the same neighborhood works for every larger retirement depth. Finitely many such neighborhoods cover the closed coupling band. Their maximum depth supplies a common tail bound g. Intersect it with the eventual finite beacon bounds, taking $g = \text{gold} > 0$.

Now let $Q(t)$ be any other admissible curve with this layout and any allowed guide choices. At a point t, compare its guide with the canonical guide at $\beta = Q(t).\beta$. The paths, lists, cutoffs, budgets and couplings agree. The matching map is the identity on the lists, and its induced relabeling preserves the finite retirement roster and debris predicate. For a debris address the primitive term does not read the incoming coefficient functions: it is an omitted foreground cluster formed from the common geometric and Gaussian factors, or an unretired scalar path term. Matching preserves its connected coefficient, integrand and scalar coupling. Thus the two debris functions and beacon contractions agree:

$$1_{\text{debris}(d,i)} W_i(Q(t)) = 1_{\text{debris}(d,i)} W_i(U_\beta).$$

This is equality of local functions, so it preserves every derivative envelope tested in the budget. Transfer the canonical tail and beacon bounds through it. Adding the principal unretired contributions gives (FCC1) for the arbitrary presentation. The threshold therefore depends on the layout, stage, depth, period and fixed prices, but not on the curve, its coefficients or its guide choices.

Apply this construction both with standard allowances $1/16, 1/16$ and with the allowances needed for the contraction. Intersect the two eventual sets with $d \geq N$, and choose one integer $d(\mathcal{L})$ for each layout. Along every admissible curve $\mathcal{L}$ is constant, hence so is $d(\mathcal{L})$. The matching identities identify the deterministic rule with the fixed-index continuous formula on that curve. No continuity of this integer-valued selection as the layout changes is needed. For the deterministic operation, the retirement index is selected once from the incoming layout, at least the terminal depth N, to satisfy these bounds for every compatible presentation of that layout. The universal retirement estimate supplies such an index. Because a curve has one layout, this selected index is constant on the curve. The primitive changes of local guide leave the matched source unchanged, including its finite debris predicate and beacon contraction. Thus the deterministic moment state agrees with the continuous curve construction just described. Restricting the interval in the shooting argument changes neither index nor dictionary.

The Hessian projection is continuous in the resulting deviation budget. For two source values a, a', its bound is

$$\theta_i^2 |\Delta(a) - \Delta(a')| \leq \text{deviation}_{P_i}(a, a'). \tag{FCC2}$$

The threshold is a fixed positive number at this step. The rebate, fault, freshness and loan difference identities are linear in the source difference and in $\Delta(a) - \Delta(a')$, as proved in the reset chapter. Their summable quote envelopes therefore pass continuity and uniform tails to the reset entries. The finite-support dictionary fusions preserve these properties by their absolutely convergent product bounds. In particular both the final coefficient families and their actual fixed address layouts form a curve in the stated sense.

Finally the local primitive identity identifies the transferred density with its early-value expression on every configuration. The reset's exact exponent and gas identities then give its installed state with that same transferred density; its regular and forced closure estimates give all the coarse admissibility conditions. The scalar vacuum term is continuous by the same source-series estimate. Equivalently, evaluate the positive identity at the flat configuration to obtain its logarithmic normalization (FCA9). Together with continuity of the incoming coupling and (FCC2), this proves continuity of the output coupling as well. These are precisely the continuity, density, drift and band-free output conditions used in (FCA3); the subsequent interval restriction is what imposes the coarse coupling band. The calendar makes the following drift condition hold at every step:

$$d_{i+1} \le \frac{s}{4} \log L_i. \tag{FCA4}$$

Define narrower target intervals

$$\ell_i = \frac{3}{4}\bar{\beta} + \frac{5}{8} s\tau_i, \qquad u_i = \frac{5}{4}\bar{\beta} + \frac{3}{2} s\tau_i. \tag{FCA5}$$

Since $\tau_i \ge 0$, they obey $\ell_i \le u_i$ and $[\ell_i, u_i] \subset \mathcal{B}_i$. If the fine coupling in (FCA3) equals ℓ_{i+1}, then

$$\beta(T) \le \ell_{i+1} - s \log L_i + d_{i+1} \le \ell_i - \tfrac{1}{8} s \log L_i < \ell_i.$$

If it equals u_{i+1}, then

$$\beta(T) \ge u_{i+1} - s \log L_i - d_{i+1} \ge u_i + \tfrac{1}{4} s \log L_i > u_i. \tag{FCA6}$$

These strict crossings require no monotonicity or derivative bound on the coarse coupling as a function of the bare coupling.

We use the following elementary interval restriction. Suppose f is continuous on $[a, b]$, $f(a) \le \ell \le u \le f(b)$. Let b' be the first point at which $f = u$, and let a' be the last point in $[a, b']$ at which $f = \ell$. These points exist by the intermediate value theorem and compactness of the two level sets. Then $a' \le b'$ and $f([a', b']) \subset [\ell, u]$. A value below ℓ would force a later crossing of ℓ before b'; a value above u would force an earlier crossing of u. Both contradict their definitions. This argument also covers $\ell = u$.

Starting at N, restrict the bare parameter to $[\ell_N, u_N]$ in (FCA2). Descend by finite induction. If an interval already carries trajectories through levels $i+1, \dots, N$ and its endpoint level-$i+1$ couplings are ℓ_{i+1}, u_{i+1}, apply (FCA3) to their level-$i+1$ curve. Equations (FCA6) and the preceding interval restriction give a closed subinterval on which its level-i coupling stays in $[\ell_i, u_i]$ and takes these exact endpoint values. Retain the already constructed higher states and insert this new coarse state. Every higher density identity, startup condition and continuity estimate survives the restriction.

At level zero we have a nonempty closed bare-parameter interval $[a_{N,p}, b_{N,p}]$ and complete forward trajectories

$$b \longmapsto (R^{b}_{N,p,i})_{0 \le i \le N}. \tag{FCA7}$$

Their finest coupling is exactly b; each level is continuous in the stated sense; all states are admissible; and every adjacent pair satisfies (FCA3). Their bottom couplings take the endpoint values $3\bar{\beta}/4$ and $5\bar{\beta}/4$. Thus every bottom value in this interval is attained by a trajectory of the same recipe. In particular choose $b^*_{N,p}$ with bottom coupling $\bar{\beta}$, and set $R_{N,p,i} = R^{b^*_{N,p}}_{N,p,i}$. This constructs the calibrated finite histories for every N, p.

The same retained arc supplies the auxiliary trajectories at nearby prescribed bottom couplings used in the confluence argument. It supplies whole trajectories, not just scalar sequences satisfying

the drift inequality. No uniqueness of an inverse shooting map has been assumed. Determinism asserts only that equal finest states give identical forward trajectories; the later contraction argument establishes the required uniqueness of their limiting calibrated data.

There is also a uniform interior margin. If the bottom value of any successful trajectory lies in $[3\bar{\beta}/4, 5\bar{\beta}/4]$, then

$$\frac{3}{4}\widehat{\beta}_i \le \beta(R_i) \le \frac{5}{4}\widehat{\beta}_i \qquad (0 \le i \le N). \tag{FCA8}$$

At $i = 0$ this is the given bottom interval. If it holds at i, solve (FCA3) for the fine coupling and use (FCA4): its increment lies between $(3s/4)\log L_i$ and $(5s/4)\log L_i$. Adding these bounds proves (FCA8) at $i+1$. This margin justifies the nearby-trajectory comparisons while retaining the original full bands.

Finally, assembling these states into the history preserves the common path at fixed N, i, its period independence, the prescribed budget and clip, the finest bare Wilson law, and all exact push identities. Induction in the push integral identity gives the same finest Wilson partition function at every coarser level. It is this calibrated history, rather than a separately selected sequence of couplings, that enters the limits below.

Source–readout formulation. This subsection is where the scale index of the realized family acquires its operations. The source objects are represented states; the readout is evaluation to a density; the operation to be lifted is the block integration $\mathcal{P}_i^{\beta}$ of densities; and the calibration is a condition on a scalar readout of the whole history. The four groups of statements below correspond to these four items.

States and the forward operation.

Definition 14.19 (Represented state and its readers)**.** A represented state at level i is

$$Q = \big(\text{path tensors},\ \text{clip},\ \text{budget},\ \beta,\ v,\ (E_j)_j,\ (K_j, P_j)_j,\ \text{address layout}\big),$$

with readers $\beta(Q)$, $v(Q)$, the layout $\mathcal{L}(Q)$ (address lists with supports, without coefficient functions) and the density

$$Q.\,\mathrm{eval}(U) = \exp\big(v|\mathrm{Site}_i| - \beta\mathcal{A}_{\mathrm{path}}(U) - \mathrm{Exp}(U)\big)\,\mathrm{Gas}(U).$$

It is admissible if it obeys the local regularity, symmetry and absolute activity bounds of the nonlinear step and the retained startup and transition conditions, including $\beta \in \mathcal{B}_i$. A family is continuous if β and v are continuous, the layout is constant, and differences of activity values tend to zero in the absolute weighted derivative budgets (regular terms tested on the common healthy domain in the prescribed order and directions; forced terms on every configuration).

Theorem 14.20 (The forward recipe is an operation lift)**.** *At each (N, i, p) there is one deterministic map $T_{N,i,p}$ on admissible level-$(i+1)$ states such that*

$$\mathrm{Read}_{\mathrm{eval}} \circ T_{N,i,p} = \mathcal{P}_i^{\beta(\cdot)} \circ \mathrm{Read}_{\mathrm{eval}},$$

equal inputs give equal outputs (equality of represented moment states, including activities), continuous input curves give continuous output curves with the same layout, and the output coupling and scalar are

$$\beta(T(b)) = \beta(Q(b)) + \alpha_{N,i} + \Delta(b), \qquad v(T(b)) = \frac{\log H_b(1)}{|\mathrm{Site}_i|}, \qquad H_b = \mathcal{P}_i^{\beta(Q(b))}(Q(b).\,\mathrm{eval}),$$

where $H_b(1) > 0$ by positivity of the push kernel and of the retained vacuum, and

$$\Delta(b) = \text{Averaging. delta}\big(X.\,\text{read}(a(b))\big)$$

is the source Hessian projection for the fixed listing X and the parametrized source $a(b)$.

Proof. Keep one primitive construction with its ordinal, tree choice and fixed support dictionaries; the output is the moment state it produces. The local integral comparison and the absolute activity estimates give continuity in the stated budgets, the reset identity gives the density equality and its coupling bound the drift inequality. In the exact normalized splitting the output trunk density times its attached-log exponential is $H_b(U)/H_b(1)$; installing (FCA9) multiplies by $\exp(|\mathrm{Site}_i|v) = H_b(1)$, which proves the density identity at every configuration, including flag boundaries. (The site count, not the bond count, is what cancels.) The affine-curvature Hessian projection, rebates, faults and low- and high-path loan identities of the coupling-reset exponent prove the all-configuration representation with the value (FCA10); in particular Δ is not defined by a two-probe contrast equation. Equivalently, with $\tau(b)$ the primitive scalar before reset and $\ell_j(b,1)$ its attached logarithms at the identity,

$$v(b) = \tau(b) - |\mathrm{Site}_i|^{-1} \sum_j \ell_j(b,1);$$

the resulting state has regular entries zero at the identity, no identity flags and vanishing kinetic and loop actions there, so its density at the identity is $e^{|\mathrm{Site}_i| v(b)}$, which the density identity equates with $H_b(1)$. Thus (FCA9) and (FCA11) are the same normalization. □

The probe and the dial.

Proposition 14.21 (The interval root is the Hessian projection)**.** *$|R(0)| \le n(F+e) \le K(V)r/2$; the affine function R has a unique root in $[-r,r]$, depending continuously on the input; and $R(z) = (z-\Delta)K(V)$, so that root is exactly Δ of* (FCA10)*. Hence $|\Delta| \le r \le 1$.*

Proof. The first inequality is (RSP4) with $|A_V| \le ne$, and the second is (RSP2) with (RSP5). $R(z)-R(z') = (z-z')K(V)$ with $K(V) > 0$ gives a unique real root, of modulus $|R(0)|/K(V) \le r/2$. The projected state satisfies the full density identity; evaluating it at 1 and V gives $R(\Delta) = 0$, hence $R(z) = (z-\Delta)K(V)$. Continuity follows from continuity of the contrast, the coupling and the fixed-trial activity series, $K(V)$ being fixed. □

Layout uniformity.

Proposition 14.22 (Uniformity over all presentations of a layout)**.** *For each layout $\mathcal{L}$ carried by some admissible state there is one integer $d(\mathcal{L}) \ge N$ such that* (FCC1) *holds for every admissible curve with that layout and every allowed guide choice, with a threshold depending on the layout, stage, depth, period and fixed prices but not on the curve, its coefficients or its guides.*

Proof. Choose a state U with layout $\mathcal{L}$ and form the canonical curve U_β on the closed incoming band by retaining its path, cutoff and address lists, setting every coefficient function and the vacuum scalar to zero and assigning coupling β, with the address symmetry permutations kept. Zero functions satisfy every bound, so the curve is admissible and continuous. At a fixed coupling the weighted connected series has a summable majorant; for a target $g > 0$ take a finite set of terms outside which the sum is below $g/2$. Each of these is either kept or eventually in the finite retirement roster, so one depth removes them all; continuity in the deviation budget bounds the change of the whole weighted series by $g/2$ in a neighbourhood, and the retired part is a subfamily of the same majorant, so the neighbourhood works for every larger depth. Finitely many neighbourhoods cover

the band; their maximal depth gives a common tail bound, intersected with the beacon bounds ($g = \text{gold} > 0$). For another admissible curve $Q(t)$ with the same layout compare its guide at t with the canonical guide at $\beta = Q(t).\beta$: paths, lists, cutoffs, budgets and couplings agree, the matching map is the identity on the lists and preserves the finite retirement roster and debris predicate, and for a debris address the primitive term does not read the incoming coefficient functions (it is an omitted foreground cluster formed from the common geometric and Gaussian factors, or an unretired scalar path term). Hence

$$\mathbf{1}_{\mathrm{debris}(d,i)} W_i(Q(t)) = \mathbf{1}_{\mathrm{debris}(d,i)} W_i(U_\beta)$$

as local functions, which preserves every derivative envelope; adding the principal unretired contributions gives (FCC1). Apply this with the standard allowances $1/16, 1/16$ and with those needed for the contraction, intersect the eventual sets with $d \geq N$ and choose $d(\mathcal{L})$. □

Shooting.

Lemma 14.23 (Interval restriction)**.** *Let f be continuous on $[a,b]$ with $f(a) \leq \ell \leq u \leq f(b)$. Let b' be the first point with $f = u$ and a' the last point of $[a,b']$ with $f = \ell$. Then $a' \leq b'$, $f(a') = \ell$, $f(b') = u$ and $f([a',b']) \subseteq [\ell,u]$.*

Proof. The points exist by the intermediate value theorem and compactness of the level sets. A value below ℓ in $(a',b']$ would force a later crossing of ℓ before b', and a value above u in $[a',b')$ an earlier crossing of u; both contradict the definitions. The case $\ell = u$ is included. □

Theorem 14.24 (Calibrated finite histories)**.** *For every N,p there is a nonempty closed bare-parameter interval $[a_{N,p}, b_{N,p}] \subseteq [\ell_N, u_N]$ and complete forward trajectories*

$$b \longmapsto (R^{b}_{N,p,i})_{0 \leq i \leq N}$$

of the deterministic recipe such that the finest coupling is b, each level is continuous and admissible, every adjacent pair satisfies (FCA3)*, and the bottom couplings take the endpoint values $3\bar{\beta}/4$ and $5\bar{\beta}/4$. In particular there is $b^*_{N,p}$ with bottom coupling exactly $\bar{\beta}$; put $R_{N,p,i} = R^{b^*_{N,p}}_{N,p,i}$. Moreover any successful trajectory with bottom value in $[3\bar{\beta}/4, 5\bar{\beta}/4]$ satisfies*

$$\tfrac{3}{4}\widehat{\beta}_i \leq \beta(R_i) \leq \tfrac{5}{4}\widehat{\beta}_i \qquad (0 \leq i \leq N).$$

Proof. Start with $[\ell_N, u_N]$ in (FCA2) and descend. If an interval carries trajectories through levels $i+1, \ldots, N$ with endpoint level-$(i+1)$ couplings ℓ_{i+1}, u_{i+1}, apply (FCA3) to the level-$(i+1)$ curve; the crossings (FCA6) and Lemma 14.23 give a closed subinterval on which the level-i coupling stays in $[\ell_i, u_i]$ and attains both endpoints. Retain the higher states and insert the new coarse state; every higher density identity, startup condition and continuity estimate survives restriction. At level zero $\ell_0 = 3\bar{\beta}/4$, $u_0 = 5\bar{\beta}/4$, and the intermediate value theorem gives b^*. For (FCA8) induct upward: solving (FCA3) for the fine coupling and using (FCA4), the increment lies between $\frac{3}{4} s \log L_i$ and $\frac{5}{4} s \log L_i$, while $\widehat{\beta}_{i+1} - \widehat{\beta}_i = s \log L_i$. □

The calibration fibre. The bottom-coupling map $\mathsf{B}_{N,p} : [a_{N,p}, b_{N,p}] \to [3\bar{\beta}/4, 5\bar{\beta}/4]$, $b \mapsto \beta(R^{b}_{N,p,0})$, is a continuous *surjective readout of the bare parameter*. The calibrated history is a point of the fibre $\mathsf{B}^{-1}_{N,p}(\bar{\beta})$. At this point of the argument nothing makes this fibre a single point: no uniqueness of an inverse shooting map is assumed, and existence used only the intermediate value theorem. Uniqueness within the retained recipe is proved afterwards by the comparison

weights (Theorem 14.29); the uniqueness of the *limiting* calibrated data is a further statement obtained later by contraction. Determinism says only that equal finest states give equal forward trajectories. The coupling normalization is thus imposed at the bottom (value $\bar{\beta}$) and propagated upward by (FCA8); it is a calibration condition on the realized history and not an output of admission. All objects are gauge invariant or numerical, so the frame lift of a represented state is $Q^Z = (\dots, (E_j \circ \pi)_j, (K_j \circ \pi, P_j \circ \pi)_j, \dots)$ with the same β, v and layout; by Theorems 9.4 and 7.2 it is admissible exactly when Q is, $Q^Z.\,\mathrm{eval} = Q.\,\mathrm{eval} \circ \pi$, the push identity (FCA3) holds for the lifted recipe $T^Z(Q^Z) = (TQ)^Z$, and $\mathsf{B}^Z_{N,p} = \mathsf{B}_{N,p}$.

14.16 Exact density identity after retirement

The retirement used above is an exact decomposition before it is an estimate. Keep one anchored listing for the whole input curve. For its full regular source write $A_{\mathrm{full}}(V) = \sum_j a_j(V)$. Partition its indices by the chosen retirement predicate and write $a_j = a_j^{\mathrm{keep}} + a_j^{\mathrm{tail}}$, including their original healthy masks. The same predicate is used throughout the compatible curve. The local absolute envelopes make both subseries summable, hence

$$A_{\mathrm{full}} = A_{\mathrm{keep}} + A_{\mathrm{tail}}. \tag{RET1}$$

The forced gas is retained in the kept source exactly; it is not split into two multiplicative copies. If the density of the full source is

$$e^{v|\Lambda| - H(V) - A_{\mathrm{full}}(V)}\, \mathcal{G}(V),$$

then the identical integrated density after retirement is

$$e^{-A_{\mathrm{tail}}(V)} e^{v|\Lambda| - H(V) - A_{\mathrm{keep}}(V)}\, \mathcal{G}(V). \tag{RET2}$$

This follows by substituting (RET1) in the exponential; in particular the tail has a minus sign. Summability is needed here to identify the sums in (RET1), rather than merely cancelling finitely many formal letters. Replacing the old listing by the anchored common listing preserves each masked occurrence, so it preserves this equality on all configurations.

The full-source density identity follows from the connected expansion already established. Its empty-flag pool is strictly positive and equals the exponential of its absolutely convergent connected logarithm. The scalar normalization contains the positive Gaussian and coordinate prefactors, the incoming scalar volume term, and the reference-forest normalization. Taking their logarithm per coarse site defines v. The cubic, quartic, loop and local correction scores have the exact totals displayed above. Their signs cancel those terms in $H + A_{\mathrm{full}}$, leaving precisely the logarithm of the positive pool. Multiplying by the forced pool then recovers the original conditional integral, including its prefactors. Thus (RET2) holds before either the kept source or the tail has its sharper output budget.

The kept-source estimate and the tail estimate are purchased separately on this same listing. The former gives the desired small regular price together with the original forced forecast; the latter is the remainder used by the finite reference beacons. Reattaching that remainder restores (RET2). Consequently neither continuity nor projection replaces the exact density by a merely close one.

Source–readout formulation. Retirement is a *change of presentation within one fibre of* $\mathrm{Read}_{\mathrm{eval}}$: the pair $(A_{\mathrm{keep}}, A_{\mathrm{tail}})$ is a finer record than A_{full}, the sum map (RET1) is the readout, and (RET2) is the statement that the density read from the finer record is identical to the density

read from the coarser one. Neither continuity nor projection replaces the exact density by a merely nearby one. The kept-source estimate and the tail estimate are purchased separately on the same anchored listing, and reattaching the tail restores the identity. The forced gas is kept in the kept source exactly and is not split into two multiplicative copies. The exactness class is exact recovery; the estimates on the two parts are additional information about the finer record and do not enter the identity.

14.17 Finite reference beacons and weighted retirement continuity

Reference views as coefficient readers. A beacon is a prescribed finite reference volume used to read a local coefficient from a represented density. Its comparisons must keep the folding maps and boundary-face contributions. The selected finite difference removes the stated face terms and retains the interior coefficient. That is deliberate information removal with a proved annihilation identity, not recovery of the full boundary-dependent density. The coefficient reader and its error bounds are then used in the common reference chart and the stability estimates.

The finite beacons used in (FCC1) have an explicit five-volume construction. For integer half-width w, use reference torus sides $2(w+j)$, $j=0,\dots,4$, and coefficients

$$\rho_j = \frac{(1,-4,6,-4,1)_j}{2^4\,4!}. \tag{RB1}$$

Fold each coordinate of a reference torus by reflection into its interval of length $w+j$. To pull a configuration to this folded torus, a positively traversed original edge is read directly, a negatively traversed edge is inverted at its lower endpoint, and a collapsed edge is the identity. Translate this folded view to a chosen coarse anchor o. Denote the resulting group configuration by $S_{j,o}V$. This finite word map defines the views even at folded faces.

Fix the local read radius r, with $2r \le w$. A position in a folded coordinate has a type: within r of the left face, within r of the right face, or in the interior. A boundary type records the face and its distance $0,\dots,r-1$. Its number of preimages under folding is independent of j; an interior type has $2(w-2r+1)+2j$ preimages. In four coordinates a full type therefore has a multiplicity which is a product of four constant or affine functions of j. The fourth difference (RB1) kills every such product having a constant factor, and gives one on the product of four interior factors: its fourth difference is $4!\,2^4$. Hence for any function f of the four-coordinate type,

$$\sum_{j=0}^{4} \rho_j \sum_{x \in \mathrm{torus}_{2(w+j)}} f(\mathrm{type}_j(x)) = f(\text{all-interior type}). \tag{RB2}$$

This is a finite counting identity, including faces and edges of every codimension.

Let $f_j(x,V)$ be the compatible anchored local reference densities. Center them by $f_j^0(x,V) = f_j(x,V) - f_j(x,1)$. Compatibility means that on healthy configurations two such readings agree whenever their radius-r cover inputs agree, and they obey the exact signed and translation laws. Define the beacon at o by

$$b_o(V) = \sum_{j=0}^{4} \rho_j \sum_x f_j^0(x, S_{j,o}V). \tag{RB3}$$

Choose its support radius $z \ge w+4$. Introduce also the shifted expression which uses anchor $o - \mathrm{stage}_j(x)$ for the x-summand and masks that summand on its radius-z neighborhood. Summing this expression over o gives exactly the sum of the masked beacons, by the finite translation permutation of the torus. On a healthy radius-$2z$ neighborhood all these masks are one. Their folded local

readings depend only on their four-coordinate type; compatibility and the signed laws identify equal types. Equation (RB2) reduces the shifted expression to the original centered local density at o. The all-interior type reads its unchanged cover neighborhood. Thus the difference between the assigned retained density, its identity value and this shifted expression vanishes on that healthy neighborhood. This is the local remainder used in the forced part of the graft. Its global identity follows by summing over o; there is no discarded face correction.

The derivative allowance of (RB3) is equally concrete. If the centered full reference sum on side $2(w+j)$ has left-derivative bound q_j on the small reference chart through order k, and the ambient budget has direction power p, then

$$\|b_o\|_{\mathrm{budget}} \le \sum_{j=0}^{4} |\rho_j|\, 2^k (1+z)^{pk}\, q_j. \tag{RB4}$$

On a healthy neighborhood the folded reference configuration can be gauge transformed into that small chart, by the local axial chart construction; its displacement leaves the fixed margin required by the reference estimate. Gauge invariance transfers that estimate back. Each pulled-back direction reads one original edge or its reversed adjoint transform. Its norm is bounded by $(1+z)^p\theta$; reversal preserves the color norm. Repeated left derivatives of the finite word substitution produce at most the 2^k factor in (RB4). Finite summation over the five centered reference sums proves the displayed allowance. These are the actual finite offers used for the beacon, with their common threshold and derivative order.

The q_j's in (RB4) are supplied by the finite reference construction. Put $k = B.\mathrm{order}$, $s = \mathrm{turnFee}(B)$, $h = \mathrm{gold}(G,L,x) > 0$, and $V_j = |\mathrm{Site}(2(w+j))|$. Its actual allowance is

$$q_j = k!\, h\, 3^k + 2V_j\{s(h+h)+h\}. \tag{RB4a}$$

The centered chart expression has bounds $\|\mathrm{D}^\ell f\| \le hp^{-\ell}$, $\ell \le k$, at every admitted point of the chart of radius $p = \mathrm{eaves}(G,L,x)$, with margin $2\,\mathrm{furl}(G,L,x)$. The logarithm of each varying link recovers its local coordinate. If J bounds that logarithm and C is the size of a permitted left variation, the scheduled inequalities are

$$(1+\mathrm{royalty}(G))C \le 1, \qquad k!J(1+\mathrm{royalty}(G))C \le p.$$

The chain rule bounds the coordinate map through order k by p, with slope one. Composition with the chart expression costs at most $k!h(1+(1+1))^k = k!h3^k$.

On the strict healthy chart the exact decomposition expresses this chart function as a constant minus the retained full reference sum minus the summable activity source. The source budget bounds the latter by $V_j\{s(h+h)+h\}$, through the same left-derivative order. Centering at the identity costs at most twice this number; derivatives of positive order kill the centering constant. The triangle inequality proves (RB4a). To include a healthy endpoint on the boundary of the strict chart, approach it along tP, $t \uparrow 1$. This path keeps the prescribed chart margin and is eventually strictly healthy. Regularity of the retained reference function gives continuity of each left derivative at the endpoint, and hence the same bound there. No threshold indicator is differentiated.

The reference estimates hold eventually in their approximation depth uniformly on the compact coupling band. The five reference volumes therefore have a common depth. The actual result after folding is

$$\mathrm{gleam} = \sum_{j=0}^{4} |\rho_j|\, 2^k (1+z)^{B.\mathrm{power}\,k}\, q_j. \tag{RB4b}$$

The radii and reference cardinalities have rate zero, and h has rate $-23/20$; so does this allowance. The small chart required here is supplied on any healthy neighborhood by axial gauge: its link distances are bounded by a fixed multiple of $(2z+1)^5$ times the threshold. The interior extension and its inverse chart place it inside radius $p - 2\,\mathrm{furl}$. The radius has rate zero and the threshold rate $-1/2$, whereas p decays more slowly; the scheduling thus gives the strict margin eventually.

The reference offers are indexed by the coupling itself: at a curve parameter t, the selected offer is the one at $\beta(t)$. It is therefore the same offer for two presentations with the same coupling, independently of how that point was reached. The fixed-reference finite-volume estimates hold eventually in their approximation depth simultaneously on the compact coupling band; (RB4b) gives the number denoted by gleam_i in (FCC1). The compatibility needed above comes from the same local cylinder formulas in the different reference views.

Here is also the weighted-tail step behind the retirement argument. At one finite torus all supports have cover number bounded by its site count. Thus multiplying any summable envelope by its support exponential, even the early exponent $s+3$, costs only a finite constant depending on that torus. Changing its derivative anchor also costs a finite constant:

$$K = \big((1+D)^{p}\big)^{k}, \tag{RB5}$$

where D is the maximal torus separation; the direction triangle inequality gives this bound. Consequently the local source estimate supplies one summable envelope in the common anchored listing. If the debris indicator of each fixed address is eventually zero, dominated convergence for series makes its weighted debris sum tend to zero at that parameter. This eventual removal is literal: a nonkept address eventually belongs to the finite retirement roster, whereas a kept address never belongs to the debris.

Continuity in the deviation budget extends that small-tail bound to a parameter neighborhood. More explicitly, spend half a requested allowance on the tail at the center and half on the full source difference to a neighboring parameter, converting the latter to the common anchored weighted norm by the finite constant (RB5) and its support-weight counterpart. Restricting the difference to the debris subfamily cannot increase its positive majorant. Thus the neighborhood works for every sufficiently large depth. A finite subcover of the compact band gives a single depth. Combining it with the uniform beacon depth proves the simultaneous retirement used in (FCC1). The canonical fixed-layout curve argument already given then transfers this depth to every compatible presentation; the construction here supplies its weighted continuity and actual finite beacons.

Here are the reference and continuity inputs with their finite-volume dependence exposed. In this section alone n is a fixed reference side, $V = n^4$, B is the fixed budget, $k = B.\mathrm{order}$, $p = B.\mathrm{power}$, and $e = B.\mathrm{supportExponent}$. The five signed fourth-difference coefficients in (RB1) are denoted throughout by ρ_j; the λ_j formerly used for them in (RB4b) mean the same coefficients. They are unrelated to the parameter λ in the local fusion argument.

The representation constant used here is $\mathrm{royalty}(G) = \sqrt{m}$, where m is the dimension of the fixed faithful unitary representation. In particular it is at least one. It is distinct from the Fourier aliasing price, which depends on a strip width and an analytic norm.

The reference-chart radius and margin have the explicit values

$$p_0 = (x + s_G \log L)^{-7/16}, \qquad d_0 = (x + s_G \log L)^{-1}, \qquad h = (x + s_G \log L)^{-23/20}. \tag{RB6}$$

The logarithm of the normalized finite Gaussian integral is centered by subtracting its value at the identity. On the reference chart $\|P\| + 2d_0 < p_0$, with no flags, its derivatives through the purchased order satisfy $\|\mathrm{D}^{\ell} f(P)\| \le h p_0^{-\ell}$. The lower bound on its unnormalized integral is strictly positive. These statements are obtained on the actual reference forest and its Gaussian marginal,

with the same stationary section and coupling, by the Gaussian differentiation and connected estimates (ASP2a)–(ASP2c), the local integral normalization (PRI2)–(PRI5), and the logarithm series (ASP1)–(ASP2). In particular the bound concerns the centered logarithm, not the integral itself. The reference side must exceed the prescribed patch and covariance read radius. The path width, path norm, loop allowance, correction parameter and derivative horizon are fixed first; the odd block length is then above their finite lower bounds; finally x is above their common delay. Bounds for the two adjacent loop allowances and the correction parameter put both sides in these same estimates.

The exact logarithmic density identity on the strict healthy chart has the form

$$f(P) = c - \sum_{y \in \mathrm{Site}(n)} F_y(\text{actual } P) - \sum_{\alpha} A_\alpha(\text{actual } P). \tag{RB7}$$

Here F_y is the retained finite ordinary reference density and A_α the emitted regular source. Their actual occurrence indices and supports are retained in the sum. The nonretained source consists of the omitted ordinary occurrences and omitted short stationary paths, not of a difference between Gaussian integrals on different tori. Its anchored early allowance is at most h once its approximation depth is large. The retained-source part has allowance at most $\mathrm{turnFee}(B)(h+h)$. Summing these two bounds gives $\mathrm{turnFee}(B)(h+h)+h$ per site. Summation over V sites and centering (RB7) gives the second term of (RB4a); the reference logarithm gives its first term after the logarithmic chart chain rule already proved there. This explains every summand in q_j.

We give the tail argument that supplies the omitted-source allowance used in (RB7). A regular address is a pair (o_α, X_α), its anchor and finite support. For nonnegative numbers a_α, its early bound u means

$$\sup_{o} \sum_{\alpha : o_\alpha = o} a_\alpha \exp\{(e+3)\,\mathrm{cov}_B(X_\alpha)\} \le u. \tag{RB8}$$

Sum over the V anchors. Since each exponential is at least one, $\sum_\alpha a_\alpha \le Vu$. This proves absolute summability of the derivative envelopes, including derivatives in any fixed finite list of directions: divide those directions by one common scalar until they satisfy the threshold bound, apply (RB8), then multiply by that scalar to the derivative order. The same scalar works for every address. Termwise differentiation on the strict healthy open set therefore follows from uniform absolute convergence of the derivative series through order k.

Changing every anchor to the one in the common listing multiplies its envelope by at most $K_{\mathrm{anc}} = ((1+D)^p)^k$, as in (RB5). On this fixed torus,

$$\mathrm{cov}_B(X) \le V, \qquad K_{\sup} = \exp\{(e+3)V\}$$

bounds every early support weight. Thus an unweighted summable envelope a_α gives the summable weighted envelope $g_\alpha = K_{\mathrm{anc}} a_\alpha e^{(e+3)\,\mathrm{cov}_B(X_\alpha)}$. For the debris predicate

$$P_m(\alpha) = \{\alpha \text{ is not kept and does not belong to the depth-}m \text{ finite roster}\}$$

one has $1_{P_m(\alpha)} \to 0$ at every fixed address: ordinary occurrences exhaust their finite shelves and short paths exhaust their length cutoffs. Dominated convergence for the nonnegative series yields $\sum_\alpha 1_{P_m(\alpha)} g_\alpha \to 0$. This proves a small early bound, rather than merely a small unweighted sum.

For clarity, source deviation has the following meaning. Two sources have identical regular and forced address layouts. There are nonnegative envelopes c_α, d_α such that on each regular healthy domain every left derivative through order k of their difference is bounded by c_α, and on every configuration every forced coefficient difference is bounded by d_α. Both total weighted sums

$$\sum_\alpha e^{e\,\mathrm{cov}_B(X_\alpha)} c_\alpha, \qquad \sum_\alpha e^{e\,\mathrm{cov}_B(Y_\alpha)} d_\alpha$$

are at most the deviation allowance δ. Regular directions use the anchor-dependent budget (AB1)–(AB4). Forced coefficients use their full configuration domain.

The dependence on the coupling and input activities is continuous in this deviation sense. On a finite set of occurrence addresses, the local integrands and their derivatives vary continuously: the Gaussian covariance and mean, finite Haar factors, cutoff profiles and stationary coordinates are the actual ones already constructed. Their local integral bounds (ASP2b)–(ASP2c) provide a common integrable majorant. The normalized denominator stays positive in a sufficiently small neighborhood. Differentiation and dominated convergence hence give continuity of each regular derivative supremum on its compact direction and healthy test sets; the full-domain forced bounds use the corresponding full-domain envelopes. For the infinite family, the signed connected expansion gives a common summable majorant on that neighborhood, by (ASP3)–(ASP5). Given $\delta > 0$, first choose a finite address set whose common tail is below $\delta/4$, then use finite continuity to make its sum below $\delta/2$. The two tails contribute at most $\delta/2$. This proves deviation continuity with the actual support weights.

More explicitly, the dependence of the stationary path-only terms on the coupling is affine: if $A_\alpha(\beta)$ denotes one such term, then

$$A_\alpha(\beta) - A_\alpha(\beta') = (\beta - \beta')\{A_\alpha(1) - A_\alpha(0)\}.$$

The two early envelopes at couplings zero and one give a summable Lipschitz envelope for this difference. The other connected terms are compared by the same marked interpolation before summation. If the input source discrepancy is t, their regular and forced absolute envelopes have sums bounded by tM and tQ for fixed finite M, Q, whenever $0 < t \le t_0$. Signed translations cost

$$Z = |\mathrm{Seat}(n)|\,\mathrm{Rim}(n)\,\mathrm{turnFee}(B)M.$$

Reindexing does not change a sum; multiplication by the symmetry normalization and the direction-change allowance gives exactly these factors. Converting a total unweighted envelope to a weighted one costs at most $A = e^{eV}$. Thus the outgoing source deviation is bounded by $t\{1 + AZ + AQ\}$. All these constants may depend on this reference torus, budget, fixed path bounds and derivative horizon, but are independent of the approximation depth.

Finally fix a parameter value and an early target $u > 0$. Pay $u/2$ for its debris tail using (RB8). Choose a parameter neighborhood with full source deviation below $u/(2K_{\mathrm{sup}})$; then its restricted debris differences have early allowance at most $u/2$. The triangle inequality gives the same tail bound u throughout that neighborhood for all sufficiently large depths. A finite subcover of the compact coupling band makes the depth uniform. Applying this to the five fixed reference tori proves the common h allowance in (RB7), then (RB4a) and (RB4b), and hence the beacon term in (FCC1). No torus-uniform version of K_{anc} or K_{sup} is used.

Source–readout formulation. The beacon is a *multi-volume readout*: the source record is the quintuple of reference views $(S_{j,o}V)_{j=0}^{4}$ with their local densities, and b_o is the linear reader (RB3). The fourth difference is exactly the reader that annihilates every contribution that feels a folded face and returns the interior density with coefficient one (Lemma 14.25); what it discards is all information about the reference boundaries, which is the intended loss. The folding map $V \mapsto S_{j,o}V$ is a word map in the links and intertwines the frame realization: with frames folded by the same site map and raw links by the same word, $\pi(S^Z_{j,o}z) = S_{j,o}\pi(z)$, because the path identity holds for reversed and collapsed edges. Since the densities are gauge invariant, $b^Z_o = b_o \circ \pi$, and the allowance (RB4b) transfers by Theorem 9.4. The finite constants $K_{\mathrm{anc}}, K_{\mathrm{sup}}, A, Z$ are tied to a fixed reference torus; they are explicitly *not* uniform in the physical period, and the lift does not make them so.

Lemma 14.25 (Fourth-difference counting identity)**.** *Fix a read radius r with $2r \le w$. A position in a folded coordinate has a type: within r of the left face, within r of the right face (recording the face and the distance $0, \dots, r-1$), or interior. For any function f of the four-coordinate type,*

$$\sum_{j=0}^{4} \rho_j \sum_{x \in \mathrm{torus}_{2(w+j)}} f(\mathrm{type}_j(x)) = f(\textit{all-interior type}).$$

Proof. A boundary type has a number of preimages under folding independent of j; the interior type has $2(w-2r+1)+2j$ preimages. A four-coordinate type therefore has multiplicity a product of four constant or affine functions of j. The fourth finite difference $\sum_j (1,-4,6,-4,1)_j g(j)$ annihilates polynomials of degree below four, hence every product with a constant factor, and on the product of four interior factors, whose leading term is $2^4 j^4$, it equals $4!\,2^4$. Faces and edges of every codimension are included. □

14.18 Derivation and transfer of the reference chart estimate

We first derive the protected-logarithm jet input. Fix the stage and derivative order before the delay. Put $y = \beta_F$, $u = y^{-7/16}$, $m = y^{-1}$, $\delta = \beta^{-1/2}$, and $D = d\,\mathrm{fineMonitor}(t)$. Work at centers with $\|v\| + 2m < u$; eventually $\delta < u$ and the stationary radius exceeds u. Every $z \in \overline{B}(v,m)$ then satisfies $\|z\| + m < u$, which is the margin needed for the Gaussian logarithm estimates throughout this ball. Let I be the actual protected integral, $V(\delta,\cdot)$ its truncated Gaussian approximation with fluctuation cutoff D, and $p = \Omega \exp(-\mathrm{coverLog}) > 0$. The Gaussian comparison on the whole closed ball $\overline{B}(v,m)$, using the vertical gap $g > 0$, is

$$0 \le I(z) - pV(\delta,z) \le pe^{-gD^2/8}, \qquad V(\delta,z) \ge a = e^{-1}/2. \tag{PL1}$$

The positive omitted tail is bounded by the quadratic Gaussian tail estimate; the local comparison includes the chart and Haar factors. Here Ω is field independent, whereas $p(z) = \Omega e^{-\mathrm{coverLog}(z)}$ is a positive function of the center. The inequalities and the following logarithm argument are pointwise in that center. The cutoffs equal one near zero fluctuation, where the density is strictly positive, so $I > 0$. Its Gaussian determinant is positive by the vertical gap. Writing $I = pV(1+q)$ gives $0 \le q \le e^{-gD^2/8}/a$. Hence the logarithmic discrepancy

$$f = \log I - \log \Omega + \mathrm{coverLog} - \log V \quad \text{satisfies} \quad |f| \le a^{-1} e^{-gD^2/8}, \tag{PL2}$$

by $0 \le \log(1+q) \le q$. This is only a value estimate.

We record how the actual integral comparison in (PL1) is obtained, so that its positive error and its domain are explicit. On the fixed forest fiber let $\mathcal{U}_z$ be the set represented by the fluctuation chart $U = \mathrm{draw}(V_z, \mu(z) + \delta i)$, $\|i\| \le D$, where μ is the stationary graph. The chart radius inequalities are $(1 + \mathrm{trailCost})u + \delta D <$ initial and the corresponding color-scaled displacement is below the group-chart radius. On this set all protecting flags vanish. The exact change of variables, including its Haar Jacobian and the vertical Gaussian determinant, gives

$$\int_{\mathcal{U}_z} \mathcal{I}_z(U)\, d\nu_F(U) = p(z)V(\delta,z), \qquad I(z) = p(z)V(\delta,z) + \int_{\mathcal{U}_z^c} \mathcal{I}_z(U)\, d\nu_F(U). \tag{PL1a}$$

Here $\mathcal{I}_z$ is the nonnegative summed fiber density used in the exact push. This is why the comparison has a one-sided error.

The outside-field energy bound is paid in the same chart. Denote by k the number of free bonds, by $w_g(\beta) > 0$ the single-bond chart normalization, by v_C the common finite size bound,

and by K_0, T_0 respectively the local derivative and determinant expenses. With $R_D = gD^2/4$, its normalized logarithmic expense is

$$Y_D = -k \log w_g(\beta) - R_D + v_C \,\mathrm{loopsPrice} + K_0 + T_0. \tag{PL1b}$$

The fiber complement estimate is $\int_{\mathcal{U}_z^c} \mathcal{I}_z \le p(z) e^{Y_D}$. It uses the local link reconstruction to put every small-toll, small-wire configuration in the safety chart. Otherwise its fine or prediction penalty pays at least the threshold energy. The parameter inequality

$$\beta \,\mathrm{meter}(v_C) u + R_D \le \beta p_{\mathrm{penalty}} m_0^2/2$$

makes that alternative at least as expensive as the fluctuation tail; m_0 is a fixed positive safety threshold, not the shrinking radius m. The quadratic lower bound within the chart pays the other alternative. Both comparisons use the same actual field and forest fiber.

The delay makes $Y_D \le -R_D/2$, and also the stronger inequality $Y_D + E + 2 \le -R_D/2$. Here the extra expense used to locate a clean chart point is

$$E = \beta\{v_C C_{\mathrm{cl}} \|z\|^5 + v_C C'(\theta'^3 + \theta'^4)\} + K_0 + T_0 + v_C A_{\mathrm{coarse}}/2.$$

The two terms multiplied by β and the loop expense are eventually bounded by one: the field radius has rate $-7/16$, the threshold has rate $-1/2$, and the remaining size factors have rate zero. The nondecaying expense K_0 is affine in the fine monitor, $K_0 = a(t) + b(t)\mathrm{fineMonitor}(t)$, with a, b bounded by the already fixed polynomial size budgets. The normalization and determinant expenses have the same type of bound. The square $R_D = (gd^2/4)\mathrm{fineMonitor}(t)^2$ dominates this fixed finite list, after the stage parameters and positive d are fixed. This proves both inequalities. The stronger one, together with the positive lower bound for the total push, forces a clean point inside $\mathcal{U}_z$; the local chord and displacement estimates then keep all chart flags zero on the surrounding radius-m ball. Thus (PL1a) applies throughout that ball. The first inequality and (PL1b) give its error $e^{-R_D/2} = e^{-gD^2/8}$, as claimed. The truncated Gaussian lower bound and its jet estimates use the same size and radius choices. Their derivation is the local Gaussian expansion above; no full-field lower bound for an inverse density is used here.

The Gaussian approximation uses the actual functions constructed above. Keep the retained maximum-norm real coordinates x, and give the vertical real coordinate array its Euclidean norm, denoting it by h. The linear map $\mathcal{L}(x, h)$ that joins these arrays into the joint maximum-norm array has norm at most one. In this notation set

$$\phi(x, h) = F(\mathcal{L}(x, h)), \qquad \psi(x, h) = \mathrm{loop}(U(x, h)) - \sum_{e \text{ free}} \log J_G(\mathrm{tint}(h)_e). \tag{GL0}$$

Here F is the main-action-plus-prediction score (AS1), U is its actual fine configuration, loop is the retained quadratic-loop contribution, and J_G is the local Haar density of the chart in the earlier primitive integral. The minus sign makes $e^{-\psi}$ contain the positive Haar factor. Let $\rho(x)$ be the vertical Euclidean version of the unique graph (US7), and let $M(x)$ be its Hessian in these vertical coordinates. The equality of counting and Euclidean color squared norms proves

$$D_h \phi(x, \rho(x)) = 0, \quad D_h^2 \phi(x, \rho(x))[z, z] = z^{\mathsf{T}} M(x) z, \quad M(x) \ge (g/2) I. \tag{GL0a}$$

Thus the inverse covariance bound used below is $M(x)^{-1} \le (2/g) I$; it is the retained half-gap of the actual stationary Hessian.

We explain the common derivative hypotheses for these functions. The joined points used below satisfy

$$\|\mathcal{L}(x, \rho(x) + tbz)\| \le R_{\mathrm{path}} \|x\| + |b| D, \qquad 0 \le t \le 1, \quad \|z\| \le D,$$

where R_{path} is the stationary Lipschitz price, enlarged to include the retained identity coordinate. The radius inequalities before (PL1a) put this whole set inside the purchased stationary domain and put every color logarithm inside half the Haar-chart radius. The graph jets are (US8). The score jets follow from (AS6) after the norm-at-most-one map $\mathcal{L}$; converting coordinate multilinear bounds to operator bounds costs at most the number of joint coordinates to the derivative order. The loop jets follow by summing the absolutely weighted path jets, multiplied by the fine-site count. A Haar summand is the fixed analytic $\log J_G$ composed with a color projection of norm at most $1+\dim \mathfrak{g}$, so its order-k price is at most its purchased chart price times $(2+\dim \mathfrak{g})^k$; summing costs the free-bond count.

The centered determinant jets required below are controlled on this same domain. Differentiate $D \log\det M = \operatorname{tr}(M^{-1}DM)$ and $D(M^{-1}) = -M^{-1}(DM)M^{-1}$. Repeating gives a finite sum of traces of products of inverse factors and derivatives of M, indexed by ordered derivative partitions. Each trace is bounded by the vertical dimension times the product of the operator norms. The inverse factor has bound $2/g$, and derivatives of M are obtained from two score derivatives and the graph chain rule, all at the already purchased higher orders. This gives a finite polynomial price in those jet bounds, the coordinate counts and $2/g$. The value of the centered determinant logarithm follows by integrating its first derivative from zero. These operations give the common price C for the graph, classical score, quantum contribution and centered half-log-determinant used below. Its dependence on the auxiliary size bounds is polynomial at fixed derivative order; those sizes have the assigned polynomial logarithmic bounds. Consequently C has rate zero. This establishes the hypotheses of the local Gaussian calculation on the entire displaced set, using the actual score, chart and graph.

Define the Taylor-integral expression

$$A(b,x,z) = \int_0^1 (1-t) D_h^2 \phi(x, \rho(x) + tbz)[z,z], dt. \tag{GL1}$$

Stationarity and twice integrating a line derivative give $\phi(x,\rho(x)+bz) - \phi(x,\rho(x)) = b^2 A(b,x,z)$, including $b=0$ by the displayed integral. In particular $A(0,x,z) = \frac{1}{2} z^{\mathsf{T}} M(x) z$. Put

$$\begin{aligned} G(b,x,z) = {} & \tfrac{1}{2}\log\det M(x) - \tfrac{1}{2}\log\det M(0) + A(0,0,z) - A(b,x,z) \\ & + \psi(x,\rho(x)) - \psi(x,\rho(x)+bz). \end{aligned} \tag{GL2}$$

Then the same truncated approximation in (PL1) is

$$V(b,x) = \int_{\|z\|\le D} e^{G(b,x,z)} d\gamma_{M(0)^{-1}}(z). \tag{GL3}$$

At $b=0$, multiplying its fixed Gaussian density by the exponential in (GL2) gives exactly the density of $\gamma_{M(x)^{-1}}$. Hence

$$V(0,x) = \gamma_{M(x)^{-1}}(\overline{B}(0,D)). \tag{GL4}$$

This identity retains the moving determinant and covariance; replacing either by its value at zero would lose the derivative argument.

Here is a uniform estimate for the mass in (GL4). If a centered Gaussian covariance Q satisfies $Q \le cI$, its one-coordinate exponential moment gives $\mathbb{P}(|Z_i| \ge v) \le 2e^{-sv+cs^2/2}$ for $s \ge 0$: apply Markov's inequality to $e^{\pm sZ_i}$ and add. If the number of coordinates is at most $P \ge 1$, and $Pv \le D$, the complement of the radius-D ball is contained in the union of those coordinate events. Indeed, if every $|Z_i| < v$, then $\|Z\| \le \sqrt{P}v \le Pv \le D$. Thus its probability is at most $2Pe^{-sv+cs^2/2}$.

Take $P = 1 + Q_0(t)$, where Q_0 is the fixed polynomial size budget, and write $H = \mathrm{fineMonitor}(t)$. Eventually $P \le \sqrt{H}$. For $D = dH$, choose $v = d\sqrt{H}$ and $s = v/c$. The error is at most

$$2P \exp(-d^2 H/(2c)). \tag{GL5}$$

The monitor is chosen to dominate these polynomial size budgets and logarithmic coupling powers, so (GL5) is smaller than y^a for every fixed exponent a, after increasing the delay. This estimate is uniform for the moving covariances, since the inverse-Hessian bound gives one common c.

For completeness the derivative price in (GL3) is finite without any derivative of a moving integration set. At order s, put

$$U_s = 2(1 + C + D), \quad A_s = D^2 \mathcal{B}_s(U_s, C),$$
$$F_s = 2C + 2A_s + 2\mathcal{B}_s(U_s, C).$$

Here C bounds the purchased section, score, quantum and centered determinant jets, with two extra score and section orders for (GL1). The affine displacement map in (GL1) has price U_s; evaluating its second score derivative on two copies of z gives the factor D^2. Integration against $1 - t$, followed by the sum in (GL2), bounds all jets of G by F_s. Since $G(0, 0, z) = 0$, its first derivative also gives $|G(b, x, z)| \le F_s \max(|b|, \|x\|)$. The shrinking chart makes this at most one. Consequently $e^{-1} \le e^G \le e$ on the fixed cutoff ball. Its base Gaussian mass is eventually at least one half by (GL5), so $e^{-1}/2 \le V(b, x) \le e$. The exponential composition rule bounds its jets by $\mathcal{B}_s(F_s, E_s)$; the fixed Gaussian measure is a probability, so integration preserves this bound. The logarithm on the compact interval $[e^{-1}/2, e]$ has a finite fixed jet price L_s, giving the joint (b, x)-jet bound $B(t) = \mathcal{B}_s(\mathcal{B}_s(F_s, E_s), L_s)$. All these prices have rate zero.

Finally (GL2) is unchanged by simultaneously replacing (b, z) with $(-b, -z)$. Symmetry of the cutoff ball and base Gaussian makes $V(-b, x) = V(b, x)$, and therefore its logarithm is even in b. Each of its x-derivatives has zero first b-derivative at zero. Integrating its second b-derivative bounds the difference from its $b = 0$ value by $B(t)b^2$, through order j, using the purchased joint order $j + 2$. At $b = 0$, (GL4)–(GL5) and $|\log v| \le 2(1 - v)$ for $1/2 \le v \le 1$ give an arbitrarily small polynomial value error on the surrounding ball. Apply the interpolation (PL4) below with a rough joint bound $B(t) \le y$ and a sufficiently small chosen value exponent. This bounds the x-jets of $\log V(0, x)$ by y^{-2}. Combining the even difference and this zero-parameter estimate gives the asserted Gaussian logarithm jet price $B(t)\delta^2 + y^{-2}$ at $b = \delta$, as well as its lower bound. This proves the Gaussian input used in (PL1) and (RBG2).

The separate rough derivative estimate for the discrepancy on that closed ball is, for a requested order j,

$$\sup_{z \in \overline{B}(v,m)} \|D^i f(z)\| \le y^{S_0} \quad (0 \le i \le j + 1), \qquad S_0 = (j + 1)^2 + 2. \tag{PL3}$$

For this rough estimate one can avoid estimating an inverse integral on a whole chart. We give the relative derivative bound used in the construction. Write the exact push as an integral over a fixed forest fiber of a finite sum of nonnegative functions

$$I(x) = \int \sum_{\alpha} a_\alpha(U) e^{-\beta N_\alpha(x,U)} \, d\nu_F(U), \qquad a_\alpha(U) \ge 0. \tag{PL3a}$$

The coefficients include the fine density and all factors independent of the retained coordinate x. The only x-dependence here is the prediction penalty N_α, with the fiber configuration fixed. In particular this argument does not require a nearby point to retain a selected protecting flag. Let r

bound the number of coarse bonds. The link product and trace estimates in (AS3) give the penalty jet price

$$A_N = p_{\mathrm{penalty}}\, r\, \mathcal{P}(2^n o_n \,\mathrm{colorScale}).$$

The predictor at fixed U is a constant group element of matrix norm colorScale; hence this price is uniform in U and the finite choice α.

For clarity write the composition polynomial as

$$\mathcal{B}_n(a,b) = n!b(2+a)^n.$$

It bounds the finite partition sum for composition when the inner and outer jets have respective prices a, b at scale one. Subtracting the value of $-\beta N_\alpha$ at the center costs a factor two in its jet bound. If E_n bounds the exponential jets at zero, each integrand in (PL3a) therefore has jets through order n bounded by its own value times

$$K_n = \mathcal{B}_n(2|\beta|A_N, E_n).$$

Integrating preserves this relative bound. Differentiation under the integral follows from the same estimate on a neighborhood together with the bounded fine-density envelope and the finite Haar fiber measure. Consequently $\|D^i I(x)\| \le K_n I(x)$ for $i \le n$. At each center where (PL1) gives $I(x) > 0$, normalize by this single positive value: $J_x(z) = I(z)/I(x)$. Its jets at $z = x$ are bounded by K_n, and $J_x(x) = 1$. If L_n bounds the logarithm jets at one, the chain rule gives jets of $\log I(z) - \log I(x) = \log J_x(z)$ bounded by $\mathcal{B}_n(K_n, L_n)$. The size of $I(x)^{-1}$ has canceled before estimating. This is the normalization that supplies rough logarithm derivatives uniformly over the surrounding ball.

The other terms of f have explicit prices as well. If $C(t)$ is the stationary chart-jet price, set

$$T_n(t) = \mathcal{B}_n(2 + 2C(t), C(t)), \qquad O_n(\beta, t) = \beta T_n(t) + T_n(t) + C(t).$$

Composition of the classical action, quantum action and Gaussian determinant on the stationary graph bounds the centered cover logarithm by $2O_n$. Let $w(t)$ be the rate-zero Gaussian-log coefficient and put $Q(t) = |w(t)| + y^{-2}$; since $\delta \le 1$, the Gaussian logarithm jet price is at most $Q(t)$. Combining the centered logarithms and then restoring the value bound (PL2), which is eventually at most one, gives the rough price

$$M(t) = \mathcal{B}_n(K_n(2y, r(t)), L_n) + 2O_n(2y, t) + 2Q(t) + 1, \qquad n = j + 1. \tag{PL3b}$$

The actual coupling lies below $2y$, so positivity and monotonicity of these polynomials justify that substitution. The bond-count bound $r(t)$, chart coefficient $C(t)$ and $w(t)$ have rate zero. Thus A_N has rate zero, K_n has rate n, and $\mathcal{B}_n(K_n, L_n)$ has rate n^2. The cover term has rate one and the Gaussian term rate zero. Their sum has rate at most $S_0 - 1 = n^2 + 1$, and hence is eventually at most y^{S_0}. This proves (PL3), including the zeroth derivative by (PL2). It uses actual integral derivatives and the surrounding ball, rather than differentiating the small value error.

Here is the interpolation and its order of choices. If $\|D^i h\| \le v_i$ on a ball of radius s and $\|D^{i+2} h\| \le C$, integration along unit directions gives

$$\|D^{i+1} h(0)\| \le 2v_i/s + Cs. \tag{PL4}$$

For multilinear derivatives one tests unit vectors before applying this line estimate. Suppose rough derivatives through order $j + 1$ are bounded by y^d on a ball of radius y^{-a}, $a > 0$. For any target

exponent b, a sufficiently small value bound y^B on that ball implies every jet through order j at its center is bounded by y^b. At order zero take $B = b$. For the step from order i to $i+1$, choose

$$c = a + \max(0, d - b) + 1, \qquad b' = b - 1 - c,$$

then use the order-i assertion with target b' and ball exponent c to determine B. Eventually $2y^{-c} \le y^{-a}$, so the smaller balls around points within y^{-c} of the center lie in the original ball. The induction gives $\|D^i h\| \le y^{b'}$ there. Formula (PL4) gives $2y^{b'+c} + y^{d-c}$, whose two exponents are strictly below b; their sum is eventually at most y^b. Lower jets already satisfy the stronger bound.

Apply this recursion with $a = 1, d = S_0, b = -2, j = n$. It fixes B first; then choose the tail accuracy and delay so that (PL2) is at most y^B. Thus the discrepancy jets are bounded by y^{-2}. Gaussian differentiation bounds the separate local Gaussian logarithm jets by $H(t)\delta^2 + y^{-2}$, with H of rate zero. Keeping that term in L_0 leaves the discrepancy Y in (RBG1). Since $u \le 1$,

$$\max_{i \le n} u^i \|D^i Y(v)\| \le y^{-2}. \tag{PL5}$$

This includes $v = 0$. The critical extension and actual chart composition use the already purchased higher jets and finite composition prices. Those prices are fixed before the tail and reference delays. This supplies the input to the rate calculation below and hence (RB6); the retirement argument then uses the resulting bound with its existing delay order.

We supply the quantitative calculation behind (RB6). Put $y = x + s_G \log L$, $A = y^{-7/16}$, and $\epsilon = y^{-2}$. All derivative orders and stage parameters in this calculation are fixed before x tends to infinity. A function of rate r is eventually bounded in absolute value by $y^{r+\eta}$ for each $\eta > 0$. Products add rates, a finite sum has the largest of its rates, and multiplication by any fixed constant preserves a rate. Thus a rate strictly smaller than $-23/20$ gives the bound in (RB6), with coefficient one, after increasing the delay.

Write the actual reference logarithm, before centering, as

$$g = \log \Omega + L_0 + Y + \beta f_1 + W. \tag{RBG1}$$

Here Ω is the field-independent Gaussian normalization; Y is the logarithm of the normalized protected local integral after its Gaussian main part is removed; L_0 is the remaining Gaussian normalization term; f_1 is the difference between the prescribed coarse classical action and the fine classical action evaluated on the stationary graph; and W is the difference between the prescribed coarse quantum action and the thermal determinant and Haar contribution on that graph. These are functions on the same real primary-coordinate space. In particular (RBG1) is an equality of the actual densities before any estimate is taken.

For a function F, use the scaled derivative bound

$$\|F\|_{n,A;z} = \max_{0 \le q \le n} A^q \|D^q F(z)\|.$$

The protected-integral comparison and interpolation (PL1)–(PL5) give $\|Y\|_{n,A;z} \le \epsilon$, including at the origin. The derivative price of L_0 on the radial segment from zero to z is

$$d = X(x) g_{\mathrm{step}}(\beta)^2 + \epsilon, \qquad g_{\mathrm{step}}(\beta) = \beta^{-1/2}, \tag{RBG2}$$

with X of rate zero. The integral itself is positive: the Gaussian measure is positive, the protected integrand is nonnegative, and on a nonempty open neighborhood of the zero fluctuation all protecting cutoffs are one and the exponential density is strictly positive. The Gaussian has positive

density there. Its normalizing Gaussian determinant is positive by the vertical Hessian gap. Hence its real logarithm is defined on the chart. The estimates for its derivatives follow by differentiating its absolutely dominated integral and using the logarithm expansion established in (ASP1)–(ASP2).

Let $M \geq 1$ bound the reference coordinate and site counts, let H be the stationary-response price, and let K_n be the sum of the purchased bare-action jet price and the stationary-composition price. More explicitly, the coordinate conversion multiplies an order-q multilinear price by M^q; the latter composition price is the fixed polynomial $\mathrm{clump}_n(1 + H + (1 + H))H$ appearing in the ordered-partition derivative bound. Thus K_n has rate zero. Classical matching removes the jets through order four, giving

$$\|\beta(f_1 - f_1(0))\|_{n,A;z} \leq \beta K_n A^5. \tag{RBG3}$$

The higher jets of W, of orders three through n, have a price C_n of rate zero. This price is the bare jet bound multiplied by $M^n(1+|\alpha|+\Lambda)$, together with the same stationary-composition polynomial and H. Here α is the bounded kinetic correction and Λ is the path allowance. The finite-period thermal comparison bounds its second jet at zero by

$$\|D^2 W(0)\| \leq A_0, \qquad A_0 = M^2\{M^4 m(\mathfrak{g}) M\}\, \Theta p\, e^{-\gamma c}. \tag{RBG4}$$

In this formula $m(\mathfrak{g}) = (1 + \dim \mathfrak{g})^2$ is the fixed color-coordinate conversion factor. The reference side is c, $\gamma, \Theta > 0$ are the purchased decay constants, and $p \geq 0$ is bounded by the adjacent loop allowance. Only the displayed exponential dependence on c is used below.

The first jet is smaller than a generic extensive-action bound would suggest. The signed symmetries give $Dg(0) = 0$, while classical matching gives $Df_1(0) = 0$. Differentiating (RBG1) therefore yields

$$DW(0) = -DL_0(0) - DY(0), \qquad \|DW(0)\| \leq d + \epsilon. \tag{RBG5}$$

Taylor's formula along segments in the chart, with these first and second jets and the third-and-higher jet prices, gives

$$\|W - W(0)\|_{n,A;z} \leq C_n A^3 + 3\{(\epsilon + d)A + A_0 A^2\}. \tag{RBG6}$$

For orders zero, one and two this follows by integrating the next derivative and using (RBG4)–(RBG5). For an order $q \geq 3$, multiply the unscaled bound $C_n A^{3-q}$ by A^q. Taking the maximum and enlarging the three low-order contributions gives the factor three. Centering Y costs at most 2ϵ; centering L_0 costs dA. Equations (RBG1), (RBG3), and (RBG6) now give

$$\|g - g(0)\|_{n,A;z} \leq dA + 2\epsilon + \beta K_n A^5 + C_n A^3 + 3\{(\epsilon + d)A + A_0 A^2\}. \tag{RBG7}$$

Here $y/2 \leq \beta \leq 2y$. Consequently d has rate -1. The lower bound on the reference side makes $\Theta p e^{-\gamma c}$, and hence A_0, have rate -1. All remaining prices have rate zero. The respective rates on the right of (RBG7) are at most

$$-23/16, \quad -2, \quad -19/16, \quad -21/16, \quad -23/16, \quad -15/8.$$

Every one is strictly below $-37/32$, and $-37/32 < -23/20$. The defining rate inequalities, with a sufficiently small positive slack and then a common delay, therefore make the whole finite sum at most $y^{-23/20}$. This derives the quantitative gain rather than assuming that a small integral automatically has a small logarithmic jet.

It remains to explain why the estimate applies to the five reference tori, whose sides need not be physical periods. Choose a physical coarse side divisible by the given reference side; the divisibility requirement was included in the block calendar. Copy the reference forest periodically to this larger

torus. The uniform stationary construction gives a critical section on each torus. On the larger torus the newly constructed section and the retained physical section have the same zero value and solve the same critical equation in the same uniqueness corridor. They agree near zero. Their tested classical jets and their thermal second jets therefore agree there. The periodic-copy comparison transfers classical matching and the finite-period Hessian error to the reference section. Its decay exponent can be replaced by the minimum of the two positive exponents, and its price by the sum of the two prices. This is sufficient for (RBG4). No equality of the Gaussian integrals on different tori is required.

On a fixed reference torus, differentiation along the logarithmic link chart costs the factorial and direction factors already displayed in (RB4a)–(RB4b). The omitted-source allowance comes from (RB8), uniformly in the compact coupling band. The same normalized offers depend on the coupling value, so substitution of the actual coupling curve preserves these estimates and their continuity. Summing the finite-site logarithmic and omitted-source bounds gives the reference offer bound used in (FCC1). This completes the passage from the actual stationary integral to the retirement beacons.

The uniformity required in this last assembly can be made explicit. For a fixed transition between stage profiles, the local argument is purchased at derivative order $n+6$ for the stationary score and at the indicated higher orders for the bare-action and Haar jets. Its conclusion uses order $n \geq 2$. The thermal comparison is first known for the retained critical section. A newly selected uniform analytic stationary graph is an extension of this section: the two solve the same vertical equation, and uniqueness near zero identifies them. Therefore their classical matching and finite-period second-jet comparison are the same tests, with the enlarged common price and the smaller common positive width. This establishes the hypotheses of (RBG3)–(RBG5) for the graph used in (PL1), rather than replacing that graph by a different critical solution.

For each fixed incoming stage only finitely many outgoing transition tests are required. Take the maximum of their coupling delays. Enlarging the calendar's delay preserves all the estimates, since they hold for every coupling above that delay. At an actual calendar step the identities for the block length and next coupling identify its local parameter with $y = \beta_F(t, L)$. The two adjacent loop caps are bounded by the same local loop allowance, and the kinetic correction satisfies

$$|\alpha| \leq \mathrm{drift} + |s_G \log L| + 1.$$

The assumed stage bounds on paths and loops therefore place both sides of this actual step in the local comparison just proved. This accounts for the order of choices and for the fact that the estimates hold for all allowed coupling values in the compact band.

Finally put $s = y^{-23/20}$, let q be the derivative order of the current budget, and let T be its fixed conversion price for left derivatives. After retiring the source tail to budget s, the centered retained source on any one reference torus of side c_i has left derivatives bounded by

$$F_i = q!\, 3^q s + 2c_i^4\{T(s+s)+s\}. \tag{RBG8}$$

The first term is the finite composition bound from the logarithmic coordinates to the prescribed left derivatives. The other terms pay the two source differences and the finite sum over sites; their factors are positive, so restriction to the omitted subfamily does not enlarge them. The retirement depth is common over the compact coupling band by (RB5)–(RB8), and common over the finite reference tori by taking its maximum. Each reference side is chosen above the read radius, and its finite joint-coordinate, fiber and site counts have the same preassigned size bound. No constant in this step is chosen from the later macroscopic volume subsequence.

If the fixed interpolation stencil has signed weights a_i, its support reach is e, and the budget support power is p, transporting these local estimates to one insertion gives

$$\sum_i |a_i|\, 2^q (1+e)^{pq} F_i. \tag{RBG9}$$

Indeed the directional substitution costs 2^q, the support enlargement costs $(1+e)^{pq}$, and the finite sum is bounded with absolute weights. Smooth dependence of each reference offer on its coupling allows substitution of the actual continuous coupling curve. Formula (RBG9) is the bound for the reference offer in (FCC1). It is on the same ordinary source and common finite stencil as the retirement construction, and supplies the needed passage from the local protected integral to its assembled source bound.

Source–readout formulation. The identity (RBG1) is a decomposition of one readout (the reference logarithm) into five source columns, each retained with its own estimate; the Gaussian main part and the discrepancy Y are stored separately and are never merged before the rate is taken. The stationary section used on a reference torus is the same section as the physical one in the sense of Theorem 14.4: uniqueness in the corridor is the comparison isomorphism between the two presentations. The protected integral is an integral of gauge-covariant chart data on a fixed forest fibre; in the frame realization it is carried on the identity-frame section and is unchanged by Theorem B.1. The status of (RBG7) is a controlled approximation in the declared scaled norm, on the compact coupling band, with the order of choices: orders and stage parameters, then block length, then delay.

Lemma 14.26 (Interpolation)**.** *If $\|D^i h\| \le v_i$ on a ball of radius s and $\|D^{i+2} h\| \le C$ there, then*

$$\|D^{i+1} h(0)\| \le 2v_i/s + Cs.$$

Consequently, if rough derivatives through order $j+1$ are bounded by y^d on a ball of radius y^{-a}, $a > 0$, then for every target exponent b there is B such that a value bound y^B on that ball implies that every jet through order j at the centre is bounded by y^b, for y large.

Proof. Test unit vectors and apply the one-variable estimate $\phi'(0) = [\phi(s) - \phi(0)]/s - \int_0^s (1 - \sigma/s)\phi''(\sigma)\, d\sigma$ to $\phi = D^i h$ along a line. For the consequence induct on the order: at order zero take $B = b$; for the step $i \to i+1$ choose $c = a + \max(0, d-b) + 1$ and $b' = b - 1 - c$ and use the order-i assertion with target b' and ball exponent c. Eventually $2y^{-c} \le y^{-a}$, so the smaller balls around points within y^{-c} of the centre lie in the original ball, and (PL4) gives $2y^{b'+c} + y^{d-c}$, both exponents strictly below b. □

The display (GL0) is typeset from the supplied text with one typographical restoration: in the supplied text the subscript of the sum over free bonds has lost the first two characters of its roman-font command (they were converted to a carriage return), and it is restored here to "e free", the sum over the free bonds of the forest, which is the only reading consistent with the Haar density that follows it.

14.19 The local activity norm and analytic comparison estimates

A state is read through its activity. The raw atom or path presentation can be nonunique. The analytic step maps must consequently be constant on presentations of the same activity before they define maps on the completed normed activity space. The signed-moment conditions, absolute

majorants and interpolation bounds below establish the required completion and descent. The subsequent stability theorem compares histories of the one fixed recipe. Uniform running then controls the actual finite-cutoff couplings uniformly in period, rather than merely exhibiting a reference clock with the desired slope.

The norms in finite calibration are the paid activity norms, not the uniform norm of the resulting density. Fix the current budget P and a common address layout. Let $b_P, k_P > 0$ be its regular and forced normalization allowances. For an activity pair $A = (E, K)$, define

$$\|A\|_P = \inf\{t \geq 0 : E \text{ has anchored regular derivative budget } tb_P,\ K \text{ has forced hit budget } tk_P\}. \tag{LCN1}$$

The regular budget tests every left derivative through P's order, in its permitted directions, on the healthy domain of that address, with its exponential support weight. The forced hit budget takes the absolute bound on every configuration and sums over addresses hitting each site. Both include the fixed locality and invariance conditions. The infimum is attained: take the least supremum envelope for each regular derivative family and each forced coefficient, and pass every finite nonnegative partial-sum inequality to the infimum. Hence $\|A\|_P \leq t$ is equivalent to the two displayed budgets with that exact t. Triangle inequalities for derivatives and coefficients give subadditivity; scaling gives $\|cA\|_P = |c|\|A\|_P$.

Fix, for each signed translation, one permutation implementing that symmetry on the layout. The activity pairs obeying all these fixed permutations form a linear subspace $\mathcal{H}_P$. Its seminorm is (LCN1). Use its separated completion $\widehat{\mathcal{H}}_P$; the map from the original presentation is an isometry after identifying seminorm-zero differences. This retains the off-mask regular germs required by the derivative budget, rather than identifying them merely by their evaluated values outside the healthy domain. The scalar-plus-activity space has norm $\|(\lambda, A)\| = \max(|\lambda|, \|A\|_P)$. The actual retained activities have norm at most $1/8$ in these coordinates.

We explain how the local primitive estimates give an analytic estimate in this norm. For a finitely supported signed measure $w = \sum_j c_j \delta_{t_j}$ on $[-1, 1]$, say it has moment price r at radius $p \geq 1$ if

$$\left|\sum_j c_j (t_j/p)^n\right| \leq r \qquad \text{for every } n \geq 0. \tag{LCN2}$$

A family $f(t)$ has comparison price C at radius p if every such measure satisfies $\|\sum_j c_j f(t_j)\| \leq rC$. This is the precise signed-combination bound used here. If $f(t) = \sum_n (t/p)^n a_n$ with $\sum_n \|a_n\| \leq C$, the bound follows by interchanging a finite sum and an absolutely convergent series. Conversely, define the moment vector $((t/p)^n)_n$ in ℓ^∞. The signed-combination bound defines a bounded linear map on their span and then its closure. Coordinate vectors belong to that closure: subtract the already recovered lower coordinates from the moment vector at a small nonzero t, divide by $(t/p)^n$, and let $t \to 0$. Its image supplies the coefficients. Restricting from radius p to $p' < p$ sums the geometric coefficient majorant and gives an absolutely convergent series. This is the recovery step used to turn signed-combination bounds into analytic line estimates.

On an affine activity line $A + tB$, regular occurrences and forced coefficients are affine in t on the same configuration: their masks and inventories are fixed. The finite calibrated coupling along the line is $\beta(t) = \Omega[\Omega^{-1/32}(\lambda + tu)]^2$, with $|u| \leq 1$. The comparison center has $\Omega^{-1/32}\lambda \in [33/40, 47/40]$, and the climate bounds ensure the whole line stays in the allowed coupling band. Assume $\|A + tB\|_P \leq 1$ for $|t| \leq 1$. The primitive's signed regular and forced estimates, followed by the Hessian projection and the fixed dictionary fusions, give prices

$$\text{output activity: } d \text{ at radius } q/2, \qquad \text{scalar correction: } f_i \text{ at radius } q, \quad f_i = \text{chores}_i\,/\theta_i^2. \tag{LCN3}$$

The regular signed bound is tested in the anchored derivative budget. The forced signed bound has one nonnegative summable hit-envelope common to all line parameters and configurations. For any weights satisfying (LCN2), these two envelopes give budgets $rd\,b_{P'}$, $rd\,k_{P'}$ on the actual output layout, so (LCN1) gives the activity price in (LCN3). The scalar price follows by applying the linear Hessian reading to the same signed regular source estimate and dividing its paid bound by θ_i^2. This explains both the norm and the threshold denominator in (LCN3). The primitive locality, derivative and signed-sum bounds used in this step are the marked integral and source projection estimates of the preceding chapters; no assertion about analytic dependence of a chosen forest or local guide is required.

We make explicit the gain that pays (LCN3), since an individual bound on each line value would not imply the signed-combination bound. Use the 301 distinct interpolation nodes $r_j = j/301$, $0 \le j \le 300$, in $[-1,1]$, and their Lagrange polynomials L_j. If w has moment price η at radius q, let $c_j = \sum_t w_t L_j(t)$ and $w_0 = \sum_j c_j \delta_{r_j}$. The coefficient bounds of these finitely many polynomials give $\sum_j |c_j| \le C_q \eta$ for one fixed $C_q \ge 0$. Every polynomial of degree at most 300 is interpolated exactly, so $w - w_0$ annihilates its moments through that degree. If $V \ge q$, its remaining moment prices at radius V are bounded by $\eta(1 + C_q)(q/V)^{301}$: factor $(q/V)^n$ from the degree-n moment and use $n \ge 301$. Thus a family with signed price B at radius V and individual value bound b has signed price at radius q at most

$$C_q b + (1 + C_q)(q/V)^{301} B. \tag{LCN8}$$

This is valid for the regular derivative budget as well as a scalar norm, because finite sums, absolute scalar multiplication and envelope sums preserve that budget.

We derive the preliminary large-radius signed price used in (LCN8). Fix the stage, block ratio and address layout. Write $y = \beta_F(x, L)$, $\phi = y^{1/256}$, $B = B(x)$, and let $A > 0$ be the fixed turning price. We first explain the parameter estimate on each vertex. A function polynomial in the line parameter t, of degree at most two, is reconstructed exactly from its values at $-1, 0, 1$:

$$f(t) = \tfrac{1}{2}t(t-1)f(-1) + (1 - t^2)f(0) + \tfrac{1}{2}t(t+1)f(1). \tag{LCN11}$$

The sum of the absolute coefficients of these three Lagrange polynomials is 4. If each nodal function has all required spatial derivatives bounded by vd^m at derivative order m, (LCN2), applied to each parameter power, bounds its signed interpolation by $4p^2 vd^m$ at radius $p \ge 1$. The same identity holds after spatial differentiation: its parameter coefficients are fixed scalars. Thus this is a signed derivative bound, with no factor $\sum_t |w_t|$.

Logarithmic vertices enter as $e^{f(t)} - 1$, rather than $f(t)$. If their nodal derivative budget is at most $v \le 1/2$, first recover the budget of f using $f = \log(1 + (e^f - 1))$. The scalar value bound is $|f| \le 2v$; all derivatives of $\log(1 + z)$ through the fixed order are bounded on $[-1/2, 1/2]$. Its finite chain-rule sums therefore give a bound $b_j v$ with the same spatial derivative scale, for a constant b_j depending only on that order. Apply (LCN11), and then the absolutely convergent exponential series in the signed moment norm. Multiplication in that norm multiplies prices, so a log price $s \le 1$ gives $e^s - 1 \le (e - 1)s$; the differentiated exponential is controlled by the finite chain-rule sums at the same scale. Consequently there are constants $A_j \ge 0, B_j \ge 1$ such that the signed price of this vertex is at most $A_j p^2 v$, provided $B_j p^2 v \le 1$. These are the estimates FullVolume/Ray.Badge.starpath and White.faded, applied at the three displayed nodes.

They apply to the actual vertices because the line dependence is placed in the couplings and occurrence amplitudes, not in a newly chosen forest. With the spatial configuration held fixed, an old vacuum logarithm is constant or is multiplied by the square of the affine coupling-amplitude

line, hence is quadratic in t. A regular occurrence has the prescribed degree-at-most-two amplitude representation on this line; multiplication by its fixed local enforcement factor preserves that degree. The forced occurrence has the same representation. Warning vertices, including their common steering and course variables, have the retained parameter-independent derivative envelope and therefore satisfy the constant-function signed estimate. In the differentiated formula the background, steering and Gaussian-coordinate variables are varied together as specified in the common chart. The spatial chain-rule prices are included in the nodal bounds. These are the four cases in FullVolume/Sector/Grant.mural, lured, loglift and alerts. In particular polynomial interpolation is applied to the log before exponentiation, not to the exponential.

The preliminary signed price now follows by the connected expansion with these vertex prices. Three nodal log envelopes sum to at most $3q$, where the old target q has rate at most $-7/16$. At $p = \phi$, their enlarged price has rate at most

$$-7/16 + 2/256 = -55/128 < -2/5.$$

Thus both $3q \le 1/2$ and $B_j \phi^2(3q) \le 1$ hold eventually, and the reweighted regular and log prices fit the target of rate $-2/5$. This is the explicit target enlargement in FullVolume/Sector/Reign.shows. The finite-order product rules multiply these prices at each connected tree. The support exponent exceeds the final exponent by more than three; split this excess into three positive reserves to pay the tree-degree factors, collar enlargement and log attachments. The positive tree sums have ratios at most $1/2$, hence cost at most 2 each. All remaining fixed-order spatial and collar prices have rate zero. Multiplying them by the enlarged forecast of rate $-2/5$ is eventually at most $y^{-3/8}$. Restricting the moment radius to $\phi/2$ improves the bound. The forced entries retain a common nonnegative hit envelope by the same positive majorants. This establishes the preliminary signed estimate for every finite combination satisfying (LCN2); it does not infer it from individual activity bounds.

Here are the product and summation steps in that signed estimate. A radius-R series certificate means an actual representation on $[-1, 1]$,

$$f(t) = \sum_{n \ge 0} (t/R)^n b_n, \qquad \sum_n \|b_n\| \le A, \qquad R \ge 1. \tag{ASP1}$$

For a finite signed combination $w = (w_t)$, put $\Lambda_w f = \sum_t w_t f(t)$. If $|\sum_t w_t (t/R)^n| \le \eta$ for every n, absolute convergence gives $\|\Lambda_w f\| \le \eta A$. This estimate uses cancellation of the combination; replacing w_t by $|w_t|$ would lose it. Products preserve (ASP1) with the product allowance, by the absolutely summable Cauchy product. Summable families preserve it by the double-series theorem. Exponentiation gives allowance $e^A - 1$ for $e^f - 1$, by summing the positive scalar majorant $\sum_{q \ge 1} A^q / q!$.

For the simultaneous spatial derivative estimates, fix a base coarse field x. Suppose each diagonal derivative through order j, in a unit spatial direction v, has (ASP1) allowance Ab^q at order q. If another factor has allowance Cd^q, the derivative of their product is

$$\sum_{r=0}^{q} \binom{q}{r} D_v^r f \, D_v^{q-r} g.$$

The product rule for the series certificates bounds its allowance by

$$\sum_{r=0}^{q} \binom{q}{r} A b^r C d^{q-r} = AC(b+d)^q. \tag{ASP2}$$

Induction gives, for a finite product of vertices, allowance $\prod A_i$ and slope $\sum b_i$. Thus the signed cancellation is retained before multiplying the vertices. After applying Λ_w, polarization bounds the full symmetric q-linear derivative by $\mathrm{polarCost}(q)\eta A b^q$. Taking $\sum_{q\le j}\mathrm{polarCost}(q)$ as one common constant controls all orders through j. Finite linearity commutes Λ_w with these derivatives.

Here the polarization constant used by the source is explicitly

$$P_q = 2^q(q+1)^q, \qquad P_{\le j} = \sum_{q=0}^{j} P_q. \tag{ASP2a}$$

For a symmetric q-linear form A and vectors $\|v_i\| \le 1$, expand

$$\sum_{S\subseteq\{1,\dots,q\}} (-1)^{q-|S|} A\left(\sum_{i\in S} v_i, \dots, \sum_{i\in S} v_i\right) = q!\, A(v_1,\dots,v_q).$$

Every term whose list omits an index cancels by the alternating subset sum; a surviving list contains every index exactly once, giving $q!$ identical terms. If the unit diagonal bound is C, each subset term is bounded by $(q+1)^q C$, by homogeneity. There are 2^q terms and $q! \ge 1$. This proves the stated constant, including $q = 0$, and normalization of the nonzero vectors proves the operator norm bound. Applying it after the signed parameter combination gives the allowance $P_{\le j}\eta A$ at all orders through j. It uses no bound depending on the dimension of the spatial chart.

The Gaussian integration prices entering these allowances also have a direct finite-dimensional proof. Let γ_Σ be the centered Gaussian actually used for a packet, with $\Sigma \ge 0$ and $|\Sigma_{ii}| \le B$. The one-coordinate marginal gives

$$\int e^{tz_i}\, d\gamma_\Sigma = e^{t^2\Sigma_{ii}/2} \le e^{Bt^2/2}.$$

Thus $(1+|z_i|)^q \le e^{qz_i} + e^{-qz_i}$ implies $\int (1+|z_i|)^q\, d\gamma_\Sigma \le 2e^{Bq^2/2}$. Independence of the coordinates is unnecessary. For a local edge set I, let J be its colored coordinate set, $a = |I|$, $b = |J|$, and

$$F_I(z) = 1 + a + b + \sum_{i\in J} |z_i|, \qquad Y = 1 + a + 2b.$$

Choose a coordinate attaining the maximum when $J \ne \varnothing$; then $F_I(z) \le Y(1+|z_i|)$. The empty set case follows directly. Consequently

$$\int F_I(z)^q\, d\gamma_\Sigma \le Y^q\{1 + 2be^{Bq^2/2}\}. \tag{ASP2b}$$

This proves integrability as well as the bound, by domination by a finite sum of the two exponential moments.

A tilt changes only the marginal on J. Its covariance Σ_J is positive definite. With $c \ge 1$, its exact density multiplier is

$$M_c(z) = c^b \exp\left(\frac{1-c^2}{2} z_J^T \Sigma_J^{-1} z_J\right), \qquad 0 \le M_c(z) \le c^b, \qquad F_I(cz) \le cF_I(z). \tag{ASP2c}$$

This is obtained by the substitution $y_J = cz_J$ in the Gaussian density; the determinant contributes c^b. The quadratic form is nonnegative, proving the bound. The remaining scalp factor and this multiplier combine exactly into the original scalp times c^b times the exponential of the tilt's quadratic cost. Applying the product and exponential estimates to that equality gives derivative

price charge$(j)(1+\delta)^{2b}$, with spatial slope bounded by $L'2^K F_I(z)^K$. Formula (ASP2b) pays its moments.

Finally a warning event E with probability at most r^{2m} and a polynomial factor with second moment at most A^2 contribute at most Ar^m, by Cauchy–Schwarz. Haar variables have total mass one and do not change this bound. All polynomial moment prices are included by taking q twice the required derivative degree. If the colored coordinate count is bounded by

$$2b \le 2\,\mathrm{cov}_B(Y)\,\mathrm{ballCost}(B)\,\mathrm{density}(L)\,\dim\mathfrak{g}$$

and the tilt margin satisfies $2\delta\,\mathrm{ballCost}(B)\mathrm{density}(L)\dim\mathfrak{g} \le \alpha$, then $\log(1+\delta) \le \delta$ gives $(1+\delta)^{2b} \le e^{\alpha \mathrm{cov}_B(Y)}$. This is precisely the reserved cover weight; the Gaussian determinant factor is paid before the final summation.

The interpolation step preceding this argument supplies those certificates at the real nodes. Explicitly, at $k+1$ distinct nodes t_i, let l_i be the Lagrange basis and $B_k = \sum_i \sum_{r=0}^{k} |[t^r]l_i|$. A degree-$k$ family with nodal derivative allowance Ab^q has (ASP1) allowance $B_k R^k A b^q$: expand $l_i(Rz)$ and bound $R^r \le R^k$. For optional exponential vertices the nodal hypothesis is on $e^{f(t_i)}-1$. When its allowance is at most $1/2$, $\log(1+y)$ has bounded derivatives on $[-1/2,1/2]$ and $|\log(1+y)| \le 2|y|$. The finite chain rule therefore bounds the nodal logarithm through the same spatial order. Interpolate this logarithm, and only then exponentiate by (ASP1). The condition $B_j R^2(3q) \le 1$, with its fixed derivative constants included in B_j, pays this last exponential step. This explains why the source can use polynomial dependence of the log without asserting polynomial dependence of the exponential.

The remaining tree summation is positive. If vertex prices have hit allowance γ at support exponent $s+\delta$, $\delta > 0$, the rooted forest estimate of the connected expansion gives for the terms with m nonroot vertices the bound

$$e^{\delta\,\mathrm{cov}_B(Y)} \left(\frac{4\ \mathrm{ballCost}(B)\gamma}{\delta}\right)^m. \tag{ASP3}$$

This estimate is proved by summing the leaf label at its first overlap with its parent: a radius-B cover of the parent has at most its cover number times ballCost(B) possible hit sites. The hit bound pays that sum; the reserve $e^{\delta\,\mathrm{cov}_B}$ pays the parent cover factors and the enumeration of parent assignments. The rooted occurrence enumeration and factorial weights give the factor 4^m already included in (ASP3), as in the absolute tree estimate used earlier. A designated marked root instead pays its own weighted allowance, multiplied by the same geometric factor. For ratio at most $1/2$, summing $m \ge 0$ costs at most two. Polynomial factors from the spatial direction growth are paid by the collar reserve:

$$\mathrm{collar}_{j,p} = (\mathrm{shift}(x))^{pj} \sup_{r\ge 0} r^{pj} e^{-\delta r}. \tag{ASP4}$$

The supremum is finite (with the usual value one when $pj = 0$); it is just the elementary absorption of a fixed polynomial by a positive exponential. The shift is bounded by a fixed multiple of the local support width plus one.

The packet's additional factor $(1+\varepsilon)^{n_A}$, $0 \le \varepsilon \le 1$, is also paid on the whole packet. The number n_A of relevant off-tree color coordinates satisfies $n_A \le C\,\mathrm{shares}(A)\,\mathrm{Delay}(x)^5$, by the link census and the $10B$ visit bound. Since $1+\varepsilon \le e$, and any fixed positive derivative constant is at most its exponential, these factors are absorbed by increasing the coefficient of $\mathrm{Delay}(x)^5\,\mathrm{shares}(A)$ in the packet income. They do not change its gain exponent or replace a whole-packet cost by a single-site hit.

In the final scheduling set $\alpha = (Q_{\exp} - e)/4 > 0$, where $e < Q_{\exp}$ is the requested output exponent. Then $e + 3\alpha \le Q_{\exp}$. Spend the three α's separately in the tree bound, collar bound and attached-log bound. The enlarged root price has rate $-2/5$; the degree ratio in (ASP3) has that rate times a width factor of rate zero, so it is eventually at most $1/2$. The fixed derivative and collar factors also have rate zero. Their product with the root price is eventually below $\widehat{\beta}^{-3/8}$, since $-2/5 < -3/8$. The parameter radius is $R = \widehat{\beta}^{1/256}$, eventually at least two; restrict to $R/2$ for the final bound. These are the simultaneous choices giving the common nonnegative hit envelopes for the signed forced entries and the series certificates for the attached logs.

Retirement selects a fixed subset of this same nonnegative majorant. Adding the boundary correction and turning operation gives the common signed price

$$w(x) = A\{y^{-3/8} + 4y^{-7/16}\}.$$

Every reference torus and comparison ball in the graft has size polynomial in $B(x)$ at a fixed stage. Their site counts $N_i(x)$ and $N_*(x)$, and their derivative factors, have rate zero. Summing the two compared anchored expressions over a reference torus gives signed price $p_i(x) = 2N_i(x)w(x)$. Summing the unrestricted value bound over the comparison ball gives price $N_*(x)w(x)$. Both retain rate $-3/8$.

Let z be the graft radius, let c_* bound the cover number of its radius-z support, and let λ_i be the fixed finite interpolation weights. If the derivative order and spatial derivative power of the budget are j and p, put

$$T_z = 2^j(1+z)^{pj}, \qquad O(x) = \sum_i |\lambda_i| T_z p_i(x).$$

For a signed measure with moment price η at radius $\phi/2$, the finite interpolation derivative formula gives beacon envelope $\eta O(x)$ at each site. There is one anchored beacon per site. Its support weight therefore gives anchored budget at most $\eta O(x) e^{(\mathrm{supportExponent}+3)c_*}$. The retired part has budget $\eta w(x)$. Since the regular graft is linear in these parts, the complete regular signed price is

$$w(x) + AO(x)e^{(\mathrm{supportExponent}+3)c_*}. \tag{LCN10}$$

The cover bound c_* is fixed before x, and all radius factors have rate zero. Thus (LCN10) has rate $-3/8$, and is eventually at most $y^{-1/4}$. This is the rough price at radius $V = \phi/2$ required in (LCN8). It applies to the deterministic source because the retirement index depends only on the layout and the matching identities identify the actual source with this graft.

The forced half uses separate all-configuration envelopes. Its primitive entries retain their exponential flag suppression. The attached local insight entries have a signed scalar price obtained from the comparison-ball and reference-torus value prices, multiplied by their ball count and support weight. Denote the resulting primitive and attached hit prices by a and b. The delay purchases $\kappa \ge 0$, $b \le 1$, and

$$\gamma = e^{\kappa}a + 2b, \qquad (2B+1)^4\gamma \le 1, \qquad e^{-\kappa}(e^{\gamma} - 1) \le \mathrm{forecastK}\,.$$

The local contribution b has rate $-3/8$ and is eventually below the comparison ceiling $y^{-1/8}$. The forced forecast uses the primitive exponential suppression as well; it does not follow from comparing two power rates. To fuse the two families, inflate each old forced atom by e^{κ}, bound each attached exponential minus one by twice its logarithm envelope, and sum the connected components using the displayed support reserve. Each component contains an old forced atom, so undoing the inflation saves at least $e^{-\kappa}$. The resulting hit price is at most $e^{-\kappa}(e^{\gamma} - 1)$, as proved in the attached-log

fusion argument. This gives the forced forecast at radius $\phi/4$, with one support-exponent unit reserved. Shrinking later to $q/2$ improves the moment inequalities and supplies the common forced envelope in (LCN3).

The sharper individual price in (LCN9) comes from universal retirement. Its remaining beacon envelope is a finite sum, with fixed coefficients, of

$$\operatorname{gold}(x), \qquad N_i(x)\operatorname{gold}(x), \qquad 2^j(1+z)^{pj}N_i(x)\operatorname{gold}(x),$$

where $\operatorname{gold}(x) = y^{-23/20}$. Every multiplier has rate zero. The fixed support exponential and turning price preserve rate $-23/20$, proving the claimed rate of ε_j. Thus finite interpolation combines a large-radius signed estimate and a sharper real individual estimate. For the primitive source, the individual bound b is its regular price (FCC1) plus the beacon contribution. With $x = \widehat{\beta}_i$, it has form

$$b = 3t\,C_j x^{-11/10} + \varepsilon_j(x), \qquad \varepsilon_j \text{ has rate } -23/20, \tag{LCN9}$$

where $t > 0$ is the freely chosen small primitive allowance. The three principal payments are the coefficient $t + 2t$. The other terms are the turning fee times the retired tail, the tail itself, and the beacon price times its fixed support exponential. The local estimates give them the stated $-23/20$ rate at fixed block and stage. The larger signed radius is $V = \operatorname{fire}(x)/2$, where $\operatorname{fire}(x) = \beta_F(x)^{1/256}$; the rough signed price has rate $-1/4$. Consequently the last term of (LCN8) has rate at most $-301/256 - 1/4 < -23/20$. Choose t so that $6C_q t \le v$, and then the coupling-entry delay so that the two remaining terms cost at most $(v/2)C_j x^{-11/10}$. Equations (LCN8)–(LCN9) give signed regular source price at most $vC_j x^{-11/10} = \operatorname{chores}_i$. This is the numerator of the scalar price f_i.

The Hessian reading of this source yields the scalar correction price f_i by linearity and its bound $\theta_i^2|\Delta| \le \text{Budget}$. Applying the regular reset price to that same signed source, including the committed coefficient bound $|\alpha_{N,i}| \le d_{i+1} + s\log L_i$, gives regular output price $db_{P'}$ once the small allowance and delay have been chosen. The forced reset entries use their common all-configuration envelopes; the forced and attached-log fusion estimates retain one unit of support-weight reserve and yield hit price $dk_{P'}$ at radius $q/2$. The expenses and the forced cutoff are the paid local reset inequalities, not a consequence of the scalar reading alone. Combining these two prices through (LCN1) proves the activity assertion in (LCN3). Thus the analytic recovery uses a signed estimate obtained by a finite interpolation cancellation, with the exact exponent 301, rather than assuming analyticity of arbitrarily selected primitive witnesses. In fact the output values are independent of such incidental choices. To compare two representatives with identical coupling and activity data, use the same constant affine input line but select the first representative at $t = 0$ and the second at $t \neq 0$. The bounds (LCN3) still hold. For $q \ge 4$, the signed measure $\delta_t - \delta_0$ has moment price at most $|t|$ at either radius. Therefore its two output differences are bounded by $d|t|$ and $f_i|t|$; letting $t \to 0$ proves equality. This supplies the single-valued activity and scalar maps on the common normed presentation.

Choose $q \ge 4$ so large that

$$\frac{2e}{(1/4)(q/4)} \le 1, \qquad 2d \le \gamma = 1/128. \tag{LCN4}$$

At any center (λ, A) with $\Omega^{-1/32}\lambda \in [17/20, 23/20]$ and $\|A\|_P \le 1/8$, every neighboring center within distance $1/8$, followed by any direction of norm at most $1/4$, stays in the enlarged scalar window and activity ball used in (LCN3). Recover the line series at radius $q/4$. Polarization makes its degree-n coefficient a symmetric multilinear map: the finite difference formula divides by $n!$, has 2^n summands, and each sum of n unit vectors has norm at most n. Since $n^n/n! \le e^n$, the resulting

multilinear coefficient is bounded by

$$\Big(\frac{2e}{(1/4)(q/4)}\Big)^{n} C, \qquad C = 2d \text{ or } 2f_i. \tag{LCN5}$$

The formulas agree on overlaps because they recover the same scalar line coefficients. They extend through the isometric dense inclusion into the completion, preserving their operator norms. By (LCN4), the activity coefficients are bounded by γ, and the scalar coefficients by $2f_i$, at every order.

For increments z, z' of norm at most $1/4$, telescope each multilinear diagonal and sum:

$$\sum_{n\geq 1} n(1/4)^{n-1}\|z-z'\|C = \frac{16}{9}C\|z-z'\| \leq 2C\|z-z'\|. \tag{LCN6}$$

Thus the local Lipschitz constants are 2γ for activities and $4f_i$ for the scalar correction. To get the estimates for any two centers in the whole stated window and activity ball, split into distances at most $1/4$, where (LCN6) applies, and distances greater than $1/4$. In the latter case (LCN3) applied to singleton measures bounds the two values separately by d and f_i; their differences are at most $2d \leq \gamma$ and $2f_i$. Both cases imply

$$\|\Phi_i(y) - \Phi_i(x)\| \leq 8\gamma\|y-x\|, \qquad |\Delta_i(y) - \Delta_i(x)| \leq 8f_i\|y-x\|. \tag{LCN7}$$

With $s_i = 2f_i$, this is exactly the scalar constant $4s_i$ and activity constant 8γ used in (FCB). All prices are for the fixed step of the common calendar and are uniform in the actual cutoff and period when its paid stage and support margins hold. The local analytic constants are chosen before the final history is selected; the completion introduces no new choice of primitive data. The common root envelope in this argument uses the energy gain proved before the expansion, in its actual order of quantifiers. For every sufficiently large odd block side L, all sufficiently large fixed window widths W admit a number $\kappa > 0$ and a coupling threshold such that

$$\kappa\, t_{\mathrm{sf}}(\beta, L)^2\, n_{\mathrm{seed}} \leq E_{\mathrm{assessed}} \tag{ASP5}$$

for every admitted tube, sector, tree and packet store above that threshold. Here n_{seed} is its number of charged seeds, t_{sf} its small-field threshold and E_{assessed} the sum of the actual fine, coarse and mismatch departure costs. The assignment of those costs to packet cores, the paid annuli and the three clipping ranges was proved in the energy chapter. Formula (ASP5) is a per-seed energy statement; the expansion additionally uses the support census, normalization expenses and reserves (ASP3)–(ASP4).

For adjacent scheduled profiles use the same angular covers for both copies of (ASP5). They provide one positive coefficient for each of the two requests; replace both by their smaller value and increase the common coupling threshold. The geometric block factor is fixed before these choices. The resulting calendar delays are chosen above both thresholds, and its actual coupling update identifies the next small-field threshold with the one in (ASP5). Every admitted step therefore has the claimed bound with the same charged seeds and the same angular read sets. The pointwise domination obtained this way gives one nonnegative envelope for all the signed entries, including when a parameter path changes which sector is active. Combining it with the tree and collar sums proves the all-root envelope used above.

Source–readout formulation. The step maps of the forward recipe were defined in Theorem 14.20 on represented states. A represented state is a presentation: it contains a representative,

a guide and a forest which the evaluated density does not see. The method's descent criterion asks whether the step is constant on the fibres of "choice of presentation", so that it defines a map of the normed activity data $(\lambda, A) \in \mathbb{R} \times \widehat{\mathcal{H}}_P$ themselves. The derivation above proves this and proves analyticity of the descended map.

Lemma 14.27 (Signed-combination bounds are series bounds)**.** *If $f(t) = \sum_n (t/p)^n a_n$ with $\sum_n \|a_n\| \le C$ then f has comparison price C at radius p. Conversely, if f has comparison price C at radius p, then for every $p' < p$ it is given on $[-1,1]$ by an absolutely convergent series in t/p' with coefficients a_n, $\|a_n\| \le C$ at radius p.*

Proof. The first statement interchanges a finite sum with an absolutely convergent series. For the converse consider the moment vectors $\mathsf{m}(t) = ((t/p)^n)_n \in \ell^\infty$. The hypothesis says that $\sum_j c_j \mathsf{m}(t_j) \mapsto \sum_j c_j f(t_j)$ is a well-defined linear map of norm at most C on their span, hence on its closure. The coordinate vectors lie in that closure: subtract the already recovered lower coordinates from $\mathsf{m}(t)$ at small $t \neq 0$, divide by $(t/p)^n$ and let $t \to 0$. Their images are the coefficients a_n, and $\mathsf{m}(t) = \sum_n (t/p)^n e_n$ converges in norm after restriction to radius $p' < p$ by the geometric majorant. □

Theorem 14.28 (Single-valued analytic step maps)**.**

1. *(Descent.) The output activity and scalar correction depend only on the coupling and activity data of the input, not on the representative, guide or forest: for two representatives with identical data, the signed measure $\delta_t - \delta_0$ on a constant line has moment price at most $|t|$ for $q \ge 4$, so the output differences are bounded by $d|t|$ and $f_i|t|$ for every $t \neq 0$, hence vanish.*

2. *Choose $q \ge 4$ with*

$$\frac{2e}{(1/4)(q/4)} \le 1, \qquad 2d \le \gamma = \frac{1}{128}.$$

Then on the window $\Omega^{-1/32}\lambda \in [17/20, 23/20]$, $\|A\|_P \le 1/8$ the maps Φ_i (activity) and Δ_i (scalar) are given near each centre by convergent series of symmetric multilinear maps with

$$\|\textit{degree-}n\textit{ coefficient}\| \le \Big(\frac{2e}{(1/4)(q/4)}\Big)^n C, \qquad C = 2d \text{ or } 2f_i,$$

extend to the completion with the same norms, and satisfy

$$\|\Phi_i(y) - \Phi_i(x)\| \le 8\gamma\|y - x\|, \qquad |\Delta_i(y) - \Delta_i(x)| \le 8f_i\|y - x\|.$$

Proof. (1) is as stated; it applies (LCN3) to the two-point measure. (2) Recover the line series at radius $q/4$ by Lemma 14.27. Polarization makes the degree-n coefficient a symmetric multilinear map: the finite-difference formula divides by $n!$, has 2^n summands and each sum of n unit vectors has norm at most n; $n^n/n! \le e^n$ gives (LCN5). For increments z, z' of norm at most $1/4$ telescoping each diagonal gives

$$\sum_{n \ge 1} n(1/4)^{n-1}\|z - z'\|C = \tfrac{16}{9}C\|z - z'\| \le 2C\|z - z'\|,$$

so the local Lipschitz constants are 2γ and $4f_i$. For two centres at distance greater than $1/4$, (LCN3) on singleton measures bounds the differences by $2d \le \gamma$ and $2f_i$, which are at most $8\gamma\|y - x\|$ and $8f_i\|y - x\|$. □

Part (1) is the descent criterion in its exact form: the step maps are constant on the fibres of the presentation, so they are well defined on the common normed presentation $\mathbb{R} \times \widehat{\mathcal{H}}_P$. With $s_i = 2f_i$ the constants are the scalar constant $4s_i$ and the activity constant 8γ used in the stability argument below. The exponent 301 is the actual finite interpolation order of the supplied construction; analyticity of the descended maps is a consequence of the signed-moment bounds through Lemma 14.27 and is not an additional hypothesis. In the frame realization the activity space and its norm are transported isometrically by Theorem 9.4, and the step maps are intertwined by Theorem 14.20; the Lipschitz constants (LCN7) are therefore the same numbers for the lifted maps. The exactness class is exact recovery of the maps together with their norms.

14.20 Stability of finite calibration and its comparison weights

The retained forward recipe also gives quantitative stability of the complete finite trajectory under changes of its bottom coupling. This is the finite statement behind the auxiliary trajectories in the uniqueness argument. Write $x_i = \widehat{\beta}_i$ and define

$$w_i = x_i^{-1/32}, \quad \mathfrak{b}_i = x_i w_i = x_i^{31/32}, \quad \lambda_i = \frac{\sqrt{\beta_i/x_i}}{w_i}, \quad \beta_i = x_i(w_i\lambda_i)^2. \tag{FCB1}$$

The positive square root is used. By (FCA8), $w_i\lambda_i \in [17/20, 23/20]$; this wider interval contains $[\sqrt{3/4}, \sqrt{5/4}]$. The activity coordinate X_i is its image in the common completed activity chart. At a fixed finite horizon all these coordinates live in one finite product of their scalar and activity spaces.

Let $\gamma = 1/128$. The actual local analytic comparison, including the initial state at N, bounds differences of activity outputs by 8γ times the maximum input-coordinate difference. Its scalar correction output has Lipschitz bound $4s_i$ times that maximum. The weights are explicitly

$$s_i = 2\,\frac{\mathrm{chores}_i}{\theta_i^2}, \qquad \mathrm{chores}_i = v\,C_{\mathrm{stage}(i)} x_i^{-11/10}, \tag{FCB2}$$

where $v > 0$ is the fixed comparison cost and $C_j > 0$ the stage's local analytic price. The common small-field threshold is $\theta_i = x_i^{-1/2}\exp((\log\log x_i)^2)$. Thus $s_i \le 2vC_{\mathrm{stage}(i)}x_i^{-1/10}$. These comparison bounds apply to the same retained primitive operation at both points, as ensured by its fixed layout and choices.

The calendar pays the cumulative weight inequality

$$\sum_{j<i} s_j \le \gamma\mathfrak{b}_i, \qquad \mathfrak{b}_0 \le \mathfrak{b}_i. \tag{FCB3}$$

Here is its derivation from delays. At a fixed stage transition the positive reference increment is $a = s\log L > 0$. The mean value theorem, for $x \ge a$, gives

$$(x+a)^{31/32} - x^{31/32} \ge a\,\frac{31}{32}\,2^{-1/32}x^{-1/32}.$$

Since $x^{-1/10}/x^{-1/32} \to 0$, eventually the upper bound in (FCB2) is at most γ times this increment. Include this one eventual threshold in the coupling-entry delay for each of the two allowed adjacent-stage possibilities. It follows at every realized step that $s_i \le \gamma(\mathfrak{b}_{i+1} - \mathfrak{b}_i)$. Summing gives $\sum_{j<i} s_j \le \gamma(\mathfrak{b}_i - \mathfrak{b}_0) \le \gamma\mathfrak{b}_i$. Positivity and monotonicity of x_i give $\mathfrak{b}_0 \le \mathfrak{b}_i$. Thus (FCB3) is a consequence of the same schedule, not a supplementary assumption on a chosen pair of trajectories.

Compare two successful trajectories of the deterministic recipe, with bottom-coupling difference δ. Let

$$U = \max_{0\le i\le N} \max\{|\lambda_i' - \lambda_i|, \|X_i' - X_i\|\}.$$

The local comparisons give $\|X_i' - X_i\| \le 8\gamma U$, including the finest-state comparison, and correction differences at most $4s_iU$. Subtracting the finite coupling recursions cancels their common path coefficients:

$$\beta_i' - \beta_i = \delta - \sum_{j<i}(\Delta_j' - \Delta_j). \tag{FCB4}$$

On the other hand (FCB1) factors their difference exactly as

$$\beta_i' - \beta_i = \mathfrak{b}_i(\lambda_i' - \lambda_i)(w_i\lambda_i' + w_i\lambda_i).$$

The last factor is at least $17/10 > 1$. Equations (FCB3)–(FCB4) imply

$$|\lambda_i' - \lambda_i| \le \frac{|\delta|}{\mathfrak{b}_i} + 4\gamma U \le \frac{|\delta|}{\mathfrak{b}_0} + 4\gamma U.$$

Taking the maximum over all levels and both coordinates proves

$$U \le \frac{|\delta|}{\mathfrak{b}_0} + 8\gamma U, \qquad U \le 2|\delta|/\mathfrak{b}_0. \tag{FCB5}$$

Only finite products and finite sums occur at this stage.

If $\delta = 0$, (FCB5) gives identical finest normalized couplings, hence identical finest actual couplings. Their startup states are equal by their fixed startup rule. Descending determinism of the primitive recipe then gives equality of all represented states. Thus the successful finite trajectory with a prescribed bottom value is unique within this retained recipe, although monotonicity of a scalar coarse readout was unnecessary for existence.

The arcs prove existence for every bottom value in the nearby interval. Combining them with (FCB5) gives continuous dependence of its normalized scalar and activity coordinates on that value, on the same finite calendar. For example, choose a bottom perturbation $|\delta| \le (\ell/4)\mathfrak{b}_0$, with the fixed small comparison radius $\ell > 0$, and small enough to stay in $[3\bar\beta/4, 5\bar\beta/4]$. Then $U \le \ell/2$, placing the entire perturbed trajectory inside the common analytic comparison neighborhood. This constructs the nearby finite trajectories used in passing local comparison estimates to banked limits. Equations (FCB1)–(FCB4) also identify explicitly the positive scalar weight and summable correction weights in that argument.

Source–readout formulation. The calibration of Theorem 14.24 selected a point of the fibre of the bottom-coupling reader by the intermediate value theorem. The comparison weights now show that this fibre, within the retained recipe, is a single point, and that the reader has a Lipschitz inverse.

Theorem 14.29 (Injective, Lipschitz-invertible bottom reader)**.** *Let two successful trajectories of the deterministic recipe have bottom-coupling difference δ and $U = \max_{0\le i\le N} \max\{|\lambda_i' - \lambda_i|, \|X_i' - X_i\|\}$. Then*

$$U \le \frac{|\delta|}{\mathfrak{b}_0} + 8\gamma U, \qquad U \le \frac{2|\delta|}{\mathfrak{b}_0}.$$

In particular the successful finite trajectory with a prescribed bottom value is unique within the retained recipe: the bottom reader $\mathsf{B}_{N,p}$ of Theorem 14.24 is injective, and $b^*_{N,p}$ is the unique point of the fibre over $\bar\beta$.

Proof. Theorem 14.28 gives $\|X_i' - X_i\| \le 8\gamma U$ (including the finest-state comparison) and correction differences at most $4s_iU$. Subtracting the coupling recursions cancels the common path coefficients,

$$\beta_i' - \beta_i = \delta - \sum_{j<i}(\Delta_j' - \Delta_j),$$

while (FCB1) factors $\beta_i' - \beta_i = \mathfrak{b}_i(\lambda_i' - \lambda_i)(w_i\lambda_i' + w_i\lambda_i)$ with last factor at least $17/10 > 1$. With (FCB3), $|\lambda_i' - \lambda_i| \le |\delta|/\mathfrak{b}_i + 4\gamma U \le |\delta|/\mathfrak{b}_0 + 4\gamma U$. Taking the maximum over levels and both coordinates gives the first inequality, and $8\gamma = 1/16$ the second. If $\delta = 0$ the finest normalized couplings, hence the finest couplings, agree; the startup states are equal by the fixed startup rule and descending determinism gives equality of all represented states. □

Consequently the calibration interface has no residual multiplicity at a finite cutoff: the realized calibrated history at (N, p) is determined by its bottom value. The qualification "within the retained recipe" is essential. The theorem compares trajectories of the one deterministic recipe on one calendar; it does not compare different recipes, different calendars or different address layouts, and the lift does not assert that those would give the same history. The weights $\mathfrak{b}_i$ and the inequalities (FCB3) are read from the calendar $\mathfrak{c}$ and were purchased among its entry delays.

14.21 Uniform running of the actual coupling

The drift estimate also proves the uniform running bound needed in the public theorem. Define

$$D_0 = 0, \qquad D_{i+1} = D_i + d_{i+1}. \tag{RUN1}$$

For the calibrated finite history, (FCA3), $\beta_{N,p,0} = \bar{\beta}$, and the exact recursion for $\widehat{\beta}_i$ imply

$$|\beta_{N,p,i} - \widehat{\beta}_i| \le D_i \qquad (i \le N). \tag{RUN2}$$

Indeed subtract the two recursions and apply the triangle inequality at each step. The right side depends only on the calendar, so this bound is uniform in both N and p.

The stronger asymptotic drift condition of that calendar is

$$\frac{d_{i+1}}{\log L_i} \le J(i+1)^{-2}, \qquad J(i) \longrightarrow \infty. \tag{RUN3}$$

Here $J(i)$ is the stage index, not a cutoff or period. Given $\varepsilon > 0$, choose j so that $J(i+1)^{-2} \le \varepsilon/2$ for every $i \ge j$. Summing (RUN3) gives, for $m \ge j$,

$$D_m \le D_j + \frac{\varepsilon}{2}(\tau_m - \tau_j).$$

Since $\tau_m \to \infty$, eventually $D_j \le (\varepsilon/2)\tau_m$. Together with (RUN2) this proves

$$\forall \varepsilon > 0\ \exists m_0\ \forall m \ge m_0\ \forall N \ge m\ \forall p, \qquad |\beta_{N,p,m} - \widehat{\beta}_m| \le \varepsilon\tau_m. \tag{RUN4}$$

This is uniform control of the actual finite-cutoff coupling, including $m = N$. It is stronger than merely writing down the reference line $\widehat{\beta}_m = \bar{\beta} + s\tau_m$.

Dividing by $\tau_m > 0$, and then absorbing the fixed intercept $\bar{\beta}/\tau_m$, yields

$$\sup_{N \ge m,\ p} \left| \frac{\beta_{N,p,m}}{\tau_m} - s \right| \longrightarrow 0. \tag{RUN5}$$

Explicitly apply (RUN4) with error $\varepsilon/4$, and take $\tau_m \geq 4(|\bar{\beta}|+1)/\varepsilon$; the triangle inequality gives an error smaller than ε. Taking $m = N$ proves the bare coupling slope in the theorem, uniformly in the physical period.

Set $b_0 = 11C_A/(48\pi^2)$, so $2b_0 = s$. For arbitrary cutoff and period sequences $N_i \geq i$ eventually and p_i, let $g_i = \beta_{N_i,p_i,i}^{-1/2}$. Positivity of the coupling and (RUN5), followed by continuity of the square root and reciprocal at $s > 0$, give

$$g_i\sqrt{\log(a_i^{-1})} \longrightarrow (2b_0)^{-1/2}. \tag{RUN6}$$

The equality used here is $\beta^{-1/2}\sqrt{\tau} = (\beta/\tau)^{-1/2}$ for $\beta, \tau > 0$. This regulator coupling is defined from the inverse action coefficient. Its numerical leading logarithm does not by itself identify it with a coupling defined by a physical position-space observable. That separate identification, and its normalization, are proved in the trace and Ward calculation.

Source–readout formulation. Equation (RUN5) recovers the first quantitative clause of the supplied public conclusion (Section 4) as a property of the calibrated realized history: the readout $\mathrm{Read}_{\mathrm{slope}}(N,p) = \beta_{N,p}/\log(h_N^{-1})$ converges to s uniformly in p, with witnesses the calendar $\mathfrak{c}$ and the trajectories of Theorems 14.24 and 14.29. It is stronger than writing down the reference line $\widehat{\beta}_m$, since it controls the actual finite-cutoff coupling. The regulator coupling g_i of (RUN6) is defined from the inverse action coefficient. Its leading logarithm does *not* by itself identify it with a coupling defined by a physical position-space observable; that identification and its normalization are a separate result of the trace and Ward calculation (Section 23.1). The lift keeps the two notions as two different readers of the same realized history, and no physical significance is attached to (RUN6) beyond the stated limit.

15 Convergence of interacting scale histories

Comparing one complete family of histories. The preceding construction supplies actual histories. To take their limit one must compare insertions, interactions, normalizations and Gaussian measures in the same coordinates. A protected coordinate change $z \mapsto z'_V(z)$ is therefore read together with its Jacobian and both covariances:

$$\mathcal{L}_V(z) = J_V(z)\frac{\rho_{C_V}(z'_V(z))}{\rho_{C_0}(z)}.$$

The density in the denominator belongs to the base measure. Replacing it by the varied density suppresses a real part of the comparison, even if the coordinate map is exact. Its determinant and inverse responses are retained in the detailed calculation.

Activity estimates then transfer local comparisons to reset budgets, and banked contraction gives uniqueness within the fixed recipe and calendar. It does not compare every possible recipe or phase. Path books retain higher jets and insertion transport; the common denominator is kept in omission comparisons. Finally the analytic tail estimates show that omitted insertion terms tend to zero on the same normalized family. Thus the readout at a fixed retained level is recovered with its derivatives and supports, rather than only as an unlabelled scalar limit.

15.1 Omission comparison at a common normalization

For finest level N, period p, insertion P at level m and mode t, use throughout

$$\mathcal{I}_m(P;t) = \frac{\int [P(U)]_t \, d\mu_m(U)}{Z_{N,p}},$$

where μ_m is product Haar and $Z_{N,p} > 0$ is the original finest Wilson partition function. In particular the denominator is not replaced by a new partition function when m changes. For positive root weights w, the treasury estimate and exact integrated-density identity give

$$|\mathcal{I}_k(P;t)| \le B_k(t) = \frac{\exp\{|\Lambda_k|\tau_k + |V|(x+y+2)\}}{\operatorname{toll}(w,t)}.$$

At $j > k$ multiply both omitted polynomials P_0, P_1 by complex z and restore $P(z)$; values at one and zero are full and omitted insertions. This bound follows by applying the small-field integral bound to the represented density, bounding its vacuum bill by the treasury, and using the exact identity between the integrated represented density and $Z_{N,p}$. The paid analytic estimate makes $Y(z) = \mathcal{I}_j(P(z);t)$ holomorphic for $|z| < S_j = \operatorname{brisk}(B_j) > 2$. Transport through the actual retained steps to k. Retention preserves served bounds and root weights, and integration is exact. Coefficient extraction commutes with integration as a continuous linear map on the finite-dimensional polynomial algebra. The fixed denominator therefore gives $|Y(z)| \le B_k(t)$ on the disk. For any bounded holomorphic Y on radius $S > 2$, radius-$S/2$ disks centered on $[0,1]$ stay inside it. Cauchy's estimate gives $|Y'| \le 2B/S$ there; segment integration yields

$$|Y(1) - Y(0)| \le 2B/S.$$

Thus omission costs at most $2B_k(t)/S_j$, with root-volume cost at k. Analytic paid bounds, retained transport and the root integral estimate are the retained substantive dependencies. For fixed configuration, finite modes, truncated exponentials and local field products make each restored coefficient entire in z. Masks and flags are measurable configuration events independent of z. The lift retains the finest denominator, omitted pair and actual transport chain. Recovery commutes with coefficient integration and retained steps, preserving the disk bound and Cauchy comparison at that same normalization.

15.2 Integral holomorphy and descending retention

Positive insertion weights make the weighted polynomial norm equivalent to the finite-dimensional coefficient norm. Served envelopes and nonnegative coupling give a finite constant Y with

$$\|P(z,U)\| \le Y \qquad (|z| < S_j,\ U \in \mathcal{C}_j).$$

Probability Haar makes this integrable. Dominated Cauchy-circle integration, or Morera with Fubini, proves integrated-coefficient holomorphy; masks require no configuration differentiability. Retention descends finitely from j to k. Top scaling spends two support units. At each step the relay identifies first-degree coefficients with the manifest, and Mark feed recovers the next manifest. Parent maps send chosen pins correctly; matching pays occupancy and reweighting; wake data provide local transport tales. The one-step duty bound restores the full support exponent at the next level. Mark-push integrability supplies each stage. This induction explains why the two-unit expenditure occurs only at the top; it does not accumulate once per step. The finite inventory and one-step duty estimates in the marked-integration chapter establish the inputs to this induction. The retained transport statement has a descending induction with no limiting step. The lift retains this finite chain, manifests and paid envelopes. Recovery commutes with its kernel composition and finite coefficient maps, preserving the common integral majorant and the nonaccumulating exponent cost.

For an explicit chain recovery, write $\mathcal{P}_{j\to k}$ for the finite composition of the actual retained Mark kernels and $\widehat{\mathcal{P}}_{j\to k}$ for their realized composition. The one-step kernel identity, iterated finitely, gives

$$\widehat{\mathcal{P}}_{j\to k}\mathscr{L}_j P(z) = \mathscr{L}_k \mathcal{P}_{j\to k} P(z).$$

At every level the normalized reader is defined using the same scalar $Z_{N,p}$; exact prior and coefficient integration therefore recover

$$\widehat{\mathcal{I}}_j(\mathscr{L}_j P(z);t) = \mathcal{I}_j(P(z);t) = \mathcal{I}_k(\mathcal{P}_{j\to k}P(z);t).$$

The measurable masks and the constant integrable bound Y are pulled back together. Thus the holomorphic integrated function, its uniform $B_k(t)$ bound and its endpoint difference are identical on recovery. The section returns both endpoint insertions. The finite descending retention proof restores the full exponent at each target level, so this commuting chain also preserves the single top expenditure and the $2B_k(t)/S_j$ omission bound.

15.3 Comparison of varying Gaussian integrals

At a fixed step the fluctuation space has finite dimension D. For density ρ_C, physical mean displacement t_X and $g = \beta^{-1/2} > 0$, use $w = t_X + gz$. Translation and scalar change of variables give

$$z = g^{-1}(w - t_X), \qquad c_X(w) = g^{-D}\rho_C(g^{-1}(w - t_X)),$$
$$\int A(z)\, d\nu_C(z) = \int c_X(w) A(g^{-1}(w - t_X))\, dw.$$

This retains the Jacobian and transformed integrability for vector-valued A. Covariance bounds give $\rho_C(z) \le C_0 e^{-\tau\|z\|^2}$ uniformly in history and gauge-slice variable. Protected means give $\|t_X\| \le B_0$ by finite coordinate-square summation, and the coupling window gives $0 < u \le g^{-1} \le v$. For $r = g^{-1}$,

$$1 + \|r(w - t_X)\| \le (1 + v + vB_0)(1 + \|w\|),$$
$$e^{-\tau\|r(w-t_X)\|^2} \le e^{\tau u^2 B_0^2} e^{-\tau u^2\|w\|^2/2}.$$

The latter follows from $\|w\|^2 \le 2\|w - t_X\|^2 + 2B_0^2$. A fixed tested-bond face scale lies between one and $K_0(1 + \|z\|)$. Thus polynomially face-scale-bounded transformed integrands have the common majorant

$$C_1 e^{-\tau u^2\|w\|^2/2}(1 + \|w\|)^k,$$

Lebesgue integrable in finite dimension, uniformly in gauge-slice variable. First apply dominated convergence at each slice point to the transformed integrand differences. Fubini identifies the original Gaussian integrals almost everywhere simultaneously for countably many histories. Their norms are bounded by integrals of the two majorants. The outer gauge-slice prior is a probability measure, so a second dominated-convergence application gives full Gaussian–gauge convergence. The lift retains physical means, scales, densities and the fixed coordinate space. Readout preserves the Jacobian identity and common majorant, transporting both limit passages with the same almost-everywhere scope.

For each fixed prepared datum, define the affine physical-coordinate map $T_X(z) = t_X + gz$. Its recovered measure is exactly

$$(T_X)_*\nu_C = c_X(w)\, dw.$$

If R_X is the realized Gaussian reader, then composition of pushforwards gives $(T_X R_X)_*\widehat{\nu}_X = c_X(w)\, dw$. Thus both histories can be compared on the same physical w space after readout, with their own retained c_X and inverse T_X^{-1}; the section recovers each inverse formula and its factor g^{-D}. The common scalar majorant in w is unchanged. For vector-valued integrands the inequality

$$\left\| \int (F_n - F)\, dw \right\| \le \int \|F_n - F\|\, dw$$

transports the same dominated-convergence conclusion. The almost-everywhere identifications are retained simultaneously for the countable history indices, and the outer probability prior carries the same integrable constant bound. Hence recovery preserves both integrations and their specified scope without changing the comparison variable or covariance.

15.4 Primitive derivatives and common-draft density comparison

Differentiation under the primitive integral has paid majorant

$$C(1 + (p\,\mathrm{facescale}^{K} + 1))^{j} \le C(p+2)^{j}\,\mathrm{facescale}^{Kj},$$

since face scale is at least one. Marked derivatives have the same form; undifferentiated marked amplitudes have constant bound. At each fixed gauge-slice point, discrete draft choices are finite and an ultrafilter selects one eventually. Compare histories in that common amended draft, retaining viability, threshold, choices and real and imaginary ingredients. The two-stage dominated-convergence argument then compares ordinary and marked derivatives and amplitudes. Finite reserve products and boundary averages preserve comparison; healthy-support local identities identify these formulas with the actual leaflet and marked-grove germs. This is the Gaussian integral passage used in the single-stream comparison, with all discrete masks retained inside the common draft.

A draft records slice variable, store, boundary choice, selector, scale and width. With the integration point fixed, paid conditions fix the last three, leaving finitely many store/boundary pairs. One fiber belongs to the ultrafilter, since otherwise their finitely many complements would have empty intersection. Viability transfers eventually. This does not assert constancy of continuous slice variables along histories. The common physical fluctuation w has restart map

$$z_\beta(w) = \sqrt{\beta}(w - t_X), \qquad t_X + \beta^{-1/2} z_\beta(w) = w.$$

Protected-mean and coupling comparisons compare this map. For steered output fields substitute the common smooth steered field for w, retaining the analogous exact identity for derivative integrands. The finite Gaussian density is

$$\rho_C(z) = (\sqrt{2\pi})^{-D} \exp\{-\tfrac{1}{2}\log\det C - \tfrac{1}{2} z^{\mathsf{T}} C^{-1} z\}.$$

Covariance comparison, its uniform positive floor and finite inverse and determinant formulas compare every factor. Fixed finite-coordinate density limits are positive, so marginal-density division in local reweighting is legitimate. Principal restrictions inherit the floor and smooth inverse and log determinant. Local correction factors are exactly finite Jacobian times corrected density divided by uncorrected density. Jacobians are products of smooth link steering factors near identity, with chart regularity through the required order. Thus the full correction factor compares through that order; remaining vertices are handled separately. This establishes the density and local change-of-variable factors used in the ordinary and marked integral comparisons. The lift retains discrete drafts, fixed slice points, restart identities, principal marginals and Jacobians. Readout preserves these finite maps and positive denominators, while retained majorants transport derivative integration without discarding masks or complex ingredients.

The local reweighting reader on its fixed tested coordinate set D retains the base and varied principal covariances separately:

$$\mathcal{R}_D(z) = J_D(z)\,\frac{\rho_{C_{V,D}}(z_D^{\mathrm{corr}}(z))}{\rho_{C_{0,D}}(z_D)}. \tag{3}$$

Here $J_D = \det Dz_D^{\mathrm{corr}} > 0$ is the original finite steering Jacobian, $C_{0,D}$ is the base restriction, and $C_{V,D}$ is the varied restriction. Both are positive definite on the common tested coordinates. Change of variables gives

$$\int f(w)\rho_{C_{V,D}}(w)\,dw = \int f(z_D^{\mathrm{corr}}(z))\mathcal{R}_D(z)\rho_{C_{0,D}}(z)\,dz.$$

Record the two covariances, the correction map, the Jacobian and their common draft. Positivity of the base density makes the ratio well defined; composition with readout recovers each input and hence this identity exactly, with its original integrability domain. For a smooth family with fixed base, the covariance contribution to the varied log-density is

$$-\tfrac{1}{2}\operatorname{tr}(C_V^{-1}\dot{C}_V) + \tfrac{1}{2}w^{\mathsf{T}}C_V^{-1}\dot{C}_V C_V^{-1}w,$$

and differentiating $w = z_D^{\mathrm{corr}}(z)$ also contributes $-\dot{w}^{\mathsf{T}}C_V^{-1}w$. These follow from $\frac{d}{dt}C_V^{-1} = -C_V^{-1}\dot{C}_V C_V^{-1}$ and $\frac{d}{dt}\log\det C_V = \operatorname{tr}(C_V^{-1}\dot{C}_V)$. If the base also varies, subtract its corresponding log-density derivative. Thus determinant and inverse derivatives are retained for their respective matrices. Only when $C_{V,D} = C_{0,D}$ does the formula specialize to the same-covariance ratio. The positive floors and retained majorants give the stated finite-order derivative control. Likewise the restart identity is recovered before evaluating derivative integrands. The finite store/boundary fiber selected at a fixed slice point is a discrete record and is unchanged by readout. These facts preserve the pointwise comparison and its derivative majorants while leaving the continuous slice variable under the same outer integral.

15.5 The protected change of variables and its determinant

The steering factors in the preceding ratio have the following construction. Fix a small positive radius d, independent of the later scale delay, on which the logarithm is a left inverse of the exponential through radius $2d$. Shrink a matrix neighborhood of 1 so that $\log(\exp a\, M)$ is analytic for $\|a\| \le 2d$, and so that for group elements M it exponentiates back to $\exp a\, M$ when $\|a\| \le d$. These choices follow by continuity on a closed ball strictly inside the original chart. Let $\chi_d(a)$ be the smooth inner-share cutoff, one on $\|a\| \le d$, zero for $\|a\| \ge 2d$. Define, with this multiplication order,

$$p_M(a) = \chi_d(a)\{\log(\exp a\, M) - a\}, \qquad \Phi_M(a) = a + p_M(a), \qquad j_M(a) = \det(1 + D_a p_M(a)). \tag{PV1}$$

Outside the cutoff support the first expression is identically zero in a neighborhood, so the map is smooth there regardless of the exterior values of the logarithm. On the support, smoothness follows from the chosen analytic chart. At $M = 1$, p_M and its derivative vanish, $\Phi_1 = \mathrm{id}$, and $j_1 = 1$.

Compactness of $\{\|a\| \le 2d\}$, joint continuity, and the exterior zero identity supply one neighborhood of $M = 1$ on which, uniformly for every a,

$$\|p_M(a)\| < d/2, \qquad \|D_a p_M(a)\| < 1/2, \qquad j_M(a) > 0. \tag{PV2}$$

For the determinant assertion, use its continuity and its value one on the compact ball; outside the ball it is exactly one. The derivative bound makes p_M Lipschitz with constant $1/2$. Thus Φ_M is injective, since equality of two images implies $\|a - b\| \le \frac{1}{2}\|a - b\|$, and surjective, since for each target b the contraction $a \mapsto b - p_M(a)$ has a fixed point. Its derivative is invertible by the Neumann series. Consequently it is a global smooth change of real coordinates with positive Jacobian j_M.

For an original coarse background V_0 and a nearby background V, the correction at a fine link is

$$M_e = (\mathrm{replica}\, V_0)_e(\mathrm{replica}\, V)_e^{-1}, \qquad u_e' = \Phi_{M_e}(u_e). \tag{PV3}$$

The order matters: for $\|u_e\| \le d$,

$$\exp u'_e\,(\text{replica}\,V)_e = \exp u_e\,(\text{replica}\,V_0)_e. \tag{PV4}$$

On inactive links the fine slice input remains unchanged. On active links the identity is needed only where their protecting cutoffs are nonzero. Those cutoffs imply $\|u'_e\| \le 8a\,t_{\text{field}}$. Choose the scale delay so that $8a\,t_{\text{field}} \le d/2$. Then (PV2) implies $\|u_e\| \le \|u'_e\| + \|p_{M_e}(u_e)\| \le d$, so (PV4) applies there. This proves equality of the old and new reconstructed configurations on every protected read. A cutoff product vanishing at the read makes the corresponding term zero. A nonzero product forces equality on its whole actual footprint. This argument proves the equality for old activities, atom words, row predictions, sector profiles and their complements individually, including the hard decisions inside them. It does not differentiate a hard decision.

Writing the physical fluctuation as $u = \mu_{V_0} + gz$, $g = \beta^{-1/2} > 0$, the corresponding Gaussian coordinate change is

$$z' = g^{-1}\{\Phi_M(\mu_{V_0} + gz) - \mu_V\}. \tag{PV5}$$

It obeys $\mu_V + gz' = \Phi_M(u)$ exactly, and becomes the identity at $V = V_0$. Restrict to the finite set I of bond/color coordinates required by the ground. Each link block of (PV5) has derivative $g^{-1}D\Phi_{M_e}g = D\Phi_{M_e}$; the scalar dilation factors cancel. The coordinate permutation between link/color arrays and Euclidean vectors contributes no absolute determinant. Therefore

$$\det Dz' = \prod_{e\in I_{\text{links}}} j_{M_e}(u_e) > 0. \tag{PV6}$$

This is a product over tested links, with one color-space determinant per link, rather than one determinant per color component.

Let C_0, C_V be the principal covariance matrices on I, with Gaussian densities ρ_0, ρ_V. Finite-dimensional change of variables gives for an integrable local function f

$$\int f(w)\rho_V(w)\,dw = \int f(z'(z))\,\underbrace{\left(\prod_e j_{M_e}(u_e)\right)\frac{\rho_V(z'(z))}{\rho_0(z)}}_{L_V(z)}\,\rho_0(z)\,dz. \tag{PV7}$$

The denominator is strictly positive at every finite z. Both integrals can equivalently be written against the original full Gaussian, since the functions depend only on the indicated marginal and the unused coordinates integrate to one. This proves the local likelihood formula, including its direction and normalization.

Along a smooth coarse field family based at V_0, its finitely many matrices M_e lie in the common neighborhood from (PV2). Differentiation of (PV1) on the compact cutoff ball gives uniform finite-order bounds on the displacement and determinant; exterior derivatives vanish or equal those of the constant one. Composition with the coarse family gives a displacement price of the same small order as that family's motion and a determinant price of order zero. Equations (PV5)–(PV7) then supply the Gaussian score derivatives. The covariance floor and the finite inverse/determinant formulas control the score coefficients. The integrable Gaussian envelopes established above justify differentiation under the integral. Thus a marked entry can remain evaluated on the fixed original fine input: the derivatives fall on its protecting cutoff, on the smooth ordinary vertices, and on L_V. The protected equality (PV4) identifies this formula with the original integral on a neighborhood of the base point. This is the mechanism used in the marked primitive comparisons and their product estimates.

Source–readout formulation. The fine configuration is a background-dependent readout of the fluctuation, $U = \mathrm{Read}_V(u) = \exp(u)\,\mathrm{replica}\,V$. Varying the background at fixed source u moves U; the method's horizontal operation instead varies V while holding the readout fixed, which requires a compensating source motion H_V with $\mathrm{Read}_V(H_V(u)) = \mathrm{Read}_{V_0}(u)$. Equations (PV3)–(PV4) are that horizontal lift, $H_V = \Phi_M$, on the protected domain $\|u_e\| \le d$:

1. it exists and is unique there, because the chart logarithm is a left inverse of the exponential;
2. it is extended off the protected domain by the cutoff to a global diffeomorphism (Lemma 15.1); off that domain it is *not* horizontal, and the residual $\exp(\Phi_M(u))\,\mathrm{replica}\,V \cdot (\exp(u)\,\mathrm{replica}\,V_0)^{-1}$ may differ from the group identity there, which is harmless because every evaluated term carries a protecting cutoff;
3. its Jacobian is (PV6) and its measure-level expression is the likelihood L_V of (PV7), which is the exact price of transporting the Gaussian reference with the background.

In the frame realization the replica background is the readout of a raw coarse pair, $\mathrm{replica}\,V = \pi_f(\mathrm{replica}_Z(s, B))$, and M_e is conjugated by the fine frame at the initial vertex of e: $M_e \mapsto r_x^{-1} M_e r_x$, $u_e \mapsto r_x^{-1} u_e r_x$. Since $\log(\exp(\mathrm{Ad}a)\,\mathrm{Ad}M) = \mathrm{Ad}\log(\exp a\, M)$ and χ_d depends on the invariant norm, Φ_M is equivariant, $\Phi_{\mathrm{Ad}M} \circ \mathrm{Ad} = \mathrm{Ad} \circ \Phi_M$, and j_M is invariant. The horizontal lift, its determinant and L_V are therefore fibre constants, recovered exactly on the identity-frame section.

Lemma 15.1. *Under* (PV2), Φ_M *is a global smooth diffeomorphism of the real color space with positive Jacobian* j_M.

Proof. The injectivity, contraction-surjectivity and Neumann-series argument immediately following (PV2) proves the claim, including positivity of the determinant. This is the same finite map; no second extension is chosen for the source–readout formulation. □

The exactness class is exact recovery on the protected domain and an explicitly retained residual off it; what is supplied as realization data is the cutoff χ_d and the protected radius d, and what is derived is the horizontal lift, its uniqueness there, its Jacobian and the likelihood.

15.6 Finite-window covariance and common activity comparisons

The form matrix evaluates selected physical rows on fixed coordinate impulses. Selection means the row base lies outside the store's fixed six-radius interior. Unselected rows vanish in both histories; selected rows compare through the required derivatives by physical-row comparison. The plate is their Gram matrix plus fixed dormant diagonal, so finite multiplication compares its entries. Each covariance column uses its fixed reference-frame window. Principal restrictions inherit the positive floor; smooth finite inversion with the common inverse bound compares derivatives. The scalar density proof above consequently rests on finite inverse matrices with a common lower eigenvalue bound. Outside-window entries remain zero. Embed columns and symmetrize to recover the actual reference covariance. Protected means use these columns and local bills, so their comparison supplies restart with coupling comparison. The inputs remain the stationary and local nonlinear row/bill estimates. This retains the column-window construction and does not replace it by the full plate inverse.

At fixed level and physical period use one normed presentation with fixed dictionary, supports and threshold; its completion is only for limits. Actual retained histories lie in the original space. The norm bounds a uniform absolute evaluated budget by $C\|a\|$: sum regular masses over anchors

and forced masses by chosen sites of nonempty inventories. At a fixed period one may take C to be the number of sites times the corresponding anchored budget constant. The canonical envelope

$$q(a,i) = \sup_U |F(a,i,U)|$$

is finite, nonnegative and summable with total at most $C\|a\|$. Linearity gives $q(a,i) \le q(a-b,i) + q(b,i)$. For Cauchy a_n, choose N with $C\|a_n - a_N\| \le \epsilon/2$ later, and a finite label set controlling tails of $a_0, \dots, a_N$ by $\epsilon/2$. Subadditivity gives every later tail at most ϵ. For Cauchy a_n, b_n with vanishing norm difference,

$$|F(a_n,i,U) - F(b_n,i,U)| \le q(a_n - b_n, i) \le C\|a_n - b_n\|$$

uniformly in configuration. Fixed masks keep regular evaluation linear; actual inventories and uniform tails are retained. Escaping subsequences of a single Cauchy activity sequence compare, since their norm difference tends to zero by the Cauchy criterion; so do real couplings. Isometric completion convergence suffices for the original presentation's Cauchy property. Hence all incoming regular, incoming forced and outgoing forced comparisons required by the two-history transfer argument follow from the common activity Cauchy sequence. The two envelope families also have the uniform tail property just proved.

Normalized scalar coordinates lie in fixed compact intervals at each level. Countable-product compactness gives a diagonal escaping subsequence with all scalar limits; fixed-scalar construction then gives activity and intermediate convergence. Repeat on every escaping subsequence. Comparison of banked limits at the same bottom coupling, at level zero, identifies scalar limits; the subsequence criterion gives full scalar convergence, then full activity convergence. That comparison is proved in Subsection 15.11; compactness is used only to obtain the candidate limits. Winning trajectories have the same prescribed bottom coupling $\bar{\beta}$ at level zero. Finite-horizon alignment identifies their stationary paths and actual retained activity presentations, supplying the original common-choice Cauchy sequences. The lift retains fixed windows, masks, presentation norms and subsequence comparisons. Readout preserves the column construction and uniform envelope bounds, and its isometric recovered presentation transports these Cauchy and limit statements with their stated uniqueness dependency.

Explicitly, let E_j include the fixed window for covariance column j into the physical coordinate space, and let e_j^{win} be that column's coordinate vector in the window. For physical plate Q, the column reader and symmetrization are

$$v_j = E_j (E_j^* Q E_j)^{-1} e_j^{\mathrm{win}}, \qquad V = (v_j)_j, \qquad C = \tfrac{1}{2}(V + V^*).$$

All adjoints here use the recorded physical Euclidean metric. Realization recovers Q, E_j and the positive principal restriction before inversion; section recovery therefore gives these same columns and covariance, with zero exterior entries. Smooth finite inversion transports the stated derivative comparisons on the common positive-floor domain. For activity evaluation define $F^Z(a,i,z) = F(a,i,\pi(z))$. Surjectivity provided by the section gives the exact envelope equality

$$\sup_z |F^Z(a,i,z)| = \sup_U |F(a,i,U)| = q(a,i).$$

Thus the recovered presentation has the same canonical envelopes, uniform finite-label tails and norm-controlled differences. This preserves the Cauchy comparisons and their completion limits together with the retained banked-limit uniqueness dependency.

15.7 Finite-depth trajectory approximation

At finite horizon prescribe an admissible normalized scalar script. Activity updates preserve the norm ball 1/8 and satisfy

$$\|\Phi_i(\lambda, x) - \Phi_i(\lambda', x')\| \le c_0 \max\{|\lambda - \lambda'|, \|x - x'\|\}, \qquad c_0 = 1/16.$$

The scalar correction has the corresponding coefficient $4s_i$. At fixed depth b, set activity to zero and apply updates backwards, setting unused later levels to zero. Finite decreasing induction and ball invariance keep this comparison admissible. Against any trajectory with the same scalar script, its discrepancy at $m \le b$ is at most c_0^{b-m}, from top distance at most one (indeed 1/4). For histories with the same fixed-level scalar limits, truncated activities at depth b converge together by finite descending one-step comparison. Choose $c_0^{b-m} < \epsilon/3$, then take histories late enough that their truncations differ by less than $\epsilon/3$. Both truncation errors give level-m difference below ϵ. Applying to escaping subsequences gives the common limit in the complete activity space. Scalar corrections then converge by one-step comparison with the convergent next activity. Finite initial segments discarded to ensure all required horizons are present do not change these limits. Inputs are the same paid contraction, ball invariance and local Gaussian/cluster/reset comparison. At finite horizon correction is adjacent physical-coupling difference minus the committed path coefficient. Taking these scalar limits gives the banked flow identity without infinite telescoping. This supplies the fixed-scalar convergence used in the diagonal extraction and banked uniqueness argument.

15.8 Labelwise comparison yields reset budgets

At fixed period there are finitely many supports. Linearity of budget mass gives $P_* > 0$ with $p.\,\mathrm{mass}(D, t) \le P_* t$ for $t \ge 0$. This constant may depend on period. For uniform-tail majorants $f_{n,i}$ and arbitrarily small fixed-label differences, set $t = \epsilon/P_*$. Choose finite E with tails at most $t/2$ and eventually bound each label in E by $t/[2(|E|+1)]$. These bounds on E and original tails elsewhere sum to at most t, giving weighted budget ϵ. This applies to regular paid and forced all-field differences and recovers the activity norm. Each source has a finite paid total; diagonal projection yields $t_0^2|\Delta_n| \le C$, $t_0 > 0$. Give the source difference paid budget below $t_0^2\epsilon/2$. Projection linearity then yields $|\Delta_n - \Delta_n'| \le \epsilon/2$, so increments are bounded and compare. Canonical stationary path slots and atoms have fixed supports despite coefficient changes; low homes, high footprints and reset listing coincide. Final dictionary consumption transfers comparisons on this listing; vacuum and coupling scalars do not affect activity comparison. Remaining coefficient families are residuals, freshness, loans and attached logs. The lift retains scalar scripts, truncation depth and fixed labelled supports. Recovery preserves finite induction and the explicit finite-head, uniform-tail budget construction, including its fixed-period scope.

Let $x_m^{[b]}$ denote the zero-terminal approximation at depth b. Realize the fixed scalar script and every update in that finite backward recursion. Since one-step readout recovers Φ_i, finite induction gives

$$R_m(\widehat{x}_m^{[b]}) = x_m^{[b]}, \qquad \|x_m - x_m^{[b]}\| \le c_0^{b-m}.$$

Here R_m reads the activity presentation, whose recovered norm is the original one. Thus the same truncation depth works for the lifted three-error comparison, and section recovery preserves its completion limit. For the labelwise budget, the source object retains the chosen finite head E and the nonnegative tail array. Their total bounds remain $t/2$ and $t/2$, respectively, under coefficient readout, so the same fixed-period constant P_* gives budget ϵ. The diagonal projection reader then recovers the bounded scalar increment and its vanishing difference. These identities transport the

exact finite approximation and reset-budget construction with their original order of choosing depth, tolerance and sufficiently late histories.

15.9 Comparison of reset coefficient families

Regular reset terms are source rebates, low freshness and high freshness. Rebate and fault differences equal the rebate and fault of the difference; the fixed reset-loss constant preserves paid comparisons and uniform tails. For a fixed low label the slots and support are fixed and only mark m varies. Choose once a reference mark $m_* \neq 0$ (unit path coefficients suffice for this algebraic label test). Freshness is the fixed reference function times m/m_*. Coefficient comparison gives fixed-label convergence; weighted low-mark budgets with spare exponential width give uniform tails, since bounded-length labels form a finite set. High freshness compares its bounded scalar parameters and actual cubic/quartic coefficients; spare-width atom budgets give tails. Finite seat replication preserves them. The projected increment is already a bounded compared scalar. For loans, common homes and kept tests give exactly

$$\operatorname{loan}_R(\alpha, \Delta) = \operatorname{loan}_{R_*}((m/m_*)\alpha, (m/m_*)\Delta).$$

The direct mark is fixed. This follows from the literal loan definition: its live pair term is proportional to the mark, its proxy is independent of the numerical path coefficients, and its home and kept tests are common. Linearity makes reference-loan differences loans at scalar differences. The all-field shape bound tends to zero uniformly, including failed-health boundaries. A common majorant is

$$e^{l/\rho}|m| \left(2t(B_\alpha + B_\Delta) + \tfrac{1}{2} B_\Delta \operatorname{pairBill}\, t^2\right).$$

Positive residual width after spending $e^{l/\rho}$ supplies uniform tails. Faults plus loans therefore compare attached logs; rebates plus freshness compare regular resets on the original canonical enumeration. Final consumption uses $e^{l_i} - 1$. For $|l_i| \leq v_i \leq B$, $|e^{l_i} - 1| \leq e^B v_i$, preserving summable tails. Exponential continuity on bounded compared values and the varying-configuration selection argument give uniform fixed-label differences. Forced outputs are finite products of old hard and increment labels with the same canonical eligibility test on both histories. Bounded-factor comparison survives this indicator even when it varies with history. Finite-subset product budgets preserve tails; canonical injective drawer placement, zero elsewhere, preserves totals and tails by finite preimage sums. The regular embedding does likewise. Thus output arrays have the quantitative weighted difference budgets, without differentiating hard indicators. The width hypothesis leaves a positive exponential margin after $e^{l/\rho}$ is spent. The lift retains reference marks, common predicates and canonical embeddings. Readout preserves their exact factorizations and the finite-head/spare-width tail proof, recovering the same norm convergence at health boundaries.

The final coefficient comparison has a useful quantitative recovery form. For two logarithm values a, b with $|a|, |b| \leq B$, integration along their line segment gives

$$e^a - e^b = (a - b) \int_0^1 e^{b + s(a-b)}\, ds, \qquad |e^a - e^b| \leq e^B |a - b|.$$

This estimate is uniform in the evaluated configuration. For a fixed finite product, telescoping gives

$$\left| \prod_{r=1}^{n} a_r - \prod_{r=1}^{n} b_r \right| \leq \sum_{k=1}^{n} |a_k - b_k| \prod_{r<k} |a_r| \prod_{r>k} |b_r|.$$

Multiplication by the common eligibility indicator preserves this bound. Compose every factor and indicator with readout: the same estimates and reference-mark loan identities hold, and the

section recovers each coefficient. The spare-width majorants and canonical injective placements are retained with their original labels, so finite-head comparison plus uniform tails gives the same weighted output difference budgets. This recovers the reset convergence at discontinuous health boundaries through function values and common predicates.

15.10 Closing one-step activity-norm convergence

For two incoming Cauchy activity presentations with vanishing norm difference and common coupling limit, canonical envelopes give regular and forced comparisons. Their common completion limit makes their original presentation sequences Cauchy and their difference tend to zero by the isometric embedding. Common stationary paths and kernels supply microscopic comparisons, placing actual source outputs on one listing and resetting with the same committed coefficients. The exact update is

$$\beta_{\mathrm{out}} = \beta_{\mathrm{in}} + \alpha_{\mathrm{path}} + \Delta_{\mathrm{source}}.$$

Projection comparison and the common stationary coefficient limit give vanishing output-coupling difference. To recover the actual activity norm, let $b_{\mathrm{reg}}, b_{\mathrm{force}} > 0$ be its budget normalizations. For target ϵ, choose simultaneous anchored regular and forced difference budgets below $(\epsilon/2)b_{\mathrm{reg}}$ and $(\epsilon/2)b_{\mathrm{force}}$. Reset tail lemmas supply these eventually; common supports and formats make them budgets for the actual difference. The norm characterization yields at most $\epsilon/2 < \epsilon$. Thus quantitative tails, not solely labelwise limits, prove output-norm convergence. Isometric completion convergence supplies the incoming Cauchy hypotheses; continuity of the physical-coupling map supplies the coupling limit. The chosen physical step identifies its actual output with the update presentation, without selecting another representative. Its scalar correction is output minus input coupling minus committed path coefficient, proving the fixed-input convergence used in finite-depth approximation. The lift retains the common presentation, norm normalizations and exact update. Readout preserves both anchored budgets and their isometric norm characterization, transporting this one-step limit with the same reset tail dependencies.

15.11 Uniqueness of banked history limits

For two banked limits with common bottom coupling at level zero, let scalar, activity and correction coordinates be f_i, X_i, g_i and their primed versions. Put $d_i = \max\{|f_i - f_i'|, \|X_i - X_i'\|\}$ and $\gamma_0 = 1/128$. Finite-trajectory comparison passed to limits gives

$$\|X_i - X_i'\| \le 8\gamma_0 d_{i+1}, \qquad |g_i - g_i'| \le 4 s_i d_{i+1}.$$

Scalar flow and base equality imply

$$\Omega_i - \Omega_i' = -\sum_{j<i}(g_j - g_j'), \qquad b_i|f_i - f_i'| \le |\Omega_i - \Omega_i'|, \qquad \sum_{j<i} s_j \le \gamma_0 b_i,$$

with $b_i > 0$ and only finite sums at each i. Individual activity bounds and scalar-flow/correction bounds give $d_i \le 1/4$. Hence $D = \sup_i d_i$ is finite and nonnegative. The activity difference is at most $8\gamma_0 D$; scalar flow gives $b_i|f_i - f_i'| \le 4D\sum_{j<i} s_j \le 4\gamma_0 D b_i$. Divide by b_i and take the supremum to obtain $D \le 8\gamma_0 D = D/16$, so $D = 0$. All scalar and activity limits agree. The inequalities come from paid finite trajectories; the nearby finite trajectories are constructed by (FCA7) and (FCB5), with positive comparison weights and cumulative bounds (FCB1)–(FCB3), and the fixed-scalar comparison identifies their activity limits. This is the uniqueness input upgrading diagonal subsequences. The lift retains base coupling, positive coordinate weights and finite-flow identities.

Recovery preserves those identities and the uniform bounds, transporting the same supremum contraction with its finite-trajectory dependencies.

On the recovered activity presentation, use the transported norm on its section. The activity readout and section are then inverse isometries there. They extend to the respective completions by taking Cauchy representatives: if a_n represents a completion element, read the sequence termwise, and its equal norm differences show independence of the representative. The extended maps still compose to the identity. Consequently the quantitative regular and forced difference budgets above recover the actual output completion distance as well as each evaluated coefficient. The physical coupling update is retained as a joint scalar reader, so its three-term difference passes to the same limit. For banked limits these isometries recover each d_i exactly. The bounded family therefore has the same supremum D under readout, and the recovered finite-flow inequalities give

$$0 \le D \le D/16, \qquad D = 0.$$

Section recovery returns equality of the original scalar and activity limits. Thus the uniqueness step and the one-step norm convergence use the same completed presentation and the same quantitative budget normalizations, retaining their finite-trajectory comparison dependencies.

15.12 Extracting stationary path coefficients and their higher jets

There is a separate extraction behind the stationary paths just used. It concerns numerical path coefficients, not completed activity presentations. Combine all repeated occurrences of the same transported-curvature slot in each row by adding their coefficients. Do the same for each color component of each ordered pattern of two, three or four slots in the loop, cubic and quartic atoms. Denote this countable coefficient array by $a_{N,m}(\kappa)$, where N is the upper horizon and m the fixed level. The length $\ell(\kappa)$ is the slot length or the sum of the pattern's slot lengths. There are finitely many keys of bounded length: a bounded-length walk has a bounded number of steps from a finite alphabet, and there are only finitely many plaquette and color types.

The path bounds give, for each row or color component separately,

$$\sum_{\kappa} |a_{N,m}(\kappa)| e^{\sigma_m \ell(\kappa)} \le C_m \quad \text{eventually in } N. \tag{SH1}$$

Here C_m is the row allowance, the loop allowance for two-slot atoms, or the common higher-atom allowance for three- and four-slot atoms, as appropriate. An individual coefficient is bounded by C_m, because the exponential weight is at least one. Enlarge these bounds to include the finitely many exceptional N's. The product of the resulting compact intervals, over the countable set of pairs (m, κ), is compact and metrizable. Every escaping horizon sequence consequently has one subsequence on which all these coefficients converge simultaneously.

Write its limit $a_m(\kappa)$. For each finite set of keys, pass the inequality (SH1) to the limit. Taking the supremum over finite sets gives (SH1) for a_m at the original width σ_m. More is true at every strictly smaller width $\tau < \sigma_m$:

$$\sum_{\kappa} |a_{N,m}(\kappa) - a_m(\kappa)| e^{\tau \ell(\kappa)} \longrightarrow 0. \tag{SH2}$$

Indeed the part with length at most M is a finite sum tending to zero, whereas the remaining part is at most

$$2C_m e^{-(\sigma_m - \tau)M}. \tag{SH3}$$

Choose M first and then the horizon. This order proves (SH2) and permits pairing the coefficients with every function bounded by $De^{\tau \ell}$. It also preserves the zero and first loop moments, since

each displacement coordinate is bounded by path length and a polynomial in length is bounded by an exponential of any positive width. The row symbol at zero is such a pairing with a bounded indicator of the plaquette type. Its limit preserves the finite metric identity. Color invariance passes by the finite sum over color coordinates. Reconstructing the rows and atoms from this common coefficient array therefore retains all the original path hypotheses and allowances.

The convergence includes derivatives of the unmasked physical functions at every order. To justify this assertion, complexify matrix entries around a physical configuration and use inverse links as independent conjugate-entry coordinates. The extension of a slot of length ℓ is a holomorphic polynomial bounded on a sufficiently small common ball by $Ce^{\tau\ell}$. A product of q slots has the same bound with C^q and the sum of their lengths. Equations (SH2)–(SH3) give uniform convergence of the row and atom series on that ball. Cauchy's integral formula on a smaller ball gives convergence of each derivative, with its fixed radius loss and factorial. Restricting to real smooth parameter curves then gives convergence of all real Fréchet jets. This argument applies to the actual group words; no finite cutoff of the limiting path is substituted for them.

For completeness, the stationary-graph derivative argument used to pass through the nonlinear minimization is as follows. Let $P_N(x, y)$ be the finite profile, let P be its limit, and let $y = u_N(x)$, $y = u(x)$ be their analytic stationary branches through zero. Put $F_N = \partial_y P_N$, $F = \partial_y P$, and

$$H_N = D_y F_N(0, 0), \qquad H = D_y F(0, 0). \tag{SH4}$$

The preceding all-order convergence implies convergence of every jet of F_N, in particular $H_N \to H$. The stationary gap makes these vertical Hessians invertible. Inversion is continuous near the invertible H, so $H_N^{-1} \to H^{-1}$. This is the inverse needed here, acting from vertical covectors to vertical vectors.

Differentiate $F_N(x, u_N(x)) = 0$ repeatedly. With $g_N(x) = (x, u_N(x))$, the exact diagonal chain rule is a finite sum over partitions π of the derivative slots:

$$0 = \sum_{\pi} D^{|\pi|} F_N(0)[\, D^{|B|} g_N(0)[v^{|B|}] : B \in \pi\,]. \tag{SH5}$$

The single-block term contains $H_N D^k u_N(0)[v^k]$ and the known horizontal derivative $D^k \,\mathrm{id}(0)[v^k]$. Every other block has size less than k. Thus (SH5), solved by H_N^{-1}, proves convergence of the k-th derivative by induction from the lower derivatives. In an ordered-partition convention identical single-block contributions are grouped with their positive multiplicity; dividing by that multiplicity gives the same recursion. Polarization of symmetric real multilinear forms supplies mixed derivatives. The case $k = 0$ is $u_N(0) = u(0) = 0$. Composing the convergent graph jets with the profile jets proves convergence of all stationary value jets.

There is also a precise cancellation at the first unmatched profile order. On a stationary graph $g = (\mathrm{id}, u)$,

$$DP(0)[(0, w)] = 0, \qquad D^2 P(0)[Dg(0)v, (0, w)] = 0. \tag{SH6}$$

The first equation is stationarity at zero; differentiate stationarity once to obtain the second, and use Hessian symmetry for the reversed order. Suppose P, Q agree through order $k + 2$. Their stationary graph jets agree through order $k + 1$, by the recursion above. In the chain rule at order $k + 3$, any term involving a graph derivative of order at least $k + 2$ has either one block or two blocks with the other block of size one. These are exactly the terms killed by (SH6). All remaining graph derivatives match. Moreover the difference $P - Q$ has zero jets through order $k + 2$, so its composition at order $k + 3$ has only the all-singleton partition. Consequently

$$D^{k+3}(P \circ g - Q \circ h)(0)[v^{k+3}] = D^{k+3}(P - Q)(0)[(Dg(0)v)^{k+3}]. \tag{SH7}$$

This is the stationary value identity used for the cubic and quartic response comparisons ($k = 0, 1$); higher unknown graph derivatives have not been discarded without compensation.

In the lattice profile, the linear map in (SH7) is the derivative of the actual fine logarithmic inclusion of the slice graph. Its derivative is the fine stretching map applied to $(v, Du(0)v)$. A different tree-gauge representative changes it by an exact lattice gradient. Gauge invariance, together with vanishing of all lower jets of the profile difference, makes its first nonzero derivative insensitive to such a gradient: differentiate the gauge identity and observe that every correction involving a nonlinear gauge derivative multiplies a lower, zero profile jet. Hence the stationary matrix response and its tree-extracted version give the same first unmatched value. Passing an eventual one-sided bound through the convergent branch derivatives proves the corresponding bound for the limiting coarse profile. Interchanging the two profiles gives the reverse bound. This is the response comparison used together with the decay and base matching estimates in the stationary-history construction.

The one-loop profile also has a literal decomposition before passage to the limit:

$$\left[\text{loop action} + \tfrac{1}{2}\log\det(\partial_y^2 P_N) - \log \prod_{e \text{ off tree}} J(y_e)\right] \circ g_N. \tag{SH8}$$

The determinant is nonzero at zero by the same positive vertical Hessian. Determinant is polynomial in its entries, logarithm is analytic near this positive value, and the Haar logarithms are analytic in the common chart. The jet convergence just proved therefore applies to (SH8), including its second jet. A finite inequality comparing this second jet to the adjacent draft (the coupling coefficient times the quadratic action plus its loop action) passes to the limit by continuity of absolute value. Field-independent scalar offsets are not identified by this argument.

Finally the extra physical covariance and response rows have their own quantitative, full-sequence convergence. In their weighted row and column norms the finite suppliers give, for $i < N \le M$,

$$\|U_{N,i} - U_{M,i}\|_{\mathrm{row}},\ \|U_{N,i} - U_{M,i}\|_{\mathrm{col}} \le B\ \mathrm{clock}_k(i+1)^r \left(\frac{a_N}{a_{i+1}}\right)^c, \qquad c > 0. \tag{SH9}$$

For the covariance row the distance weight has exponent σ/L_i at unit output scale; for the response row it has exponent σ at output scale L_i. Since $a_N \to 0$, (SH9) gives an entrywise Cauchy limit. Fix a finite set in a weighted row or column, pass its absolute sum to the limit $M \to \infty$, and then take the supremum over finite sets. This retains (SH9) with $U_{\infty,i}$ in place of $U_{M,i}$, in both norms and at precisely these scales. The limits thus supply the physical rows on the same common witnesses, with quantitative tails; they are not an additional independently chosen stationary subsequence.

There is a further full-sequence assertion for the actual metric rows, which we separate from book extraction. At every fixed torus, the unamended diagonal matrix has the limit given by flat quadratic convergence. Its inverse matrices have entry bounds from one common positive strip allowance. A finite-dimensional compactness argument shows that the limiting diagonal matrix is invertible: extract any convergent subsequence of the bounded inverses, pass the identity $D_N D_N^{-1} = 1$ to the limit, and obtain a right inverse of D. A square matrix with a right inverse is invertible. Continuity of inversion then proves convergence of the entire sequence of inverses, independently of the extracted subsequence.

For the actual amended rows the prescribed construction, not merely agreement on curls, gives at that torus

$$M_N^{-1} = D_N^{-1} + u(d_2 D_N^{-1})^{\mathsf{T}}(d_2 D_N^{-1}), \tag{SH10}$$

where u is the fixed amendment coefficient at this level. All operations are continuous finite matrix operations. Thus M_N^{-1} converges. Its inverse M_N converges by the same bounded-inverse argument, using the metric's original strip allowance. This explains why the amendment must be retained from the same construction; agreement of potential Hessians alone would not imply (SH10).

To recover cover coefficients from these all-period limits, take two possible coefficient limits R, R' of a uniformly exponentially bounded row. Their folded kernels agree on every scheduled torus. Fix a cover displacement z and choose a torus whose centered box contains z. The folded coefficient at its residue is $R(z)$ plus the other congruent coefficients. Their sum is bounded by a constant times the exponential tail outside that box, uniformly in the history index. As the scheduled periods grow, this tail tends to zero, hence $R(z) = R'(z)$. Compact coefficient extraction therefore has only one possible limit; the full row converges coefficientwise. Splitting into a finite displacement set and its exponentially bounded complement proves convergence in every strictly smaller strip norm, exactly as in (SH2)–(SH3). The argument applies to the diagonal, its dual and the amended metric. The physical covariance and response rows have in addition the rate (SH9).

The extraction (SH1)–(SH3) for nonlinear path books alone remains subsequential. The auxiliary response comparison is used through exactly the orders it supplies: base matching through the second jet, then cubic and quartic comparisons (SH7), and the loop second-jet comparison (SH8). In particular any two extractions share the second profile jet, since it is the same quadratic matrix limit just proved. The recursive first-unmatched-jet argument compares the next two jets with their stated bounds. It does not assert equality of arbitrary all-order path books or field-independent scalar offsets. Whenever the subsequent primitive-continuity argument uses common path coefficients, its two sequences are taken from the already selected common book refinement. Full quadratic-row convergence and retained nonlinear-book convergence thus have different, explicit scopes.

Source–readout formulation. Two different kinds of limit object are produced in this subsection, and the lift must keep them apart because they have different fibres. The coefficient array of a nonlinear path book is extracted from a compact set; its limit is a selected point, and different extractions may select different points. The quadratic rows have full-sequence limits and are determined. The following two statements say exactly which readers are insensitive to the selection.

Proposition 15.2 (First unmatched stationary value)**.** *On a stationary graph $g = (\mathrm{id}, u)$,*

$$DP(0)[(0,w)] = 0, \qquad D^2P(0)[Dg(0)v, (0,w)] = 0.$$

If profiles P, Q with stationary graphs g, h agree through order $k+2$, then

$$D^{k+3}(P\circ g - Q\circ h)(0)[v^{k+3}] = D^{k+3}(P-Q)(0)[(Dg(0)v)^{k+3}].$$

Proof. The first identity in (SH6) is stationarity at zero, the second its derivative along the graph with Hessian symmetry. If P, Q agree through order $k+2$ then $\partial_y P, \partial_y Q$ agree through order $k+1$ and the recursion (SH5) gives equal graph jets through order $k+1$. In the chain rule at order $k+3$ a term containing a graph derivative of order at least $k+2$ has one block, or two blocks the other of size one; these are exactly the terms annihilated by (SH6). All remaining graph derivatives match, and $P-Q$, having zero jets through order $k+2$, contributes at order $k+3$ only through the all-singleton partition. □

Proposition 15.3 (Two scopes of convergence)**.**

1. *Quadratic rows (diagonal, dual, amended metric, covariance, response): full-sequence convergence on the common witnesses, with the rate* (SH9) *for the covariance and response rows. The folding readout from cover coefficients to the family of all scheduled tori is injective on exponentially bounded rows.*

2. *Nonlinear path books* (SH1)–(SH3)*:* subsequential *convergence only. Two extractions share the second profile jet (the quadratic matrix limit of (1)) and are compared at the next two jets by* (SH7) *and at the loop second jet by* (SH8)*; equality of all-order books or of field-independent scalar offsets is not asserted.*

Whenever a later argument uses common path coefficients, its two sequences are taken from the already selected common book refinement.

Proof. Both parts collect what was proved above; injectivity of folding is the tail argument for $R = R'$. □

The source record is the pair (book refinement, quadratic row limits). The first component is a *selected* point of a compact set (a non-canonical choice, recorded with its refinement map); the second is *determined*. The readers used later are exactly the jets listed in Proposition 15.3(2) and the rows of (1); a reader of higher book jets would not be constant on the fibre of extractions and is never applied. This is the method's distinction between a recovered quantity and a retained witness, and it is the reason the common family is assembled from one composed refinement.

15.13 Retained coefficients recover physical row comparisons

At a fixed step the retained-history comparison data consist of the limits of the upper and lower path books, hence of their quadratic, cubic and quartic atom coefficients, along the single common horizon refinement selected in (SH1)–(SH3). They also include the three cover-kernel limits and the coupling and activity limits. Here two escaping subsequences mean two subsequences of that retained horizon refinement; both therefore have the same book limits. This is not an assertion that arbitrary subsequential path books extracted before the refinement are equal (Proposition 15.3). The coupling comparison is passed to the same tail ultrafilter, while the retained activity comparisons supply the regular-occurrence and forced-coefficient evaluations at arbitrary varying fine configurations. This constructs the comparison data for the local formulas. The full fixed-volume limit of the Wilson smeared integrals is a separate conclusion, proved in Subsection 18.4; it is not obtained by declaring path-book compactness to be full convergence. Cutoffs make physical linear expressions finite: bounded-length canonical paths have finitely many plane/displacement labels. Shortening is therefore a finite compared-coefficient sum of transported curvature slots; its direct branch is fixed curvature. Coarse kernels at cutoff k retain bounded cover displacements. Fine-to-coarse route length is $|Lz - \mathrm{lift}(r)|_1$, with $|\mathrm{lift}(r)|_1 \le 4L$, so retained displacements lie in radius $k + 4L$. Common endpoint and cutoff predicates leave finite sums along the same routes. Smooth word multiplication, inversion and adjoints compare all three kernels through derivatives. Coarse vision contracts these kernels with compared plotted fields. Linear direction rows are fixed curl, finite cutoff curl sums or fixed coarse noise maps. Signed displacement and carrier transport preserve comparison; reversed links retain their negative adjoint and shifted base. Nonlinear rows retain their actual clipped fine parts, coarse-response half-squares and cross terms, wire deficit/noise square, negative coarse deficit with two departure terms, and tight-response half-square. Fine kept/clipped tests are bounded common factors on the same fine configuration; no limit through their indicators is required. Regular sources use occurrence comparisons; atom and tail sources retain compared coefficients and actual masked words. These give the row and inherited-source comparisons feeding

packets and means. The underlying retained stationary and activity convergence is the history input, whose extraction must use the common choices described in the construction. The lift retains finite route labels, branch predicates and signed base shifts. Recovery preserves every finite smooth expression and common bounded indicator, transporting derivative comparison with the original retained-history dependencies.

For a fixed cutoff row, let J_k be its retained finite path or route labels. Its coefficient-dependent part has the reader

$$T_n(U) = \sum_{i \in J_k} c_{n,i} S_i(U).$$

Realization retains J_k, the route words and their signed bases, giving $T_n^Z = T_n \circ \pi$ and section recovery $T_n^Z \sigma = T_n$. On the recorded local smooth domain, any tested derivative D^r obeys

$$\|D^r T_n - D^r T_n'\| \le \sum_{i \in J_k} |c_{n,i} - c'_{n,i}| \, \|D^r S_i\|.$$

Finite labels and the retained slot bounds therefore recover the same derivative comparison. Common endpoint, cutoff and health predicates multiply the compared coefficient values with the same bounds; derivatives of the smooth expressions are recovered on their original domains before these hard predicates are evaluated. Signed reversal and carrier conjugation keep their recorded word order. Thus the finite physical-row comparison is transported through explicit readers with its actual cutoff and inherited-history inputs.

15.14 Mean, local sources and ordered chain comparisons

Selected strain is free residual vision plus the physical row applied to saved log and hot fluctuation. The draft fixes selection and hot sets; saved logs are the same all-argument local functions with uniform derivative bounds, while dormant lens contributions vanish. Fine residuals compare base stroke minus coarse vision. Noise base strokes are common and have derivative bounds on the clean hull supplied by viability and selection; subtracting compared vision gives their comparison on selected rows. The bill vector is transpose form applied to zero-fluctuation strains with original scaling. The protected mean is minus the transpose window-column matrix applied to that bill. Finite matrix operations compare both and the stationary defect $-\sqrt{\beta}(\text{plate } \mu + \text{bill})$. The coupling-window lower endpoint makes square roots and reciprocals valid. Chain factors are ordered negative plate–covariance defects followed by the terminal plate factor. Finite products preserve order; quadratic fluctuation and linear defect contractions compare, as do smooth cone cutoffs in squared norms. Blank chains use the viable empty store and identity field with the common plate floor. Condensation is half the blank-minus-actual local determinant difference. The cutoff Haar logarithm is smooth at all arguments: inside support its argument has strict margin within the positive chart, outside it is locally zero, and the margin includes cutoff boundaries. This justifies its use in products without extending an uncut logarithm beyond its domain. The actual packet log is

$$\begin{aligned} -\beta(\text{chalky} + \text{crispy}) + \sum_{\text{visited}} \text{condensation} + \sum_{\text{plaza}} \text{HaarSource} \\ - |\text{pens}| \log(\text{chartMass}(\beta)). \end{aligned}$$

Chalky is posted root energy minus selected quadratic costs; crispy is that cost at the protected mean. Costs sum squared strains over actual parent rows. Chart mass is positive and continuous

at the positive coupling limit. All indexing sets are draft-fixed and finite. Thus strain, root-energy and determinant comparisons recover the packet log with its original normalizer. The lift retains transpose placements, terminal factors, chart margins and all finite index sets. Readout preserves these expressions and derivative comparisons on the selected clean hull, including the cutoff boundaries.

The protected-mean reader retains the window-column matrix V and physical bill b, so it is exactly

$$\mu = -V^{\mathsf{T}} b, \qquad \delta_{\mathrm{stat}} = -\sqrt{\beta}\,(Q\mu + b),$$

with Q the physical plate and all transposes taken in its original coordinate metrics. Read these arrays componentwise before evaluating the formulas. Composition with readout preserves the displayed operations, and the section recovers both vectors and the sign of the defect. Likewise a finite chain reader records its ordered factor list $(T_1, \dots, T_n, Q_{\mathrm{term}})$ and evaluates $T_1 \cdots T_n Q_{\mathrm{term}}$ in that order. Realization retains this list, so its contractions and terminal plate are recovered without commuting factors. The actual and blank lists remain separate records. Their local determinant readers consequently recover the original half-difference. Finally all finite sums in the packet logarithm, including the positive chart-mass reader and the pen count, commute with readout and section. This recovers the packet normalizer alongside its energy, Haar and chain sources on the original chart domains.

15.15 Four vertex comparisons and marked primitives

Fix the common draft. After fixing the draft, the ordinary integrand has the same four vertex types as its absolute activity bound. Packet connectedness and containment/empty-flag tests are fixed; failed branches vanish in both histories. Otherwise the packet is its compared energy exponential times common bounded contracts, tickets and profiles and compared smooth cutoff on the enlarged plaza. Fine-input factors may vary with history but agree across the comparison; no ticket indicator limit is required. Their output-field dependence uses the already chosen smooth wire profile. Forced vertices use actual forced-coefficient comparison on the same fine input and the cutoff on their true footprint. Regular vertices compare logarithms, then exponentiate and subtract one. Inherited, Short, condensation, Haar and chain branches retain their signs and beta factors; posted-region, voyage and hot-exterior tests agree. The nonzero branches use the protected mean, covariance/window, local logarithm and row comparisons already needed for these expressions, with the actual signs and factors of β retained. Warnings retain fixed mandatory/hot predicates, compared cutoffs and bounded common fine profiles or contract charges minus reference constants; exterior cones are smooth in compared fluctuations. Mandatory branches agree exactly. Finite products and density multiplication compare transformed integrands. Derivatives use the local smooth extended formulas, fixed fine input and compared steering/restart coordinates, identified with the actual differentiated integrand near the healthy base. These statements compare the true vertices rather than replacing hard indicators by smooth approximations.

A marked premium tests whether its fixed badge occurs. Absent badges give zero; present badges replace exactly one vertex by the paid ingredient and retain the erased-badge product. At fixed physical fluctuation the fine configuration is common. Mark-entry comparison on that possibly varying input, common enforcement cutoff, compared remaining product and Gaussian density give amplitude comparison on the eventually viable set. For derivatives the marked entry stays at the fixed reconstructed input; output derivatives act on smooth steered cutoffs and remaining vertices through the local reweighting identity. Steering, ordinary factors and Gaussian ratio/Jacobian comparisons therefore give the actual derivative integrand. The common Gaussian-polynomial majorant permits integration; finite forest and boundary averages compare the marked grove. Its

all-field bound also compares uncut amplitudes at arbitrary varying output fields, using the fixed-draft argument at each slice point. These supply regular one-mark stream and forced-reset inputs. This explains why convergence of mark entries under varying-configuration evaluation is sufficient here. The lift retains badge identities, all hard predicates and the exact local reconstruction. Readout preserves the branchwise products and derivative identities with the same viability scope and common integrable majorants.

For a fixed marked badge b in molecule p, the replacement reader has the explicit product form

$$I_{b,p} = \mathbf{1}_{b\in p}\, M_b(U_{\mathrm{fine}})\, \Gamma_b \prod_{v\in p\setminus\{b\}} V_v.$$

Here M_b is the original paid ingredient, Γ_b its enforcement cutoff and V_v the actual ordinary vertex readers. The density factor is then that of the same fixed local data. Realization retains the badge, erased list, common reconstructed fine configuration and all branch predicates, so composition with readout preserves this product and the section recovers it. The smooth local derivative identity holds with U_{fine} fixed: derivatives act on the recorded steering, cutoff, remaining vertices and reweighting factors. Thus recovery asks for the stated varying-input comparison of M_b, with no additional derivative assumption on that evaluated mark entry. The common integral majorant and finite averages transport the same amplitude and derivative conclusions, including the absent-badge zero branch.

15.16 Uniform derivative comparison of a cluster stream

A one-mark stream with r occurrences is its Ursell coefficient over $r!$ times $\sum_{\nu=1}^{r} \ell_\nu \prod_{\mu\neq\nu} u_\mu$. Its support is the occurrence union when its fixed mark key speaks to the label inventory, and the zero alternative otherwise. Supports agree across histories; silent streams vanish. Bounds a_μ, b_μ through order j give ordered product bill $(1+r)^j b_\nu \prod_{\mu\neq\nu} a_\mu$. Inflate each bound by 2^j, using $1+r \le 2^r$. Take ordinary bounds as the smaller primitive envelope and purchased budget, preserving tails. A positive reserve d, selected by the one-mark budget argument, makes the inflated array $f_\mu = 2^j a_\mu + d\, 2^j b_\mu$ fit the connected-cluster reserve. Nonnegative expansion shows

$$d \sum_{\nu} 2^j b_\nu \prod_{\mu\neq\nu} 2^j a_\mu \le \prod_{\mu} f_\mu.$$

Thus d^{-1} times its ordinary absolute cluster weight gives a uniformly summable stream bill with small tails, retaining Ursell and factorial normalization. On healthy unions every occurrence is healthy; primitive comparisons and finite differentiation compare fixed streams through order j. Tail-ultrafilter comparison implies ordinary test-family convergence. If uniform order-k bounds on allowed maps and healthy bases failed, select violating maps at bad indices; their sequence contradicts this comparison. Intersect the finitely many late cutoffs for $0 \le k \le j$. For arbitrary parameter directions use

$$Lz = (k+1)^{-1} \sum_{\nu=1}^{k} z_\nu \widehat{v}_\nu, \qquad \widehat{v}_\nu = v_\nu/\|v_\nu\| \ (v_\nu \neq 0),$$

with zero for zero directions. In maximum norm $\|L\| \le 1$ and $L((k+1)\|v_\nu\| e_\nu) = v_\nu$. Pullback preserves allowed bounds; a test tolerance $b/(k+1)^k$ yields full multilinear operator bound b. Separate primitive bills supply smoothness at healthy bases; masks are not differentiated. At fixed period choose finite Y bounding $1 + \mathrm{dist}(x,y)$. Conversion to anchored directions costs at most $Y^{p.\,\mathrm{power}\,j}$, paid by shrinking tolerance first. Identity ensures the healthy test set is nonempty. Treat

real and imaginary streams separately, retain normalization and embed by zero in the common enumeration. The lift retains occurrence supports, allowed maps and normalizations. Recovery preserves this finite differentiation and contradiction argument, with its fixed-period conversion and summable tails.

For an occurrence address $\omega = (i_1, \dots, i_r)$, retain the exact stream reader

$$S_\omega = \frac{\varphi(\omega)}{r!} \sum_{\nu=1}^{r} \ell_{i_\nu} \prod_{\mu \neq \nu} u_{i_\mu}.$$

The speaking predicate, ordered occurrences and marked position stay in the record. The algebra-map identity gives $S_\omega^Z = S_\omega \circ \pi$ and section recovery, including repeated labels and silent streams. For any allowed physical parameter map a, its section lift $\sigma \circ a$ satisfies $S_\omega^Z \circ \sigma \circ a = S_\omega \circ a$ on the same healthy domain. Thus all tested derivatives and the multilinear rescaling by L are identical on recovery. The paid norm on section directions is transported from that original allowed class. The retained nonnegative inflated-product bound consequently supplies the same summable stream envelopes and tails. Applying the finite uniform-test argument and then the fixed-period anchored conversion recovers the original paid derivative comparison, with its Ursell coefficient and occurrence factorial unchanged.

15.17 Microscopic comparisons through projection

Regular comparison uses common supports, uniformly summable paid tails and arbitrarily small fixed-label derivative differences. Forced comparison uses uniform absolute tails and convergence at every varying configuration sequence. The latter implies uniform smallness by selecting violating configurations if necessary, without mask continuity. Finite-head/tail splitting proves dominated-series comparison: finite label differences vanish and both uniform tails can be made arbitrarily small. This permits varying evaluations, barcode sums, fixed injective zero extensions and scalar combinations. Products compare by $fg - f'g' = (f - f')g + f'(g - g')$; total absolute bounds times tail budgets control their tails. For the fixed weighted local section projection Π of price C_{read}, collected core is seed plus projected bulk sum, and

$$|\Pi(f - g)|_1 \leq C_{\mathrm{read}}\, \mathrm{paidBound}(f - g).$$

Linearity and summable projected tails give core comparison when seeds compare. Common-threshold residual and rebate operations are linear; their fixed price controls enlarged-support regular residuals and all-field rebates of differences and preserves tails. Canonical zero extension keeps regular comparison. The new forced profile is old profile plus the threshold hook of collected rebates. Hooks sum support classes times compatible forced products. Common address data preserve discrete tests; compared numerical factors, finite products and summable product budgets preserve forced comparison after multiplying rebates and summing. The keeper barcode embedding preserves convergence and tails. Thus local projection and collection recover the formatted output comparison. The lift retains common support data, projection and keeper maps; readout commutes with their linear operations and absolute series, preserving the paid and all-field bounds without differentiating indicators.

The core collection reader has the explicit convergent expression

$$c_{\mathrm{out}} = c_{\mathrm{seed}} + \sum_i \Pi f_i.$$

The source object records the same weighted projection and its output coordinates. Define its realized projection by first recovering the regular germ; section recovery then gives exactly Πf_i.

The bound $C_{\mathrm{read}}\,\mathrm{paidBound}(f_i)$ is unchanged and summable, so finite partial sums followed by their limits recover c_{out}. For two inputs and a finite head E, its difference satisfies

$$|\Delta c_{\mathrm{out}}|_1 \le |\Delta c_{\mathrm{seed}}|_1 + C_{\mathrm{read}} \sum_{i\in E} \mathrm{paidBound}(\Delta f_i) + C_{\mathrm{read}} \sum_{i\notin E} (b_i + b_i').$$

The retained uniform tails and fixed-label comparisons recover each term of this bound. Residual, rebate and hook readers retain their original support predicates; the canonical keeper map is recorded with its inverse on its image and zero extension elsewhere. Their linear identities and summable product majorants consequently give section recovery of the full formatted output, including its forced barcode profiles.

15.18 Common shot inputs and all-configuration comparison

For fixed capture x and frame index i, the shot merges primary insertion, regular remainder labels and forced barcode labels from the same input book. Common orientation fixes threshold, addresses and profiles; keeper stowing gives a common forced ingredient. Regular input prices are $\psi p.\,\mathrm{mass}(D_j, v_j)$ and forced prices $\psi p.\,\mathrm{mass}(K_j, c_j)/\kappa_1$, with $\kappa_1 > 0$. Capture wealth makes these summable. Primary tariffs use the larger heading and heading-plus-core price; the same array bounds output cores and remainders. Actual regular ingredients are real/imaginary fixed masked functions at their logarithmic keys, with weight at most v_j. Forced ingredients are common stowed functions at gas keys, with weight at most c_j; zero inputs give zero outputs. Nonconstant primary terms use the fixed masked section; constant frame terms give identical scaled units and zero forced profile, independent of keeper presentation. These are genuine fixed ingredients across the two histories, although their auxiliary preparation data may vary.

Fix scale, block L, degree d and dictionary. Their couplings approach each other, their regular and forced activity coefficients have the weighted comparison already proved, and their stationary kernels and path responses have the common limits. The established activity, coupling, kernel and path comparisons supply Gaussian, mean, saved-field and vertex comparisons. Common-chart conversion preserves each ingredient, wing and weight exactly, even if preparation data vary. Each fragrance address has the same support E_α, containing zero. Mixed marked-integral comparison gives arbitrarily small fixed-address paid derivative differences for both real and imaginary bulk terms, with common threshold/order and uniform marked-cluster tails. Its hypotheses are now the identical ingredient just identified, the compared Gaussian data and vertices, and their common summable derivative envelopes. We have therefore proved the bulk comparison, including both regularity on the common support and the summable tails needed for projection. Multiplication by L^d, recombination and zero extension preserve this bulk comparison. Forced output first restores missing ordinary logs, then marked slots, then resets on common shelves. Ordinary reset comparisons and marked integral comparisons, finite subsets, exponentials and summable product bounds preserve tails. The same path, threshold and rebate operations give

$$F_{n,\alpha}(U_n) - F'_{n,\alpha}(U_n) \longrightarrow 0 \quad \text{for every configuration sequence } U_n. \tag{HL1}$$

If uniform convergence failed, choose violating configurations on a bad subsequence and arbitrary ones elsewhere, contradicting (HL1). Uniform tails then give convergence of summed profiles by finite-head/tail splitting. The lift retains fixed input ingredients, wing/address supports and common shelves. Recovery preserves exact ingredient equality and the paid majorant, transporting (HL1) and its uniform consequence without mask continuity.

For each fixed ingredient label j, write M_j for its common real or imaginary input function. Its realized reader is $M_j^Z = M_j \circ \pi_f$, so the two histories have identical input readers even when their

auxiliary preparations differ. Section recovery gives $M_j^Z \sigma_f = M_j$. Retain the same logarithmic or gas key, wing, tariff and fragrance support with this function. The recovered summable tariff therefore bounds both its output core and remainder with the original forced scale κ_1. For the forced output let $\Delta F_{n,\alpha} = F_{n,\alpha} - F'_{n,\alpha}$. Surjective coarse readout, supplied by its section, gives

$$\sup_z |\Delta F_{n,\alpha}(\pi_c(z))| = \sup_U |\Delta F_{n,\alpha}(U)| \longrightarrow 0.$$

Thus (HL1), its uniform consequence and its uniform-tail summation are recovered on precisely the original all-configuration domain. Real and imaginary coefficients retain their separate readers and fixed L^d normalization before recombination. The constant-frame branch likewise recovers its identical unit and zero forced profile through either keeper.

15.19 Summing local insertion comparisons

Real and imaginary forced comparisons recombine exactly as

$$L^d(F_n + iG_n) - L^d(F'_n + iG'_n) = L^d[(F_n - F'_n) + i(G_n - G'_n)].$$

The primary seed already compares; linear projection, residual/rebate bounds and common keeper extension give core convergence and arbitrarily small collected remainder on the original input book. Local insertion differences keep common coefficient supports and orientation; append the negative second labelled remainder to the first to form a legitimate difference. If each pair has summable common bound f_i, its difference costs at most $2f_i$. For target $r > 0$, choose finite E with $\sum_{i \notin E} 2f_i \le r/2$ and eventually use $r/[2(|E| + 1)]$ on each index in E. Total price is at most r, so the paid merge gives that remainder bound for the full difference. Absolute convergence and common enumeration identify it with the difference of merged insertions. Core sums converge by the same finite-head/tail split. Finite frame and prescribed motion-chip sums preserve comparison of shot inputs captured on the same coordinate axes. Site marks therefore have vanishing core difference and arbitrarily small common remainder. Each actual coefficient readout is its site's core; recorded capture is capture under the common output keeper. The keeper difference identity makes the difference of captures the capture of the formatted difference, whose norm is bounded by its arbitrarily small remainder price. Thus both coefficient-isomorphism outputs and recorded captures compare to zero, supplying the fixed-input transfer comparisons. This proves the required forced comparison for the raw complex preparation. The lift retains labelled negative remainders, merge enumeration, frame chips and keeper identities. Readout preserves absolute merges and finite sums, recovering both output comparisons with the same quantitative remainder budget.

For the merged difference retain two disjoint labelled copies, one for each history, and put the minus sign on the second coefficient reader. Finite partial merges then satisfy the exact identity

$$\mathscr{L}\left(\sum_{i \in E}(M_{n,i} - M'_{n,i})\right) = \sum_{i \in E}(\mathscr{L} M_{n,i} - \mathscr{L} M'_{n,i}).$$

The summable $2f_i$ majorant and the retained merge bounds pass this identity to the full countable merge and recover it through the section. The chosen finite-head prices and tail prices sum to at most r on both sides. Applying the same keeper difference reader gives

$$\|\operatorname{capture}(M_n) - \operatorname{capture}(M'_n)\| \le \operatorname{remainderPrice}(M_n - M'_n) \le r$$

eventually. The recovered finite frame and motion sums preserve this inequality. Core readers recover the coefficient outputs separately, so both components of the transfer comparison have exactly the original vanishing bounds and common support format.

15.20 Transfer comparison metric

In the fixed finite coefficient frame let $|x|_1 = \sum_\nu |x_\nu|$. Gram data contain complex-linear captures g_P with $\|g_P x\| \le a|x|_1$. Define $P \sim_\epsilon Q$ by $\epsilon \ge 0$ and $\|g_P x - g_Q x\| \le \epsilon a|x|_1$. This may be nonseparating; it is symmetric, triangular and universally bounded by two. Coefficient-map comparison is $|(S - T)x|_1 \le \epsilon|x|_1$. Finite coordinate-axis bounds imply both uniform comparisons. Adjacent actual integrated transfers have flow Φ_n and isomorphism S_n, with

$$|x|_1 \le b|S_n(P)x|_1, \quad |(S_n(P) - S_n(Q))x|_1 \le c\epsilon|x|_1, \quad \Phi_n(P) \sim_{\epsilon/2} \Phi_n(Q).$$

At fixed input, flows and maps are Cauchy. Outside eventual admission the zero capture and identity coefficient map complete total sequences. For normalization write $g_X V = T$, $g_Z W = U$, with map differences bounded by c_0, and $|x|_1 \le b|Vx|_1$. For $k = V^{-1}i$,

$$g_X i - g_Z i = (Tk - Uk) + g_Z(Wk - i).$$

The terms cost $c_0 b|i|_1$ and $ac_0 b|i|_1$. Thus $X \sim_\ell Z$ if $(1 + a)c_0 b \le \ell a$, using only tested coefficient space. Escaping-subsequence comparisons on finitely many axes imply uniform map bounds, then flow bounds by this identity. Failure of a double-tail estimate would select bad $u(n), v(n) \ge n$, contradicting those subsequence comparisons. For Cauchy inputs P_n, insert a fixed late P_N between output flows: outer errors contract by half and the middle is small by fixed-input convergence. For coefficient maps the outer errors cost c times input error. Input tolerance $\epsilon/[4(c + 1)]$ and middle tolerance $\epsilon/2$ give the uniform comparison. Finally, if an eventual pairwise tail bound failed, one could choose bad indices $u(n), v(n) \ge n$; both escape to infinity and contradict the assumed subsequence convergence, which turns the subsequence statement into an actual double-tail Cauchy estimate. The lift retains capture maps, coefficient isomorphisms and this possibly nonseparating relation. Recovery preserves the tested-space inverse normalization and finite-axis bounds, transporting contraction and the actual double-tail Cauchy property.

The comparison relation itself has an exact recovered model. Let J be the section realization of the insertion-data seminormed space, with its transported seminorm, and define $\widehat{g}_P = Jg_P$. Then

$$\|\widehat{g}_P x - \widehat{g}_Q x\|_Z = \|g_P x - g_Q x\|,$$

so $P \sim_\epsilon Q$ is equivalent to the lifted comparison for the same ϵ. The coefficient frame and its ℓ^1 norm are retained unchanged; readout recovers S_n, V^{-1} and each finite-axis test. Consequently the inverse-normalization identity has exactly the original two bounds and threshold $(1 + a)c_0 b \le \ell a$ after recovery. The relation continues to be possibly nonseparating. Its triangle and contraction inequalities, fixed late input, and bad-index subsequence construction are recovered with the same errors, establishing the same double-tail Cauchy statements on the actual integrated transfer data.

15.21 Fixed-level convergence of transported insertions

At fixed physical period integrated Gram data satisfy eventually $W_{i,n} = T_{i,n}.\,\mathrm{flow}(W_{i+1,n})$. Flows halve comparison error and preserve the two-part Cauchy property. All admitted inputs compare within two. Replacing the input d levels up by fixed admissible data therefore costs at most $2\,2^{-d}$. For fixed d finite transfers give a Cauchy approximation. Choose this error below a quarter tolerance and use triangle inequality to prove fixed-level Gram Cauchy comparison, using only finitely many eventual identities. For coefficient transport suppose $x_n = S_n y_n$ eventually, x_n Cauchy, S_n uniformly Cauchy on bounded inputs and $|v|_{\mathrm{coeff}} \le b|S_n v|_{\mathrm{coeff}}$. Bounded x_n imply bounded y_n, and

$$S_n(y_n - y_m) = x_n - x_m + (S_m - S_n)y_m.$$

Both terms tend uniformly to zero on tails, and the inverse bound proves y_n Cauchy. Finite-dimensional norm equivalence gives ordinary convergence. Apply adjacent actual relations normalized by the reference inverse. The lowest coefficient is eventually fixed terminal y; Gram comparison and each paid inverse bound inductively give Cauchy coefficients at every fixed level, without an infinite future product bound. For captures,

$$g_n x_n - g_m x_m = g_n(x_n - x_m) + (g_n - g_m)x_m.$$

Uniform capture bounds control the first term and Gram comparison on bounded coefficients the second, giving captured-data Cauchy sequences. Escaping horizons eventually include every fixed level with original common choices and prescribed stationary paths. History conditions retain lawful, scalar, activity and path budgets. Bounded coefficients times fixed normalization give uniform insertion payment. Exponentials and forced profiles are measurable sums of masked regular terms, with continuous pullback and the prescribed conjugation or identity. The lift retains reference inverses, actual adjacent maps and capture identities. Recovery preserves finite-depth approximation and these Cauchy estimates, together with fixed-level payment and measurability for integration.

The recovered finite-level inverse argument can be written quantitatively. If $|y_m|_{\mathrm{coeff}} \le M_y$ and $\|S_n - S_m\|_{\mathrm{coeff}} \le \varepsilon_{n,m}$, the actual adjacent identity gives

$$|y_n - y_m|_{\mathrm{coeff}} \le b(|x_n - x_m|_{\mathrm{coeff}} + \varepsilon_{n,m} M_y).$$

Readout retains S_n and the coefficient norm, so the same inequality holds for the realized coefficient readers and is recovered by the section. Similarly the transported capture seminorm gives

$$\|g_n x_n - g_m x_m\| \le a|x_n - x_m|_1 + \varepsilon'_{n,m} a|x_m|_1$$

when the Gram comparison error is $\varepsilon'_{n,m}$. Both right sides vanish on double tails at each fixed level. The realized finite-depth approximation retains its $2\,2^{-d}$ error, and every proof at a chosen depth uses the same finite set of eventual history identities. Thus coefficient and capture convergence are recovered on the original escaping horizons, with the stated fixed-level payment bounds and measurable readout functions available for the ensuing integral limit.

15.22 Uniform fixed-level insertion integrands

Use eventual uniform boundedness and uniform Cauchy comparison over the entire configuration set. Eventual equality, fixed pullbacks, finite sums and products preserve this property; continuous maps preserve it on bounded finite-dimensional ranges by uniform continuity on compact balls. For captured data x, its masked regular sum $\mathcal{L}_x$ and barcode forced profile $\mathcal{F}_{x,B}$ satisfy

$$|\mathcal{L}_x(U)| \le \|x\|, \qquad |\mathcal{F}_{x,B}(U)| \le k\|x\|.$$

Paid health bounds and masks bounded by one prove the first; restriction of the forced budget proves the second. Absolute convergence makes both linear in x, including differences under fixed masks. Capture Cauchy therefore gives uniform remainder and profile Cauchy bounds. Each placed core axis is continuous on compact finite-lattice configuration space and bounded. Its mask cannot increase that bound. Finite-axis expansion gives a uniform coefficient-Lipschitz bound, also for the constant axis. Core Cauchy thus implies uniform masked-core comparison. Format and capture identities identify actual exponential entries with core plus regular remainder, and actual forced profiles with captured profiles. Fixed motions, required conjugation and normalization preserve these comparisons at every fixed anchor. No new normalization is chosen. Eventually

measurable uniformly Cauchy functions have Cauchy probability integrals, since integral differences are bounded by uniform differences. Finite polynomial restoration preserves comparison by finite-dimensional continuity; the common positive unmarked denominator permits quotient limits. The lift retains these capture identities, masks and probability priors. Literal readout preserves the full-domain sup bounds and hence normalized fixed-level expectations without continuity across masks.

For the realized fixed-level readers, surjectivity of π gives

$$\sup_z |f_n(\pi(z)) - f_m(\pi(z))| = \sup_U |f_n(U) - f_m(U)|.$$

Thus uniform Cauchy comparison, including every fixed mask, is exactly preserved. Prior push-forward recovers each coefficient integral and its uniform-difference bound. To state the quotient step quantitatively, let N_n be an integrated restored coefficient and $D_n \geq d_* > 0$ its unmarked denominator. If $|N_n| \leq M_N$, then

$$\left|\frac{N_n}{D_n} - \frac{N_m}{D_m}\right| \leq \frac{|N_n - N_m|}{d_*} + \frac{M_N |D_n - D_m|}{d_*^2}.$$

The recovered numerator, denominator and lower bound are identical, so the same estimate transports the normalized limit. Section recovery returns the full-domain insertion functions before integration, with their fixed translation, conjugation and normalization. This establishes recovery of fixed-level expectations using the original masks and law.

15.23 Exact unmarked density and component regrouping

Fix a forest, probability Haar in its gauge coordinates and common output. Packed addresses record store, finite warning set and labelled lists. The prescribed store layout maps these to molecules; roster, warning ownership and lists give a left inverse, and recover every layout-admitted molecule. Omitted addresses have zero activity. Summable vertex tariffs bound finite-subset products by the exponential of their sum, justifying integration and finite boundary averaging. Restricting to the injectively encoded addresses preserves summability. Neutrality with fixed forest makes $O_X = \mathrm{levy}_X \, \mathrm{deducted}_X$ independent of store/boundary replacement; each of b boundary choices has tribute O_X/b. Integrating the expansion and then summing the finite boundary choices gives the store's exact cut entry. Full sector coverage yields

$$\mathrm{push}(R; V) = O_X \sum_p \mathrm{receipt}_X(p).$$

This is a density relative to output probability Haar. Partition actual enlarged supports into connected components. Grounds are disjoint, and union reconstructs the molecule. Compatibility, exact flag coverage and component layout/eligibility/flag-read tests are equivalent to union approval: flags in another ground cannot enter a component. Plant output flags into component data and apply its own eligibility. Local data and selected regular keys agree on its ground with union data; locality and disjoint-ground independence, including boundary averages, therefore give

$$\mathrm{receipt}_X\left(\bigcup_{p\in A} p\right) = \mathbf{1}_{\{\text{compatibility,exact flag coverage}\}} \prod_{p\in A} a_X(p).$$

Failed union approval either fails coverage or makes a component factor zero, so all zero cases are included. Union is the inverse of component decomposition; every compatible family outside the

image has zero contribution. Absolute reindexing proves

$$\operatorname{push}(R;V) = O_X \sum_{A \text{ finite}} \mathbf{1}_{\{\text{compatibility,exact flag coverage}\}} \prod_{p\in A} a_X(p).$$

Gaussian chart masses and kernel normalizers are positive; remaining scalar factors are exponentials or positive powers, hence $O_X > 0$. This extensive scalar remains outside the gas before coupling reset. The frozen data in this identity are the store and the boundary choice after the gauge slice is fixed; the threshold, width and selector belong to the fixed paid chart. The comparison of these expressions uses the common drafts, principal covariance matrices and Gaussian densities constructed earlier in this section, together with the bill and protected-mean comparison and the four vertex cases. For a marked product it uses the same comparison with its finite marker coefficient retained. The next subsection states how these addresswise comparisons pass through the common source, its products and the reset; the signed line estimate (LCN11) gives the corresponding parameter control. No varying discrete approval predicate is differentiated. In the realized record the frozen pair (store, boundary choice) is a column of the primitive datum of Section 13.24, and the change of background between compared expressions is the horizontal lift of Section 15.5. The lift retains address inverses, planted data and scalar normalization. Recovery preserves the component bijection and exact flag tests; absolute majorants justify reindexing and recover the unnormalized density identity.

Let $\mathcal{A}$ be the permitted packed-address set and $\mathcal{M}$ the molecule addresses with nonzero receipt. Record the encoding e and roster-reading inverse d. On these data $de = I$ and $ed = I$; discarded addresses carry the prescribed zero reader. The connected-component map and inverse union are another pair of discrete inverse readers, retaining warning ownership, ground incidence and exact flag coverage. Their realizations therefore recover these identities before summation. For the realized output readout π_c, the retained absolute majorant permits the exact pullback

$$\operatorname{push}(R;\cdot)\circ\pi_c = (O_X\circ\pi_c) \sum_{A \text{ finite}} (\mathbf{1}_{\text{permitted}}\circ\pi_c) \prod_{p\in A} (a_X(p)\circ\pi_c).$$

The scalar reader retains neutrality and the original boundary count, so its b contributions are still O_X/b and its positive extensive factor stays outside the gas. Section recovery returns the unnormalized density relative to the same probability Haar prior, including all coverage-failure zero cases. This supplies exact recovery of the address and component regroupings together with their normalization.

15.24 Comparison of the complete common source

Fix period, order j and budget B of order at most j at the common temperature, with addresses and trial tori fixed first. Regular comparison means common sparse summable B-seminorm envelopes and addresswise seminorm convergence; forced comparison uses uniform norms instead. These are stronger than pointwise comparison of the total action; the distinction permits the subsequent products and reset operations. Local Gaussian comparison along every healthy configuration/test sequence implies uniform derivative comparison by a violating-sequence argument. At fixed torus choose $Y \geq 1 + \operatorname{dist}(x,y)$ and pay conversion $T = Y^{B.\,\text{power}\,j}$ by using tolerance ϵ/T. Sparse stream envelopes multiplied by T remain valid. The forced stream uses order zero and its own sparse integral envelopes. In particular, the ultrafilter statements used to express local Gaussian limits here imply ordinary sequential convergence, since every free ultrafilter on every bad subsequence would give the same contradiction. Path terms have coefficients $-v_n, -\beta_n, -\beta_n, -1$ on low, cubic,

quartic and loop terms. Weighted path arrays compare and scalars have common bounded limits. The identity

$$a_nF_n - a'_nF'_n = a_n(F_n - F'_n) + (a_n - a'_n)F'_n$$

preserves sparse envelopes and addresswise convergence. Frame transport and averaging cost fixed $\mathrm{turnFee}(B)\,\mathrm{Rim}(c)$; divide tolerance accordingly. Thus the negative log stream and path terms give full regular-source comparison, with hard-source forced comparison on the same canonical enumeration. The enumeration used to place these terms in the common dictionary is the same for both histories. Finite trial panels eventually lie in one finite label set and cutoffs exceed all its labels, so panels agree. Removing selected regular terms and summing their site contributions preserves comparison with fixed selection weights. Trial pullback through the padded shape, of radius z, maps a healthy neighborhood into the all-healthy trial torus and costs

$$M = 2^{B.\,\mathrm{order}}(1+z)^{B.\,\mathrm{power}\,B.\,\mathrm{order}}.$$

Use tolerance ϵ/M. Finite trial/site sums with fixed ray weights give ornament regular comparison and failed-trial forced comparison. Site correction is selected regular contribution minus its identity value minus failed trial. Healthy identity evaluation and subtraction give its uniform comparison. The complete source consists of unselected regular plus ornament, and original hard minus frame-averaged site correction, on common disjoint labels. The lift retains trial panels, padded maps, weights and enumerations. Recovery preserves these exact assemblies and fixed-parameter seminorm costs, recovering comparison of the labelled source rather than only its total function.

The complete-source reader retains each summand's tag in the disjoint union: negative logarithmic stream, low/cubic/quartic/loop path term, unselected regular term, ornament, original hard term and site correction. Coefficientwise lifting preserves their scalar signs and finite frame weights. For a trial shape map θ, its recovered pullback is simply

$$(f \circ \theta) \circ \pi = f \circ (\theta \circ \pi).$$

The source domain retains the same healthy-neighborhood preimage, and section recovery returns the original trial reader. Its derivative bound therefore uses the same M, while frame transport uses the same $\mathrm{turnFee}(B)\,\mathrm{Rim}(c)$. The eventually common panel is kept as a fixed finite address set before removal or summation. Its identity-value reader is recovered at the identity section. Hence the selected-minus-identity-minus-failed site correction and the two complete-source parts are recovered term by term, with their original sparse envelopes and seminorm tolerances. The disjoint labels preserve comparison of the full source arrays through these assemblies.

15.25 Marked products through forced reset

A marked family $u_{n,i}(t,V)$ has sparse bounds on $|u_{n,i}| + |\partial_t u_{n,i}|$ and comparison of value and derivative along arbitrary configuration sequences at prescribed parameters. Compared unmarked logarithms become constant marked families. Their factors $e^{\ell_{n,i}} - 1$ compare by mean-value bounds on a common bounded interval, with a fixed exponential envelope multiplier. Extend all families by zero through the same embeddings. A forced atom is a finite product of these factors times the common eligibility indicator. The identity

$$\partial_t \prod_{i\in A} u_i = \sum_{i\in A} (\partial_t u_i) \prod_{k\in A\setminus\{i\}} u_k$$

compares both atom and marked derivative. The shared zero-one eligibility factor preserves differences without continuity assumptions. Sparse product bounds control finite-subset sums; further

widening is the same zero extension. Evaluate ∂_t at zero to obtain the forced marked-reset comparison, bounded by its marked envelope. The lift retains parameterized factors, shared indicators and embeddings. Recovery commutes with finite product differentiation and zero extension; sparse majorants preserve the summed comparison used in the two-history output limit.

For a fixed forced address A, write its eligibility as $\chi_A(V)$. The exact parameterized reader and its derivative are

$$H_A(t,V) = \chi_A(V) \prod_{i\in A} u_i(t,V), \qquad \partial_t H_A = \chi_A(V) \sum_{i\in A} \partial_t u_i \prod_{k\neq i} u_k.$$

The recorded predicate is independent of t. Composition with physical readout therefore commutes with this derivative at each fixed input; section composition recovers both expressions, including $t = 0$. For the two histories the same χ_A multiplies their value and derivative differences, so its absolute value at most one preserves the comparison. The zero-extension maps retain each address and its inverse on the image. The sparse product majorant then controls the same summed readers after realization, giving exact recovery of the forced marked reset and its derivative envelope on the original configuration domain.

15.26 Analytic control of omitted insertion terms

One estimate needed in the continuum comparison is obtained by scaling all restored insertion errors with a single complex parameter. For holomorphic f on $|z| < S$, $S > 2$, bounded by B, Cauchy's derivative estimate on radius-$S/2$ disks centered on $[0,1]$, each of which lies strictly inside the radius-S disk, gives

$$|f(1) - f(0)| \le 2B/S. \tag{IC1}$$

Retain the normalized moment

$$\mathrm{Stamp}_{j,t}(P) = \frac{\int (P(U))_t \, d\,\mathrm{prior}_j(U)}{Z_N},$$

with the same original positive Wilson denominator at all levels. For served $P = \mathrm{restore}(P_0, P_1)$, set $P(z) = \mathrm{restore}(zP_0, zP_1)$ and $f(z) = \mathrm{Stamp}_{j,t}(P(z))$. Then $P(1) = P$, and $P(0)$ removes both errors. Retained estimates give holomorphy and integrability on $S = \mathrm{brisk}(B_j) > 2$, spending two exponent units without changing geometric support. Transport to fixed $k < j$ preserves served bounds and integration. Continuous finite-dimensional coefficient extraction commutes with that integral. The root estimate gives

$$|f(z)| \le \mathcal{B}_{k,t} := \frac{\exp\{|\,\mathrm{Site}_k\,|\,\mathrm{treasury}_k + n(x+y+2)\}}{\mathrm{toll}(w_k, t)}.$$

Here n counts insertion slots and the toll uses positive root weights. All volume dependence is at the fixed reporting level k; there is no exponential cost involving the increasingly fine level j. Applying (IC1) proves

$$|\,\mathrm{Stamp}_{j,t}(P) - \mathrm{Stamp}_{j,t}(\mathrm{restore}(0,0))| \le \frac{\mathcal{B}_{k,t}}{\mathrm{brisk}(B_j)/2}. \tag{IC2}$$

The lift retains error inputs, supports, common denominator and the actual transport chain. Recovery preserves their complex scaling and coefficient integrals, so (IC1)–(IC2) hold with the same reporting-level treasury and exponent margin.

The uniform omission estimate uses exact transport and the common analytic disk; it does not require intermediate mixed-pin representations to converge. Explicitly lift the restored family coefficientwise as $\widehat{P}(z) = \mathscr{L}_j P(z)$. The retained readout gives

$$\widehat{\mathrm{Stamp}}_{j,t}(\widehat{P}(z)) = \frac{\int [\widehat{P}(z)]_t \, d\mu_j^Z}{Z_N} = \mathrm{Stamp}_{j,t}(P(z)).$$

This identity holds throughout the same open disk: the paid majorant supplies every integral and the recovered error scaling leaves masks and supports fixed. Its two endpoint values are recovered by the section. The actual finite retained-kernel composition preserves this entire scalar function while moving its bound to reporting level k. Thus the holomorphic derivative estimate, endpoint difference and factor $\mathrm{brisk}(B_j)/2$ are identical after realization. The normalization reader keeps Z_N fixed, and the bound keeps the explicit $|\,\mathrm{Site}_k\,|$ cost. Recovery of (IC2) consequently uses the analytic family itself and its exact transport, with no additional convergence assumption on intermediate mixed-pin representations.

16 Radial jets, calibrated labels and the free Maxwell comparison hierarchy

From local coefficients to usable field labels. The first objects are finite Taylor coefficient spaces. Curl reads their linearized curvature, and radial reconstruction provides an explicit section on closed homogeneous two-forms. The Maxwell equation imposes a further condition on that curvature. Polynomial labels are then sampled by finite local cylinders, with calibrated coefficient maps and a free comparison hierarchy. Keeping polynomial position degree, engineering weight and field count distinct is necessary for the later detector and scaling arguments. The free hierarchy is a supplied comparison sector; its presence does not assert that the interacting hierarchy is free.

Before the detector on physical coefficient space can be constructed, the supplied construction builds the label algebra, its local lattice representatives, the calibrated coefficient changes and the complete free comparison hierarchy with its shell charges. This section gives that construction in full, from the radial Taylor variables to the elimination of higher-degree vector and angular defects; each subsection is followed by its source–readout formulation. The labels are the first place where the recovery problem is posed for *fields* and not only for measures: a label is a source record, its lattice cylinder is a realization, and its Taylor readings are the readout.

16.1 Radial Taylor variables and the free equation space

The jet variables in this chapter are Taylor coefficients, with the factorial already divided out. At order $j \geq 0$, a radial potential coefficient is a family A_i of symmetric $(j+1)$-linear maps from $\mathbb{R}^4$ to the real color space. Its associated homogeneous potential is $a_i(x) = A_i(x, \ldots, x)$, and the constraint is

$$\sum_i x_i a_i(x) = 0 \quad \text{for every } x. \tag{RJ1}$$

This finite-dimensional real space is the space called Taylor in Labels/Jet. Its polynomial-position degree is $j+1$, its engineering weight is $j+2$, and its field-count degree is one. Polynomial labels inherit their weight and field count by addition over their factors; constants have both degrees zero. Complex labels are obtained by complexification. No field equation has yet been imposed on these radial variables.

Define the curvature coefficient by the homogeneous polynomial

$$Q_{il}(x) = \partial_i a_l(x) - \partial_l a_i(x). \tag{RJ2}$$

On symmetric tensors this means freezing one input at e_i, subtracting the reversed expression, and multiplying by $j+1$. The curvature is antisymmetric and satisfies

$$\partial_a Q_{bc} + \partial_b Q_{ca} + \partial_c Q_{ab} = 0, \tag{RJ3}$$

since the second derivatives commute. Conversely let Q be a homogeneous degree-j antisymmetric two-form satisfying (RJ3). Set

$$a_i(x) = \frac{1}{j+2} \sum_k x_k Q_{ki}(x). \tag{RJ4}$$

Antisymmetry proves (RJ1). Differentiating (RJ4), subtracting the reversed derivatives, and using (RJ3) gives

$$\partial_i a_l - \partial_l a_i = \frac{2Q_{il} + \sum_k x_k \partial_k Q_{il}}{j+2} = Q_{il},$$

where the last equality is Euler's identity for degree j. Conversely, for a radial potential, differentiation of $\sum_k x_k a_k = 0$ gives $\sum_k x_k \partial_i a_k = -a_i$. Hence $\sum_k x_k Q_{ki} = (j+1)a_i + a_i = (j+2)a_i$. Thus curl and (RJ4) are inverse linear maps. Equality of homogeneous diagonal polynomials implies equality of their symmetric tensors by polarization, so these are identities of the coefficient spaces themselves. These are the maps and inverse identities in Maxwell/Exterior.

The free solution space consists of those Q which additionally satisfy

$$\sum_a \partial_a Q_{ab} = 0 \quad \text{for every } b. \tag{RJ5}$$

The free gauge-jet map sends this subspace to radial potentials by (RJ4). It is injective because curl is its left inverse. In particular, saying that a radial coefficient is in its range is exactly saying that its curvature satisfies the free equation (RJ5); Bianchi is already built in by (RJ2). Taking normalized Taylor coefficients of a smooth Maxwell solution on an open set preserves (RJ3) and (RJ5): differentiate these equations at the point, commute the derivatives, and divide by the Taylor factorial. Conversely the polynomial attached to an element of the solution space obeys these equations everywhere. This justifies the solution-jet substitutions used in the equation-ideal argument below.

Color conjugation acts on these coefficients by the adjoint action. A rotation g acts by

$$a_i(x) \longmapsto \sum_k g_{ik} a_k(g^{-1}x).$$

The radial constraint, curl, Bianchi and (RJ5) are preserved, directly by orthogonality and the chain rule. Thus the reconstruction (RJ4) intertwines both actions. The polynomial ideal is the kernel of restriction from radial coefficients to this free solution range, taken at each finite degree horizon. A choice of projection onto that range is a finite-dimensional linear choice; the polynomial telescoping argument below shows explicitly how its kernel is generated by the linear equation coordinates. This explains both the projection P_j used there and the radial reconstruction used in the Gaussian covariance comparison.

Source–readout formulation. The radial Taylor variables are a normal form for a quotient, and the source–readout content of the subsection is the identification of that quotient with a concrete section. The source object is a homogeneous potential; the readout is its linearized curvature (RJ2); the fibre of the readout is the space of linearized gauge transformations; and the radial reconstruction (RJ4) is a section whose image is the radial gauge. The following theorem states this with the gauge parameter written explicitly; the explicit parameter is not printed in the supplied text and is verified in the proof.

Theorem 16.1 (Radial section of the curl readout)**.** *Let $\mathcal{A}_j$ be all homogeneous $\mathfrak{g}$-valued potentials of degree $j+1$ (no radial constraint), $\mathcal{Q}_j$ the closed homogeneous two-forms of degree j,* $\operatorname{curl} : \mathcal{A}_j \to \mathcal{Q}_j$ *the map* (RJ2) *and $\rho : \mathcal{Q}_j \to V_j$ the map* (RJ4)*. Then*

1. *ρ takes values in V_j,* $\operatorname{curl}\rho = \mathrm{id}$ *on $\mathcal{Q}_j$ and* $\rho\operatorname{curl} = \mathrm{id}$ *on V_j;*
2. *for arbitrary $a' \in \mathcal{A}_j$, $\rho(\operatorname{curl} a') = a' - \partial\lambda$ with $\lambda = (j+2)^{-1}\sum_k x_k a'_k$, a homogeneous scalar of degree $j+2$;*
3. *hence* curl *is onto, its fibres are the orbits of $a' \mapsto a' + \partial\lambda$, and V_j meets each orbit exactly once.*

Proof. Antisymmetry of Q gives (RJ1) for ρQ. Differentiating (RJ4), subtracting the reversed derivative and using (RJ3), $\partial_i a_l - \partial_l a_i = [2Q_{il} + \sum_k x_k \partial_k Q_{il}]/(j+2) = Q_{il}$ by Euler's identity in degree j. For (2), $\sum_k x_k(\partial_k a'_i - \partial_i a'_k) = (j+1)a'_i - \partial_i(\sum_k x_k a'_k) + a'_i$, again by Euler's identity, which is $(j+2)(a'_i - \partial_i\lambda)$. If a' is radial then $\lambda = 0$, which is the second half of (1). (3) follows: two potentials with the same curl have the same image under $\rho\operatorname{curl}$, so they differ by a gradient, and a radial gradient $\partial\mu$ has $x \cdot \partial\mu = (j+2)\mu = 0$. Equality of homogeneous diagonal polynomials is equality of symmetric tensors by polarization. □

The exactness class is quotient equivalence: potentials modulo the fibre are in bijection with closed two-forms, with ρ as inverse on the radial slice. The free equation space of the title is then a subspace of the readout space, and membership in it is a property of the curvature jet, independent of the potential chosen over it.

16.2 Polynomial labels, coherent frames and local sections

The polynomial variables in this construction are coordinates of real radial jets, rather than coordinates of arbitrary connections. Write V_j for the real space of radial homogeneous potentials of polynomial degree $j+1$, described in (RJ1)–(RJ4), and choose a real basis of each V_j. If $z_{j,a}$ are its coordinate functions, the algebra of labels before imposing gauge invariance is

$$\mathcal{P} = \mathbb{C}[z_{j,a} : j \geq 0,\ 1 \leq a \leq \dim_{\mathbb{R}} V_j]. \tag{PF1}$$

Each polynomial uses finitely many variables. Its evaluation on a real jet $J = (J_j)_{j\geq 0}$ substitutes the real coordinates of J_j, regarded as complex numbers. Such evaluations separate polynomials: reduce to the finitely many variables present, and successively use the fact that a nonzero one-variable polynomial cannot vanish at every real number.

Give $z_{j,a}$ the bidegree $(j+2, 1)$. Thus the bidegree of a monomial z^m is

$$\left(\sum_{j,a}(j+2)m_{j,a},\ \sum_{j,a} m_{j,a}\right) = (r, f). \tag{PF2}$$

The first entry is engineering degree and the second is number of fields. The horizon-d label space consists of the adjoint-invariant polynomials whose monomials have $r \le d$. It is finite dimensional: only variables of weight at most d occur and every exponent is at most d. Adjoint transformations and orthogonal transformations mix coordinates within a fixed V_j; therefore they preserve both entries of (PF2). Taking a homogeneous component preserves adjoint invariance and the horizon. Conjugation of coefficients preserves these spaces as well, because evaluation takes place on real jets.

There is a basis b_i, with $b_0 = 1$, consisting of real polynomials of individual bidegrees (r_i, f_i). Indeed decompose into the finitely many bidegrees, and then into real and imaginary parts. The resulting real homogeneous elements span the complex space; choose an independent spanning subfamily containing 1. At successive horizons extend the previous independent family inside the same spanning set. This gives coherent frames: previous basis elements remain present as elements, although their nonzero index numbers need not remain fixed. The only elements with field number zero are constants. Hence $f_i > 0$ for $i \neq 0$.

Coefficients $c = (c_i)$ mean the polynomial $\sum_i c_i b_i$; use $\sum_i |c_i|$ as a coefficient norm. There are two different scaling operations:

$$J_j \mapsto h^{j+2} J_j, \qquad J_j \mapsto t J_j. \tag{PF3}$$

On b_i these multiply the reading by h^{r_i} and t^{f_i}, respectively. In particular coefficient scaling by field number is the diagonal map $c_i \mapsto t^{f_i} c_i$, invertible for $t \neq 0$, and fixes the unit. The coefficient action of $s \in O(4)$ is obtained by applying the tensor action to $\sum_i c_i b_i$ and taking its coordinates. This action is an action on the polynomial label space. Its compatibility with the physical quotient requires the separate rotation argument below; it does not follow merely from this definition.

Here is the actual construction of a local lattice representative of each label. At horizon d put

$$A_i^J(x) = \sum_{j<d-1} J_{ji}(x), \qquad a_{x,i}^{J,h}(t) = h A_i^J(h(x + te_i)). \tag{PF4}$$

An empty sum is zero. Along this edge solve the matrix group equation

$$u(0) = 1, \qquad u'(t) = u(t) a_{x,i}^{J,h}(t), \tag{PF5}$$

where the color is embedded in the skew-adjoint matrix Lie algebra. Set the link equal to $u(1)$. Existence does not assume a global logarithm. In an exponential chart, differentiating $\log(\exp b \exp v)$ at $v = 0$ converts (PF5) to a smooth ordinary differential equation in b. Solve locally, left translate to any initial value, and glue. Solutions agreeing once agree throughout an overlapping interval: the derivative of uv^* is zero. The times reachable from zero form an open set with open complement, by the same local extension and gluing argument; connectedness of the real line gives the solution at every time. This also proves uniqueness.

For a finite collection of edges these links depend analytically on the finite jet J near zero. One can see this without differentiating an infinite configuration map. The integral equation in matrix path space is $u = 1 + \mathcal{V}_J u$, where $\mathcal{V}_J$ is integration followed by multiplication by the connection. Near $J = 0$ its norm is less than one, so $(1 - \mathcal{V}_J)^{-1} = \sum_{n \ge 0} \mathcal{V}_J^n$ is analytic. Uniqueness identifies it with (PF5). Differentiating at zero gives the edge tangent $\int_0^1 A_i^J(x + te_i)\, dt$. Products and inverse links therefore have tangents equal to their signed edge sums.

A sampling slot consists of an oriented elementary plaquette based at an integer point and a finite path from the origin to that point. Evaluate its curvature and transport the color back along that path. A gauge transformation changes every slot by the same adjoint transformation at the

origin. At the identity configuration, the connector derivatives cancel: differentiating ZPZ^{-1}, with $Z = P = 1$, leaves just P'. Consequently the linearization of the slot is the circulation of the integrated polynomial potential around the plaquette.

These linear samples separate the truncated radial jets. For one homogeneous component $A \in V_k$, define

$$N_{ij}(x,t) = \int_0^1 \int_0^1 \partial_i A_j(x + tse_j + tre_i)\, dr\, ds - \int_0^1 \int_0^1 \partial_j A_i(x + tse_i + tre_j)\, dr\, ds. \tag{PF6}$$

At $t = 1$ the fundamental theorem of calculus identifies this with the square circulation. At $t = 0$ it is $\partial_i A_j - \partial_j A_i$. Homogeneity gives $N_{ij}(tx,t) = t^k N_{ij}(x,1)$. If all integer square samples of a sum of components through k vanish, multiply the rescaled component of degree ℓ by $t^{k-\ell}$. Choose integer x_t with $tx_t \to x$. The resulting finite sum is zero for $t > 0$; continuity at zero leaves only $N_{ij}^{(k)}(x,0)$. Thus the top curl vanishes. Radial reconstruction (RJ4) makes the top potential zero. Descending induction eliminates all components. No interchange of the two iterated integrals in (PF6) is required.

Only finitely many slots are needed. Among kernels of finite sampling families choose one of minimal dimension. Appending any further slot cannot lower that dimension, hence cannot lower the kernel. A vector in this kernel is therefore killed by every slot and, by the preceding argument, is zero. Write $X = \prod_{j<d-1} V_j$, $Y = \mathfrak{g}^m$, and $T : X \to Y$ for the resulting injective linearization.

There is an adjoint-equivariant left inverse $C : Y \to X$. Equip Y with the sum of the invariant color inner products. The range of T is invariant under the adjoint action in both directions, so its orthogonal projection commutes with that action. Compose this projection with T^{-1} on its range. This proves the equivariance of C without needing an invariant inner product on X.

Let $B : X \to Y$ denote the nonlinear slot readings of the links (PF5) at $h = 1$. Then $B(0) = 0$, $DB(0) = T$, and $D(CB)(0) = 1$. The finite-dimensional analytic inverse function theorem supplies a local inverse g^{-1} of $g = CB$. Define

$$R(y) = g^{-1}(Cy). \tag{PF7}$$

It is analytic near zero, $R(0) = 0$, and $R(B(x)) = x$ near zero. For each fixed gauge transformation its equivariance follows by applying g to both candidate expressions, using equivariance of B, C, and using injectivity on a sufficiently small common neighborhood. The neighborhood is allowed to depend on that transformation.

Replace this shared recovery map by its Taylor polynomial

$$R_{\le d}(y) = \sum_{q=0}^{d} \frac{1}{q!} D^q R(0)[y, \dots, y], \qquad P_i(y) = b_i(R_{\le d}(y)). \tag{PF8}$$

All derivatives through order d of the first expression equal those of R; differentiating each homogeneous term proves this directly, including its factorial. Expanding multilinear forms in real panel coordinates makes P_i an actual complex polynomial. Its evaluation on real panels is real, and it is adjoint invariant. Because $R_{\le d}(0) = 0$, each substituted coordinate has no constant term. Every monomial of b_i has exactly f_i factors. Hence every monomial of P_i has at least f_i slot factors. The substitution is not truncated again: terms of degree greater than d are retained.

The geometric scaling identity for (PF5) is

$$U(J,h) = U(\mathrm{trunk}_d(h^{j+2} J_j), 1). \tag{PF9}$$

It follows from homogeneity in (PF4) and uniqueness of the edge equation. Along this curve the slot panel is $B(\mathrm{trunk}_d(h^{j+2}J_j))$. Equations (PF7)–(PF8), and the chain rule through order d, therefore give

$$\frac{d^q}{dh^q}P_i(\text{slots of } U(J,h))\Big|_{h=0} = \frac{d^q}{dh^q}b_i((h^{j+2}J_j)_j)\Big|_{h=0}, \quad 0 \le q \le d. \tag{PF10}$$

This is equality of finite Taylor readings, not equality at nonzero mesh.

To impose the signed lattice symmetries, extend P_i linearly in the coefficient vector, precompose configurations by each signed coordinate permutation, apply the inverse tensor action to the coefficient input, and average over this finite group. Equation (PF10) is preserved: the holonomy of the transformed connection is the transformed holonomy, including reversal and inversion of negatively oriented edges, by (PF5) and uniqueness. The tensor action preserves field number. A transformed slot is a signed slot; combining all these finite slot families into one list and substituting signed variables therefore preserves the lower bound f_i on monomial degree. The averaged section fixes the unit and obeys the exact signed symmetry and reality laws.

All paths and squares used are finite. Enclose them and their signed images in one origin-containing symmetric integer box. Each transported curvature slot is a polynomial in matrix entries of its links and their conjugates: inverse unitary links use conjugate transpose, and curvature uses its fixed linear projection of the skew part. Thus the section is a finite polynomial cylinder, gauge invariant, continuous everywhere and smooth along link variations. On a torus it is placed by translating and reducing this finite stencil. Formula (PF10) is used when the stencil does not wrap; the cylinder itself is defined even when it wraps.

Finally fix a derivative order and the direction power of (AB4). On this fixed stencil each transported curvature slot, including its normalized ordered left derivatives, has an envelope bounded by $C\theta$ on the healthy domain. The zeroth estimate is the healthy plaquette bound and adjoint isometry; differentiated plaquettes and connector products have bounded coefficients on the compact group and each allowed direction contributes θ times a fixed stencil constant. For a differentiated product of at least f_i slots the Leibniz rule distributes the derivatives; an undifferentiated slot pays θ, and a differentiated slot pays at least one θ. Further derivatives only improve this power since $\theta \le 1$. Finite sums and a maximum over the finite basis give a constant A, independent of torus size, placement and the remaining budget parameters, such that the section of b_i has normalized envelope

$$A\theta^{f_i}. \tag{PF11}$$

This is the local section estimate used in training and reset budgets. It includes nonlinear terms of (PF8); it is not an estimate only of the linearized sampler.

The actual finest-lattice insertion includes two coefficient changes in a specified order. Let $\theta_N > 0$ be the healthy-field threshold at level N, let $S_d(c; o, U)$ be the section just constructed, and let $\mathcal{T}_{d,N,p}$ be the terminal coefficient isomorphism of the selected history. Put

$$I_{d,N,p}(v; o, U) = a_N^{-d} S_d\Big((\theta_N^{-f_i}(\mathcal{T}_{d,N,p}^{-1}v)_i)_i; o, U\Big). \tag{PF12}$$

Thus calibration is applied first, threshold scaling second, and engineering normalization last. The exponent in the threshold scaling is the field number f_i, not the engineering degree r_i. A general horizon label contains several bidegrees, so neither diagonal scaling may be replaced by one scalar on that entire space.

Although the coefficient vector in (PF12) depends on N, p, its stencil is the single stencil of S_d. Linear combination and multiplication by constants do not enlarge it. Consequently (PF12) is continuous, gauge invariant and supported on that translated stencil for every cutoff, including

cutoffs at which it wraps around the torus. The finite-Taylor normalization and the calibrated symmetry statements are used only after the reference level fits below N; this distinction is needed when constructing a family defined at every cutoff.

At such a cutoff the terminal map commutes with conjugation and signed orthogonal actions. Its inverse does too, by applying the terminal map to the two candidate values. Threshold scaling commutes with those actions because they preserve field number. Together with the corresponding identities for S_d, these facts prove the reality and signed covariance of (PF12). If e_0 denotes the coefficient vector of the constant polynomial, the physical unit at level m is $a_m^d e_0$. The based-history identity is

$$\mathcal{T}_{d,N,p}(a_N^d e_0) = a_{k_d}^d e_0. \tag{PF13}$$

Substituting its inverse in (PF12), using $f_0 = 0$ and $S_d(e_0) = 1$, proves that the reference physical unit gives the constant cylinder one. Translating the placement translates precisely the same coefficient vector; no new history or new calibration is chosen at the new site.

Source–readout formulation. The chain is

$$J \xrightarrow{\text{realize (PF5)}} U(J,h) \xrightarrow{\text{slots}} B \xrightarrow{R} J,$$

with $R \circ B = \mathrm{id}$ near zero (PF7): on the image of the realization the slot panel recovers the jet exactly, as analytic germs. The polynomial section replaces R by $R_{\le d}$ and thereby recovers the label reading only through Taylor order d in the mesh (PF10); the loss is everything of higher order, and it is intended (those terms have engineering weight above the horizon). The realization map $J \mapsto U(J,h)$ is not onto configurations and is not gauge invariant; the readout B is adjoint covariant at the origin and the labels are adjoint-invariant polynomials, so $P_i \circ B$ is gauge invariant. In the frame realization of Theorem 7.1, $S_d(c; o, \pi(r,A)) = S_d(c; o, A)$ (Proposition 9.1 and gauge invariance), the envelope (PF11) is a fibre constant (Proposition 9.3) and the insertion (PF12) lifts as $I^Z = I \circ \pi$ with the same stencil. The three coefficient operations of (PF12) are recorded separately in the realized tuple, in their order.

Lemma 16.2 (Finitely many slots separate truncated radial jets)**.** *There is a finite slot family whose linearization $T : X = \prod_{j<d-1} V_j \to Y = \mathfrak{g}^m$ is injective, and an adjoint-equivariant left inverse $C : Y \to X$.*

Proof. If all integer square samples of a sum of components through degree k vanish, multiply the rescaled component of degree ℓ by $t^{k-\ell}$ and choose integer x_t with $tx_t \to x$; the finite sum is zero for $t > 0$, and continuity at $t = 0$ leaves only $N_{ij}^{(k)}(x, 0)$. The top curl vanishes, so by Theorem 16.1 the top potential vanishes, and descending induction removes all components. Among kernels of finite sampling families choose one of minimal dimension; appending any slot cannot lower it, so a vector in it is killed by every slot and is zero. For C, equip Y with the sum of invariant color inner products; the range of T is invariant, its orthogonal projection commutes with the adjoint action, and C is that projection followed by T^{-1} on the range. No invariant inner product on X is needed. □

The lemma is the finite-dimensional separation statement on which the chain rests: finitely many lattice slots already determine a truncated radial jet, and the left inverse can be chosen compatibly with the adjoint action, so that the gauge covariance of the readout is not broken by the choice of slots. The exactness class of the label reading is controlled approximation in the mesh (exact through Taylor order d), built on an exact local section.

16.3 Calibrated coefficient changes and exact solutions

The inverse of the local slot sampler in (PF8) is an inverse on truncated radial jets. The coefficient inverse required for a physical insertion is a different map. We construct it in the norms and units in which it is used. Fix a degree horizon d, its coherent frame, and a reporting level $k \geq k_d$, where k_d is the reference level for that frame. All assertions in this paragraph are uniform in $N \geq k$ and the period p, with k fixed.

Let $\mathcal{T}_{d,N,p}$ be the calibrated terminal coefficient isomorphism of the insertion history. If v is its reference coefficient, transport $\mathcal{T}_{d,N,p}^{-1} v$ along that same history to level k and take its primary coefficient. Denote the result by

$$C_{d,k,N,p} v, \qquad A_{d,k,N,p} v = a_k^{-d} C_{d,k,N,p} v. \tag{PC1}$$

At the reference level $C_{d,k_d,N,p} v = v$, by the normalization of the history. At an adjacent step its coefficient estimates give, in the frame norm $|\cdot|_1$,

$$|C_i v|_1 \leq R_{i+1} L_i^d |C_{i+1} v|_1, \qquad |C_{i+1} v|_1 \leq R_{i+1} |C_i v|_1, \tag{PC2}$$

where $R_{i+1} \geq 1$ is the actual column allowance and $L_i \geq 1$ the block side. These are bounds on the primary coefficients of the original transported entries. They do not replace the primary coefficient by its class in the physical quotient.

Define the finite number

$$O_{d,k} = \prod_{i<k} R_{i+1} L_i^d \geq 1. \tag{PC3}$$

Iterating the first inequality of (PC2), starting at the reference level, and enlarging the product by its factors before that level, proves

$$|v|_1 \leq O_{d,k} |C_{d,k,N,p} v|_1 = O_{d,k} a_k^d |A_{d,k,N,p} v|_1. \tag{PC4}$$

Iterating the second inequality gives a finite upper bound as well. Thus C and A are injective and hence bijective on the finite coefficient space, with both bounds uniform in the remaining cutoff and period at this fixed level. No bound on $O_{d,k}$ as $k \to \infty$ is asserted.

The readout map is constructed by polynomial substitution followed by Gaussian contraction. Start with the linear map sending a reference coefficient v to the local polynomial representative of its transported insertion. Rename its finitely many slot variables as the actual link-coordinate variables, multiply each by the threshold factor, and embed them in the one common stencil. Let $M(v)$ denote the resulting polynomial. For its actual symmetric covariance Q, Gaussian contraction means the finite linear operator

$$\mathcal{W}_Q = \exp\left(\frac{1}{2} \sum_{\alpha,\beta} Q_{\alpha\beta} \partial_\alpha \partial_\beta \right). \tag{PC5}$$

The exponential terminates on each polynomial. Apply $\mathcal{W}_Q$, substitute the linearized slot expressions in the radial-jet variables, and retain monomials of engineering weight at most d. The resulting polynomial is invariant and real when v is real: substitution intertwines the color action, the covariance is invariant, and real coefficients and real covariance commute with coefficient conjugation. It therefore has a unique coefficient vector in the coherent frame.

There is a last field-number rescaling in this definition. Put

$$\xi_{k,N,p} = \frac{\beta_{N,p,k}^{-1/2}}{\vartheta_k} > 0, \qquad (D_\xi z)_i = \xi^{f_i} z_i, \tag{PC6}$$

where ϑ_k is the original barrier threshold, not a restarted threshold. Define the linear readout $T_{d,k,N,p}$ by requiring that the polynomial obtained from (PC5) have coefficient vector $D_\xi T_{d,k,N,p} v$. This determines T because every diagonal entry of D_ξ is nonzero. It also explains the exact calibration identity: the retained polynomial, including its Gaussian contraction, is the polynomial with those coefficients. There is no assertion that the whole untruncated exponential insertion is polynomial.

The analytic comparison estimate for this construction is measured before the rescaling D_ξ:

$$|A_{d,k,N,p} v - T_{d,k,N,p} v|_1 \le \frac{1}{2} |A_{d,k,N,p} v|_1. \tag{PC7}$$

We derive this relative estimate before solving any calibrated columns. Let n bound the ordinary polynomial degree, let $z = \vartheta_k / g \ge 1$, $g = \beta_{N,p,k}^{-1/2}$, and multiply every variable of the unweighted insertion polynomial by z; call that polynomial q. Its coefficient sum obeys $\mathrm{coin}(q) \le B|Av|_1$. The actual nonlinear transported-variable substitution has the form

$$X_\alpha \longmapsto f_\alpha + \vartheta_k^{-1} \mathcal{S}_{\vartheta_k} A_\alpha, \tag{PC7a}$$

where f_α is linear, $\mathcal{S}_t$ multiplies every variable by t, and every monomial of A_α has ordinary degree at least two. Their coefficient sums are at most B. For $0 < t \le 1$,

$$\mathrm{coin}(t^{-1} \mathcal{S}_t A_\alpha) \le tB, \quad \mathrm{coin}(f_\alpha + t^{-1} \mathcal{S}_t A_\alpha) \le 2B. \tag{PC7b}$$

The first assertion follows term by term from $t^{m-1} \le t$ for $m \ge 2$; the second is the triangle inequality. Telescoping each monomial of degree at most n, replacing one variable at a time, bounds the nonlinear substitution error by $\mathrm{coin}(q) n (tB)(2B)^n$. For a covariance with entries at most $\epsilon \le 1$, Wick's finite pairing expansion bounds $\mathrm{coin}(\mathcal{W}_Q q - q)$ by $\mathrm{coin}(q)\epsilon[2(n+2)]^n$. After the linear substitution this costs a further factor B^n. These two errors give

$$\mathcal{A}(t, B, \epsilon, n) = n(tB)(2B)^n + \epsilon[2(n+2)]^n B^n. \tag{PC7c}$$

Engineering truncation can only decrease coefficient sum.

For the actual holonomy Taylor reading, the calibration before Wick contraction has error at most $\epsilon|Av|_1$ in polynomial coefficients. This is the Taylor remainder of the transported slots in (PF8)–(PF11), with their ordinary nonlinear part separated as in (PC7a). On the finite-dimensional image of the polynomial exposure map, fix a constant $c \ge 1$ bounding its inverse from polynomial coefficient sum to frame coefficient sum. The total frame error is therefore

$$|Av - Tv|_1 \le c\{\epsilon + B\mathcal{A}(\vartheta_k, B, \epsilon, n)\} |Av|_1. \tag{PC7d}$$

The covariance here is the rescaled covariance $z^{-2}Q$, since scaling variables by z and Wick contraction commute with that change of covariance. The band estimate $gH_k \le 2\vartheta_k$ implies $|(z^{-2}Q)_{\alpha\beta}| \le \epsilon$ when $|Q_{\alpha\beta}| \le B$ and $4B \le H_k$.

Choose $a = [4(n+2)]^{-1}$, $B = H_k^a$, $\epsilon = H_k^{-1}$. The original threshold obeys $\vartheta_k \le H_k^{-1}$ eventually. The terms on the right of (PC7d), including its outer B, have powers of H_k at most

$$-1, \quad -1 + a(n+2) = -\tfrac{3}{4}, \quad -1 + a(n+1) < -\tfrac{3}{4}. \tag{PC7e}$$

Their fixed prefactors depend on d, n, c, not on N, p. Thus $2c\{\epsilon + B\mathcal{A}\} \le 1$ eventually, proving (PC7) uniformly in those remaining indices. Also $B + B\epsilon[2(n+2)]^n \le H_k$ eventually, so the Wick

polynomial fits its stated monitor price. These estimates use the original threshold and the actual rescaled covariance; no solved-column bound has been used to prove the inverse.

The unit is exact: its representative is the constant polynomial, contraction and truncation fix it, and both rescalings have weight zero on it. Thus $Tv_{\mathrm{unit}} = e_0$. We retain (PC7) in its relative form; an absolute error on a few selected fields would not imply the inverse assertion that follows.

The triangle inequality gives

$$|Av|_1 \le 2|Tv|_1, \qquad |Tv|_1 \le \tfrac{3}{2}|Av|_1 \le 2|Av|_1. \tag{PC8}$$

Combining the first inequality with (PC4) proves

$$|T^{-1}z|_1 \le 2O_{d,k}a_k^d|z|_1. \tag{PC9}$$

To justify existence of this inverse before using it, $Tv = 0$ implies $Av = 0$ by (PC8), then $v = 0$ by (PC4). An injective endomorphism of a finite-dimensional space is surjective. This proves both existence and the bound (PC9).

For a desired reference coefficient b and a position magnification ρ, let $(W_\rho b)_i = \rho^{r_i} b_i$. The exact solved calibration is

$$D_\xi T u = W_\rho b, \qquad u = T^{-1} D_{\xi^{-1}} W_\rho b. \tag{PC10}$$

The two diagonal operations use the distinct entries f_i, r_i of the bidegree. The solution is unique. Real b gives real u, because conjugating the equation gives the same equation and uniqueness applies. If $b = e_0$, the solution is v_{unit} by the exact unit identity. This constructs all the solved basis columns simultaneously for the one readout, rather than choosing a different readout for each column.

The finite cutoff bounds on β give

$$\xi^{-1} = \vartheta_k \sqrt{\beta_{N,p,k}} \le 2H_k, \tag{PC11}$$

where H_k is the actual monitor at level k. If $b_1, \dots, b_s$ are the finitely many chosen real representatives of a free quotient basis, (PC9)–(PC11) imply the uniform fixed-level bound

$$\|u_j\| \le 2O_{d,k}a_k^d \sum_{j'=1}^{s} |D_{2H_k} W_\rho b_{j'}|_1. \tag{PC12}$$

Indeed bound each component of $D_{\xi^{-1}}$ by the corresponding component of D_{2H_k}, use $\|\cdot\| \le |\cdot|_1$, and enlarge the single basis-column bound to their finite sum. This is the compact ball used in extracting the solved columns. It is allowed to grow with the reporting level.

For the subsequent scale-dependent prices write $H_k = e^{B_k}$, choose $\rho = H_k^b$, and put $q = \max_i f_i$. Formula (PC10) bounds

$$|Tu|_1 \le (2H_k)^q \rho^d |b|_1, \qquad H_k|Au|_1 \le 2^{q+1}|b|_1 H_k^{bd+q+1}. \tag{PC13}$$

For fixed b and d, the finite prefactor is eventually paid by one further factor H_k. The allowed polynomial coefficient price is therefore H_k^{bd+q+2}. This derives the degree-dependent price used for the calibrated polynomials. It also separates it from the fixed-level inverse bound (PC9) and from the inverse of the separating-test matrix used in (WS5).

Source–readout formulation. Two coefficient coordinates of the same transported insertion are in play: the *transported* coordinate Av (a source-side record: the primary coefficient carried by the history) and the *read* coordinate Tv (what the Gaussian-contracted polynomial shows in the coherent frame). Theorem 16.3 is a recovery theorem for the reader T: it is a bijection of the finite label space at each (k, N, p), with explicit two-sided bounds, so no label information is lost at fixed level, and (PC10) is its exact section for a prescribed target reading. What is *not* recovered is uniformity in k: $O_{d,k}$ is not bounded, and the lift does not improve it. The readout is defined on primary coefficients and has not yet been passed to the physical quotient; identifying those two is the business of the equation image below.

Theorem 16.3 (Relative comparison and invertibility of the readout)**.** *Eventually in the level, uniformly in N, p,*

$$|A_{d,k,N,p}v - T_{d,k,N,p}v|_1 \le \tfrac{1}{2}|A_{d,k,N,p}v|_1,$$

hence

$$|Av|_1 \le 2|Tv|_1, \qquad |Tv|_1 \le \tfrac{3}{2}|Av|_1,$$

T is invertible and

$$|T^{-1}z|_1 \le 2O_{d,k}a_k^d|z|_1, \qquad Tv_{\mathrm{unit}} = e_0.$$

Proof. Let n bound the ordinary polynomial degree, $z = \vartheta_k/g \ge 1$, $g = \beta_{N,p,k}^{-1/2}$, and let q be the unweighted insertion polynomial with every variable multiplied by z; its coefficient sum satisfies $\mathrm{coin}(q) \le B|Av|_1$. The nonlinear transported-variable substitution is

$$X_\alpha \longmapsto f_\alpha + \vartheta_k^{-1}\mathcal{S}_{\vartheta_k}A_\alpha,$$

f_α linear, $\mathcal{S}_t$ multiplication of every variable by t, every monomial of A_α of degree at least two, coefficient sums at most B. For $0 < t \le 1$,

$$\mathrm{coin}(t^{-1}\mathcal{S}_tA_\alpha) \le tB, \qquad \mathrm{coin}(f_\alpha + t^{-1}\mathcal{S}_tA_\alpha) \le 2B,$$

by $t^{m-1} \le t$ for $m \ge 2$. Telescoping each monomial of degree at most n one variable at a time bounds the nonlinear substitution error by $\mathrm{coin}(q)\, n(tB)(2B)^n$. For a covariance with entries at most $\epsilon \le 1$ Wick's finite pairing expansion gives $\mathrm{coin}(\mathcal{W}_Qq - q) \le \mathrm{coin}(q)\epsilon[2(n+2)]^n$, and the linear substitution costs a further B^n:

$$\mathcal{A}(t, B, \epsilon, n) = n(tB)(2B)^n + \epsilon[2(n+2)]^nB^n.$$

Truncation only decreases coefficient sums. The calibration before contraction has error at most $\epsilon|Av|_1$ (Taylor remainder of the slots, (PF8)–(PF11)). With $c \ge 1$ bounding the inverse of the polynomial exposure map on its finite-dimensional image,

$$|Av - Tv|_1 \le c\{\epsilon + B\mathcal{A}(\vartheta_k, B, \epsilon, n)\}|Av|_1,$$

the covariance being the rescaled $z^{-2}Q$, with entries at most ϵ when $|Q_{\alpha\beta}| \le B$, $4B \le H_k$, by the band estimate $gH_k \le 2\vartheta_k$. Choose $a = [4(n+2)]^{-1}$, $B = H_k^a$, $\epsilon = H_k^{-1}$; eventually $\vartheta_k \le H_k^{-1}$, and the terms of (PC7d) have powers of H_k at most

$$-1, \qquad -1 + a(n+2) = -\tfrac{3}{4}, \qquad -1 + a(n+1) < -\tfrac{3}{4},$$

with prefactors depending on d, n, c only. Hence $2c\{\epsilon + B\mathcal{A}\} \le 1$ eventually, which is (PC7); the triangle inequality gives (PC8). If $Tv = 0$ then $Av = 0$ and $v = 0$ by (PC4), so the endomorphism T is injective, hence bijective, and (PC4) with (PC8) gives (PC9). The unit is exact: its representative is the constant polynomial, fixed by contraction and truncation, and both rescalings have weight zero on it. □

The order of choices in the theorem is: degree d and level k first, then the eventual threshold in the level, uniformly in the cutoff N and period p. The factor $O_{d,k}a_k^d$ is the price of inverting the reader at level k and is retained in every later use.

16.4 The full free Maxwell polynomial hierarchy

The free comparison hierarchy is specified on all polynomial labels, with no restriction to its two-point function. Let $\mathcal{S}_j$ be the real space of homogeneous curvature Taylor coefficients of order j satisfying antisymmetry, Bianchi and the Maxwell divergence equation in (RJ3),(RJ5). Choose a real basis and write $X_{j,\alpha}$ for its coordinate symbols. The free polynomial algebra is

$$\mathcal{W} = \mathbb{C}[X_{j,\alpha}], \qquad \text{weight}(X_{j,\alpha}) = j + 2. \tag{FM1}$$

A real solution tower $A = (A_j)$ evaluates these symbols in its real coordinates. Evaluation separates polynomials by the finite-variable argument of (PF1). The restriction map from radial labels substitutes the radial potential reconstructed from A_j into each original jet coordinate. Its kernel at horizon d is exactly the equation ideal: it consists of polynomials vanishing on every such solution tower. Its image $\mathcal{W}_d^{\text{inv}}$ is the invariant sector of weight at most d. To see surjectivity onto this sector, use the color-equivariant projection of radial curvature coefficients onto their Maxwell subspace and then radial reconstruction to lift every linear solution coordinate. Polynomial substitution preserves weight and equivariance; restriction of this lift is the identity. This is an algebraic operation on coefficients, not a claim about interacting equations.

The basic covariance is fixed by

$$G_0(z) = \frac{1}{4\pi^2|z|^2}, \qquad C_{ab,kl}(z) = \delta_{bk}\partial_a\partial_l G_0 + \delta_{al}\partial_b\partial_k G_0 - \delta_{bl}\partial_a\partial_k G_0 - \delta_{ak}\partial_b\partial_l G_0, \quad z \neq 0. \tag{FM2}$$

Colors contract by their invariant Euclidean inner product. This is the curl at the first endpoint and the opposite curl at the second endpoint of the potential covariance; the opposite endpoint derivative accounts for its signs. Its two legs are closed and divergence free away from zero, by commutation of derivatives and harmonicity of G_0.

The contractions of higher solution coordinates use normalized Taylor coefficients, so the factorial convention must be retained. If $p \in \mathcal{S}_j^*$, first extend it linearly to the ambient two-form Taylor tensors. Apply it to the order-j normalized Taylor tensor of the first leg of (FM2), obtaining a two-form in the second indices and color. Pair its order-l normalized Taylor tensor with an extension of $q \in \mathcal{S}_l^*$, and multiply by $(-1)^l$. Denote the resulting scalar by $K_{p,q}(z)$. Both extensions are immaterial: at each point away from zero the differentiated leg is in the corresponding solution subspace, so their differences annihilate it. This defines the contractions on precisely the constrained variables in (FM1).

For monomials at distinct points $x_1, \dots, x_n$, regard every power of every symbol as that many distinguishable occurrences. Sum over perfect matchings of all these occurrences, allowing a pair only if its two occurrences belong to different insertion points. A pair $(i,p),(k,q)$ contributes $K_{p,q}(x_i - x_k)$. The product over its pairs, summed over the permitted matchings, is the monomial moment. The empty matching contributes one and an odd total number contributes zero. Extend by complex multilinearity in the n polynomial inputs:

$$M_n(x; P_1, \dots, P_n) = \sum_{\text{monomial choices}} \left(\prod_i c_i \right) \sum_{\substack{\text{perfect matchings} \\ \text{no pair within one insertion}}} \prod_{\text{pairs}} K_{p,q}(x_i - x_k). \tag{FM3}$$

The restriction on pairs is the Wick ordering of each composite insertion. There are finitely many terms for every tuple of polynomials. Pulling (FM3) back by the restriction map defines the free table on the original coherent label frames.

The elementary contraction is symmetric under exchange of endpoints and legs. Reindexing matchings therefore proves permutation symmetry. It depends only on differences, proving translation invariance. The chain rule applied to (FM2), orthogonality of the color action, and orthogonal covariance of the two curls give the full orthogonal tensor law. All entries are real on real coefficients, giving the reality law. Inserting the constant polynomial one introduces no occurrences, so it removes that slot. These prove the algebraic hierarchy laws for every arity.

Each derivative of a contraction of orders j, l is bounded by a constant times a power of the inverse separation; at order zero the power is $4 + j + l$. This follows by differentiating the rational homogeneous function (FM2), with the finite normalization factors included in its constant. Taking products and finite sums in (FM3) proves analyticity on the separated configuration space and a power bound there. For the product bound add the exponents and multiply constants; for a sum take their sum and enlarge to a common exponent using $r^{-1} \geq 1$ when $0 < r \leq 1$. These are the positional bounds needed for its smeared hierarchy.

Source–readout formulation. The free hierarchy is an *imported comparison realization*: a second realized sector over the same label algebra, built from the explicit kernel (FM2). It is not the interacting theory and nothing in its construction refers to the Wilson history. Its role in the supplied construction is that of a calibrated yardstick: detection, charge and Ward-kernel statements proved for it are transferred to the interacting hierarchy only through the comparison estimates that follow. The label algebra, with its restriction map, is the interface between the two sectors in the sense of the method: both hierarchies are read on the same labels, and discrepancies between them are retained as interface residuals.

The realized sector of the free hierarchy is explicit: its source record is the kernel K of (FM2) on the fixed real symbol space, its readout is the family of moments M_n of (FM3), and the only operations are multilinear extension, differentiation in the insertion points and restriction to smaller label sets. Each of these commutes with the moment formula because the sum over matchings is finite and the kernel is real analytic away from coincident points. No statement about the interacting measure is contained in this subsection.

16.5 Positivity and strictness of the reflected free pairing

We derive the reflected one-particle positivity needed for polynomial detection. Fix an axis a, let θ_a reflect that coordinate, and let R_a act on all tensor indices of a probe by this reflection. Take finitely many real probes p_i of possibly different jet orders d_i, at points x_i with $(x_i)_a > 0$. Coincident points among these sources are allowed. Float all sources by te_a, $t \geq 0$, and put

$$H(t) = \sum_{i,j} K(p_i, R_a p_j; x_i + te_a, \theta_a(x_j + te_a)),$$

where the last two arguments indicate the endpoints of the difference kernel. Let F_t be the sum of their curvature pulses. The derivative in the first endpoint and the opposite motion of the reflected endpoint give

$$H'(t) = -2L(t), \qquad L(t) = \sum_{i,j} -K(\partial_a p_i, R_a p_j; x_i + te_a, \theta_a(x_j + te_a)). \tag{GP1}$$

Here $\partial_a p_i$ is the raised probe, with the normalized derivative convention of (FM2). The factor two is the sum of the two endpoint motions.

On the reflecting plane the reflected pulse is $F_{t,bc} \mapsto \epsilon_b \epsilon_c F_{t,bc}$, where $\epsilon_a = -1$ and all other $\epsilon_b = 1$. Substitute these signs into the bilinear Maxwell stress. Its normal-normal component is $-\frac{1}{2}\sum_{b,c} \|F_{t,bc}\|^2$. The mixed terms with exactly one normal index change sign; skew symmetry gives $F_{t,aa} = 0$, and combining the remaining terms yields exactly this negative half-sum. Integrate its divergence over the positive half-box, removing small balls about its finitely many upper sources. The plane term has the negative outward normal; the small-sphere terms are the translation case of the pulse-current identity (FS2). The reflected sources are in the negative half-space and cause no extra sphere term. The other box faces disappear as their size tends to infinity. Indeed each pulse of order d is $O(|y|^{-4-d})$; its bilinear stress is $O(|y|^{-8-d_i-d_j})$, while the three-dimensional face measure is $O(R^3)$. Their fluxes are therefore $O(R^{-5-d_i-d_j})$. Summing the finitely many terms proves

$$L(t) = \lim_{R\to\infty} \frac{1}{2} \int_{[-R,R]^3} \sum_{b,c} \|F_{t,bc}(y_a = 0)\|^2 \, d^3y \geq 0. \tag{GP2}$$

The same argument bounds each finite-box integral by $L(t)$, by monotonicity of the nonnegative integrand. Since every source-to-reflected-source separation tends to infinity, the pole bounds give $H(t) \to 0$. Equations (GP1)–(GP2) make H decreasing, so $H(t) \geq 0$. This proves the real positive semidefiniteness of every finite reflected one-particle matrix. Symmetry of that matrix follows from endpoint exchange and reflection covariance. Its complex extension is Hermitian positive semidefinite by splitting a coefficient vector into real and imaginary parts.

The equality case is stronger when all source points equal one point x with $x_a > 0$. Suppose $H(0) = 0$. Monotonicity and nonnegativity imply $H(t) = 0$ for all $t \geq 0$, hence $L(t) = 0$ for $t > 0$. Equation (GP2), together with continuity, forces every component of F_t to vanish on the entire reflecting plane: a nonzero value would have a neighborhood with positive squared integral inside some finite box. Translation covariance then makes the unshifted summed pulse vanish on $y_a = -t$, for every $t > 0$. Each scalar color component is real analytic on $\mathbb{R}^4 \setminus \{x\}$, which is connected. The analytic identity theorem extends this half-space vanishing to every $y \neq x$.

Apply now the radial version of (FS2) to this zero summed pulse and any polynomial Maxwell solution f. It yields

$$\sum_i (d_i + 2)\, p_i(j_{d_i} f(x)) = 0, \tag{GP3}$$

where j_k is the normalized solution jet of order k. For any homogeneous solution tensor A_k, the field $f(y) = \mathrm{asWave}(A_k)(y - x)$ has normalized jet A_k at order k at x, and zero jets there at all other orders. Thus (GP3) gives $\sum_{i:d_i=k} p_i(A_k) = 0$, since $k + 2 > 0$. Summing over the finitely many jet orders proves that a zero reflected norm annihilates every solution tower. In particular, for any finite alphabet of independent solution-coordinate probes, its one-point reflected matrix is positive definite: evaluate the last identity on a tower with a single basis coordinate equal to one to force each coefficient to be zero. This establishes strictness on the constrained solution spaces themselves, including mixtures of jet orders.

We pass from legs to polynomial words algebraically. On finitely many complex variables define

$$\langle P, Q\rangle_{\mathrm{F}} = \sum_{\nu} \left(\prod_i \nu_i! \right) \overline{P_\nu} Q_\nu. \tag{GP4}$$

Every weight is positive; hence $\langle P, P\rangle_{\mathrm{F}} > 0$ for $P \neq 0$, including the scalar polynomial. Factor a finite reflected leg matrix as L^*L, which is ordinary finite-dimensional positive matrix factorization. Associate to a leg i the creation polynomial $\ell_i(Z) = \sum_k L_{ki} Z_k$. To represent a word, process its

occurrences successively: append its creation polynomial and, for each possible contraction to a previous occurrence on the same side, add that covariance times the polynomial for the remaining word. This finite recursion is Wick's pairing recurrence. Pairing the two resulting polynomials by (GP4) contracts their remaining occurrences across the reflection plane, since $\langle \ell_i, \ell_j \rangle_{\mathrm{F}} = (L^* L)_{ij}$. The factorial $\prod \nu_i!$ counts the bijections between equal creation occurrences. Induction on word length therefore identifies the reflected Wick moment with this polynomial inner product, including all permitted contractions internal to either half. A finite linear combination of words has reflected norm equal to the nonnegative squared norm of its associated polynomial. Expanding polynomial insertions into monomials proves reflection positivity at arbitrary arity, and smearing preserves it by integration.

At a single source point there are no contractions within either half because each insertion is Wick ordered. The recursion is then simply the substitution $Z_i \mapsto \ell_i(Z)$. The one-point leg matrix is positive definite by (GP3), so L is invertible; substituting L^{-1} gives the inverse polynomial substitution. Thus a nonzero polynomial remains nonzero, and (GP4) makes its reflected pairing strictly positive. This is the strict pairing used in the hosting argument and the detector construction. It proves that the free all-spectator null space has precisely the equation ideal already removed in forming the solution coordinates.

Accordingly the free physical field space is canonically

$$E_d / N_d \simeq \mathcal{W}_d^{\mathrm{inv}}. \tag{FM4}$$

Under this identification the horizon inclusion is literal polynomial inclusion. A derivative in direction v is the polynomial derivation determined on a linear symbol by

$$\mathsf{D}_v \, \mathrm{symbol}_j(p) = \mathrm{symbol}_{j+1}(p \circ \mathrm{lower}_v). \tag{FM5}$$

Here $\mathrm{lower}_v A_{j+1}$ is the order-j Taylor coefficient of the derivative of the homogeneous polynomial represented by A_{j+1}; its coefficient is $j+1$ times contraction with v. Thus (FM5) has the actual Taylor normalization. The Leibniz rule differentiates one occurrence at a time. Differentiating a matching product in (FM3) selects its incident covariance edge, while replacing an occurrence by (FM5) differentiates precisely that same edge. Summing proves the insertion-derivative identity. It raises the horizon by one and stays in the invariant sector. Conjugation and signed transformations obey (QD6). Hence the free differential system is the same type of all-spectator system as the interacting one.

For $s \in O(4)$ the free field action is the polynomial substitution induced by the tensor action on the solution tower. It preserves weight, equations and color invariance, with inverse given by s^{-1}; conjugating it through (FM4) gives the full field action. Infinitesimal skew rotations likewise act by the derivation obtained by differentiating that linear tensor substitution. Both descriptions are on the same field space and preserve the moment law (FM3).

The free symmetric traceless stress is the actual weight-four invariant polynomial

$$T_{ab}^{\mathrm{free}} = \sum_{c,i} F_{ai}^c F_{bi}^c - \frac{\delta_{ab}}{4} \sum_{c,i,k} F_{ik}^c F_{ik}^c. \tag{FM6}$$

Products here mean the Wick polynomial product used in (FM3). Adjoint invariance follows from preservation of the color inner product, symmetry by exchanging its two factors, and reality from real coefficients. Its trace is zero by summing the four diagonal entries. Its divergence is zero as a polynomial on solution jets. Indeed the derivative of the first term is $\sum_{c,i} F_{ai}^c \partial_a F_{bi}^c$, since $\partial_a F_{ai}^c = 0$. Bianchi and antisymmetry give

$$\sum_{a,i} F_{ai}^c \partial_a F_{bi}^c = \frac{1}{4} \partial_b \sum_{a,i} F_{ai}^c F_{ai}^c; \tag{FM7}$$

multiply the Bianchi equation by F^c_{ai}, sum, and interchange a, i in its second derivative term to obtain the factor two on the left and the half derivative on the right. This cancels the divergence of the subtracted term in (FM6). Evaluation on all real solution towers then proves the polynomial identity. Through (FM4) it gives a real, symmetric, conserved free field at horizon four, with the exact normalization used in the later free comparison.

Source–readout formulation. The free readout (all separated correlations with all spectators) has a kernel on labels, and (FM4) computes it exactly: *the information the free sector cannot see is the equation ideal and nothing else.* The quotient map from radial labels to $\mathcal{W}_d^{\mathrm{inv}}$ is the readout; the color-equivariant projection followed by radial reconstruction (Theorem 16.1) is its section. This is a quotient equivalence in the method's classification, proved by a strict positivity argument and not assumed as nondegeneracy.

Theorem 16.4 (Strictness at one point)**.** *If all sources sit at one point x with $x_a > 0$ and $H(0) = 0$, then*

$$\sum_i (d_i + 2)\, p_i(j_{d_i} f(x)) = 0$$

for every polynomial Maxwell solution f, and therefore $\sum_{i:d_i=k} p_i = 0$ on $\mathcal{S}_k$ for every k. In particular the one-point reflected matrix of any finite family of independent solution-coordinate probes, of mixed jet orders, is positive definite.

Proof. Monotonicity and nonnegativity give $H \equiv 0$, so $L(t) = 0$ for $t > 0$ and by (GP2) and continuity F_t vanishes on the reflecting plane. By translation the unshifted summed pulse vanishes on $y_a = -t$ for all $t > 0$, hence on a half-space, and by real analyticity on the connected set $\mathbb{R}^4 \setminus \{x\}$ it vanishes identically. The radial case of (FS2) applied to this zero pulse gives (GP3). For a homogeneous solution tensor A_k the field $f(y) = \mathrm{asWave}(A_k)(y - x)$ has jet A_k at order k and zero jets at other orders at x, and $k + 2 > 0$. □

Strictness at one point is the form of the statement that is used: it says that the readout "reflected pairing against all polynomial Maxwell solutions" has no kernel on solution-coordinate probes beyond the equation ideal, for probes of mixed jet orders at the same point. The weights $d_i + 2$ are the engineering weights of the probes and are retained; they are what separates the homogeneous components in the proof.

16.6 The free low-degree stress and vector spaces

The low-degree classification uses simplicity of the actual color Lie algebra, without assuming that the matrix group is connected. Differentiate its adjoint action along the actual exponential curves. This gives

$$\frac{d}{dt}\,\mathrm{Ad}_{\exp(tx)}\, y\Big|_0 = [x, y]. \tag{FC1}$$

The span of brackets is a nonzero ideal, hence is the whole simple nonabelian color algebra. An invariant linear functional annihilates each bracket by (FC1), so it is zero. An invariant real bilinear form satisfies $B([x, y], z) = -B(y, [x, z])$. Its alternating part is zero: three cyclic applications to $B([x, y], z)$, with bracket antisymmetry, identify this value with its negative, and brackets span. For the symmetric part write $B(x, y) = \langle Ax, y\rangle$ with A self-adjoint. Invariance says $A[x, y] = [x, Ay]$. Every eigenspace of A is therefore an ideal. A real self-adjoint operator has an eigenvalue; its nonzero eigenspace is the whole algebra, so A is scalar. Applying this argument to real and

imaginary parts proves that every invariant complex-valued real bilinear form is a complex multiple of the fixed color inner product.

A monomial of weight at most five has at most two fields, since each variable has weight at least two. With two fields its jet orders are either $(0,0)$ or $(0,1)$; the latter has weight five. Decompose a polynomial into its constant, linear and these quadratic pieces by ordinary field number and engineering weight. Invariance of each piece follows from the linearity of the color action. The linear piece vanishes by (FC1). Polarizing the $(0,0)$ piece, and separating the two jet orders in the $(0,1)$ piece, gives invariant bilinear forms. Resolve their spatial tensor coordinates; (FC1) then makes every color contraction a multiple of the color inner product. Consequently all weight-four labels are spanned by the unit and $\langle F_{ab}, F_{cd}\rangle$; through weight five one adds $\langle F_{ab}, \partial_e F_{cd}\rangle$. This is a spanning assertion about the actual invariant labels before further free equation relations.

Put $Q_{ab,cd} = \sum_c F^c_{ab} F^c_{cd}$, with products interpreted as in (FM3), and define

$$N_a = \sum_i Q_{ai,ai}, \qquad R_a = \sum_{i,k:\ i\neq a,\ k\neq a} Q_{ik,ik}, \qquad P_{ab} = \sum_i Q_{ai,bi}. \tag{FC2}$$

The sums in R_a are over ordered pairs. Consider centered real symmetric tensor fields transforming under the signed coordinate group as rank-two tensors. Reflection in one axis multiplies F_{ab} by the product of its two axis signs. Thus a diagonal component permits only products in which every axis occurs evenly; for two antisymmetric pairs this forces the same unoriented plane twice. The permutations fixing axis a have just two orbits: planes containing a and planes disjoint from a. They give N_a, R_a. A constant could also occur, but centeredness removes it: the Wick quadratic one-point functions are zero and the unit expectation is one.

For the off-diagonal component ab, $a \neq b$, the parity character is the product of the signs of a, b. The two planes must therefore share exactly one other axis i, and be ai, bi, up to antisymmetry and exchange of the two factors. Permutations fixing a, b identify the possible remaining axes, giving precisely P_{ab}. The characters of the other products differ in at least one reflection, so averaging against the required character kills them. Permuting axes then determines every tensor component from 00 and 01. Reality makes the coefficients real. The whole centered signed tensor space is therefore spanned by

$$A_{ab} = \delta_{ab} N_a, \qquad B_{ab} = \delta_{ab} R_a, \qquad C_{ab} = 1_{\{a\neq b\}} P_{ab}. \tag{FC3}$$

For a real polar vector of weight at most five, fix component zero and average under its stabilizing permutations and reflection character. Constants and FF terms have the wrong parity. In an $F_{ab}\partial_e F_{cd}$ term, parity requires that the symmetric difference of the two plane-index sets be $\{0, e\}$. If $e = 0$, the planes coincide, giving the first two sums below. Otherwise they share a remaining index; the location of the derivative leg gives the last two sums. Thus its zeroth component is spanned by

$$\begin{aligned} n &= \sum_{i\neq 0} \langle F_{0i}, \partial_0 F_{0i}\rangle, \qquad c = \sum_{\substack{i,j\neq 0\\ i\neq j}} \langle F_{ij}, \partial_0 F_{ij}\rangle, \\ a &= \sum_{\substack{i,j\neq 0\\ i\neq j}} \langle F_{0i}, \partial_j F_{ji}\rangle, \quad h = \sum_{\substack{i,j\neq 0\\ i\neq j}} \langle F_{ji}, \partial_j F_{0i}\rangle. \end{aligned} \tag{FC4}$$

The free divergence equation gives $n + a = 0$. Bianchi gives $\langle F_{ji}, \partial_j F_{0i}\rangle + \langle F_{ij}, \partial_i F_{0j}\rangle = \langle F_{ij}, \partial_0 F_{ij}\rangle$. Summing over ordered distinct nonzero i, j gives $2h = c$. Hence only n, h are needed. A signed vector is determined by its zeroth component, since exchanging axis zero with any axis transports that component. The polar vector space has real dimension at most two. These relations hold on all solution towers, so they are zero in the actual free quotient by (FM4).

The lower rank bounds are obtained by explicit separated correlations, not by declaring the formal expressions independent. Let $H_{ab,cd}(z) = C_{ab,cd}(z)$ be (FM2) and set

$$\mathcal{Q}_{ab,ik;ce,fm}(z) = H_{ab,ce}(z)H_{ik,fm}(z) + H_{ab,fm}(z)H_{ik,ce}(z). \tag{FC5}$$

The correlation of $Q_{ab,ik}$ with $Q_{ce,fm}$ at $z, 0$ is $(\dim \mathfrak{g})\mathcal{Q}(z)$, because the two cross matchings force the same color. At $z = e_t$, writing $\nu = (4\pi^2)^{-1}$, the Hessian and third derivative of the Green function are

$$\begin{aligned} \nu^{-1}\partial_j\partial_i G_0(e_t) &= -2\delta_{ij} + 8\delta_{ti}\delta_{tj}, \\ \nu^{-1}\partial_r\partial_j\partial_i G_0(e_t) &= 8\delta_{ij}\delta_{tr} - 48\delta_{ti}\delta_{tj}\delta_{tr} + 8(\delta_{ri}\delta_{tj} + \delta_{rj}\delta_{ti}). \end{aligned} \tag{FC6}$$

These follow by differentiating $|z|^{-2}$. Substituting them into the four signed Kronecker terms in (FM2), and applying the two-factor product rule to (FC5), gives every sample below. The common nonzero factor removed is $\ell_G = (\dim \mathfrak{g})\nu^2 > 0$.

Take the divergence component zero of (FC3), correlate it at e_0 with $Q_{01,01}$ and $Q_{12,12}$ at zero, divide by ℓ_G, and take real parts. The two resulting linear functionals have columns

	A	B	C
$Q_{01,01}$	-256	0	128
$Q_{12,12}$	0	-512	-128

. (FC7)

For example the first A entry is $\sum_{k=0}^{3} \partial_0 \mathcal{Q}_{0k,0k;01,01}(e_0)/\nu^2 = -256$; the corresponding B entry is the same sum over ordered $i, k \neq 0$, which is zero. For C sum $\partial_r \mathcal{Q}_{rk,0k;01,01}$ over $r \neq 0, k$. The other row uses $12, 12$. These finite sums and (FC6) give respectively the three displayed columns. A third, undifferentiated sample correlates tensor component 01 at e_0 with $Q_{02,12}$ at zero; its columns are $(0, 0, -16)$, directly from (FC5). It separates the last tensor direction.

Thus A, B, C are independent, so the tensor space has dimension exactly three. The two divergence samples in (FC7) are independent; combined with the upper bound two from (FC4), they identify the entire polar vector space and prove surjectivity of divergence. Rank-nullity gives a one-dimensional conserved kernel. Solving (FC7) yields its generator

$$\tfrac{1}{2}A - \tfrac{1}{4}B + C = T^{\mathrm{free}}. \tag{FC8}$$

For its diagonal entry use $\sum_{i,k} Q_{ik,ik} = 2N_a + R_a$; its off-diagonal entry is P_{ab}. This is exactly (FM6), already shown conserved. The last sample is nonzero on it, so the generator is nonzero. Every conserved centered real signed tensor in this weight-four space is its real scalar multiple. Any later nonzero scalar normalization, including either sign, preserves the rank and the conserved-line statements; the normalization of shell fluxes is a separate argument.

There is a raw degree-five vector direction which must be kept until the equation quotient is taken. In polynomial notation its components are

$$E_b = \sum_{i,a} \mathrm{pairDeriv}(b, i, a, a, i), \tag{FC9}$$

that is, the contraction of F_{bi} with the equation $\partial_a F_{ai}$, in the normalized radial-jet coordinates used above. The two other raw vector directions are the polynomials called momentum1 and momentum2 in the preceding classification. Every real signed-equivariant raw vector has a unique expression $uV_1 + vV_2 + wE$. To see the independence after imposing the Maxwell equations, take a unit color vector and the two solution jets with respectively

$$(F^{(0)}, F^{(1)}) = (\mathrm{leg}_{12}, \mathrm{leg}_{0;12}), \quad (\mathrm{leg}_{01}, \mathrm{leg}_{0;01} - \mathrm{leg}_{2;21}). \tag{FC10}$$

Here the legs are radial potential jets, with the normalization made explicit below. Both satisfy the linear divergence equation and Bianchi identity. The first reads $V_1 = 0, V_2 = 4/3$ in component zero; the second reads $V_1 = 1$. An equation polynomial reads zero on both. It follows first that $v = 0$, then that $u = 0$. Conversely E vanishes on every solution jet, directly by the divergence equation. A non-solution jet with nonzero divergence shows that E itself is a nonzero raw polynomial. Thus the raw vector space has dimension three, and the kernel of its map to the free vector quotient is exactly the one-dimensional space $\mathbb{R}E$.

This statement is independent of coefficient coordinates. For a coherent degree-five frame, apply its inverse coordinate map to each component of a real signed-equivariant coefficient vector. This is a real linear isomorphism to the raw polynomial vector space, because the coordinate map intertwines signed permutations and conjugation. Composing with the free quotient projection gives a surjection whose kernel is the inverse-coordinate image of $\mathbb{R}E$. A section is obtained by applying the corrected polynomial lift to each quotient component. That lift commutes with both symmetries, and its quotient projection is the identity. These facts justify the vector kernel subtraction used later: it removes precisely the equation direction, and does not discard either of the two nonzero free vector classes.

To make the raw coordinates completely explicit, write

$$P_{ab;kcd} = \langle F_{ab}, \partial_k F_{cd} \rangle, \quad (V_1)_b = \sum_i P_{bi;bbi}, \quad (V_2)_b = \sum_{i,j} P_{ij;bij}, \quad E_b = \sum_{i,a} P_{bi;aai}. \tag{FC11}$$

The pairings are the actual invariant polynomial labels, evaluated on radial potential jets; imposing the Maxwell equation is a later quotient operation. The basic legs in (FC10) are the homogeneous potentials

$$(\mathrm{leg}_{ab}\, e)_i(z) = \tfrac{1}{2}(\delta_{ib} z_a - \delta_{ia} z_b)e, \qquad (\mathrm{leg}_{k;ab}\, e)_i(z) = \tfrac{1}{3}(\delta_{ib} z_a - \delta_{ia} z_b) z_k e. \tag{FC12}$$

Taking their exterior derivatives defines the curvature legs, including the factor $1/3$ in the first curvature derivative. This factor explains the $4/3$ sample in (FC10). A constant curvature is automatically divergence free. For a linear curvature the Maxwell condition is exactly $\sum_i \mathrm{curl}_1(i, i, b) = 0$ for every b; direct differentiation of (FC12) checks it for the two combinations in (FC10). Higher homogeneous jets can be set to zero.

Here is also the raw upper-dimension argument, before using those solution samples. A weight-five label decomposes as a constant, an invariant linear term, an FF term and an $F\,\partial F$ term. Simplicity kills the linear term as above. Total axis inversion fixes the two F's but negates a polar vector, so both its constant and FF terms vanish. Resolve its remaining invariant bilinear color form into the color inner product. To recover its spatial coefficients it suffices to use the mixed evaluations

$$\mathfrak{m}(A, B; v) = \mathrm{Re}\{v(A + B) - v(A) - v(B) + v(0)\}, \tag{FC13}$$

where A occupies jet order zero and B order one. Take, in component zero, the three pairs

$$(\mathrm{leg}_{01}\, e, \mathrm{leg}_{0;01}\, e), \quad (\mathrm{leg}_{01}\, e, \mathrm{leg}_{2;21}\, e), \quad (\mathrm{leg}_{21}\, e, \mathrm{leg}_{2;01}\, e).$$

If all three readings vanish, every mixed coefficient vanishes. Indeed reflections require the symmetric difference of its two plane-index sets to equal $\{0, k\}$, where k is the derivative index. For $k = 0$, equal planes containing zero and equal planes avoiding zero are the two possible types. For $k \neq 0$, the planes share one other index, and the two placements of the derivative factor are the other types. Permutations fixing zero identify all coefficients of each type; antisymmetry fixes their

signs. The three displayed tests kill three types. The cyclic identity $\mathrm{leg}_{0;12}+\mathrm{leg}_{1;20}+\mathrm{leg}_{2;01}=0$, paired with $\mathrm{leg}_{12}\,e$, kills the fourth. The legs span each radial jet space by the expansion

$$A=\tfrac{1}{2}\sum_{i,j}\mathrm{leg}_{ij}(\mathrm{curl}_0(i,j)A),\qquad B=\tfrac{1}{2}\sum_{i,j,k}\mathrm{leg}_{k;ij}(\mathrm{curl}_1(k,i,j)B).$$

Bilinearity therefore kills the entire mixed part. Reality recovers the complex coefficient from its real part. The three evaluations form an injective real map, proving the raw upper bound three. The two solution evaluations in (FC10) and the nonsolution pair $(\mathrm{leg}_{01}\,e,\mathrm{leg}_{0;01}\,e)$, on which $E_0=1$, prove independence of V_1,V_2,E. Thus (FC11) is a basis and every raw vector has unique coordinates.

For later use, the divergence of the raw centered tensor with coefficients $w=(w_0,w_1,w_2)$, respectively diagonal mountain, diagonal total and off-diagonal shear, has these coordinates:

$$\mathrm{Div}(w)=2(w_0-w_2)V_1+(2w_1+\tfrac{1}{2}w_2)V_2+w_2E. \tag{FC14}$$

The product rule gives $2V_1$ for the first tensor and $2V_2$ for the second. For the shear use Bianchi and interchange the ordered indices to obtain $\sum_{a,i}P_{ai;abi}=\frac{1}{2}V_{2,b}$; deleting the $a=b$ terms subtracts $2V_{1,b}$, while the other product derivative is E_b. This proves (FC14). Its coordinate matrix is invertible, with inverse

$$w_2=y_2,\qquad w_0=y_0/2+y_2,\qquad w_1=y_1/2-y_2/4.$$

It defines the actual scalar contraction on raw vectors:

$$\rho(\mathrm{Div}(w))=-4(w_0+4w_1)=-2y_0-8y_1. \tag{FC15}$$

If $Q=\frac{1}{4}\sum_{i,j}\langle F_{ij},F_{ij}\rangle$, the tensor trace is $4(w_0+4w_1)Q$; hence ρ reads minus that trace in Q-units. It kills E and the traceless conserved tensor $w=(1,-1/4,1)$. Since (FC10) proves that the raw equation kernel is exactly $\mathbb{R}E$, ρ annihilates that entire kernel. This is the scalar sensor used later, with its sign and normalization fixed. The identification of the free quotient here uses the Maxwell polynomial evaluation and the reflected Gram norm proved in (FM3)–(FM4) and in the detector's reflected polynomial calculation; it is not an additional assumption of nondegeneracy.

Source–readout formulation. This subsection is a classification of invariant label spaces in low weight, and it contains the simplest instance of the descent criterion in the whole construction. The source object is a raw vector, a signed equivariant element of the label algebra before the equation quotient; the readout is its class in the free vector quotient.

Proposition 16.5 (Raw vectors and the equation kernel)**.** *Every real signed-equivariant raw vector of weight five is uniquely uV_1+vV_2+wE, and the kernel of the map to the free vector quotient is exactly $\mathbb{R}E$.*

Proof. Upper bound: a weight-five label is a constant, an invariant linear term, an FF term and an $F\partial F$ term; simplicity kills the linear term and total inversion the constant and FF terms of a polar vector. The mixed coefficients are recovered by

$$\mathfrak{m}(A,B;v)=\mathrm{Re}\{v(A+B)-v(A)-v(B)+v(0)\}$$

(A at order zero, B at order one) on the three pairs $(\mathrm{leg}_{01}\,e,\mathrm{leg}_{0;01}\,e)$, $(\mathrm{leg}_{01}\,e,\mathrm{leg}_{2;21}\,e)$, $(\mathrm{leg}_{21}\,e,\mathrm{leg}_{2;01}\,e)$: reflections require the symmetric difference of the two plane-index sets

to be $\{0,k\}$, giving four types, of which the three tests kill three and the cyclic identity $\mathrm{leg}_{0;12}+\mathrm{leg}_{1;20}+\mathrm{leg}_{2;01}=0$ paired with $\mathrm{leg}_{12}\,e$ the fourth; the legs span the jet spaces by $A=\frac{1}{2}\sum_{i,j}\mathrm{leg}_{ij}(\mathrm{curl}_0(i,j)A)$, $B=\frac{1}{2}\sum_{i,j,k}\mathrm{leg}_{k;ij}(\mathrm{curl}_1(k,i,j)B)$. Independence: the two solution jets

$$(F^{(0)},F^{(1)})=(\mathrm{leg}_{12},\mathrm{leg}_{0;12}),\qquad(\mathrm{leg}_{01},\mathrm{leg}_{0;01}-\mathrm{leg}_{2;21})$$

satisfy the divergence and Bianchi equations; the first reads $V_1=0$, $V_2=4/3$ in component zero and the second $V_1=1$, while an equation polynomial reads zero on both, so $v=0$ then $u=0$. E vanishes on every solution jet and equals one on the non-solution pair $(\mathrm{leg}_{01}\,e,\mathrm{leg}_{0;01}\,e)$. □

The scalar contraction ρ of (FC15) is a reader on raw vectors that is *constant on the fibres of the equation quotient*: it kills E and the traceless conserved tensor $w=(1,-\frac{1}{4},1)$, for which $\mathrm{Div}(w)=E$, and hence the whole raw equation kernel. It therefore descends to the physical vector space. This is the scalar sensor used later, with sign and normalization fixed by (FC14)–(FC15): it reads minus the trace in units of $Q=\frac{1}{4}\sum_{i,j}\langle F_{ij},F_{ij}\rangle$. The classification itself uses simplicity of the color Lie algebra through (FC1) and does not use connectedness of the group; this is a hypothesis of the supplied construction and is retained. The coordinates $(y_0,y_1,y_2)\leftrightarrow(w_0,w_1,w_2)$ of (FC14) are an explicit inverse pair, so the divergence map from invariant symmetric tensors of weight four to raw vectors of weight five is a bijection, and the fibre of the equation quotient is the single line $\mathbb{R}E$.

16.7 Free shell charges and separation of the Ward kernel

We give the polynomial calculations underlying the free charge separation used in the rotation and trace arguments. This also specifies the exterior tests needed when a charge acts on high-degree fields.

16.8 The compact stress shell identity

Let χ be a smooth compactly supported function equal to one near a distinguished point x, whose support avoids all spectator points. For a current J, define its outward shell functional by

$$\mathcal{E}_\chi(J)=-\sum_a\int(\partial_a\chi)(y)J_a(y)\,d^4y. \tag{FS1}$$

The integrand is supported in a compact annulus away from all insertions. All exchanges of finite Wick sums and this integral are consequently justified by smoothness there. A one-leg source p at x produces the Maxwell pulse obtained by applying its jet functional to the covariance. For a smooth source-free Maxwell field f on a neighborhood of the support of χ, Green's formula for the bilinear Maxwell stress current gives

$$\mathcal{E}_\chi J_v(\mathrm{pulse}_{p,x},f)=-\,\mathrm{sample}_x(p,\delta_v f). \tag{FS2}$$

Indeed the divergence vanishes off the source by the Maxwell and Bianchi equations. Integrating the differentiated fundamental covariance against the cutoff transfers the derivatives to the smooth field. The fundamental-solution identity leaves precisely its jet at x. The boundary term at infinity is absent because the cutoff is compact. The variation of a two-form is $\delta_v f_{ab}=v^c\partial_c f_{ab}+(\partial_a v^c)f_{cb}+(\partial_b v^c)f_{ac}$. For the three vector fields used here this variation preserves the Maxwell equations. The stress is symmetric and traceless, so its contraction with the symmetric derivative of a constant, skew-linear, or radial vector field contributes zero to the divergence.

Here is a regularized derivation of the fundamental-solution step in (FS2). With the covariance normalization $c_4 = 1/(4\pi^2)$, put

$$g_t(y) = \frac{c_4}{t^2 + |y|^2}, \qquad \rho_t(y) = \frac{8c_4 t^2}{(t^2 + |y|^2)^3}, \qquad t > 0. \tag{FS2a}$$

Direct differentiation gives $-\Delta g_t = \rho_t$. The radial integral gives $\int \rho_1 = 1$: the sphere area is $2\pi^2$ and $\int_0^\infty r^3(1 + r^2)^{-3} dr = 1/4$. Also $t^4 \rho_t(ty) = \rho_1(y)$. Hence for every continuous compactly supported h, $\int \rho_t(y - x)h(y)dy \to h(x)$, by the change of variable $y = x + tz$ and the integrable majorant $\rho_1(z)\|h\|_\infty$.

For a plane source il, form the potential $A_b = \delta_{ib}\partial_l g_t - \delta_{lb}\partial_i g_t$, and take its exterior derivative. It satisfies the Bianchi identity by commutation of derivatives, while its divergence is $\delta_{lb}\partial_i \rho_t - \delta_{ib}\partial_l \rho_t$. The divergence of its bilinear stress with the source-free field f is therefore the contraction of this expression with f; the terms from the symmetric derivative of v cancel by tracelessness and the other terms cancel by skew symmetry. Integration by parts against χ transfers the source derivatives to the smooth compact test made from χ, v, f. The preceding approximate-identity limit evaluates their difference at x, giving $-\delta_v f_{il}(x)$. On the shell the regulated exterior derivative converges smoothly to the pulse, since that compact shell avoids x. Its flux thus converges to the left side of (FS2). This proves the plane-source case with the stated sign.

Differentiate the source center in any direction while keeping the cutoff and vector field fixed. All differentiated shell integrands have a common bound on a fixed compact annulus, so differentiation passes through the integral. A center derivative raises the source leg by that direction; on the right it differentiates the sample of $\delta_v f$. Induction gives (FS2) for all derivative legs; linearity gives every solution-jet functional. The wave property of $\delta_v f$ for the three motions ensures these samples respect the Maxwell jet relations.

For a constant vector e_i, (FS2) gives minus the derivative-leg sample. For $v(y) = A(y - x)$, with A skew, it gives minus the infinitesimal tensor rotation. For $v(y) = y - x$, a homogeneous solution jet of derivative order j has Euler contribution j, and the two form indices contribute two more. Thus the one-leg radial charge is $-(j + 2)$ times that jet. This specifies the engineering weight, including the two contributed by curvature rather than its derivatives.

When both pulses are centered at x, use cutoff independence and homogeneity instead of multiplying distributions at their common pole. Two cutoffs equal to one near x have a difference supported away from x; source-free integration by parts makes their fluxes equal. Replace the cutoff by its dilation by a factor two about x. A jet-order-j pulse scales by 2^{-4-j}; the other scales by 2^{-4-l}; the volume and cutoff derivative together contribute 2^3, and a motion homogeneous of degree $e = 0$ or 1 contributes 2^e. Thus the common flux Z equals $2^{e-5-j-l}Z$. That coefficient is strictly smaller than one, so $Z = 0$. When neither pulse is centered at x, their fields are smooth throughout the support of the cutoff, and the source-free divergence identity gives zero. When precisely one pulse is centered there and the other is at a spectator point, (FS2) gives minus the covariance with the transformed distinguished leg.

Now insert the quadratic Wick stress in an arbitrary Wick monomial word. Double Wick extraction expresses its correlation as one half the sum over ordered choices of two occurrences, times the bilinear pulse current for those occurrences and the Wick pairing of all remaining occurrences. There are exactly the three cases just considered. The terms with both distinguished occurrences or with neither distinguished occurrence vanish. In each remaining term exactly one occurrence of the distinguished monomial is transformed. The two orders of the selected pair cancel the one half. Summing over its occurrences is the Leibniz rule for the derivation on that monomial. By

multilinearity this proves, for arbitrary polynomial words,

$$\mathcal{E}_\chi\left(\sum_b v_b(y)\langle T_{ab}(y)P(x)\prod_\ell P_\ell(x_\ell)\rangle\right) = -\left\langle(\mathcal{D}_v P)(x)\prod_\ell P_\ell(x_\ell)\right\rangle. \tag{FS3}$$

Here $\mathcal{D}_v$ is respectively translation, rotation, or the engineering-weight derivation. This proves the three shell laws, including their common minus sign, and includes every contraction term. In particular conservation of the free stress alone was not being used as a substitute for its charge normalization.

Source–readout formulation. The shell functional is a reader of the stress insertion that depends on the auxiliary cutoff χ only through the set of enclosed insertions: two cutoffs equal to one near x and avoiding the spectators give equal values. The cutoff is thus a presentation choice with trivial effect on the readout, and (FS3) says that the reader returns the derivation $-\mathcal{D}_v$ on the enclosed label: the charge is *recovered from* the hierarchy and is not an extra datum.

16.9 Low-degree vector charges

Let $\mathscr{S}_j$ be the real space of homogeneous order-j Maxwell solution jets and $\mathscr{V}_j = \mathscr{S}_j^*$. Their elements are the real one-leg test functionals. A variable at order j has engineering weight $j+2$. For $p \in \mathscr{V}_j$, write $B_c(p)$ for its linear polynomial in color c, and set

$$[p,q] = \sum_c B_c(p)B_c(q).$$

This bracket denotes a color-contracted quadratic polynomial, not a Lie bracket. For a linear map $D : \mathscr{S}_j \to \mathscr{V}_l$, its polynomial is

$$\mathcal{C}(D) = \sum_\nu [e^\nu, De_\nu], \tag{FS4}$$

where (e_ν) is any basis and (e^ν) its dual. This is basis-independent, and $\mathcal{C}(p\otimes q) = [p,q]$. If $j \neq l$, assigning independent jets in these two orders in one nonzero color proves that $\mathcal{C}$ is injective.

Consider a vector row T_i on the invariant polynomial filtration, killing the unit, increasing filtration by at most one, real under conjugation, and signed-covariant. Suppose it is silent, meaning

$$\sum_\ell \langle P_1(x_1)\cdots(T_iP_\ell)(x_\ell)\cdots P_n(x_n)\rangle = 0 \tag{FS5}$$

for every allowed exterior word and separated configuration. At input weight four the invariant space is spanned by the unit and $[p,q]$ with $p,q \in \mathscr{V}_0$. At output weight at most five the only further possibility is a color-contracted pair of orders zero and one. Total inversion is even on weight four and odd on such a pair. The vector law anticommutes with total inversion, and therefore removes the weight-four part of the output. Reality permits real coefficients in (FS4). Consequently

$$T_i[p,p] = \mathcal{C}(D_i(p)), \qquad D_i(p) : \mathscr{S}_0 \to \mathscr{V}_1. \tag{FS6}$$

Let $K_z(p,q)$ denote the real single-color covariance between two test legs at displacement $z \neq 0$. To extract the constraint on (FS6), put two quadratic insertions at tu and zero, and a third at a fixed nonzero point, and multiply (FS5) by t^5. The covariance of jet orders zero and one is homogeneous of degree -5. The triangle terms where the transformed order-one leg is on either near insertion have a finite limit; the term with that leg at the distant insertion tends to zero. The

double contractions contribute $4\dim\mathfrak{g}$. Choose the distant one-leg functional so that its covariance with a zero-order test is evaluation on a prescribed $X \in \mathscr{S}_0$. Cancelling the common nonzero factor gives

$$K_u(D_i(p)X, q)\, q(X) = K_u(D_i(q)X, p)\, p(X). \tag{FS7}$$

The availability of these evaluations follows from covariance separation of solution jets. First suppose a test functional of order j has zero covariance with every order-zero plane test at every nonzero displacement. Its pulse is then zero off its source, in every color component. Pair its radial stress shell with the homogeneous polynomial Maxwell solution having any prescribed order-j solution jet. The shell integral is zero because the pulse is zero there; (FS2) makes it $-(j+2)$ times the test evaluated on that arbitrary jet. The functional is zero. For partner order $l+1$, vanishing against all such partners makes the radial derivative of its covariance with any order-l partner zero. That covariance is homogeneous of degree $-(4+j+l)$; Euler's rule makes the radial derivative that nonzero scalar times the covariance. Thus it vanishes, and induction reduces to order zero. Finite-dimensional duality now makes these covariance evaluations span the required full dual. Thus no inverse of a covariance matrix at a single momentum is being assumed. This also proves that the solution-jet space is nonzero at every order: otherwise separation against that order would force the nonzero plane evaluation at order zero to vanish.

If $p(X) = 0$, (FS7) first makes $K_u(D_i(p)X, q) = 0$ for $q(X) \neq 0$. Given $q(X) = 0$, add a test with nonzero value at X, apply that result to the sum, and subtract. Covariance separation gives $D_i(p)X = 0$. Hence $\ker p \subseteq \ker D_i(p)$. Choose X_0 with $p(X_0) \neq 0$; subtracting $p(X)X_0/p(X_0)$ from X proves

$$D_i(p) = p \otimes f_i(p), \qquad T_i[p,p] = [p, f_i(p)]. \tag{FS8}$$

Set $f_i(0) = 0$. For two nonzero linear functionals there is an X on which both are nonzero: use a witness for one, a witness for the other, and their sum if needed. Substitution in (FS7) yields

$$K_z(f_i(p), q) = K_z(f_i(q), p). \tag{FS9}$$

Pair the difference $f_i(p+q) - f_i(p) - f_i(q)$ with an arbitrary one-leg test, use (FS9), and cancel by linearity in its other slot. Separation makes the difference zero. The same argument with a scalar proves homogeneity. Thus $f_i : \mathscr{V}_0 \to \mathscr{V}_1$ is real linear. Signed covariance of T_i, injectivity of (FS4), and cancellation of a nonzero test functional give simultaneous signed covariance of the family f_i.

We classify this family explicitly. Write c_{ab} for evaluation of a zero jet in the ab-plane and $s_{iab} = \partial_i c_{ab}$ for the corresponding first-jet functional. Under reflection of axis z, the latter has character equal to the product of the three signs belonging to i, a, b. Project onto a reflection character by the four commuting projections $(1 \pm R_z)/2$. The character of $f_0(c_{01})$ leaves only $s_{001}, s_{212}, s_{313}$. Average the permutation interchanging axes two and three. The Maxwell relation gives $s_{212} + s_{313} = s_{001}$, so this fixed subspace is the line spanned by s_{001}. Similarly the character of $f_0(c_{12})$ leaves only $s_{012}, s_{102}, s_{201}$. Antisymmetrize the interchange of axes one and two; the Bianchi relation $s_{102} - s_{201} = s_{012}$ leaves the line spanned by s_{012}. Hence these two values have scalar coefficients a, b.

Use (FS9) with $p = c_{01}$, $q = c_{12}$, and $z = e_0 + e_2$. The explicit differentiated covariance (FM2), evaluated below after (WE4), gives the same nonzero value $-2c_4$ for s_{001}, c_{12} and s_{012}, c_{01}. Therefore $a = b$. Permutations fixing zero give all remaining plane evaluations, which span $\mathscr{V}_0$; permutations moving zero give the other directions. We conclude

$$f_i = a\partial_i, \qquad T_i[p,p] = a[p, \partial_i p]. \tag{FS10}$$

Since $\partial_i[p,p] = 2[p, \partial_i p]$, the scalar of the translation derivation is $a/2$. Squares span the symmetric quadratic polynomials by polarization, and both rows kill the unit. This proves the entire degree-four assertion, with its factor two, rather than just a classification of invariant labels.

16.10 Fixed-displacement evaluation and the triangular Wick calculation

Here are the three explicit evaluations used in (FS7)–(FS10). Put $c_4 = (4\pi^2)^{-1}$, and let $H_z = I - 2zz^{\mathsf{T}}/|z|^2$ for $z \neq 0$. Its action on two-forms is the solution action $\rho(H_z)$. If E_{ab} is the unit antisymmetric plane tensor, the differentiated covariance (FM2) gives

$$K_z(p, c_{ab}) = \kappa_z\, p(\rho(H_z)E_{ab}), \qquad \kappa_z = 4c_4|z|^{-4} > 0. \tag{WE1}$$

Indeed substitute the second derivatives of $c_4|z|^{-2}$ into the two curls: their radial terms combine to the product of the two Householder matrix entries on the antisymmetric indices, with the displayed common radial factor. This is an identity on the six-dimensional real space of antisymmetric order-zero solution tensors.

Given $X \in \mathscr{S}_0$, set $W = \rho(H_z)^{-1}X$. Antisymmetry gives the plane expansion $W = \frac{1}{2}\sum_{a,b} c_{ab}(W)E_{ab}$. Define the one fixed functional

$$w_{z,X} = \kappa_z^{-1}\frac{1}{2}\sum_{a,b} c_{ab}(W)c_{ab}.$$

Bilinearity and (WE1) now give

$$K_z(p, w_{z,X}) = \frac{1}{2}\sum_{a,b} c_{ab}(W)p(\rho(H_z)E_{ab}) = p(X). \tag{WE2}$$

Thus the evaluation witness used at displacement $-z$ in (FS7) exists at that displacement itself. No change of the distant point or sum over different displacements is necessary.

Write $\mathcal{T}_{x,y,z}(A, B, C)$ for the three-point Wick moment. At each point its two letters are Wick ordered, so a contraction cannot join two letters at that same point. In a pairing of the six letters, each point must therefore have one leg going to each of the other two points. For letters a, v at the first point, two equal letters p, p at the second, and q, q at the third, there are two choices of which first-point letter goes to the second point. For either choice there are two choices of its partner among the equal p's and two choices of the partner among the equal q's. The third edge is then forced. Hence

$$\begin{aligned}\mathcal{T}_{x,y,z}([a, v], [p, p], [q, q]) = 4\dim\mathfrak{g}\,\{&K_{x-y}(a, p)K_{x-z}(v, q)\\ &+ K_{x-z}(a, q)K_{x-y}(v, p)\}K_{y-z}(p, q).\end{aligned} \tag{WE3}$$

The color factor is exactly $\dim\mathfrak{g}$: the three edges give Kronecker deltas requiring all three summed colors to coincide. All unequal-color summands are zero and each common color gives the same real scalar expression.

Let $x(t), y(t) \to 0$, $x(t) - y(t) = tu$, $u \neq 0$, with the third point $z \neq 0$ fixed. The near covariance of two order-zero legs scales as t^{-4}, while that of orders one and zero scales as t^{-5}. Multiplication of (WE3) by t^5 therefore gives

$$4\dim\mathfrak{g}\,\{tK_u(a, p)K_{x(t)-z}(v, q) + K_{x(t)-z}(a, q)K_u(v, p)\}K_{y(t)-z}(p, q). \tag{WE4}$$

Every covariance involving the distant point is continuous in this limit. The first term tends to zero; the second tends to $4\dim\mathfrak{g}\,K_{-z}(a, q)K_u(v, p)K_{-z}(p, q)$. If the order-one letter is instead at the distant point, its two edges stay bounded and the only short edge is an order-zero covariance. Multiplication by t^5 leaves an extra factor t, so that whole term tends to zero. Expand the panel in (FS6) in a basis as in (FS4), use (WE2) for the distant test, and apply these two limits to the three

silent terms. The common nonzero factor $4\dim\mathfrak{g}$ cancels; the odd sign of the order-one covariance under $u \mapsto -u$ gives exactly (FS7).

Finally take $p = c_{01}$, $q = c_{12}$. Substitution in (FM2) gives, at every $z \neq 0$,

$$K_z(c_{01}, c_{12}) = 8c_4 z_0 z_2 (|z|^2)^{-3}.$$

Differentiation of the first leg in direction zero gives

$$K_z(s_{001}, c_{12}) = 8c_4\{z_2(|z|^2)^{-3} - 6z_0^2 z_2(|z|^2)^{-4}\}.$$

At $z = e_0 + e_2$, its squared norm is two and both displayed coordinates are one. The value is $8c_4(1/8 - 6/16) = -2c_4 \neq 0$. The order-zero covariance is symmetric in its two plane tests; differentiating that symmetry gives the same value for $K_z(s_{012}, c_{01})$. Thus (FS9) reads $-2c_4 a = -2c_4 b$, proving $a = b$ in (FS10).

Source–readout formulation. A silent vector row is, in the terms of the method, a candidate vertical direction of the free readout: an operation on labels which no separated correlation with any spectators can see when it is applied to every insertion in turn (FS5). The two subsections above compute the fibre of such directions on degree-four fields and find that it is one dimensional, spanned by translation: by (FS10) a silent row acts there as the scalar $a/2$ times ∂_i. The computation is a recovery of the Ward kernel from the hierarchy, not an assumption about it. Its inputs are the explicit kernel (FM2), the reflection characters, the Maxwell and Bianchi relations among the first jets, and one nonvanishing number: at $z = e_0 + e_2$ the mixed covariance equals $-2c_4 \neq 0$, which is what forces the two constants a and b to agree through the symmetry (FS9). The fixed-displacement Wick evaluation (WE1)–(WE4) is triangular in the filtration, and it is this triangularity that allows the conclusion at weight four to be reached without knowledge of higher-weight rows. The exactness class is exact recovery within the free comparison sector.

16.11 Detection with a fixed exterior degree bound

We first establish the collision extraction that will be used in the polynomial argument. Put $R_a(x) = |x|^{2a}$. Say that F is cleared at order a when $R_a F$, initially defined off zero, extends to a real analytic function on all of $\mathbb{R}^4$. A harmonic pole of degree r is analytic off zero, homogeneous of degree $-r$ under positive dilations, harmonic there, and cleared at order $r-1$. The differentiated single-leg covariance of jet orders k, j is such a pole with $r = 4 + k + j$. A cleared homogeneous term of degree j means a function homogeneous of degree $-j$ cleared at order $\max(j-2, 0)$. A product of poles of positive degrees a, b is a cleared homogeneous term of degree $a+b$, because the product of their analytic numerators clears it at order $(a-1)+(b-1)$.

The pole assertion for the differentiated kernels has a short direct proof. The Green function $c_4|x|^{-2}$ is homogeneous of degree -2; multiplying it by $|x|^2$ leaves the constant c_4. Away from zero its Laplacian vanishes, since the radial Laplacian in dimension four gives $f''(r) + 3f'(r)/r = 0$ for $f(r) = r^{-2}$. Suppose F is a pole of degree $r \geq 2$ with numerator $g = R_{r-1}F$. For a fixed direction v, differentiation gives the numerator identity

$$R_r D_v F = R_1 D_v g - (r-1)(D_v R_1) g. \tag{HC0}$$

Its right side is defined with the same regularity on all real space. Differentiating the dilation identity makes $D_v F$ homogeneous of degree $-(r+1)$, and commuting ordinary derivatives with the Laplacian shows it is harmonic off zero. Thus differentiation raises the pole degree by one and

retains the required clearing order. Finite linear combinations of equal-degree poles retain all these properties. The two curls in (FM2) are a finite combination of second derivatives of the Green function, so have degree four. The two jet orders add $k+j$ derivatives, with the factor $(-1)^j$ for derivatives at the second endpoint. Color pairings and arbitrary probe coordinates are fixed linear maps. This proves exactly the degree $4+k+j$ and clearing hypothesis used above. It also gives a global bound $|F(x)| \le C|x|^{-r}$: bound the continuous function on the unit sphere and use positive dilation homogeneity. All uses here are off the real collision point; no contact value or complex cone division is involved.

For a harmonic homogeneous F of degree $-r$, Euler's identity $x \cdot \nabla F = -rF$, together with the product rule for the four-dimensional Laplacian, gives

$$\Delta(R_{a+1}F) = 4(a+1)(a+2-r)R_aF. \tag{HC1}$$

For example $\Delta(|x|^2F) = 8F + 4x\cdot\nabla F$; the general formula follows either by the same differentiation or induction. If $r \ge 2$ and the pole is cleared already at order $r-2$, it is zero. Indeed, starting with its analytic numerator at that order, (HC1) repeatedly produces an analytic numerator one order lower. At each step $a \le r-3$, so the displayed coefficient is nonzero. At the last step F itself extends analytically. Its homogeneity then implies $F(x) = t^rF(tx) \to 0$ as $t \downarrow 0$. For $r = 2$, clearing at order zero gives this conclusion immediately.

Here is the exact finite expansion needed; it includes terms more singular than the pole being extracted. For integers r, M, write

$$t^r\left(F(tx) - \sum_{j<M} W_j(tx)\right) \longrightarrow K(x) \quad (x \ne 0), \tag{HC2}$$

where each W_j is a cleared homogeneous term of degree j, and K is a harmonic pole of degree r. If $F(tx) = 0$ for all sufficiently small positive t, then $K = 0$, for $r \ge 2$. To prove this, peel the finite sum from its largest degree. A nonzero term of degree $J > r$ would diverge after scaling by t^r; multiplying the convergent expression by t^{J-r} forces $W_J = 0$. Repeat until only degrees at most r remain. Degrees below r tend to zero; the limit is the degree-r term. Thus $-K = W_r$, or is zero if that degree is absent. In either case K is cleared at order $r-2$, and (HC1) makes it zero. This finite peeling, rather than a choice of a large scaling exponent, removes the more singular double contractions.

To apply this statement, let P contain jets through order j, let a pair of test legs of orders k, l, with $l < k$, be inserted at $x_0 + z$, and keep the other vertices separated. The contraction of an order-k probe p with the order-a variables of P is the differential operator

$$B_a(p,z)P = \sum_{\nu} K_z(p, e_{a,\nu})\, \partial_{a,\nu}P, \qquad B(p,z)P = \sum_{a=0}^{j} B_a(p,z)P. \tag{HC3}$$

The index ν includes the jet basis and color. Wick's rule applies the two operators to every ordered pair of vertices. Set $r = 4+k+j$ and $M = 8+k+l+2j+1$. There are four cases.

1. Both legs contract at x_0. The resulting finite sum consists of second partial derivatives of P times products of two kernels. Their degrees are $8+k+l+a+b < M$, with $a, b \le j$. They are the cleared terms W in (HC2), with harmonic part zero.

2. The order-k leg contracts at x_0, and the order-l leg at another vertex. The distant contraction is analytic at $z = 0$. Multiplication by t^r selects only $B_j(p,z)$; the terms $a < j$ have smaller pole degree and vanish. The distant contraction tends to its value at $z = 0$.

3. The order-l leg contracts at x_0, and the other leg is distant. Its largest pole degree is $4+l+j < r$, so the scaled limit is zero.

4. Both vertices are distant. Every coefficient is regular at $z = 0$, so the scaled limit is zero.

These conclusions require only continuity of the distant Wick coefficients for the scaled limit; the coefficients are in fact analytic. Summing the finitely many placements proves (HC2). In case 2, summing the distant placements is Wick's first-leg identity at x_0. Summing colors in the invariant two-leg tester therefore gives harmonic part

$$K(z) = \langle (\mathcal{C}_j(p,q;z)P)(x_0) \prod_i V_i(x_i) \rangle, \quad \mathcal{C}_j(p,q;z)P = \sum_c B(q,c)\, B_j(p \otimes c, z)P. \tag{HC4}$$

Here $B(q,c)$ is the order-l, color-c jet variable. Orthogonality of the color action makes the sum invariant. The operator replaces a variable of weight $j+2$ by one of weight $l+2$, so it strictly lowers engineering weight whenever $l < j$.

If P is a ghost at exterior cap d, and $4 + k + l \le d$, the inserted invariant pair is an allowed spectator. For every fixed nonzero direction z, all the inserted points stay separated for sufficiently small positive t. Permuting the inserted pair past P, ghost vanishing makes the left side before subtraction in (HC2) zero. Finite peeling and (HC1) now give $K(z) = 0$ for every $z \neq 0$. Since the spectator word was arbitrary, $\mathcal{C}_j(p,q;z)P$ is itself a ghost at the same cap. This is the cluster extraction used below; it retains arbitrary spectator arity while keeping each spectator's degree bounded.

The passage to arbitrary input weight needs a uniform form of polynomial separation. An exterior bound d limits the weight of each spectator field, while allowing arbitrarily many spectators. Call a polynomial a ghost at bound d if all its separated correlations with all such spectator words vanish. We prove that one finite d, depending on the group, detects polynomials of every input weight. This is stronger than a Gram statement at a fixed input cap.

First, separated single-leg covariances with fixed partner jet order span the dual of each homogeneous solution-jet space. Their annihilator is zero by the differentiated covariance separation used in (FS7), and finite-dimensional duality proves the span assertion. Consequently if every single contraction of a polynomial with those tests is zero, all its partial derivatives in the corresponding jet variables are zero. Indeed a single contraction is the sum of those partial derivatives multiplied by the covariance coordinates; linear combinations of the spanning covectors isolate each derivative.

Next we must separate colors using invariant tests. Put $q = \dim \mathfrak{g}$. For every color index $c = 0, \dots, q-1$, choose a nonzero functional $p_c \in \mathscr{V}_c$ and a solution jet X_c with $p_c(X_c) = 1$. Such jets exist at every order. Form the polynomial matrix

$$M_{cb} = B_b(p_c). \tag{FS11}$$

Its determinant is not the zero polynomial: assign the order-c jet to color c with value X_c, and all other color components at that order to zero. This evaluation makes M the identity. The argument takes place in the polynomial ring, which is an integral domain. It does not assert that M is invertible at every field configuration.

Suppose a polynomial has no jet variables above order $j+1$. Extract a mixed cluster consisting of one highest-order contracted leg and one of the low-order color tests just chosen. Vanishing of these extracted correlations says $MD = 0$, where D_b is the single contraction in color b with the highest-order variables. Multiply by the adjugate: $\det(M) D_b = 0$. Since the determinant is a nonzero polynomial, $D_b = 0$. The preceding dual-span argument makes every highest-order partial derivative zero. Over characteristic zero a variable occurring with a positive power has a nonzero

partial derivative, so the polynomial contains no variable at that order. This removes the highest jet order one layer at a time.

We spell out the bounded-degree and induction aspects of the cluster extraction. Set $j = q$, $L = j+1$. The polynomial ring on jets through order j has finitely many variables and is Noetherian. Form its ideal generated by all positive polynomial-length homogeneous invariant polynomials. Choose finitely many such generators of this ideal, and let r bound their engineering weights. They generate every invariant polynomial as an algebra: for an invariant polynomial of length $k > 0$, express it in the ideal, take its length-k component, and average the coefficients over the compact group. Averaging is a projection onto invariant polynomials and satisfies $\mathcal{R}(hg) = \mathcal{R}(h)g$ for invariant g. For a generator of positive length l, its coefficient after this projection is the averaged length-$k-l$ part of the old coefficient; it has length strictly smaller than k. Induction on length expresses it as products of the same generators. Constants are empty products. Thus every invariant polynomial homogeneous of length k on this shelf lies in the span of products $\prod_i V_i$, with each V_i invariant of engineering weight at most r, homogeneous of length k_i, and $\sum_i k_i = k$. This is the finite hosting bound; the number of factors is unrestricted.

Here is the precise coalescence argument that uses it. Suppose a ghost P has all length components below k zero. Fix $x \neq y$ and a hosted product $Q = \prod_{i=0}^{n-1} V_i$ of total length k. Place P at x and V_i at $y + tie_0$, for $t > 0$ sufficiently small. These points are distinct and avoid x. Expand all polynomials into monomials and apply Wick's rule. A monomial of P of length $l > k$ has too many letters to pair with the k exterior letters: it has no internal pair at x, so its moment is exactly zero. If $l = k$, every exterior letter must pair with a letter at x, leaving no exterior-to-exterior pair. All the resulting kernel arguments approach the nonzero vector $x - y$; their finite products therefore converge. Terms $l < k$ are absent by hypothesis. The limit is exactly the two-point Wick pairing $\mathsf{B}_{x,y}(P_k, Q)$, where P_k is the length-k component. Ghost vanishing at every positive separated t makes this limit zero. Linearity extends it from products to their span. No estimate of kernels between coalescing exterior points is needed, since the pairing count removes those kernels before taking the limit.

Induct on k. Let P_k be the first possibly nonzero length component and reflect its complex conjugate across the time hyperplane, giving $Q = \overline{P_k}^{\,\mathrm{ref}}$. The shelf, invariance, engineering cap and length are preserved by this operation. The hosting result puts Q in the allowed span. At points $-e_0, e_0$ its reflected Wick pairing with P_k is strictly positive if $P_k \neq 0$, by the polynomial Gram calculation above. Coalescence has made the same pairing zero, a contradiction. Thus $P_k = 0$. Every polynomial has only finitely many length components, so this induction proves that a ghost lying on this fixed shelf is zero. Define

$$d = \max\{r, 4 + j + L\}. \tag{FS12}$$

Induct on the engineering weight of the candidate ghost. If its highest jet order is $J > j$, cluster a pair of jet tests of orders L and $l < q$. Its weight is $4 + L + l \leq d$, and $l < L$, $l < J$. Apply (HC2)–(HC4) with highest shelf order J: the finite cutoff is $8 + L + l + 2J + 1$, and the extracted pole degree is $4 + L + J$. The extracted mixed contraction is a ghost of strictly smaller engineering weight, hence is zero by induction. The color-matrix argument then removes the highest jet layer. Repeating leaves a polynomial on the shelf through j, where the hosting tests separate it. The weight-zero case is the constant, detected by the empty word. This proves

$$\exists d\ \forall w\ \forall P \text{ of weight at most } w, \quad P \text{ a ghost at bound } d \implies P = 0. \tag{FS13}$$

The same finite bound is used before the finite-dimensional Ward readers are selected. In particular one takes their cap at least $\max(d, 4)$; lower-cap conclusions then follow by the filtered

inclusions. Arbitrarily high output weight in (FS13) is permitted, but arbitrarily high individual spectator degree has not been silently introduced.

Source–readout formulation. Detection is the statement that the free readout is faithful on the physical label space when the exterior probes are restricted to a fixed finite degree. The quantifier order is the content: one degree bound d serves all weights.

Lemma 16.6 (Finite peeling)**.** *Suppose*

$$t^r\Big(F(tx) - \sum_{j<M} W_j(tx)\Big) \longrightarrow K(x) \quad (x \neq 0),$$

with W_j cleared homogeneous of degree j and K a harmonic pole of degree $r \geq 2$. If $F(tx) = 0$ for all small $t > 0$ then $K = 0$.

Proof. Peel from the largest degree: a nonzero W_J with $J > r$ would diverge, and multiplying the convergent expression by t^{J-r} forces $W_J = 0$. Then degrees below r tend to zero and $-K = W_r$ (or $K = 0$), cleared at order $r - 2$, hence zero by (HC1). □

Theorem 16.7 (Faithful readout with bounded probes)**.**

$$\exists d\ \forall w\ \forall P \text{ of weight at most } w: \quad P \text{ a ghost at bound } d \implies P = 0.$$

Proof. Induct on the engineering weight. If the highest jet order of P is $J > j$, cluster a pair of tests of orders L and $l < q$: its weight is $4 + L + l \leq d$ and $l < L$, $l < J$. By (HC2)–(HC4) with shelf order J (cutoff $8 + L + l + 2J + 1$, pole degree $4 + L + J$) the mixed contraction is a ghost of smaller weight, hence zero, and the color-matrix argument removes the highest layer. Repeating leaves a polynomial on the shelf, where the hosting tests separate it. Weight zero is the constant, detected by the empty word. □

In source–readout terms the theorem says that the fibre of the readout "all separated free correlations against exterior words of degree at most d" over zero is trivial on the quotient by the equation ideal. A ghost would be a nonzero vertical direction of this restricted readout; the peeling lemma removes it by extracting, at the leading scaled order, a harmonic pole that must vanish. The degree bound matters for the later transfer: only finitely many probes per weight are needed, so the rank conditions transferred to the interacting hierarchy involve finite matrices.

16.12 Elimination of higher-degree vector and angular defects

Subtract from a silent vector row the scalar translation obtained in (FS10), and denote the residual row by Q_i. It is silent, kills every degree-four field, and in particular kills the free stress. Insert that stress as an additional spectator in (FS5), apply its radial shell identity (FS3), and subtract the silent identity with the engineering-weight derivative applied to the distinguished field. The terms where the row acts on a spectator cancel in pairs. The stress term is zero. What remains says that

$$[E, Q_i]P \quad \text{is a ghost,} \qquad E = \sum_\nu w_\nu X_\nu \partial_{X_\nu}, \quad w_\nu = \text{jet order of } X_\nu + 2. \tag{FS14}$$

All additional stress fields have degree four and the spectator cap is the fixed one in (FS13). Thus (FS13) gives $[E, Q_i]P = 0$.

Every polynomial is a finite sum of engineering-weight components P_k, and $EP_k = kP_k$. A map commuting with E preserves each such eigenspace: projecting its image onto weight $l \neq k$

gives $(l-k)$ times that projection equal to zero. Total inversion acts on a weight-k monomial by $(-1)^k$, because a derivative-order-j curvature variable has sign $(-1)^j = (-1)^{j+2}$. But the vector covariance law gives $RQ_i = -Q_iR$. On a preserved weight component these two rules give simultaneously $RQ_iP_k = (-1)^kQ_iP_k$ and $RQ_iP_k = -(-1)^kQ_iP_k$. Hence $Q_iP_k = 0$, and $Q_i = 0$ on the whole cap. This proves the full one-dimensional free translation-charge kernel used after (RT17).

For the internal angular row the degree-four quarter-turn argument in (RT18)–(RT19) already makes it zero on the free stress. Apply the angular version of (FS3) in the same silent identity. It makes the commutator with each angular generator a ghost, and (FS13) makes that commutator zero. On every finite filtered space, exponentiating a finite matrix gives the unique solution of its linear differential equation. Commutation with the generator therefore gives commutation with its quarter-turn. Signed covariance of the two alternating Lie-algebra slots and the plane-basis calculation following (RT19) then make every component zero at that cap. This supplies the higher-degree angular separation with the required bounded exterior tests.

Finally the analytic segment argument following (RT19) converts the collinear scene readings and all their position derivatives into the full silent identity (FS5). Taking the cap in (FS12)–(FS13), the preceding arguments identify its kernel. Finite-dimensional decreasing-kernel selection gives a finite list with exactly that kernel. Its nonzero minors remain nonzero for the convergent interacting reading matrices. This proves the rank transfer used to normalize the actual translation charge and eliminate the actual internal angular defect.

Source–readout formulation. In the free sector the Ward readout (all silent identities) has kernel exactly the one-dimensional translation line and no angular component; the charge is therefore recovered up to one real scalar, which the shell law (FS3) fixes. This is an exact statement about the imported comparison realization. Its transfer to the interacting hierarchy is a *rank transfer*: a finite reader list with nonzero minors in the free sector keeps nonzero minors along convergent interacting reading matrices. The lift records the free statement as exact, and the interacting statement as conditional on those convergences, which are supplied by the comparison estimates of the trace chapter. Under the frame realization all objects of this section are functions of gauge-invariant labels and are unchanged.

17 Physical labels and their detection

From coefficient readings to physical labels. The free comparison hierarchy and its Taylor readers have supplied a finite coefficient language. The physical detector must now be built from actual local lattice cylinders. A coefficient vector and its chosen representative are different objects. The calibrated section selects a representative, while the quotient identifies precisely the equation directions that all allowed spectator readings annihilate.

The arguments below first bound coefficients before passing to limits, then prove equation cancellation using Haar integration by parts. The triangular and equivariant constructions preserve this cancellation through the chosen section, gauge averaging and corrected whole cylinders. Trials are placed in the same physical histories; their limits cannot be assembled from individually favorable but incompatible histories. The Kelvin and Wick tests supply the separation needed for the detector. What is recovered is the complete coefficient reading and its physical image on the stated finite degree spaces. The bare representative is recoverable only after a section is fixed; quotienting does not make it unique, and a finite reader is not an inverse on all unreduced histories.

17.1 A detector on physical coefficient space

At fixed degree let E be coefficient space, I its equation ideal and $r = \dim(E/I)$. The foundation identifies $\ker \pi_i$ with an invertible image of I, so rank-nullity gives $\dim \operatorname{ran} \pi_i = r$ uniformly in cutoffs; detector notation has $r = \dim + 1$. This dimension statement does not assert that every bare coefficient survives the equations. Choose r trained vectors $e_{j,i}$ bounded by v. Their limiting response matrix has full column rank. Compactness of its unit sphere gives a positive lower norm, stable under sufficiently small entry perturbations $\delta > 0$, with left-inverse bound L. Train at $\epsilon = \delta/4$. Finite training and short-distance probe errors each cost ϵ. Replacing trials by $\pi_i e_{j,i}$ adds a third error: the difference has zero projection, and equation cancellation makes its probe eventually below ϵ, simultaneously for finite j. The projected response is within $3\epsilon < \delta$ and its columns are independent. The range dimension bound makes them spanning. With projection norm bound A, their norms are at most Av; stable inversion yields

$$\|\pi_i x\| \le ArvL\, \|L_i \pi_i x\|.$$

Choose training index first, then small positive probe scale, then a tail of the whole-mesh filter. Constants are independent of remaining cutoffs and volume. Equation cancellation uses banked insertion estimates (IC3)–(IC5) with flight exponent $-1/32$; separated reflected Wick pairing below supplies the full-rank reference. For explicit stability let $G : \mathbb{C}^r \to \mathbb{C}^s$ obey $|u| \le K|Gu|$ in sup norm and set $\delta = [2K(r+1)]^{-1}$. Column errors of $H - G$ at most δ imply

$$\|(H-G)u\| \le r\delta \|u\|, \qquad \|u\| \le K\|Hu\| + \tfrac{1}{2}\|u\|.$$

Thus $L = 2K$ suffices. Reference injectivity follows from correlation functionals separating the free quotient; their basis matrix is G. The lift retains equation ideal, projection, trained columns, response functionals and ordered choices. Recovery preserves their finite rank and stability proof on the same varying projected columns, with the banked and Wick dependencies retained explicitly.

For explicit coefficient-space recovery, let p_E read coefficient data and j_E be its section, with $p_E j_E = I_E$. Define

$$\widehat{\pi}_i = j_E \pi_i p_E, \qquad \widehat{L}_i = L_i p_E.$$

Then $p_E \widehat{\pi}_i = \pi_i p_E$, and $\operatorname{ran} \widehat{\pi}_i = j_E(\operatorname{ran} \pi_i)$. The transported section norm makes this range isometric to the physical range, preserving its dimension r and its detector inequality:

$$\|\widehat{\pi}_i \zeta\|_Z \le ArvL\, \|\widehat{L}_i \widehat{\pi}_i \zeta\|.$$

The lifted trial columns are $j_E \pi_i e_{j,i}$ and have exactly the same response entries as the original projected trials. Hence all three error allowances and the explicit stable inverse are recovered on the same columns. Equation-kernel recovery remains $\ker \widehat{\pi}_i = p_E^{-1}(\ker \pi_i)$; section composition returns the identified equation ideal. This transports rank, detector constants and the ordered training/probe/mesh choices while distinguishing coefficient projection from the gauge-field readout.

17.2 Bounding physical coefficients before limits

For difference coefficients x_i let $X_i = \pi_i x_i$, with $\pi_i^2 = \pi_i$ and detector inequality $\|X_i\| \le Z\|L_i X_i\|$. Bounded Wilson insertions and vanishing tested errors give $\|L_i x_i\| \le c$, but the equation remainder need not be coefficient small. Normalize instead by

$$f_i = (1 + \|X_i\|)^{-1}, \qquad y_i = f_i X_i, \qquad z_i = f_i(x_i - X_i).$$

Then $\|y_i\| \le 1$, $0 < f_i \le 1$, and closure under bounded scaling and subtraction of bounded families keeps z_i admissible. Idempotence gives $\pi_i z_i = 0$. Literal Wilson equation cancellation makes its tested observables vanish, eventually $Z\|L_i z_i\| \le 1/2$. Consequently

$$Z\|L_i(X_i - x_i)\| \le \tfrac{1}{2}(1 + \|X_i\|), \qquad \|X_i\| \le Zc + \tfrac{1}{2}(1 + \|X_i\|), \qquad \|X_i\| \le 2Zc + 1$$

Thus projected primary costs are fixed before limits. The unprojected remainder has zero projection, hence vanishing and fixed cost by cancellation. Subtracting it and the original approximation error gives fixed cost for the difference insertion, using the actual detector copy count. For slot correlations, a position difference in slot j changes only its factor. Compact-space continuity gives integrability, so finite subtraction commutes with Wilson integration. Iteration expresses mixed lattice differences exactly by shifted/differenced slot insertions. The detector and cancellation estimates therefore give their stated fixed-cost bounds. The lift retains idempotent projection, admissible normalized families and slot shifts. Recovery preserves this normalization and finite subtraction, transporting the absorption argument and its literal Wilson cancellation dependency without replacing it by a coefficient-norm hypothesis.

Using the coefficient section j_E, realize the normalized families as $\widehat{X}_i = j_E X_i$, $\widehat{y}_i = j_E y_i$ and $\widehat{z}_i = j_E z_i$. Their transported norms give

$$f_i = (1 + \|\widehat{X}_i\|_Z)^{-1}, \qquad \widehat{\pi}_i \widehat{z}_i = 0, \qquad \widehat{L}_i \widehat{z}_i = L_i z_i.$$

Thus the identical cancellation input and detector bound produce $\|\widehat{X}_i\|_Z \le 2Zc + 1$, recovered by p_E and the section. For a lattice shift h of slot j, retain both slot addresses and define its difference reader by finite subtraction. Coefficientwise lifting is linear, so the difference of lifted slot readers equals the lift of that difference. Exact Wilson measure pushforward then preserves the equality between the correlation difference and the integral of the differenced slot product. Iterating preserves every prescribed mixed difference and its actual detector copy count. This recovers the fixed-cost conclusion with the same admissibility and equation-cancellation dependencies.

17.3 Equation cancellation in the physical detector

At grade d, the marking identity for $y \in I_d$ gives zero Wilson expectation of its marked insertion against bounded measurable spectators outside a finite neighborhood on sufficiently large lattices. A finite basis of I_d makes radius and size thresholds uniform by taking maxima; linearity permits arbitrary complex coefficients. Fix reporting level k. Fine chambers eventually contain this neighborhood strictly. Place translated copies in the prescribed chambers, distinguishing chamber zero; all other copies form a bounded measurable exterior spectator. Therefore

$$\mathbb{E}_\beta \prod_{\text{chambers } a} f_a = 0.$$

The distinguished copy keeps its exact scale normalization, so its replicated absolute-root seminorm is zero on the whole mesh and every volume, with k fixed first. For terminal chart Z_i and calibrated invertible change P_i, a kernel family has $y_i = P_i^{-1} Z_i^{-1} x_i$ and $Z_i P_i y_i = x_i$. The source also identifies $Z_i y_i$ with the marked equation carrier. The calibrated inverse bound preserves admissible subexponential coefficient growth, with size multiplier at most two. The trained chart gives

$$\text{upper}(Z_i P_i y_i) - \text{hallmarks}(y_i) = O_{\text{root}}(\text{flight}_{-1/32}(i)). \tag{IC3}$$

Indeed expand in the fixed finite signed-orbit frame tally. Each basis comparison costs $\text{flight}_{-1/16}$; scalar bounds sum subexponentially and are eventually at most $\text{flight}_{-1/32}/\text{flight}_{-1/16}$. Finite

recombination is exact, and signed motions/conjugations commute by covariance. The hallmark seminorm is zero by chambers, hence

$$\mathrm{upper}(x_i) = O_{\mathrm{root}}(\mathrm{flight}_{-1/32}(i)) \quad (x_i \in \ker \pi_i). \tag{IC4}$$

Apply to $x_i - \pi_i x_i$ by idempotence. Projected and unprojected trials have identical limiting responses. The equation carrier is an invertible image of I_d, and the free observable kernel is exactly I_d, giving free quotient rank by rank-nullity, without assuming independence of all bare coefficients. Reference rows are linear free correlation functionals, variable quotient field in the first slot and fixed basis spectators elsewhere. Their separating family has zero common kernel, so its quotient-basis matrix is injective. Stability and (IC4) therefore act on the same projected trial columns. The lift retains marking neighborhoods, chamber copies, trained charts and finite orbit tally. Recovery preserves the local spectator identity and exact chart recombination, transporting zero root seminorm and its flight error with the original growth and limit-order hypotheses.

The chamber cancellation has an exact source integral reader. Let $F = \prod_a f_a$ be the original finite chamber product, with its distinguished copy and exterior spectator recorded. Exact Wilson pushforward gives

$$\int F \circ \pi \, d\mu_\beta^Z = \int F \, d\mu_\beta = 0.$$

The section recovers each translated copy and its scale normalization; its support addresses retain the exterior-spectator condition. Taking the same absolute-value root therefore gives the identical zero seminorm. For the chart step, retain the inverses in their stated order: $y_i = P_i^{-1} Z_i^{-1} x_i$ and $Z_i P_i y_i = x_i$. Coefficient readout and section recover these finite linear operations and the signed-orbit tally. Each flight majorant is the same scalar reader, so the subexponential coefficient payment transports (IC3) and then (IC4) with exponent $-1/32$. The same recovered projected trial columns consequently have the original cancellation and detector responses on the fixed-reporting-level whole-mesh tail.

17.4 Haar integration by parts and triangular local realization

Vary a stored positive link by $U_e \mapsto \exp(tA)U_e$, with derivative $\nabla_{e,A}$. Product Haar is invariant and

$$\nabla_{e,A}\mathcal{S} = \langle A, \mathcal{E}_e \rangle.$$

Here $\mathcal{E}_e$ is the sum of the incident plaquette curvatures, with a backward incidence transported by the inverse preceding link. For plane $i < j$, the axis-i incidence is $F_{ij}(x) - \mathrm{Ad}_{U^{-1}_{x-e_j,j}} F_{ij}(x-e_j)$; axis j gives the negative analogous i difference, with other types zero. Ordered four-factor differentiation supplies inverse-link signs. Curvature projection and skew-adjointness give normalization

$$-\operatorname{Re}\operatorname{tr}(\mathrm{colorMat}(A)\,\mathrm{mat}(g)) = \mathrm{repDim}\langle A, \mathrm{curvature}(g)\rangle.$$

For continuous W and derivative, and bounded measurable spectator S unchanged by this variation, invariance yields

$$\mathbb{E}_\beta[(\nabla_{e,A}W - \beta\langle A, \mathcal{E}_e\rangle W)S] = 0. \tag{IC5}$$

The group law extends the derivative to every t. Compactness bounds the continuous Wilson-weighted derivative; multiplying by the spectator bound and the fundamental theorem gives a uniform integrable difference-quotient majorant. Differentiate the constant Haar integral and divide by its positive partition function. Spectators need neither continuity nor gauge invariance. For $W = 1$, nonzero β gives exact equation expectation zero. Marking retains multiplier derivatives on

varied links. The local identity holds outside a common radius, above a common torus size, for all nonzero real beta. Finite sums, scalars and conjugation preserve it by taking maximal radii/sizes and using the real Wilson law. One must also arrange the desired continuum label while enforcing these exact identities: the finite graded construction supplies a linear pair $p \mapsto (Qp, f_p)$, with gauge-invariant local polynomial f_p of leading jet Qp. Strictly higher lexicographic errors make Q triangular and injective by its first nonzero graded piece; finite dimension makes it invertible. The same argument preserves descending weight filtrations under Q^{-1}. Thus $Y(p) = f_{Q^{-1}p}$ has leading jet p, retained power filtration and exact equation marking. Paired construction imposes conjugation, signed symmetry and fixed unit on Q, Q^{-1}, Y. Enclose finitely many frame-basis polynomial stencils in one centered signed-invariant box containing zero; rewrite in its coordinates and extend linearly. Basis equality recovers Y on all coefficients. The lift retains varied-link incidence, marking radii, graded ordering and this common stencil. Readout commutes with Haar integration and finite triangular inversion, preserving (IC5) and the exact leading-jet realization.

The varied-link operation has an explicit realization on frame data. Write a for the physical color direction and $e = (x, y)$ for the stored link. Keep r fixed and replace its link coordinate by

$$A_e(t) = r_x \exp(ta) r_x^{-1} A_e.$$

Its readout is $\pi(r, A(t))_e = \exp(ta)\pi(r, A)_e$, with every other physical link unchanged. Conditional Haar invariance therefore recovers the original variation. Differentiating a smooth reader composed with π gives the physical $\nabla_{e,a}$ reader exactly; the exterior spectator is unchanged under this realized variation too. Exact Wilson pushforward then transports (IC5), with its original bounded measurable spectator and positive partition function. Section recovery gives the same incidence signs and color normalization. For the finite graded realization, retain the linear matrices Q, Q^{-1} and the polynomial stencil together. Reading the leading jet before and after Q^{-1} returns p, while the exact equation marking is recovered under the same finite linear combinations. These are local operations chosen before subsequent scheduling, so recovery preserves that order of choices and the fixed stencil's derivative credits.

17.5 Graded and equivariant assembly of the fixed section

With $R = Q$, the fixed section obeys $T(\mathrm{coords}(Rp)) = f_p$, mapping the equation ideal to the exact marking class. Compact polynomial bounds supply derivative credits. Choose section and R once at grade d, before block, period, coupling, refinement or derivative-budget scheduling. This explains why the local equation cancellation survives all subsequent scheduling choices. For each weight m, let E_m be homogeneous labels and $J_m = E_m \cap I_d$. Use the ordinary linear polynomial lift on E_m. On a finite basis of J_m, positioned equation lifts give pairs (q, f) with higher-weight $q - p$, prescribed form, power bound, gauge invariance and marking identity. Extend linearly and choose algebraic projection $L_m : E_m \to J_m$. Then

$$F_m(p) = F_m^{\mathrm{eq}}(L_m p) + F_m^{\mathrm{ord}}(p - L_m p).$$

On J_m the ordinary term vanishes; in general both preserve grade and leading label. Since $J_{(0,0)} = 0$, the unit maps to $(1, 1)$. Finite homogeneous summation gives F; homogeneity of the ideal preserves marking on its entire image. For signed label action θ_s and $\phi_s(q, f) = (\theta_s q, f \circ \mathrm{pull}_s)$, average

$$F_{\mathrm{av}} = |\mathcal{S}|^{-1} \sum_{s \in \mathcal{S}} \phi_s F \theta_{s^{-1}}.$$

Each summand preserves leading label, higher weights and marking. Weights sum to one, retaining the unit; group reindexing proves equivariance. For semilinear real structure $\mathfrak{c}(q, f) = (\overline{q}, \overline{f})$, set

$$F_{\mathbb{R}}(p) = \tfrac{1}{2}(F_{\mathrm{av}}(p) + \mathfrak{c} F_{\mathrm{av}}(\overline{p})).$$

The paired semilinear composition is complex linear. Real structure commutes with signed action and preserves grades, forms and marking, so the midpoint is real and signed-equivariant with triangular leading map and fixed unit. Perform these averages before common-stencil construction or cutoff choice. The lift retains homogeneous projections and finite symmetry averages. Recovery commutes with this linear assembly and triangular map, preserving the section's exact marking property and scheduling independence.

Recover the pair-valued graded lift by leaving its label coordinate unchanged and lifting its polynomial coordinate: $(q, f) \mapsto (q, f \circ \pi)$. Record the signed pullbacks and their induced source action so that their readouts are the original signed pullbacks. Finite linearity then gives

$$\widehat{F}_m(p) = \widehat{F}_m^{\mathrm{eq}}(L_m p) + \widehat{F}_m^{\mathrm{ord}}(p - L_m p), \qquad \widehat{F}_{\mathrm{av}} = |\mathcal{S}|^{-1} \sum_s \widehat{\phi}_s \widehat{F} \theta_{s^{-1}}.$$

Section recovery returns the original pairs in both formulas. Complex conjugation of coefficient readers commutes with composition with π, so the same midpoint recovers $F_{\mathbb{R}}$. Leading labels, the unit and higher-weight differences are therefore unchanged, and the equation subspace still has its exact marking reader. The common finite stencil and triangular inverse can consequently be retained with the same grade and power filtrations, chosen before all later scheduling parameters.

17.6 Finite polynomial gauge averaging and equation-ideal generation

For a powered local polynomial f, the constant-color orbit $f_g(U) = f(\mathrm{gauge}_g U)$ spans a finite-dimensional space: fixed representation and finite polynomial degree give finitely many coefficient tensors. Gauge change of variables preserves its marking identity and spectator radius and bound. Paired orbit vectors $(1, f_g)$ are finite dimensional; continuous first-coordinate and evaluation functionals separate them and hence span the dual. Coordinate Haar integration yields an element of their span with first coordinate one and second

$$\bar{f}(U) = \int_G f(\mathrm{gauge}_g U)\, dg.$$

It is a finite linear combination of orbit pairs. Leading label, higher weights, power and marking therefore remain those of an actual finite polynomial. General gauge action on transported polynomials uses its root value; Haar right invariance proves full gauge invariance. Equivariance and uniqueness identify the color-invariant leading form in label space.

On degree-j jets choose projection P_j onto the injective free gauge-jet range. Variable substitution defines algebra map $\mathcal{P}$, and equation labels are exactly $\mathcal{P}p = 0$, since polynomial evaluations separate. Split each linear functional as $(\ell - \ell P_j) + \ell P_j$. Telescoping products against projected products expresses every kernel polynomial as a finite sum with a literal equation-linear factor. Projection preserves jet degrees, so homogeneous terms keep their bidegrees; homogeneous components remain in the ideal and no nonzero field-count-zero constant belongs to it. To lift $\ell_{\mathrm{eq}} \ell_1 \cdots \ell_r$, express the equation functional as a finite combination of linearized transported lattice forces. For each force choose other-factor stencils avoiding its varied edge; their product is unchanged under that variation, so (IC5) gives exact local silence. Combine only after obtaining each identity. Avoiding ordinary factors agree with their linear functionals modulo field-count at least two; finite product errors have count at least $r + 1$, and after the force at least $r + 2$, above target $r + 1$. Retained higher linear force jet weights also remain strictly above target. Truncation and finite combination thus give the target leading term with higher-weight remainder and exact silent lattice function. This constructs positioned raw lifts of homogeneous ideal labels with the required triangular form. Every resulting term contains one literal factor vanishing on free solution jets. The lift retains finite

orbit tensors, jet projections and individual avoiding stencils. Recovery commutes with their finite algebra and normalized Haar averaging, preserving both exact silence and the full higher-weight bookkeeping.

The equation-factor decomposition has the explicit finite identity

$$\prod_{k=1}^{n} \ell_k - \prod_{k=1}^{n} (\ell_k P_j) = \sum_{k=1}^{n} (\ell_k - \ell_k P_j) \prod_{r<k} (\ell_r P_j) \prod_{r>k} \ell_r.$$

Its coefficientwise realization preserves every product, jet degree and literal equation factor. Section recovery returns the original polynomial identity; finite monomial summation therefore recovers the full ideal construction and its higher-weight errors. For gauge averaging, each realized orbit reader is $f_g \circ \pi$. At a fixed realized input,

$$\int_G f_g(\pi(z))\, dg = \bar{f}(\pi(z)).$$

The retained finite-dimensional orbit span represents this average by the same finite polynomial coefficients. Its paired first coordinate remains one, preserving the leading label. Each force summand retains its own edge-avoiding stencil and exterior-spectator identity before the finite sum is formed. Recovery thus preserves the actual polynomial realization, exact marking and triangular jet bookkeeping together.

17.7 Force-span and edge-avoiding probes

Linearized root-transported forces along source-edge-avoiding paths have finite expression

$$\sum_{\ell=0}^{d-2} \sum_{a<b} \text{incidence}_{\nu,a,b}(F_{ab}^{(\ell)}(x) - F_{ab}^{(\ell)}(x - e_i)).$$

Carrier derivatives drop out because identity curvature is zero; incidence signs are those of (IC5). A transverse difference of face averages is the three-unit-interval integral of a transverse derivative by the fundamental theorem. Rescaling base and mesh by t, its order-ℓ homogeneous expression, with the extra transverse factor, scales by t^ℓ. If every force probe vanishes, multiply orders by $t^{j-\ell}$ and test integer addresses whose physical rescalings approach arbitrary x. At points $t\,\text{index}_t(x)$, which tend to any prescribed $x \in \mathbb{R}^4$, the resulting continuous expression vanishes. As $t \downarrow 0$, lower orders vanish and interval averages shrink, leaving $\sum_i \partial_i F_{i\nu}^{(j)}(x) = 0$. Bianchi already holds for radial-potential curvature jets, so the top jet is free. A top-jet functional annihilating the free range annihilates the common force kernel. Finite-dimensional annihilator duality expresses it as an actual finite combination of force probes. Ordinary probes avoiding one forbidden edge span the entire finite jet dual. Outside finitely many sampling addresses, both approach path and plaquette boundary can avoid the edge. Vanishing safe samples imply curvature polynomials vanish at all integer points outside that set. Scaling to nonzero x eventually avoids the exception; top homogeneous curvature vanishes, then continuity includes zero. Descending induction and injective radial reconstruction give zero jet array. Thus safe probes have zero common kernel and finite-dimensional duality gives finite representations of all ordinary functionals. Finally

$$\langle X, \text{Ad}_g\, \mathcal{E}_e \rangle = \langle \text{Ad}_{g^{-1}}\, X, \mathcal{E}_e \rangle.$$

An edge-avoiding carrier makes this color test invariant under varying e. Expand in a fixed real color basis; each scalar coordinate, avoiding multiplier and remote spectator is unchanged. Apply

(IC5) with $W = 1$ termwise and sum finitely. The locality radius includes the force edge, and a large enough side prevents periodic collision with remote edges. This completes exact positioned marking. The lift retains safe routes, incidence signs, finite probe combinations and locality radii. Readout preserves the finite-dimensional kernel argument and termwise Haar identity, recovering the same equation lift and jet filtration.

The finite probe representation is recoverable as a linear certificate. Choose a finite spanning subfamily of the force probes $\lambda_1, \dots, \lambda_m$ for the relevant annihilator, and record the coefficients of $\ell_{\mathrm{eq}} = \sum_{q=1}^{m} a_q \lambda_q$. For each forbidden edge record likewise the finite safe-probe representation of each ordinary factor. With coefficient readout p_E, composition gives

$$\ell_{\mathrm{eq}} \circ p_E = \sum_{q=1}^{m} a_q (\lambda_q \circ p_E).$$

The coefficient section recovers this exact certificate and its homogeneous jet ordering. Every lattice probe reader keeps its own carrier and sampling stencil, so edge avoidance survives readout. Its transported color test, avoiding multiplier and remote spectator are consequently unchanged under the realized link variation already constructed. Termwise recovery of (IC5), followed by the recorded finite color sum, preserves the exact marking identity with the same locality radius and torus-size restriction. Thus the finite dual certificate and its local lattice realization are recovered together.

17.8 Uniform detector-trial placement

For fixed trained spectators define the actual lattice probe

$$\mathcal{M}_i(f; x) = \mathbb{E}_{\beta_i}[f(x_0) \prod_{\nu=1}^{n} F_\nu(x_\nu)], \qquad i = (N, p).$$

Compact-space continuity makes it complex linear in the first insertion. Linearity follows by expanding the first factor under the finite Wilson probability integral; all local polynomial insertions are continuous on the compact configuration space. Forward differences there equal differences in its first position; iteration and the detector's finite averaging preserve linearity on coefficients. Trained generic placements lie inside one physical box. Choose closed generic neighborhoods with $|x_{\nu,a}| < 1$; pair-coordinate gaps have a positive minimum. Physical periods at least four preserve this reserve for every volume, including the smallest admitted torus, since differences below two cannot be closer to a nonzero period translate. Unextracted insertion estimates give uniform local continuity on bounded coefficient balls. Replace the $4(n+1)$ coordinates of nearby placements one at a time, staying inside the neighborhood. Every intermediate replacement stays within their original physical distance and within the chosen neighborhood. Each costs at most $A^{n+1}R/r$, yielding

$$|\mathcal{M}_i(f; x) - \mathcal{M}_i(f; y)| \le 4(n+1)A^{n+1}R/r. \tag{IC6}$$

Reflected labels retain uniform bounds by their finite real-linear action. A scale-b detector average displaces points by at most $2b$ per tested direction, so (IC6) makes its error arbitrarily small uniformly on the whole-mesh tail and all volumes. Placement is exact because $a_N \,\mathrm{count}(N, k) = a_k$. Unaveraged trained columns approach free reference responses by finite-level training, whose error tends to zero by the trained insertion estimates, and smooth free evaluation at approximate generic positions, whose error tends to zero by smoothness of the free hierarchy on the distinct-position configuration space. Intersect the finitely many row/column tails. Choose training error below $\delta/4$,

then averaging error below $\delta/4$ by small b, then projection error below $\delta/4$ by (IC4). One common mesh tail gives rank without extraction. Uniform trial and projection bounds give the required prelimit detector inequality. The lift retains placements, separation reserves, direction lists and exact refinement counts. Recovery preserves these correlations and finite differences, transporting (IC6) and all three errors in their specified order.

The realized detector probe is the same correlation reader:

$$\widehat{\mathcal{M}}_i(f;x) = \int \left(f(x_0) \prod_{\nu=1}^{n} F_\nu(x_\nu) \right) \circ \pi \, d\mu^Z_{\beta_i} = \mathcal{M}_i(f;x).$$

Store each prescribed physical position and its exact refinement count with the reader. The coefficient section recovers the tested insertion, and the field section recovers every placed slot. Hence (IC6) has the same separation r and coefficient-ball bound for all admitted volumes. The detector's finite probability average commutes with this equality. If R_b bounds the displacement of its sampled placements, (IC6) bounds its averaging error by $4(n+1)A^{n+1}R_b/r$, with $R_b \to 0$ for the fixed direction list. This recovers the ordered training, averaging and projection allowances on their common whole-mesh tail, including reflected columns and the exact position identity. No new placement or extracted subsequence is introduced by realization.

17.9 Complete corrected cylinders and common trial limits

For commuting square-zero markers set

$$F(\epsilon) = \sum_{A \subseteq \{1,\dots,n\}} S_{\mathrm{free}}(x_A; q_A)\epsilon_A, \quad D = \log F, \qquad H(\epsilon, U) = e^{-D(\epsilon)} \prod_a (1 + \epsilon_a P_a(U)).$$

Empty moment is one; log and exponential are finite nilpotent polynomials. Training bounds every nonempty coefficient of H in replicated root seminorm by the common $\mathrm{gap}_k \to 0$. From $FH = \prod_a (1 + \epsilon_a P_a)$,

$$\prod_{a \in A} P_a = F_A + \sum_{B \subsetneq A} F_B H_{A \setminus B}.$$

Thus moment error is bounded by the finite absolute coefficient price of F times gap_k. That price stays bounded near the fixed distinct placement, since all free submoments are continuous. This supplies training convergence in (IC6). Singleton correction is $P_a - F_{\{a\}}$; the pair is

$$(P_a - F_{\{a\}})(P_b - F_{\{b\}}) - (F_{\{a,b\}} - F_{\{a\}}F_{\{b\}}).$$

Detector rows use at most two slots and retain this actual free cumulant. The pair correction is therefore its actual free cumulant at the rescaled placement; it cannot be replaced by a product of separately centered one-point terms. For each grade and training index finitely many lifted columns lie in a compact ball uniformly in both cutoffs. Countably many grade/training/column triples permit successive extraction and diagonal refinement of both cutoff orders, preserving earlier hierarchy and projection limits. Replace placeholder limiting columns by these actual limits without changing finite-cutoff bounds or placements. The corrected-cylinder bound passed through the fixed-placement hierarchy gives convergence to the same free matrix. Stable rank and equation dimension make limiting projected columns span the physical quotient; retaining these trials under null-label removal gives the free quotient dimension at every grade. A single reference frame and separated point pair suffice. Frame images span the quotient and include a nonzero unit; extend it to a basis from those images, unit first. Real homogeneous frame and conjugation/bidegree- preserving

projection give the same properties to this quotient basis, without probe orthonormalization. The lift retains all corrected coefficients, common refinements and selected frame images. Recovery preserves finite marker convolution and the cumulant, and commutes with the retained compact coefficient limits and hierarchy comparisons.

Coefficientwise source recovery of the complete cylinder. Use the finite algebra with commuting square-zero generators,

$$\mathcal{N}_n = \mathbb{R}[\epsilon_1, \dots, \epsilon_n]/(\epsilon_1^2, \dots, \epsilon_n^2).$$

Extend the configuration lift coefficientwise by

$$\mathscr{L}_n\left(\sum_A f_A \epsilon_A\right) = \sum_A (f_A \circ \pi)\epsilon_A, \qquad [\epsilon_A]\mathscr{L}_n f = ([\epsilon_A] f) \circ \pi.$$

This unital algebra map fixes the free scalar coefficients at the actual rescaled placement. Since the augmentation ideal is nilpotent, its logarithm and exponential are finite sums, so

$$\mathscr{L}_n H = e^{-D} \prod_a (1 + \epsilon_a (P_a \circ \pi)), \qquad F\,\mathscr{L}_n H = \prod_a (1 + \epsilon_a (P_a \circ \pi)).$$

Section restriction recovers every coefficient simultaneously. For nonempty A, the normalized Wilson reader therefore gives the same finite error sum

$$\left\langle \prod_{a \in A} P_a \right\rangle - F_A = \sum_{B \subsetneq A} F_B \langle H_{A \setminus B} \rangle, \qquad \left| \left\langle \prod_{a \in A} P_a \right\rangle - F_A \right| \le \left(\sum_{B \subsetneq A} |F_B| \right) \mathrm{gap}_k,$$

with the source's replicated-root control of each reader understood. The lifted integral is identical by normalized measure pushforward. In particular, coefficient recovery retains the pair cumulant at this placement.

Write $c^{(\lambda)}_{g,k,i}$ for the finite-dimensional coefficient columns on the common cutoff refinement. The lift of these columns is the fixed linear section j_E, and recovery is p_E, with $p_E j_E = I$. Consequently, for every convergent column on the retained refinement,

$$\lim_\lambda j_E c^{(\lambda)}_{g,k,i} = j_E c_{g,k,i}, \qquad p_E \lim_\lambda j_E c^{(\lambda)}_{g,k,i} = c_{g,k,i}.$$

Here the refinement is the source's refinement of the ordered cutoff limits, retaining its earlier hierarchy limits. Enumerating all (g, k, i) gives one compatible retained family; it introduces no replacement for the prescribed cutoff order. The same section and recovery transport the selected real, homogeneous unit-first frame and its quotient dimension.

17.10 Separated Wick detector and Kelvin triangular responses

For $x \neq y$, let $B_{x,y}$ be free two-point Wick pairing. The free quotient embeds in Maxwell solution-jet polynomials. Reflect every tensor index of the conjugate polynomial through the plane perpendicular to $x - y$, giving $p^\sharp$. Reflected positivity yields $B_{x,y}(p, p^\sharp) > 0$ for $p \neq 0$. Hence $p \mapsto (B_{x,y}(p, b_0), \dots, B_{x,y}(p, b_m))$ is injective: basis expansion of $p^\sharp$ would otherwise contradict positivity. The all-spectator null space therefore has no remainder beyond the equation ideal. Choose $x = 0$, $y = (1, 1, 1, 1)$ for a generic pair. One row per basis partner, with one spectator each, gives precisely the finite detector.

To retain the joint-order positivity input, put $r(z) = \|z\|^{-2}$ and let R_z be perpendicular reflection. A homogeneous order-j real solution probe has response

$$\mathrm{reply}(P)(y) = s_j r(y)^2 R_y^{\otimes 2} P(\mathrm{inv}(y)), \qquad s_j = 4(j+1)(j+2)(-2)^j \text{ normal}.$$

Translate its inverted-argument polynomial by $-\mathrm{inv}(z)$, decompose into degrees $k \le j$ and multiply each by s_j/s_k, obtaining $T_{jk}(z)P$ with diagonal P. Their response sum is

$$r(y)^2 R_y^{\otimes 2} s_j P(\mathrm{inv}(y) - \mathrm{inv}(z)).$$

For $y = z+tv$, inversion difference is $tV(z,v,t)$ with smooth V and $V(z,v,0) = r(z)R_z v$. Homogeneity gives factor t^j, so jets below order j vanish and the normalized order-j jet is $s_j r(z)^{j+2} R_z P$. The lift retains the quotient basis, reflected partner, Kelvin factors and triangular lower-order probes. Recovery preserves their finite polynomial transformations and Taylor jets, linking the detector to the same joint-order positivity construction continued below.

17.11 Joint-order reflected Wick positivity

Polarize the Kelvin line identity to recover symmetric derivative tensors. Reflection contributes derivative sign $(-1)^j$, giving the transformed tower Gram form

$$\sum_{j=0}^{n-1} c_j(z) \langle P_j, Q_j \rangle_{\mathrm{tensor}}, \qquad c_j(z) = (-1)^j s_j r(z)^{j+2} > 0.$$

The probe change is triangular with identity diagonal and invertible; equivalently its positive Gram identity proves injectivity and finite dimension gives invertibility. Thus the original simultaneous reflected form is positive definite on the whole tower. Choose a real orthogonal tower basis e_i with squared norms $d_i > 0$. Cross-insertion Wick contractions give

$$\langle e^\alpha, e^\beta \rangle_{\mathrm{refl}} = \mathbf{1}_{\alpha=\beta} \prod_i \alpha_i! d_i^{\alpha_i}.$$

Remove one left factor: only matching right indices pair, giving $\beta_i d_i$, and induct from the empty monomial. Within-insertion contractions are absent for these Wick products. Hence

$$\left\langle \sum_\alpha p_\alpha e^\alpha, \sum_\alpha p_\alpha e^\alpha \right\rangle_{\mathrm{refl}} = \sum_\alpha |p_\alpha|^2 \prod_i \alpha_i! d_i^{\alpha_i} > 0$$

for a nonzero polynomial. This includes mixed engineering weights, field-count degrees and the constant. The lift retains tower order, triangular probe map and cross-insertion contraction rules. Recovery preserves polarization, the finite basis change and the exact factorial weights, establishing the same joint-order detector positivity.

Finite tower and detector recovery. For the finite solution-jet tower $E = \bigoplus_{j<n} E_j$, use coefficient section and recovery j_E, p_E with $p_E j_E = I$. Write $T_z = I + N_z$ for the retained triangular probe transformation. Its strictly order-lowering part satisfies $N_z^n = 0$, giving explicit source and readout maps

$$\widehat{T}_z = j_E T_z p_E, \qquad T_z^{-1} = \sum_{a=0}^{n-1} (-N_z)^a, \qquad p_E \widehat{T}_z j_E = T_z.$$

The identity for $\widehat{T}_z$ is on the horizontal coefficient image; no inverse on redundant vertical coefficients is asserted. In terms of homogeneous polynomial extraction $[\cdot]_k$, its blocks are exactly

$$T_{jk}(z)P = \frac{s_j}{s_k}[P(\cdot - \mathrm{inv}(z))]_k, \qquad 0 \le k \le j.$$

Thus the source retains the translation of the polynomial argument together with its form-index reflection. The normalized jet reader is the physical jet reader after p_E; it recovers both the zero jets below j and the specified order-j Kelvin jet, including the single opposite-point factor $(-1)^j$ in the reflected pairing.

For the simultaneous reflected form G_z define its source form by

$$G_z^Z(\zeta, \eta) = G_z(p_E\zeta, p_E\eta).$$

It recovers the entire tower form on $j_E E$. Its polynomial extension retains the Wick contraction rule, so in the retained orthogonal coordinates the source and recovered polynomial norms are both

$$\sum_\alpha |p_\alpha|^2 \prod_i \alpha_i! d_i^{\alpha_i}.$$

The empty multiindex contributes $|p_0|^2$; mixed orders and degrees remain in this same finite sum. Finally the detector reader is

$$\mathcal{D}_{x,y}^Z(\zeta) = (B_{x,y}(p_E\zeta, b_i))_{i=0}^m, \qquad \mathcal{D}_{x,y}^Z j_E = \mathcal{D}_{x,y}.$$

Since the physical detector is injective by the retained reflected-partner argument, its source kernel is precisely $p_E^{-1}(0)$ on this coefficient carrier. Horizontal restriction therefore preserves its rank, including the unit column, at the specified generic pair $(0, (1, 1, 1, 1))$.

17.12 Normalized Taylor derivation of the Kelvin response

For symmetric order-j tensor P, use

$$\mathcal{D}_P f(x) = \sum_{\nu_1, \dots, \nu_j} \frac{\partial_{\nu_1} \cdots \partial_{\nu_j} f(x)}{j!} P(e_{\nu_1}, \dots, e_{\nu_j}).$$

Solution tensors S have harmonic trace constraints. For $G = \text{normal } r$,

$$\mathcal{D}_{S_{im}} G = (-2)^j \text{ normal } r^{j+1} S_{im}(x, \dots, x). \tag{IC7}$$

Induct from $j = 0$ with normalized recursion $\mathcal{D}_P f = (j+1)^{-1} \sum_a \partial_a \mathcal{D}_{P(e_a, \cdot)} f$. Use $\partial_a r^k = -2k r^{k+1} x_a$. Tensor-polynomial derivatives sum to vanishing frozen traces; the other sum resolves $\sum_a x_a e_a = x$. The normalization cancels $j+1$, leaving the next -2 factor. Put $V_i = \sum_a x_a S_{ai}(x, \dots, x)$. Bianchi and Euler give

$$\partial_i V_m - \partial_m V_i = (j+2) S_{im}, \qquad \sum_a \partial_a \mathcal{D}_{S_{ai}} G = -2(j+1)(-2)^j \text{ normal } r^{j+2} V_i.$$

The first identity gets two copies from differentiating x_a and j from homogeneous derivatives; the second uses (IC7) and zero divergence. Contracting the two-curvature kernel with S is twice the curl of this vector, ordered by the opposite-side derivative. Differentiation yields

$$s_j r^{j+2}[S_{im} - 2r(x)(x_i V_m - x_m V_i)], \qquad s_j = 4(j+1)(j+2)(-2)^j \text{ normal}\,.$$

The bracket reflects both antisymmetric form indices by R_x; the double radial contraction vanishes by antisymmetry. Degree-j homogeneity changes $r^{j+2}S(x)$ to $r^2S(\mathrm{inv}(x))$, giving the stated Kelvin response. The lift retains normalized tensors, trace/Bianchi/divergence constraints and derivative ordering. Recovery preserves their finite contractions and induction, including every factorial and sign in (IC7) and the detector's Kelvin coefficient.

The second-point order-k sign $(-1)^k$ differentiates $x-y$ and is included once. These Green-function identities hold off the origin.

Differential recovery off the diagonal. Regard the Green function and tensor responses as functions of the physical separation $x \neq 0$, with their coefficient source j_E independent of x. For a smooth coefficient-valued response f, finite dimensionality gives

$$p_E\partial^\alpha(j_E f) = \partial^\alpha f, \qquad p_E\mathcal{D}_P(j_E f) = \mathcal{D}_P f.$$

Tensor contraction is read through p_E before the physical contraction; thus trace, divergence and Bianchi constraints have exactly their stated physical kernels. In particular (IC7) is recovered as

$$p_E\mathcal{D}_{S_{im}}(j_E G) = (-2)^j \text{ normal } r^{j+1}S_{im}(x,\dots,x).$$

On the two-point configuration domain $x \neq y$, the same reader gives

$$p_E\partial_y^\alpha j_E f(x-y) = (-1)^{|\alpha|}(\partial^\alpha f)(x-y).$$

It follows that the normalized factorial, curvature curl ordering and second-point sign are preserved by one differential readout. These maps act on the smooth off-diagonal domain used in this passage; they do not supply a new contact extension at zero separation.

17.13 Selecting trials on the physical history

Fix a finite detector panel, real reference representatives and positive comparison width b. Arbitrarily late comparison indices k supply actual metering, restarted representations, admissible paths and one-/two-insertion and marked estimates. For completeness, the passage from arbitrarily late indices to a sequence makes all the geometric limits simultaneous. For all sufficiently fine $N \geq k$ they hold for every volume p; representations/readouts may depend on N, p, while k and its bounds do not. Only insertion arities at most two are needed, since each panel row has at most one additional insertion. One finite linear label transport serves every row, retaining filtration, reality and unit. Its inverse gives real lifts $x_{N,p}$ with

$$\sup_{N,p,\nu} \|x_{N,p,\nu}\| \leq D_k < \infty. \tag{IC8}$$

Complete exceptional early refinements by unit tuples and enlarge D_k; no estimate uses them. Comparisons propagate from interval midpoint to starting width through the actual intermediate representations. Reference coupling stays between half and all its value at k, so metering remains valid. Conditioned insertion and path propagation give the starting-width bill with common shipping error $\varepsilon(k) \to 0$. This is a statement about the transported physical insertions: it requires neither an assertion of a stronger density comparison nor a newly chosen scale history. Intersect finite panel thresholds on this same history. There are finitely many panel rows and coefficient requests, so all their eventual thresholds can be met together. To make this statement include the finitely many refinements preceding the availability threshold, put every member of the tuple equal

to the unit there, and enlarge D_k to include its norm. Writing starting width $w(k)$ and positive testing scale $g(k)$, admissible indices have $w(k) \to \infty$, $g(k) \to 0$, $\varepsilon(k) \to 0$. Calendar depth proves the first: a bounded width would contradict eventual calendar increase beyond its plateau depth. Intersect the arbitrarily late supply with $k > i$, $w(k) > j$, $g(k) < \eta$, banking, grade and finite fit conditions. Recursively select increasing $k_n, w(k_n)$ and decreasing $g(k_n)$, all cofinal or tending to zero as required. Choose (IC8) lifts at each selected index. Prescribed lattice marks fit the quarter box with separation and stencil admissibility and approach fixed physical positions. Conditioned estimates for all sufficiently fine N and every p, followed by stencil realization, identify trial entries with corrected errors $\varepsilon(k_n)$ uniformly in volume. This supplies the actual trial sequence with common readout, unit and conjugation properties. The lift retains the selected physical history, finite transport and quantifier order. Recovery preserves its bounded lifts and cofinal selection, including the uniform volume scope of the detector trials.

Trial carrier and uniform-volume recovery. Retain the selected calendar index, finite label transport, admissible path, and physical marks as part of the source trial record. If $L_{k,N,p}$ is that transport and b_ν its fixed reference representatives, then on available refinements its coefficient source and recovery are

$$x_{k,N,p,\nu} = L_{k,N,p}^{-1} b_\nu, \qquad \widehat{x}_{k,N,p,\nu} = j_E x_{k,N,p,\nu}, \qquad p_E \widehat{x}_{k,N,p,\nu} = x_{k,N,p,\nu}.$$

With the section norm, (IC8) reads $\|\widehat{x}_{k,N,p,\nu}\| = \|x_{k,N,p,\nu}\| \le D_k$. The exceptional early refinements use the prescribed unit tuple on both sides. Their completion has no role in the eventual comparison estimate. For each selected k_n there is one threshold $N_0(k_n)$ for the finite panel, with quantifier order

$$\forall n \quad \exists N_0(k_n) \quad \forall N \ge \max\{k_n, N_0(k_n)\} \quad \forall p.$$

All rows use the same $L_{k_n,N,p}$ at these indices. Realization and normalized measure recovery give exactly the same trial matrix:

$$M^Z_{n,N,p} = M_{n,N,p}, \qquad |M^Z_{n,N,p;r\nu} - M^{\mathrm{ref}}_{n,N,p;r\nu}| \le \varepsilon(k_n).$$

Here the reference entry and corrected comparison are those at the prescribed lattice marks; their physical configuration limit is retained. This equality preserves the source's uniform volume bound while allowing the representations and readouts to depend on N, p.

For an explicit cofinal selection, at the nth recursive step intersect the same arbitrarily late supply with $k > k_{n-1}$, $w(k) > \max\{n, w(k_{n-1})\}$, and $g(k) < \min\{1/n, g(k_{n-1})\}$, as well as $\varepsilon(k) < 1/n$ and the stated banking, grade and fit conditions. The source's eventual limits make this intersection nonempty. This records one sequence with all required limits, on the already retained physical history. Section restriction recovers its unit, real structure, marks and every finite trial entry before compact extraction.

18 The common continuum hierarchy

Two limits with different quantifiers. At each suitable fixed physical period, transported insertions and their two families of bounds establish a full mesh limit. Denoting a tested lattice moment by $M_{N,p}(t)$, the required conclusion has the order

$$r_p(t) = \lim_{N\to\infty} M_{N,p}(t), \qquad \lim_{k\to\infty} r_{p(q(k))}(t) = S(t).$$

The first limit uses all sufficiently fine meshes. Only the outer period limit uses a selected refinement of the requested period sequence. Auxiliary compact extractions help construct coefficients; identification with the Cauchy limit prevents them from weakening the first conclusion to a mesh subsequence statement.

Smearing, dense-position extension and derivative compatibility each use the norms and test spaces specified below. They do not follow from pointwise recovery alone. The common field quotient is formed on the same extraction and its calibrated equation image. Kernel stability under the nearby finite projections ensures that an invisible direction does not reappear after an inclusion or derivative. This is the concrete congruence check required for operations on the physical label space.

18.1 Fixed-level integrals from actual transported coefficients

Fix level m, period p, degree d and coefficient y. Along the chosen increasing finest cutoffs, the transport construction gives a Cauchy sequence of primary coefficient vectors and a Cauchy sequence of captured interaction coefficients on one common dictionary at level m. The capture realization equals the same transported insertion's interaction. Thus coefficient convergence is applied to the actual transported insertion, rather than to an independently chosen representative. Coefficient estimates give uniform boundedness and Cauchy comparison over all configurations for primary entries and forced profiles. Fixed spatial pullback, required conjugation and scalar multiplication preserve these. Finite-dimensional paid coefficient norms are bounded; continuous local terms, measurable masks and convergent sums give measurability. Strip the vacuum scalar from restored density, calling it T_n. Its mode is cold density times the selected coefficient of primary polynomial exponential and background-plus-forced field polynomial. Fixed-level flux comparison and insertion comparison make all factors uniformly Cauchy. Finite polynomial operations and continuous maps on compact bounded ranges preserve this. Probability Haar integrals are Cauchy, as are stripped partition functions with a common positive lower bound. Completeness and that bound give the normalized quotient limit. The vacuum scalar cancels exactly, identifying this quotient with the common-normalization insertion integral. This is the link between the fixed-level coefficient construction and the earlier fine-to-intermediate comparison. The lift retains actual captures, dictionary and stripped factors. Recovery commutes with finite restoration and probability integration, preserving this identification and denominator.

Uniform mode convergence under the readout. At fixed (m, p, d, y) write $T_{n,A}$ for any restored mode after stripping the scalar vacuum factor, and $Z_n \geq \delta > 0$ for the corresponding stripped partition function. On the framed source put $T^Z_{n,A} = T_{n,A} \circ \pi$. Surjectivity and normalized Haar pushforward give

$$\|T^Z_{n,A} - T^Z_{l,A}\|_\infty = \|T_{n,A} - T_{l,A}\|_\infty, \qquad a_{n,A} = \int T_{n,A}\, d\mu = \int T^Z_{n,A}\, d\nu.$$

Here $\pi_*\nu = \mu$ is the fixed-level probability Haar prior, before multiplication by the restored density. The actual captured coefficients, forced flags and measurable masks are the same in both integrands. Thus $|a_{n,A} - a_{l,A}| \leq \|T_{n,A} - T_{l,A}\|_\infty$; no continuity of a mask is needed. If $|a_{l,A}| \leq C_A$, then

$$\left| \frac{a_{n,A}}{Z_n} - \frac{a_{l,A}}{Z_l} \right| \leq \delta^{-1} |a_{n,A} - a_{l,A}| + C_A \delta^{-2} |Z_n - Z_l|.$$

Both terms tend to zero by the retained uniform coefficient construction. The cancelled vacuum scalar is identical in numerator and denominator at every finite cutoff, so this recovered quotient is the original common-normalization mode integral on this fixed level and period.

18.2 Public hierarchy identifications

The public Hilbert space, word domain, vacuum and fields are the reconstructed ones. These identifications keep the lattice limits, OPE terms and reconstructed fields attached to the same table. Labels are the union of graded table fields; phases and probability measure are the character decomposition. Selected and unselected correlations are that table's evaluations, and OPE data come from its one reader. Recursive smeared-word evaluation gives the identical word vector, so vacuum moments agree without a limit interchange. Taking its inner product with the vacuum proves equality of the moments. Translation, spectral measure and inverse-affine Poincare test action are the original reconstructed objects. Thus the public covariance and spectral assertions follow by literal substitution once their reconstructed counterparts have been proved. Graded pins are injective; label covering and insertion-token representatives give the two separate surjectivity statements. Nontriviality of the field operators follows from the degree-zero unit: its image is nonzero by injectivity and the unit's nonvanishing; boundary faithfulness then supplies a Schwartz test with nonzero field operator. Resolved whole words have continuous character evaluations and barycenter equal to the unselected expectation, yielding the public mixture formula. The compression-valued moments and character readings in the reconstruction chapter provide the resolved-word placement and barycenter identities used here. The public holomorphic wave uses pinned labels. Ordered Euclidean times recover the original configuration and heat-root value; joint Schwartz tests and shifts $t_i + irh_i$ are precisely the reconstructed boundary prescription. Holomorphy, heat-root and boundary existence retain their substantive reconstruction dependencies. Its boundary theorem therefore gives the required vacuum moment. The boundary label-quotient argument in the reconstruction chapter proves faithfulness. The final public projection introduces no second quantum field system.

18.3 One family, table and composed refinement

Choose the Wilson family before observables or volume subsequences. Linear supplier floors and nonlinear component floors form one maximum before block lengths; actual width and chart bills form a later maximum, paid by delay on the same clock. This includes flat delay at $e = 0$, so the actual horizon uses the same flat kernels. The explicit order of the pre-length floors, post-length numerical bills and countable width demands is (CAL12)–(CAL18); their consumption gives the flat delay condition at entrance index zero on the run's actual horizon, as proved there, and the same fixed family is used below. For requested order f, first extract g for fixed-level parameter limits, then w for the continuum table. Use $g \circ w$; increasing sequences preserve earlier convergence and $(f \circ g) \circ w = f \circ (g \circ w)$ fixes the final volume order. On that table choose reconstructions for all characters, one equivariant OPE reader and trace test. The preceding construction estimates and the subsequent reconstruction, operator-product and trace arguments establish these services on the retained table. Retain reader coefficients and fields through all truncations, then choose contacts in its same basis. This is the quantifier order used by the public theorem and its detailed extension. Every convergence assertion from the first extraction survives the second, because a strictly increasing natural sequence tends to infinity. The lift retains these actual witnesses and their dependency order. Recovery preserves the common table and composed subsequence, together with whole-word phase, operator and boundary identifications, rather than mixing existence choices.

Recovery of the one public system and its retained refinement. Denote the reconstructed word-domain system by $(\mathcal{H}, \mathcal{D}, \Omega, \Phi, U, E)$, where E is its energy–momentum projection-valued measure. Its public readout is the identity on these actual objects. If $W_0 = \Omega$ and $W_{a+1} =$

$\Phi(\iota q_{a+1}, f_{a+1})W_a$, the public recursion has the same W_a at every step, and hence the same $\langle \Omega, W_a \rangle$. Here ι is the graded pin map. This identifies unbounded fields only on the retained common word domain. The two distinct covering maps remain

$$\{\text{insertion tokens}\} \twoheadrightarrow \{\text{graded fields}\} \longrightarrow \mathcal{L} = \bigcup_d \iota_d(\mathcal{F}_d),$$

where each ι_d is injective and the union covers $\mathcal{L}$. Their recovery retains the degree-zero unit and the source's boundary faithfulness argument.

For a whole resolved separated word W, the retained character reader and barycenter give

$$S_\chi(W) = \mathrm{ev}_\chi(W), \qquad S(W) = \int S_\chi(W)\, d\mathbb{P}(\chi).$$

The reader acts on the resolved whole word prescribed by the source. Its weak character continuity supplies the public continuity statement. The pinned holomorphic wave and joint Schwartz test are the same reconstructed objects, so the public boundary reads its already established boundary value along $(t_i + irh_i)_i$. Existence of that boundary value retains its reconstruction dependency.

For completeness, if a_N denotes any quantity whose first extraction converges, recovery on the retained final refinement gives

$$\lim_{n\to\infty} a_{g(w(n))} = \lim_{n\to\infty} a_{g(n)}, \qquad f(g(w(n))) = (f \circ (g \circ w))(n).$$

This is subsequence preservation on the same Wilson family and common table. The source record keeps the pre-length maximum of supplier floors, then the lengths, then the maximum of actual width/chart bills, and then the delay on that clock, including its flat $e = 0$ condition. On the final table it retains the character reconstructions, the single equivariant OPE reader at every truncation, and then contacts in that reader's basis. Recovery reads this ordered witness record and therefore preserves the shared family, horizon, table, volume order and OPE prescription.

18.4 Full fixed-volume mesh limits and classical derivatives

At fixed intermediate level, Confluence means eventual uniform boundedness and uniform Cauchy comparison in the fixed configuration space. Finite products and sums preserve it, as do continuous maps on common compact ranges. For eventually measurable functions on probability Haar,

$$\left|\int f_N - \int f_M\right| \le \int |f_N - f_M| \le \sup_U |f_N(U) - f_M(U)|.$$

Finite polynomial restoration therefore gives Cauchy numerator and stripped partition integrals. The finite polynomial insertion numerator is built from the scalar-normalized density, transported primary insertions and local error insertions by exactly these operations. Retain before division the fixed-level lower bound

$$\exp[-K - |\mathrm{Edges}| \dim_{\mathbb{R}}(\mathrm{Mat}(G)) \log(16/v)] > 0,$$

where v is its positive threshold and K bounds small-field action and activity costs at coupling at most $2\widehat{\beta}_j$, independently of finer $N \ge j$. The trajectory partition identity identifies this with the normalized denominator. The scalar factor cancels exactly, proving ratio convergence with the established nonlinear Confluence inputs; this is the fixed-level ratio convergence used by Rising.latter.

On a compact collision-free set, thicken compactly without collisions. FixedBounds controls two extra differences of each word. Coordinate-path telescoping bounds its oscillation by physical

distance; applying it to first differences gives a common discrete-gradient Lipschitz constant A. Subtract the affine function with gradient frozen at the start. Each residual step costs $a_N A$ times endpoint distance; telescoping again gives a uniform quadratic Taylor bound. Next-difference limits require no new subsequence. At x choose physical $q > 0$ small enough for the neighborhood and Taylor error, then put $k_N = a_N\lceil q/a_N\rceil \to q$, eventually between q and $2q$. The long difference quotient has Cauchy endpoint limits. Taylor comparison places it within $Bk_N \le 2Bq$ of the unit next difference. Choose q for error at most $\epsilon/4$, then tail quotients within $\epsilon/2$ to prove next differences Cauchy at x. Local Lipschitz bounds and address error $O(a_N)$ give oscillation control; dense-set extension yields local uniform convergence along the original mesh. Repeat for every word. For word limit f_l and next limits $g_{a,l}$, passing mesh-address approximations through Taylor yields

$$|f_l(y) - f_l(x) - \sum_a (y_a - x_a) g_{a,l}(x)| \le B|y - x|^2.$$

Division by distance proves Frechet differentiability with derivative $h \mapsto \sum_a h_a g_{a,l}(x)$. Continuous next limits identify all coordinate derivatives; iteration gives smoothness of every order, with order/compact-dependent constants and thresholds. The coefficient-history convergence and the uniform derivative estimates provide these next limits and Taylor bounds at every fixed finite order. The lift retains the fixed-volume mesh, difference words, addresses and Taylor reserves. Recovery preserves finite telescoping and uniform bounds, transporting the full-sequence limits and classical derivative identification in this order.

Recovery before differences and full-mesh differentiation. Let π_N be the framed configuration readout at mesh a_N, and let $\Delta_{a,N}$ be the physical forward difference in a position coordinate, with divisor a_N. The map π_N is independent of insertion positions. For any finite difference word l and the same positioned lattice insertion I_N, coefficientwise pullback therefore satisfies

$$\Delta_{l,N}(I_N \circ \pi_N) = (\Delta_{l,N} I_N) \circ \pi_N.$$

The normalized Wilson reader, including all retained corrections, gives the same correlation $F_{N,l}$ on the two carriers at every finite mesh. It consequently preserves both the uniform fixed-volume bounds and the physical lattice Taylor inequality

$$\left| F_{N,l}(Y) - F_{N,l}(X) - \sum_a (Y_a - X_a) F_{N,a::l}(X) \right| \le B|Y - X|^2.$$

Here X, Y are physical mesh positions in the retained compact thickening; B and the eventual mesh threshold may depend on that compact and word. For the long quotient $Q_N(q)$ at the source's addressed point, the same reader yields

$$|F_{N,a::l}(X_N) - Q_N(q)| \le 2Bq,$$

for all sufficiently fine meshes after fixing $q > 0$. Hence its pairwise Cauchy estimate is bounded by $4Bq + |Q_N(q) - Q_M(q)|$. Choose q first, then the mesh tail. This is the same full mesh sequence. The classical reader then recovers

$$\partial_a f_l = g_{a,l}, \qquad Df_l(x)h = \sum_a h_a g_{a,l}(x),$$

by dividing the retained quadratic remainder by $|y - x|$. Thus all finite word limits, their derivative identification, and the small-field partition lower bound remain attached to the same fixed physical volume.

18.5 Stencil independence, interpolation and dense-position extension

For fixed backward/forward reaches $b, f \geq 0$, $e = b + f > 0$, the normalized stencil $[F(x + fe_a) - F(x-be_a)]/(ea_N)$ averages e unit forward differences at $x-be_a+je_a$, $0 \leq j < e$. Differences commute with translation averages; iteration expresses any fixed mixed stencil as a finite average of unit words. All displacements are $O(a_N)$, giving the same derivative limit without an extra reach factor. The right-closed cube interpolant assigns a left-coordinate subset I weight $\prod_{a\in I} \theta_a \prod_{a\notin I}(1-\theta_a)$, where θ_a is measured from the right endpoint. Nonnegative weights sum to one; vertices lie within $O(na_N)$. Compact cluster convergence bounds the weighted error by the largest vertex error, proving local uniform interpolant convergence even at ties. This assertion does not require or claim that a finite-mesh interpolant is smooth at its cell boundaries; lattice differences are taken before interpolation. For dense separated positions compare fine moments p_N to fixed-level $V_{j,N}$, Cauchy by Confluence, with uniform error bounded by fixed cost over $g(j) \to \infty$. Choose positive weights $x_i = \zeta_i/(|w_i|+1)$ to pay the finite word's charged coefficients and sum its eventual weighted bounds. Choose j for error below $\epsilon/4$, then the fixed-level tail for pairwise difference below $\epsilon/4$, proving full-mesh Cauchy comparison. More generally eventual compact oscillation bounds and Cauchy values on a dense set imply local uniform convergence even for discontinuous approximants. At arbitrary x, choose nearby dense d in a compact neighborhood and use

$$|T_N(x) - T_M(x)| \leq |T_N(x) - T_N(d)| + |T_N(d) - T_M(d)| + |T_M(d) - T_M(x)|.$$

Completeness gives pointwise limits; oscillation gives continuity. Failure of compact uniform convergence supplies bad points with a convergent subsequence, contradicting oscillation, pointwise convergence and limit continuity. Apply this on fixed-volume periodic collision-free positions, using nonlinear oscillation and schedule-compatible ancestor positions. The previous Taylor arguments then identify all derivative limits. The lift retains reach stencils, cube weights and ancestor positions. Readout preserves finite averages and the same compact oscillation bounds, recovering full-sequence extension without an additional mesh refinement.

Exact stencil, interpolation and dense-position readouts. To distinguish integer addresses from physical positions, write X for a mesh address and $a_N X$ for its physical point. A fixed-reach stencil has the exact recovery formula

$$\frac{F_N(X + fe_a) - F_N(X - be_a)}{(b+f)a_N} = \frac{1}{b+f} \sum_{j=0}^{b+f-1} \Delta_{a,N} F_N(X - be_a + je_a).$$

Pullback by π_N commutes with every finite translation and difference in this identity, so mixed stencils retain their exact normalized finite average. If $v_I(x)$ are the right-closed cell vertices and $w_I(x)$ their specified weights, both carriers give the same interpolant and bound

$$\begin{aligned} \mathcal{I}_N F_N(x) &= \sum_I w_I(x) F_N(v_I(x)), \\ |\mathcal{I}_N F_N(x) - f(x)| &\leq \max_I |F_N(v_I(x)) - f(v_I(x))| \\ &\quad + \max_I |f(v_I(x)) - f(x)|. \end{aligned}$$

Nonnegativity and unit total weight prove this inequality also at cell ties, and both terms vanish uniformly on a compact collision-free set. All lattice differences in this construction precede interpolation.

For the ancestor-position comparison, write C for its common finite weighted bill and $G(j) \to \infty$ for the source's scale function (to separate it from a testing coupling). The same normalized reader gives

$$|p_N - p_M| \le \frac{2C}{G(j)} + |V_{j,N} - V_{j,M}|.$$

The weights $\zeta_i/(|w_i| + 1)$ and the finite collection of localized errors are retained in C. Fixing j before the mesh tail preserves the source's two-level argument. The resulting dense-position Cauchy values and eventual compact oscillation bounds are identical under recovery. Thus the dense-set extension lemma applies to the same full mesh sequence, even when its approximants have discontinuities.

18.6 Smearing and ordered scalar-coupling refinement

Write a lattice test sum as the integral of its piecewise constant cell function; cell volume a_N^{4n} is exactly its prefactor. Compact test support lies in a period box away from collisions. Cluster convergence bounds all sampled values uniformly there: otherwise unbounded samples have a convergent position subsequence contradicting the cluster limit. Bounded tests give one integrable majorant. Sampled points converge to physical points, so dominated convergence yields the smeared limit, also for every identified derivative cluster limit. Let $t_m = \log(a_m^{-1})$ and $\widehat{\beta}_m = \bar{\beta} + 2b_0 t_m$, with $b_0 = 11\,\mathrm{Casimir}(G)/(48\pi^2)$. For each m, histories obey $\widehat{\beta}_m/2 \le \beta_{N,p,m} \le 2\widehat{\beta}_m$ for $N \ge m$. Extend earlier N by $\widehat{\beta}_m$ only for compactness. The countable compact metrizable product over (p, m) gives a common mesh subsequence with limits $a_{p,m}$. Along the requested volume sequence, a second compact product extraction gives B_m at every level. Extensions agree eventually with actual histories: for fixed m, the auxiliary extension agrees eventually with the original coupling because the mesh index tends to infinity. Hence the chosen refinement has the stated iterated limit at every level, and later refinements preserve these ordered limits. Uniform history estimates $|\beta_{N,p,m} - \widehat{\beta}_m| \le \epsilon t_m$ pass through the ordered fixed-m limit, so $(B_m - \widehat{\beta}_m)/t_m \to 0$ and $B_m/t_m \to 2b_0$. For first $m(\mu)$ with $t_m \ge \log\mu$, the schedule condition $t_{m+1}/t_m \to 1$ squeezes $t_{m(\mu)}/\log\mu \to 1$. Thus the positive history coupling $g(\mu) = B_{m(\mu)}^{-1/2}$ satisfies

$$g(\mu)\sqrt{2b_0 \log\mu} \to 1, \qquad g(\mu) \to 0.$$

Identification with a separately defined position-space diagnostic retains its own comparison-theorem dependency. The lift keeps cell volumes, compact test domains and both refinement maps; recovery preserves dominated smearing and the ordered coupling limits, including this stated scope.

Cell integration and scalar-history recovery. For a compactly supported test φ on a collision-free n-point configuration set, retain the physical cell partition $\{C_{N,X}\}_X$. If $s_N(x) = a_N X$ on $C_{N,X}$, the lattice test reader is exactly

$$a_N^{4n} \sum_X \varphi(a_N X) F_N(a_N X) = \int \varphi(s_N(x)) F_N(s_N(x))\, dx.$$

The support is confined to one fixed compact enlargement in the period box for sufficiently fine meshes. The source's compact cluster bound and bounded test give its integrable majorant. Since the recovered lattice values agree at every sample, both carriers have the same dominated limit $\int \varphi(x) f(x)\, dx$, also for every retained difference-word limit.

For the scalar coefficients, the source history record includes all $\beta_{N,p,m}$ and the auxiliary early extension. Coordinate evaluation recovers them unchanged, so the retained two compact extractions give

$$a_{p,m} = \lim_{N\to\infty} \beta_{N,p,m}, \qquad B_m = \lim_{k\to\infty} a_{p(k),m},$$

where limits follow the selected mesh and volume refinements respectively. The closed running inequality passes through these two limits at fixed m, before $m \to \infty$. For μ sufficiently large, the source's first-crossing index gives

$$1 \le \frac{t_{m(\mu)}}{\log \mu} < \frac{t_{m(\mu)}}{t_{m(\mu)-1}} \longrightarrow 1.$$

Together with $B_m/t_m \to 2b_0$, this recovers the stated asymptotic history coupling. The readout keeps that history interpretation and the separate comparison dependency for any other coupling diagnostic.

18.7 Common compact extraction and field quotient

Retain two inputs: fixed-period centered root bounds with period-dependent constants, and separately large-period uniform bounds eventually in mesh at each period. For n insertions on compact separated positions of margin r, choose $a_m < r$ then $k \ge m$ with $6na_k < a_m$. The stencil of each insertion fits its reporting chamber at every sufficiently fine mesh, uniformly in the period: each fixed integer stencil offset times the fine spacing tends to zero, whereas $a_k > 0$ is fixed. Include finite signed/reflected images in root bound A. Choose the tiling origin leaving each point at least three reporting widths from boundaries; the inequality leaves a choice in each coordinate. Distinct points occupy distinct chambers. Pull back by prescribed motions/conjugations and apply reflection-positive chessboard to obtain

$$|\langle I_1(x_1)\cdots I_n(x_n)\rangle| \le A^n. \tag{CV1}$$

This argument applies to arbitrary separated tuples; it does not require every coordinate projection of the tuple to be separated. On the generic open set with pairwise distinct coordinate projections, choose compact thickening and projected gap reserve $4r$. Translation estimates give one-coordinate oscillation at most $A^n|h|/r$ for $|h| \le r$. Replacing $4n$ coordinates one at a time stays in the thickening and gives Lipschitz constant $4nA^n/r$. Each intermediate tuple is obtained by choosing coordinates from one endpoint or the other, so its distance from either endpoint is at most their distance. Both fixed-period and separate large-period versions hold. Count finite basis words at every degree/arity and compact generic exhaustions. Local countable Arzela–Ascoli gives one mesh subsequence and all fixed-period tables M_p. Pass uniform large-period inequalities to these limits, then take one period subsequence for table S. Multilinearity recovers all labels on these same selections. For eventual compact coordinate bounds, replace exceptional early values by a value in each compact set and extract in the countable compact metrizable product. Each coordinate's original tail agrees, preserving its limit. Additional numerical data are retained by first selecting all active (i,p) mesh limits, then using closed large-period compact sets for the outer common subsequence. All active fixed periods remain available. Uniformly bounded idempotent projections have idempotent limits. If $\|P-Q\| < 1$, $u \mapsto u - Qu$ injects $\ker P$ into $\ker Q$: its kernel would obey $\|u\| \le \|Q-P\|\|u\|$. Reverse roles to obtain equal finite kernel dimensions. Detector estimates identify limiting kernels with the exact equation-null space. Transported comparison and step-function smearing make their separated compact-test correlations vanish. The convergence-of-histories chapter establishes the confluence and comparison estimates used in this argument. The lift

retains both bound families, compact coordinates and the two refinement maps. Recovery preserves (CV1), finite-coordinate convergence and idempotency, transporting the quotient identification with the same detector and smearing dependencies.

18.8 Exact null quotient and compatible diagonal orders

Full-rank detector pairs separate the quotient: a label null against every continuum spectator has zero detector readings and belongs to the limiting kernel. Thus

$$\{\text{continuum-null labels of degree at most } d\} = \ker P_d. \tag{CV2}$$

This finite-dimensional quotient retains real and signed structures and is the detector/differentiation quotient. Transported finite differences agree with the raw generic table. Taylor and next-difference arguments extend smoothly across coordinate ties while full positions remain distinct. Density of generic tuples makes that extension unique, yielding smooth T with the same M_p and separated smearing limits. Raw compactness alone supplies no tie value. The previously established full fixed-volume mesh Cauchy limits identify the auxiliary subsequence with these M_p; restoring the full mesh changes none of them. Only the outer period subsequence remains. For nested refinements F_j, the diagonal $t_j = F_j(j)$ is increasing: $F_{j+1} = F_j \circ g_j$ and $g_j(j+1) \geq j+1$. For each fixed stage i, its tail lies in the range of F_i. Apply separately to mesh and period orders, preserving every fixed-stage bound and limit on the same final table. If each successive order refines its predecessor, take its diagonal mesh and period sequences separately. Finally, countably many further stages can be imposed without combining incompatible choices. The derivative construction removes the coordinate-projection restriction on the raw table. The lift retains null kernels, quotient structures and nested subsequence maps. Recovery preserves (CV2), unique smooth extension and both diagonal orders, keeping all later services attached to this table.

Compact table and exact quotient recovery. For each finite degree carrier E_d keep the coefficient section and recovery j_d, p_d fixed during extraction, with $p_d j_d = I$. The framed projection and its recovered limit are

$$\widehat{P}_{N,p,d} = j_d P_{N,p,d} p_d, \qquad \widehat{P}_d = j_d P_d p_d, \qquad \widehat{P}_d^2 = \widehat{P}_d, \qquad p_d \widehat{P}_d j_d = P_d.$$

Finite-dimensional convergence commutes with these fixed maps in the inner mesh and outer period limits separately. In the section norm, $\|\widehat{P} - \widehat{Q}\| = \|P - Q\|$ on the horizontal image, so the source's less-than-one projection comparison preserves the stabilized physical kernel dimension. Redundant vertical coefficients remain in $\ker \widehat{P}_d$ and carry no extra physical quotient dimension. The complete quotient readout is explicitly

$$\overline{p}_d : E_d^Z / \ker \widehat{P}_d \longrightarrow E_d / \ker P_d, \qquad [\zeta] \longmapsto [p_d \zeta], \qquad \overline{j}_d : [u] \longmapsto [j_d u].$$

Indeed $\ker \widehat{P}_d = p_d^{-1}(\ker P_d)$, which makes both maps well-defined and mutually inverse; their compositions use $p_d j_d = I$ and $\zeta - j_d p_d \zeta \in \ker p_d$. The detector and separated smearing readers factor through these maps. By (CV2) their common physical null space is exactly the stated equation-null kernel. Real structure and signed motions are transported through the same quotient isomorphism.

The table reader acts coordinatewise on the retained basis-word correlations and additional numerical data. Its extraction record keeps both input regimes: fixed-period bounds with constants depending on p, and separate large-period bounds uniform in those periods with mesh thresholds

allowed to depend on p. The normalized integral identity preserves (CV1), and preserves the generic-position estimate

$$|S_N(\mathbf{x}+he)-S_N(\mathbf{x})|\le A^n|h|/r,\qquad |h|\le r,$$

with the same constants in the applicable regime. Thus the countable compact coordinates, their exceptional early completions, and all active fixed-period limits are recovered on the same two selected orders. In particular later period selection does not discard the fixed-period tables outside that selected period sequence.

Finally retain the nested mesh and period maps as part of the source record. For each fixed stage i and $j\ge i$ there is a tail index $h_i(j)$ such that $t_j=F_i(h_i(j))$, with $h_i(j)\to\infty$. Every fixed-stage eventual estimate and limit therefore survives each diagonal, separately in mesh and period. The unique smooth extension from generic tuples is then recovered by the same derivative reader on collision-free configurations. Since the full fixed-period mesh limits were already established, this auxiliary extraction reads those same M_p; the final theorem retains only the selected outer period order.

18.9 Uniformly bounded symmetric projections, the calibrated equation image and the differential quotient

The construction of the common compact extraction and the field quotient is completed by three further passages of the supplied text. They are given in full, in the order of the input. The first constructs the finite-cutoff projections and their uniform bound.

Here is the construction and uniform bound for those finite-cutoff projections. At degree d, let $K_{N,p}$ be the image of the free equation ideal under the exact calibrated label map already constructed. This is a subspace of the single fixed coefficient space E_d, although it varies with N,p. Choose once a linear coordinate isomorphism $W:\mathbb{C}^{\dim E_d}\to E_d$, with Euclidean norm on its source, and let π_K be orthogonal projection onto $W^{-1}K$. Put

$$F_K=I-W\pi_KW^{-1}. \tag{CVP1}$$

It kills K, and $x-F_Kx\in K$ for every x. Moreover

$$\|F_K\|\le\|I\|+\|W\|\|W^{-1}\|, \tag{CVP2}$$

independently of K, since $\|\pi_K\|\le 1$. Thus no conditioning estimate on a moving basis of $K_{N,p}$ is required.

When K is invariant under the signed label action V_s and coefficient conjugation, average F_K to

$$Q_K=\frac{1}{|\mathcal{S}|}\sum_{s\in\mathcal{S}}V_sF_KV_{s^{-1}},\qquad P_Kx=\tfrac12\{Q_Kx+\overline{Q_K\bar x}\}. \tag{CVP3}$$

Both maps still kill K and differ from the identity by a K-valued map. For the second statement, average the differences $x-V_sF_KV_{s^{-1}}x$, each of which belongs to K, and then use conjugation invariance. These two properties imply $P_K^2=P_K$ and $\ker P_K=K$: apply P_K to $x-P_Kx\in K$, and conversely use that same inclusion when $P_Kx=0$. Reindexing the finite group sum proves signed equivariance; the real average proves conjugation equivariance. Its norm is bounded by (CVP2) times the fixed finite average of $\|V_s\|\|V_{s^{-1}}\|$. The real average does not enlarge that bound, because conjugation is isometric in the frame coordinates. At the finitely many cutoffs where the calibrated symmetry has not yet been established, use (CVP1) instead. This supplies a uniform bound at every cutoff and the required exact symmetries on the admitted tail.

The projection maps are therefore idempotent and uniformly bounded before extraction. Their limits are idempotent as well, at each fixed period and in the outer limit. Their kernel dimensions stabilize in both limits. Indeed, if P, Q are idempotents and $\|P - Q\| < 1$, the map $u \mapsto u - Qu$ sends $\ker P$ into $\ker Q$ injectively: if $u = Qu$ and $Pu = 0$, then

$$\|u\| = \|(Q - P)u\| \le \|Q - P\|\,\|u\|, \tag{4}$$

so $u = 0$. Interchanging P and Q gives injections in both directions and therefore equal kernel dimensions in the fixed finite-dimensional coefficient space. Norm convergence of the projections makes their distance less than one on the admitted tail in each limit. This is the kernel stability used below; it does not identify kernels without the detector and equation-image arguments.

The second passage restores the full-mesh order after the compact extraction.

In the present construction the full fixed-volume mesh limits were already established by the two-level Cauchy argument. The auxiliary mesh subsequence just used therefore converges to those same M_p. To see explicitly that this restores the required smeared order, fix a token word and one compactly supported separated test t. For each suitable period p, let $R_{N,p}(t)$ be its lattice smeared moment. The full-mesh result gives $R_{N,p}(t) \to r_p(t)$ as $N \to \infty$. The extracted table gives, along its mesh subsequence $\phi(N)$, a limit $D_p(t)$, followed by the stated volume limit of $D_p(t)$. Cofinality of ϕ also gives $R_{\phi(N),p}(t) \to r_p(t)$. Uniqueness of a complex limit therefore identifies $D_p(t) = r_p(t)$ at each suitable p. Substitute this equality into the retained volume-limit identity. The result has the full mesh order followed by precisely the same volume subsequence as the final table. This argument restores the smeared convergence assertion; it does not claim that every auxiliary structure originally extracted on a mesh subsequence has been extended to all meshes.

The third passage, which the input places under two headings of its own, specifies the calibrated equation image and then the inclusions and derivatives on the physical quotient.

18.10 The calibrated equation image and its silent representatives

We specify the finite-cutoff change of coefficients used in the preceding projection construction. This also explains why its kernel directions are silent in continuum tests. Fix a degree, its coefficient frame, and a second sound local section V, such as the marked equation section. Write e_j for the coordinate vectors and $c \ge 1$ for the fixed cost of signed motions and coordinate decomposition in this frame. The local insertion transport constructs columns u_j satisfying

$$|e_j - u_j|_{\mathrm{coeff}} \le (2c^2)^{-1}. \tag{CVR1}$$

The columns are combined by the signed and conjugate averaging already used for local sections, rather than by an unsymmetrized matrix. Explicitly, if $\chi_{s,j}(x)$ are the averaged coordinates of the inverse motion of x, then

$$Px = \sum_{s,j} \chi_{s,j}(x)\, su_j, \qquad x = \sum_{s,j} \chi_{s,j}(x)\, se_j, \qquad \sum_{s,j} |\chi_{s,j}(x)| \le c|x|_{\mathrm{coeff}}. \tag{CVR2}$$

Here a conjugating motion conjugates the coordinate as well, so the combined map is complex linear. The fixed motion bound is $|sv|_{\mathrm{coeff}} \le c|v|_{\mathrm{coeff}}$. Subtract the two sums and apply (CVR1):

$$|x - Px|_{\mathrm{coeff}} \le \tfrac{1}{2}|x|_{\mathrm{coeff}}, \qquad |P^{-1}y|_{\mathrm{coeff}} \le 2|y|_{\mathrm{coeff}}. \tag{CVR3}$$

Indeed the first inequality makes P injective; finite dimensionality makes it surjective, and applying the first inequality to $P^{-1}y$ gives the second. These bounds hold uniformly in the period on the eventual mesh tail.

For precision about the purchase of (CVR1), put $X = V.\mathrm{stencil} \cup T.\mathrm{stencil}$, where T is the original section. At stage N, write h_N for the support exponent, q_N for the number of columns, and $b_N \geq 0$ for the purchased derivative-credit bound of V. The transport estimate gives

$$L_{N-1}^{1/8}|e_j - u_j|_{\mathrm{coeff}} \leq 2(1+q_N)Y_N, \quad Y_N = 2 + e^{81h_N}(b_N + q_N + 2|X|). \tag{CVR4}$$

The delayed length requirement is precisely $2(1+q_N)Y_N \leq L_{N-1}^{1/8}/(2c^2)$, which proves (CVR1). It is bought together with $b_N \leq \widehat{\beta}_N$, not inferred from a limit after selecting the columns. The common error price can be taken to be

$$E_N = 4(1+q_N)Y_N + e^{81(h_N+2)}Z_N\zeta_{d,N}/u_N, \quad Z_N = (a_V + 2a_T)\sum_j \theta_N^{-f_j}, \tag{CVR5}$$

where a_V, a_T are the fixed coefficient costs of the two sections, $\theta_N > 0$ is the field threshold and f_j the field-number degree of coordinate j, $u_N > 0$ is the purchased insertion allowance, and $\zeta_{d,N} > 0$ its degree-dependent margin. The sum is the finite price of undoing the threshold dilation. Each factor in (CVR5), including u_N^{-1}, has the subexponential bound supplied by the common calendar. Finite sums and products preserve that bound by dividing a requested exponent among their finitely many factors.

At any fixed admitted reporting indices m, k, the rooted discrepancy between the insertion of e_j through V and that of u_j through T, including every signed or conjugate motion, is at most

$$C_{d,m,k}E_N e^{-s_N/8}, \qquad C_{d,m,k} = \exp\{(a_m/a_k)^4\tau_k + 5\}\frac{a_k^{-d}}{\zeta_{d,k}/4}e^{s_k/8}. \tag{CVR6}$$

Here τ_k is the explicit vacuum treasury derived in the denominator estimates above. These are fixed finite constants after the reporting indices are chosen. Expand any subexponentially priced coefficient row by (CVR2). The finite number of terms and their subexponential coordinates, together with (CVR5), absorb at most $e^{s_N/16}$ eventually. Thus the entire discrepancy has rooted bound tending to zero, with a remaining exponential factor $e^{-s_N/16}$. Its fixed stencil, gauge invariance and continuity make it eligible for the separated spectator product and smearing argument below.

Let $\mathcal{T}$ be the terminal coefficient isomorphism and B the marked embedding of free labels into terminal coefficients. The calibrated equation image is exactly

$$K_{N,p} = \mathcal{T}P\mathcal{T}^{-1}B(I_d). \tag{CVR7}$$

If $x \in K_{N,p}$, set $y = P^{-1}\mathcal{T}^{-1}x$. Then $\mathcal{T}y \in B(I_d)$. The actual insertion through V with coefficient y is therefore exactly zero against physically separated spectators on a sufficiently fine mesh, by the local marking identity. The uniform radius for that assertion was proved by the finite-basis maximum in the equation-lift construction. Formula (CVR3) preserves the subexponential price of y, and (CVR6) compares this exactly silent insertion with the original insertion of x. Consequently every subexponentially priced row in (CVR7) tends to zero against every separated test. This is the kernel cancellation used in (CV2) and (QD3); it follows from the actual calibrated map and a local identity, rather than from dimension counting alone.

18.11 Inclusions and derivatives on the same physical quotient

The differential system is constructed using all spectator tests on this quotient. Let E_d be the finite coefficient space and N_d the null subspace in (CV2). A test reading means: put the distinguished label at any chosen slot, put arbitrary other labels in all remaining slots, multiply the correlation

by a smooth compactly supported test whose support lies in the separated configuration space, and integrate. Denote it by $\mathcal{L}_\psi(b)$, including the spectator data in ψ. The family of all these linear functionals has common kernel N_d. For a smooth table this follows from uniqueness of continuous kernels as distributions on the open separated set; a nonzero value has a neighborhood and a compact bump detecting it.

At finite mesh a_N, a derivative insertion is the actual forward difference

$$\delta_a I_b(o) = a_N^{-1}\{I_b(o + e_a) - I_b(o)\}. \tag{QD1}$$

In the finite torus sum, translating the distinguished summation variable gives the exact summation-by-parts identity

$$a_N^{4n} \sum_r \psi(a_N r)\, \langle \delta_a I_b(r_i)\, \text{spectators}\rangle = a_N^{4n} \sum_r \frac{\psi(a_N r - a_N e_{i,a}) - \psi(a_N r)}{a_N}\, \langle I_b(r_i)\, \text{spectators}\rangle. \tag{QD2}$$

Translation is a permutation of the finite torus, so this is true even at its seam. Once the physical period contains the compact support and its one-step thickening, both potentially nonzero terms have their actual cover coordinates. Taylor's theorem makes the test quotient converge uniformly to $-\partial_{i,a}\psi$, on one fixed separated compact. The separated correlation bound controls the remainder in the cell integral. Take the mesh limit at that fixed suitable period first, and then the retained volume limit. This proves the derivative test limit in the same mesh-then-volume order as the hierarchy; no uniform count of lattice points over all periods is needed.

The coefficient approximations used for both inclusion and (QD1) have subexponential top-coordinate cost: for every $\epsilon > 0$, their pullback to the terminal frame is eventually bounded by $\exp(\epsilon s_N)$, where s_N is the calendar scale. For bounded reference coefficients this follows by multiplying the backward column bounds: their logarithm is $\sum_{i<N} \log C_i$. The calendar gives $\log C_i \le \epsilon \log L_i/2$ after a fixed level, so the tail is at most $\epsilon(s_N - s_k)/2$; the finite head is absorbed into $\epsilon s_N/2$. The relay estimates extend this cost statement to the actual inclusion and finite-difference approximations.

Here is why their errors vanish against every test. A local rooted discrepancy bound $y_{N,p} \to 0$, with reporting spacing chosen arbitrarily small after fixing its requested tolerance, gives a separated correlation bound $A|y_{N,p}|$. Choose that reporting spacing below the physical separation, place the spectator insertions in distinct cells, and apply the reflection-positive product estimate, leaving the discrepancy in its own cell. Its compactly smeared sum is the integral of a step function, bounded by this quantity times the test supremum and the volume of one fixed enlargement of its support. Hence it tends to zero. This also applies to an exactly silent separated remainder with zero bound.

One must control approximations before assuming their coefficients bounded. Write $P_{N,p}$ for the finite projection and $x_{N,p}$ for an approximating coefficient. Normalize its null remainder by

$$z_{N,p} = \frac{x_{N,p} - P_{N,p}x_{N,p}}{1 + \|P_{N,p}x_{N,p}\|}. \tag{QD3}$$

It lies in the finite projection kernel. The normalized projected part has norm at most one and therefore the subexponential cost just proved. Multiplication by the reciprocal denominator costs at most one, and subtraction preserves subexponential cost by spending $\epsilon/2$ on each summand. The relay's equation cancellation then gives $\mathcal{L}_\psi(z_{N,p}) \to 0$ for every test. This is precisely the normalized remainder required by the detector argument: it proves boundedness and recovery of the projected limit using the stable finite detector, without presuming boundedness of the unprojected approximant. The detector identifies its limiting responses with one vector in E_e/N_e.

Apply this recovery to an inclusion from horizon d to $e \geq d$, and to (QD1) from d to $d+1$, first on a fixed basis of E_d. Choose representatives of the recovered vectors and extend linearly. This gives linear maps $K_{d,e} : E_d \to E_e$ and $D_{d,a} : E_d \to E_{d+1}$ satisfying, simultaneously for all spectator tests,

$$\mathcal{L}_\psi(K_{d,e}b) = \mathcal{L}_\psi(b), \qquad \mathcal{L}_\psi(D_{d,a}b) = -\mathcal{L}_{\partial_{i,a}\psi}(b). \tag{QD4}$$

The same approximating coefficient family is used for every spectator; separate choices for different tests would not define a field.

If $b \in N_d$, the right sides of both equations (QD4) vanish for every test, including the differentiated test, which still has compact separated support. Thus $K_{d,e}b \in N_e$ and $D_{d,a}b \in N_{d+1}$. The quotient maps

$$\iota_{d,e}[b] = [K_{d,e}b], \qquad \partial_a[b] = [D_{d,a}b] \tag{QD5}$$

are consequently independent of the chosen representative. Their pointwise kernel identities follow from (QD4): inclusion gives equality of two continuous kernels, and integration by parts gives the ordinary derivative of the smooth kernel. Integration by parts is legitimate on the separated open set because the test support is a compact subset of that set. Extend the product of test and kernel by zero outside; it is smooth with compact support, so its total directional derivative integrates to zero.

Faithfulness against all spectators now proves every compatibility law. Inclusion is injective, since its zero class has exactly the same readings as its input. Inclusions compose and the identity inclusion is the identity, since both sides have identical readings. Differentiation commutes with inclusion, because it differentiates the same smooth kernel in either horizon. Two derivatives commute by the ordinary mixed-derivative theorem. The derivative of the unit is zero: the unit removes its insertion slot and the resulting kernel is independent of that coordinate. Permuting insertion slots transfers these identities from the distinguished first slot to every slot by the linear permutation chain rule.

Finally conjugation commutes with real directional differentiation. For a signed orthogonal motion s, its coordinate derivative transforms by

$$\partial_b(f \circ s) = \sum_a s_{ab}(\partial_a f) \circ s. \tag{QD6}$$

Apply the already established reality and signed covariance identities to the kernel, differentiate on the separated neighborhood, and use faithfulness again. It follows that the quotient inclusions commute with both symmetries and that the quotient derivatives obey (QD6). This establishes the differential structure on the actual null quotient used in reconstruction, with no extra representative-dependent operation.

Source–readout formulation. This subsection is the central instance, for fields, of the method's descent theorem. The *source* is the coefficient space E_d of labels; the *readout* is the family $(\mathcal{L}_\psi)_\psi$ of all spectator readings; the equivalence it induces, $\mathrm{Eq} = \{(b, b') : \mathcal{L}_\psi(b) = \mathcal{L}_\psi(b')\ \forall\psi\}$, has classes $b + N_d$; and the physical field space is the image E_d/N_d. Theorem 18.3 verifies that this equivalence is a *congruence* for the operations of the target (inclusion, differentiation, conjugation, signed motions), which is exactly what is required for the readout to determine an autonomous field calculus. Three kinds of non-canonical data occur and are kept apart: the projections P_K (a choice of complement to a subspace; any other bounded symmetric choice gives the same quotient), the representatives $K_{d,e}b$, $D_{d,a}b$ (defined modulo N), and the column family u_j (a presentation of the calibrated map). What is canonical, and what the lift recovers exactly, is the quotient with its operations. What is lost by the readout is precisely N_d, and Proposition 18.2 identifies the

finite-cutoff shadow of that loss as the calibrated equation image (CVR7). In the frame realization the insertions are gauge-invariant cylinders, so $\mathcal{L}^Z_\psi = \mathcal{L}_\psi$ and $N^Z_d = N_d$; the reflection quotient theorem (Theorem C.1) then applies to the same classes.

The three formal statements used in this description are the following.

Lemma 18.1. *P_K is an idempotent with* $\ker P_K = K$*, signed and conjugation equivariant, with norm at most* (CVP2) *times the finite average of* $\|V_s\|\|V_{s^{-1}}\|$*.*

Proof. The paragraph following (CVP3) proves that P_K kills K, that $x - P_K x \in K$, and hence idempotence and the stated kernel. It also proves both equivariances by the finite group and real averages and gives their norm bound. The operator in the present statement is that same finite average, so those conclusions apply without a second projection choice. □

Proposition 18.2 (Silence of the calibrated equation image)**.** *Every subexponentially priced row in $K_{N,p}$ tends to zero against every separated test.*

Proof. For $x \in K_{N,p}$ put $y = P^{-1}\mathcal{T}^{-1}x$, so $\mathcal{T}y \in B(I_d)$. The insertion through V with coefficient y is exactly zero against physically separated spectators on a fine enough mesh, by the local marking identity (with the uniform radius from the finite-basis maximum of the equation-lift construction). (CVR3) preserves the subexponential price of y, and (CVR6) compares this exactly silent insertion with the original insertion of x. □

Theorem 18.3 (Operations descend to the physical quotient)**.** *There are linear maps $K_{d,e} : E_d \to E_e$ $(e \geq d)$ and $D_{d,a} : E_d \to E_{d+1}$ with, simultaneously for all spectator tests,*

$$\mathcal{L}_\psi(K_{d,e}b) = \mathcal{L}_\psi(b), \qquad \mathcal{L}_\psi(D_{d,a}b) = -\mathcal{L}_{\partial_{i,a}\psi}(b).$$

They map N_d into N_e and N_{d+1}, so

$$\iota_{d,e}[b] = [K_{d,e}b], \qquad \partial_a[b] = [D_{d,a}b]$$

are well defined on E_d/N_d, independent of representatives and of the choice of $K_{d,e}, D_{d,a}$. Inclusions are injective, compose and are the identity for $e = d$; derivatives commute with inclusions and with each other; the derivative of the unit is zero; conjugation commutes with real directional differentiation; and for a signed orthogonal motion s

$$\partial_b(f \circ s) = \sum_a s_{ab}(\partial_a f) \circ s.$$

Proof. Apply the recovery just described to a basis of E_d, choose representatives of the recovered vectors and extend linearly; the same approximating coefficient family is used for every spectator. If $b \in N_d$ both right sides of (QD4) vanish for every test (the differentiated test still has compact separated support), which gives the mapping of null spaces. The remaining laws follow from faithfulness: two classes with the same readings against all spectators are equal. For instance inclusion preserves all readings; differentiation in either horizon differentiates the same smooth kernel; mixed derivatives commute by the classical theorem; the unit removes its slot. Integration by parts is legitimate because the test is compactly supported in the open separated set. Slot permutations transfer the identities from the first slot. □

19 Phase decomposition and Wightman reconstruction

From positive readings to operators on a common domain. Reflection-positive pairings determine a seminormed word space. Zero-norm words are quotiented before Hilbert completion. Equality of the entire reflected Gram matrices under readout therefore gives an isometry on the quotients and a unitary identification of their completions. It does not turn every insertion into a bounded operator. The reconstructed fields act on the specified common invariant span of smeared field words applied to the vacuum.

The following slab, heat and boundary arguments identify that domain, the attachment map and the tube boundary distributions. Character evaluation is applied to whole compressed words, with the common phase measure retained. Full support and continuity have a specific role in upgrading certain almost-everywhere statements to every character; they are not a universal permission to remove exceptional sets. Likewise the one-dimensional invariant vacuum follows from the heat and phase arguments. No positive lower bound on the rest of the energy spectrum follows from vacuum uniqueness. These distinctions are preserved when the same Hilbert-space readings are transported to the realized family.

19.1 Bounded slab insertions and their norm derivatives

Fix an axis a, width $d > 0$, and word $w = ((x_i, v_i))_{i=1}^{n}$ of distinct points inside $0 < x_i^a < d$. A buffer $0 < r \le 1$ requires pairwise maximum-coordinate distance at least r and $r \le x_i^a \le d - r$. For degrees at most D and coefficient norms at most R, we construct

$$\|U_d(w)\| \le C^n (n/r)^{An}, \tag{BS1}$$

with $C > 0, A \in \mathbb{N}$ depending only on D, R. The empty word has bound one. On a future plant the prescribed action inserts w before that plant translated by de_a. Boundedness precedes extension to the Hilbert completion.

The centered one-cell estimate gives a threshold F, exponent P, and $J > 0$, such that one insertion at level $k \ge F$ has price Ja_k^{-P}. Include all signed rotations and reflected conjugates by enlarging the coefficient ball by their finitely many operator norms on the finitely many raw degree spaces. These constants consequently work for every reflected copy.

The decreasing schedule satisfies $a_{k+1} \ge c_0 a_k^2$. At the first $m \ge F$ below $r/32$, we have $a_m \ge c_1 r^2$; decrease $c_1 > 0$ to include $m = F$. Choose fixed sufficiently small $s > 0$ and the first $k \ge F$ below $s(r/n)^2$. The same first-crossing argument gives

$$a_k \ge c(r/n)^4, \qquad 20n a_k < a_m, \qquad k \ge m. \tag{BS2}$$

Indeed $20n a_k < 20sr^2/n \le 20sr^2 < c_1 r^2$, and strict decrease gives $k \ge m$. Exterior insertions are paid at still finer scales, chosen after this main scale; their prices may depend on the exterior word.

First let d' be an integer multiple of a_m. Alternating insertion of w and its reflection produces, in the squared norm after q repetitions, $4q$ main copies and two exterior copies. The main positions repeat the original $2n$ position patterns modulo a_m. Thus cell packing needs only $20n a_k < a_m$, independently of q. The buffer separates the main train from the translated exterior. The rooted product estimate and the fixed finite-word lattice limit give

$$\|\text{plant of the repeated word}\|^2 \le B_p [Ja_k^{-P}]^{4nq}, \tag{BS3}$$

with B_p independent of q. This is the train estimate in Next/PHASE/Trains.travel_price and Seats.replicas. By (BS2), its main price is

$$Q = [Ja_k^{-P}]^n \le [Jc^{-P}]^n (n/r)^{4Pn}.$$

We may use the latter expression for Q.

To remove the exterior prefactor, let X send algebraic future words to the reflection Hilbert space, V insert the main word, and $V^\sharp$ insert its opposite reflected word. Reflection and translation invariance give $\langle XVu, Xv\rangle = \langle Xu, XV^\sharp v\rangle$. Put $T = V^\sharp V$. Taking square roots in (BS3) and finite linear combinations gives $\|XT^q u\| \le B_u Q^{2q}$. Therefore $s_u = \sup_q Q^{-2q}\|XT^q u\|$ is finite. Moving powers of T across the pairing yields

$$(Q^{-2q}\|XT^q u\|)^2 \le \|Xu\| s_u.$$

Taking suprema gives $s_u^2 \le \|Xu\| s_u$, hence $s_u \le \|Xu\|$. Finally $\|XVu\|^2 = \langle Xu, XTu\rangle \le Q^2\|Xu\|^2$. Insertion thus annihilates null vectors, descends to the quotient, and extends uniquely with norm at most Q. This is OS/Train.producer.

For general d, use $d' = a_m\lfloor d/a_m\rfloor$. Since $a_m < r/32$, the same word lies in the narrower slab with buffer $r/2$. Compose its operator on the right with the heat contraction of width $d - d'$. On plants this is exactly the required insertion at width d, and its norm does not increase. The empty word is the heat contraction. All intermediate choices give the same operator because they agree on the dense plants. Replacing r by $r/2$ only enlarges C, proving (BS1) for one common family of operators. This is the padding and uniqueness argument of OS/Halves and Work/B01.

We next prove operator-norm regularity. Fix labels and width, and write $F(x) = U_d(w_x)$ on the open set of distinct slab configurations. For a slot-axis index $b = (i, j)$, replace its label by a representative of its physical j-derivative and call the resulting placement $F_b(x)$. Set

$$B(x)h = \sum_b h_b F_b(x). \tag{BS4}$$

Near a fixed configuration there is one positive buffer for the original word and these finitely many differentiated words. Equation (BS1) makes B locally bounded in operator norm. Between two plants, the placement is a separated correlation with a fixed exterior word. The scalar slot-derivative law gives its derivative as the matrix element of $B(x)$, also for finite linear combinations of plants.

On a convex position ball suppose $\|B(z) - B_0\| \le \eta$. The scalar mean-value estimate gives

$$|\langle u, [F(y) - F(x) - B_0(y - x)]v\rangle| \le \eta\|u\|\|v\|\|y - x\|.$$

Density gives the same estimate in operator norm. First set $B_0 = 0$: local boundedness makes every placement locally Lipschitz. Apply this also to each differentiated label array; all F_b, hence B, are norm continuous. Then set $B_0 = B(x)$ and shrink the ball. Its oscillation tends to zero, proving $DF(x) = B(x)$ in norm. Induction with differentiated labels gives smoothness of every order on the same position domain. This is the argument of Work/B02/Analysis.norm_remainder and Work/B02/Slab.smooth_place; it establishes the norm regularity used in the waves and compression-valued moments below.

Source–readout formulation. Insertion of a word is first defined on the algebraic word space (the source side); Proposition 19.1 is the congruence statement that allows it to be read on the reflection quotient: it maps the null space to itself and is bounded. The bound (BS1) is uniform in the exterior word, which is what separates an *operator* on the recovered Hilbert space from a family of matrix-element estimates. The derivative $B(x)$ is computed with the quotient derivative of Theorem 18.3, so the operator-valued map $x \mapsto U_d(w_x)$ is differentiated entirely on the readout side, with no reference to a representative. By Theorem C.1 the lifted slab operators are the unitary images of these and satisfy the same bounds.

Proposition 19.1 (Removal of the exterior prefactor)**.** *Let X send algebraic future words to the reflection Hilbert space, V insert the main word and $V^\sharp$ its opposite reflected word, so that $\langle XVu, Xv\rangle = \langle Xu, XV^\sharp v\rangle$. If $\|XT^q u\| \le B_u Q^{2q}$ for $T = V^\sharp V$ and all q, then $\|XVu\| \le Q\|Xu\|$. Hence insertion annihilates null vectors, descends to the quotient and extends uniquely with norm at most Q.*

Proof. $s_u = \sup_q Q^{-2q}\|XT^q u\|$ is finite. Moving powers of T across the pairing and using Cauchy–Schwarz, $\|XT^q u\|^2 = \langle Xu, XT^{2q}u\rangle \le \|Xu\|\,\|XT^{2q}u\|$, so $(Q^{-2q}\|XT^q u\|)^2 \le \|Xu\|s_u$; taking the supremum, $s_u^2 \le \|Xu\|s_u$ and $s_u \le \|Xu\|$. Finally $\|XVu\|^2 = \langle Xu, XTu\rangle \le \|Xu\| \cdot Q^2 s_u \le Q^2\|Xu\|^2$. □

19.2 Surjectivity of the reflection attachment

The boundary attachment J is an isometry from a complete Hilbert space, so its range is closed. For regularized words integrated against Fourier transforms of compact smooth tests, the boundary tilt identifies the vector with an attached boundary distribution on the chilled test. Explicitly the joint pairing gives $\langle A, A\rangle = \langle B, B\rangle$; regularized approximants to B and partial-boundary convergence give $\langle A, B\rangle = \langle B, B\rangle$. Thus $\|A - B\|^2 = 0$. The strict-time and independently heated constructions below supply these pairing and tilt inputs. Compact cutoffs approximate Fourier Schwartz tests. The word bound $C\|f\|_{L^1}$ and closed range extend membership to all integrated regularized words. Pairing a pointwise word with a vector orthogonal to the range gives zero smeared pairings; continuity makes the pairing zero everywhere. Double orthogonal complement and closedness place pointwise words in the range. Strictly ordered positive-time Euclidean words are regularized words: choose regularization below the first time and all gaps, then recursively apply local fields, translations and heat operators. This reproduces the reflection plant with its insertion representatives; quotient representatives agree against every future word and give the same plant. For arbitrary future words sort times by permutation symmetry, then add ϵi to the ith time. For small positive epsilon the configurations remain separated with strict positive time order and converge back. Expanding plant squared-distance into three table pairings proves norm continuity from smooth separated correlations. Since the range is closed, the unperturbed plant also lies in it. Future plants span densely by GNS, so J is onto. Here GNS denotes the Gelfand–Naimark–Segal positive-functional construction [8, 9]; the domain and quotient recovery are the ones specified above. The lift retains boundary word tests, chilled tests, representatives and heat/translation recursion. Recovery preserves the pairing identities and plant norm limits, transporting this surjectivity to the lifted completed space with its actual word domain.

Attachment recovery after Hilbert completion. Let $U_B : \mathcal{H}_B \to \mathcal{H}_B^Z$ and $U_R : \mathcal{H}_R \to \mathcal{H}_R^Z$ be the section-induced isometries onto the completed source boundary and reflection spaces. They descend through the exact null preimages before completion. Define

$$J^Z = U_R J U_B^{-1}, \qquad U_R^{-1} J^Z U_B = J, \qquad \operatorname{Ran} J^Z = U_R \operatorname{Ran} J.$$

Thus J^Z is isometric with closed range. For the source's regularized vectors A, B, its two pairing equalities give on both carriers

$$\|U_R A - U_R B\|^2 = \|A\|^2 + \|B\|^2 - 2\,\mathrm{Re}\langle A, B\rangle = 0.$$

Boundedness of U_R commutes with the vector-valued test integrals and their L^1 approximation. The orthogonal-complement reader is recovered by U_R^{-1}, preserving the passage from zero tested

pairings to zero continuous pointwise pairings. For future plants $v_\epsilon \to v$ the same map preserves the squared-distance table formula and hence the norm limit. Therefore closedness and the dense span of future plants recover surjectivity on the completed source reflection space as well. This construction preserves the supplied tilt, partial-boundary and joint pairing dependencies and uses their actual heat/translation word recursion and insertion representatives.

19.3 Explicit tested boundary formula

For word v smear its wave as $\Psi(y, f)$ against a joint Schwartz test. Set $a_i = i$ and $D = -i\partial_{\mathrm{dawn}(a)}$. The tested Cauchy–Riemann identity reads $\frac{d}{dt}\Psi(y + ta, D^k f) = \Psi(y + ta, D^{k+1} f)$. Assume $|W(x + iy)| \le Ck^{-m}$ for gaps at least k, $0 < k \le 1$. For nondecreasing y, define

$$P_m(y, f; t) = \sum_{k=0}^{m} \frac{(-t)^k}{k!} \Psi(y + ta, D^k f).$$

Adjacent derivative terms cancel, leaving $\partial_t P_m(y, f; t) = (-t)^m \Psi(y + ta, D^{m+1} f)/m!$, bounded for $0 < t \le 1$ by $(C/m!)\|D^{m+1} f\|_{L^1}$. Thus

$$\begin{aligned} B(y, f) = &\sum_{k=0}^{m} \frac{(-1)^k}{k!} \Psi(y + a, D^k f) \\ &- \int_0^1 \frac{(-t)^m}{m!} \Psi(y + ta, D^{m+1} f)\, dt \end{aligned}$$

is defined on all nondecreasing heights. For strict heights the fundamental theorem gives $B = \Psi$. Interior continuity of the finite terms and dominated convergence of the remainder prove continuity at every nondecreasing y_0. Hence every strict-height approach has limit $B(y_0, f)$, with unrestricted gap ratios. Moreover

$$|B(y, f)| \le C \sum_{k=0}^{m} \|D^k f\|_{L^1}/k! + \frac{C}{m!} \|D^{m+1} f\|_{L^1}.$$

This continuous Schwartz seminorm, linearity and label multilinearity give a tempered multilinear boundary distribution. Uniqueness makes admissible choices of m agree. Character-independent C, m give boundary character continuity and passage of barycenters to the boundary. The construction of the waves and the tested Cauchy–Riemann and polynomial gap estimates used here are given in (SW1)–(SW7) and (CR1)–(CR2) below. The lift retains the joint test, differential operator and cancellation polynomial. Recovery commutes with finite differentiation and the dominated integral, preserving the boundary distribution and its unrestricted strict-height limit.

19.4 Character integration and heat-ground identification

For an arity-n word, regularized factor bounds $Q_a p^{-d_a}$ at $p = k/8$ give, for time gaps at least $k \le 1$,

$$|W_\chi(v; x + iy)| \le \left(\prod_a Q_a 8^{d_a} \right) k^{-\sum_a d_a},$$

uniformly in character and real positions. Phase-to-unselected factor-norm comparison supplies the same constants: the factors are constructed in (SW1)–(SW3), and the phase norm comparison is (ST1)–(ST4), using the bounded slab construction (BS1)–(BS4). Strict phase values are continuous

and barycenter to the unselected value. For joint Schwartz f, $Ck^{-m}|f|$ is integrable; dominated convergence proves smeared character continuity, and probability normalization plus Fubini gives the strict smeared barycenter. The same bound, combined with the probability normalization of the phase measure, makes the double integral absolutely integrable. The common finite Taylor-ray boundary representation passes continuity through its finite strict terms and dominated remainder. Its polynomial part uses finitely many strict evaluations; its remainder has a common integrable bound on the ray parameter after enough test derivatives. Its uniform boundary bound permits dominated character integration, giving the Wightman barycenter with the same gap-order and derivative inputs. The tested boundary formula and strict-time wave estimates below provide this Taylor representation and its common polynomial-order bound.

For positive reflection heat contractions E_s with strong limit Π, the joint momentum measure gives

$$\langle x, E_s x\rangle = \int e^{-sE(p)}\, d\nu_x(p).$$

Nonnegative energy makes the integrand bounded by one and tending to $\mathbf{1}_{E=0}$. Dominated convergence and strong convergence give equal quadratic forms for Π and $P(\{E=0\})$; polarization proves equality. For attachment J, $B_s = J^* E_s J$ has boundary Boltzmann diagonal measures. Its zero-energy intertwining and $J^*J = I$ give strong limit $P_{\mathrm{boundary}}(\{E=0\})$. No rank-one premise enters this step; locality and character arguments establish vacuum rank later. Separate surjectivity realizes all reflection heat evolution on the boundary space. The lift retains whole-word character readings, common factor bounds and joint spectral measures. Recovery commutes with dominated test/character integration and spectral quadratic forms, preserving the barycenter and zero-energy projection identification without strengthening its rank claim.

Tested boundary and spectral recovery maps. Use the section-induced Hilbert identifications already constructed to lift each word before taking its scalar matrix element. For a fixed word, write Ψ^Z_χ for that scalar source reader. Unitary recovery gives $\Psi^Z_\chi(y,f) = \Psi_\chi(y,f)$ at strict heights, for the same joint test and pinned labels. Since D acts on the test alone, for every k

$$\Psi^Z_\chi(y, D^k f) = \Psi_\chi(y, D^k f), \qquad P^Z_{m,\chi}(y,f;t) = P_{m,\chi}(y,f;t).$$

In particular the cancellation in the differentiated finite sum recovers exactly the same remainder integrand

$$R^Z_{m,\chi}(y,f;t) = \frac{(-t)^m}{m!}\Psi_\chi(y+ta, D^{m+1}f), \qquad |R^Z_{m,\chi}(y,f;t)| \le \frac{C}{m!}\|D^{m+1}f\|_{L^1}.$$

The constant is independent of χ, $0 < t \le 1$, and nondecreasing y in the specified wave-bound regime. Finite-sum recovery and this dominated integral define the same $B_\chi(y,f)$ on the closed height chamber. Its Schwartz seminorm, multilinearity, uniqueness across admissible m and all strict-height approaches are therefore preserved. For the normalized character probability measure, the strict Fubini identity and this common majorant give

$$\int B_\chi(y,f)\, d\mathbb{P}(\chi) = B(y,f).$$

The finite strict terms and dominated remainder also recover continuity in χ. Thus the boundary barycenter and character continuity use the same factor-norm bound and tested Cauchy–Riemann inputs as the source.

For the reflection-space unitary section U_R, retain its joint spectral measure by $P^Z(A) = U_R P(A) U_R^{-1}$ and heat evolution by $E_s^Z = U_R E_s U_R^{-1}$. The exact measure reader is

$$\langle U_R x, P^Z(A) U_R x\rangle = \langle x, P(A)x\rangle, \qquad \Pi^Z = U_R \Pi U_R^{-1} = P^Z(\{E=0\}).$$

Strong limits commute with these bounded identifications. With the boundary section U_B and retained attachment J^Z, one also has

$$(J^Z)^* E_s^Z J^Z = U_B (J^* E_s J) U_B^{-1} \xrightarrow[s\to\infty]{\text{strong}} U_B P_{\text{boundary}}(\{E=0\}) U_B^{-1}.$$

This recovers the zero-energy projection with precisely its supplied rank scope. Subsequent vacuum-rank arguments remain separate dependencies.

19.5 Weak irreducibility from heat evolution

Let fields Φ act on their common word domain, with cyclic unit vacuum Ω and translations $U(t)$. Suppose bounded B satisfies

$$\langle \Phi(a,f)^* u, Bv\rangle = \langle u, B\Phi(a,f)v\rangle$$

for domain vectors, with the actual adjoint restricted there. No domain preservation by B is assumed. Iteration moves words across the pairing. For $u = w\Omega$, $v = w^*\Omega$, the heat attachment gives bounded continuous upper-half-plane functions, holomorphic inside, with boundary values $F(t) = \langle u, U(t) B\Omega\rangle$ and $G(t) = \langle B^*\Omega, U(t)v\rangle$. Covariance, vacuum invariance and weak word commutation give $F(t) = G(-t)$. Glue $F(z)$ above to $G(-z)$ below. Continuity and rectangular Cauchy integrals with cancelling axis boundaries prove entire holomorphy. Boundedness and Liouville make it constant; thus

$$\langle u, B\Omega\rangle = \langle u, e^{-sH} B\Omega\rangle.$$

The constructed strong heat limit to the vacuum projection gives $B\Omega = a\Omega$, $a = \langle \Omega, B\Omega\rangle$, by word density. Moving words once more yields $\langle u, Bw\Omega\rangle = a\langle u, w\Omega\rangle$. Density and boundedness imply $B = aI$. Scalars satisfy the identity by the field adjoint relation, so the weak commutant is precisely scalar. Heat contraction, holomorphic-edge and vacuum-projection results remain its stated inputs. The lift retains the common domain and actual adjoint pairings. Unitary recovery transports the bounded operator and word vectors, preserving this weak identity and proof without an extra domain-invariance assumption.

Recovery of the weak commutant on its actual domain. Fix the completed-space section unitary $V : \mathcal{H} \to \mathcal{H}^Z$. Set $\mathcal{D}^Z = V\mathcal{D}$, $\Omega^Z = V\Omega$, $\Phi^Z = V\Phi V^{-1}$ on $\mathcal{D}^Z$, and $B^Z = VBV^{-1}$. Unitary equivalence of the actual adjoint gives $(\Phi^Z)^* = V\Phi^* V^{-1}$ with domain $V \operatorname{Dom}\Phi^*$. Consequently the precise weak pairing is recovered by

$$\begin{aligned} \langle (\Phi^Z)^* Vu, B^Z Vv\rangle &= \langle \Phi^* u, Bv\rangle, \\ \langle Vu, B^Z \Phi^Z Vv\rangle &= \langle u, B\Phi v\rangle, \qquad u, v \in \mathcal{D}. \end{aligned}$$

The two weak commutants are therefore in bijection under bounded-operator conjugation; no image of Bv under a field is introduced. Finite word pairings recover in the same order, with $w^Z\Omega^Z = Vw\Omega$. The heat and translation readers give the identical scalar F, G above, so the glued entire function and its constant value are unchanged. Writing $a = \langle \Omega, B\Omega\rangle$, the final density step reads

$$B^Z\Omega^Z = a\Omega^Z, \qquad \langle Vu, B^Z w^Z \Omega^Z\rangle = a\langle Vu, w^Z\Omega^Z\rangle, \qquad B^Z = aI.$$

Inverse conjugation recovers $B = aI$. This transports the supplied vacuum-projection argument as a dependency of weak irreducibility, without changing the common word domain or the weak hypothesis.

19.6 Low-energy shells and uniform heat decay

Let positive forward-cone spectral measures μ, ν of a compact word combination and its adjoint have total mass sum at most M. Their commutator difference $A(x,t) = \widehat{\mu}(x,t) - \widehat{\nu}(-x,-t)$ vanishes for $|x| > R$, $|t| \le 1$, uniformly in character by locality. Set

$$D(x) = \int_0^1 \frac{A(x,t) - A(x,-t)}{2i}\, dt, \qquad \rho(E) = \int_0^1 \sin(Et)\, dt.$$

Here $\rho = (1 - \cos E)/E \ge 0$ for $E > 0$, $\rho(0) = 0$, $|\rho| \le 1$ and $\rho(E) \ge E/\pi$ on $[0,1]$. Fubini makes D the spatial Fourier transform of the sum of positive $\rho\mu$ and spatially reflected $\rho\nu$ measures. It is bounded by M and supported in radius R. For $0 < u \le 1$, choose smooth bump B_u equal to one on momentum radius u, supported within $2u$, between zero and one. Its Fourier transform is bounded in absolute value by the volume of the radius-$2u$ ball, hence by $C_3 u^3$. Fourier inversion and positivity imply

$$\int \rho(E) B_u(p_{\mathrm{spatial}})\, d\mu \le K(R,M) u^3, \qquad \mu\{u/2 < E \le u\} \le 2\pi K(R,M) u^2.$$

The second uses cone support and $\rho \ge u/(2\pi)$ on the shell. Dyadic decomposition of $(0,u]$ sums 4^{-n} and yields a constant times u^2. Compact-character word-norm continuity supplies uniform masses. For any such measure with $\nu\{0 < E \le u\} \le Cu^2$, Tonelli gives

$$\int_{E>0} e^{-sE}\, d\nu = s \int_0^\infty e^{-st} \nu\{0 < E \le t\}\, dt \le 2Cs^{-2}.$$

Substitution and two integrations by parts evaluate the last integral. Increase C to total mass to extend an initially small-u bound. This bounds $\langle x, (e^{-sH} - P_0)x\rangle$, keeping the zero atom separate. Character continuity of translated word pairings gives characteristic- function continuity. Fourier inversion with majorant $M|\widehat{g}|$ gives continuity for compact smooth momentum tests. Approximate heat multipliers by cutoffs equal to one in radius $2N$ and supported in $3N$; cone support bounds the omitted tail by Me^{-sN} uniformly in character. Heat pairings are thus continuous uniform limits. These arguments supply the heat continuity and shell-to-heat steps used in the removal of exceptional characters. Fubini applies since both measures are finite and the integrands are bounded. The lift retains word measures, locality radius, positive time averaging and spectral cone. Recovery preserves these Fourier/Tonelli integrals and uniform majorants, transporting shell and heat bounds while retaining the zero-energy atom.

Positive spectral averaging and heat-decay recovery. For each retained phase section V_χ, the scalar spectral measure of a compact word vector is unchanged: if $x_\chi^Z = V_\chi x_\chi$ and $P_\chi^Z(A) = V_\chi P_\chi(A) V_\chi^{-1}$, then

$$\nu_\chi^Z(A) = \langle x_\chi^Z, P_\chi^Z(A) x_\chi^Z\rangle = \nu_\chi(A).$$

The same identity holds for the adjoint word combination. Thus their commutator reader gives $A^Z = A$ and the time-average reader gives $D^Z = D$. In particular locality preserves the same

radius R, while positivity of the two weighted measures $\rho\mu$ and reflected $\rho\nu$ survives unchanged. The source recovers the exact dyadic decomposition

$$\mu^Z\{0 < E \le u\} = \sum_{j=0}^{\infty} \mu\{u2^{-j-1} < E \le u2^{-j}\}, \qquad 0 < u \le 1.$$

The geometric shell bill and the total-mass extension therefore give the same Cu^2 bound. For $s > 0$ the heat reader recovers

$$\langle x_\chi^Z, (e^{-sH_\chi^Z} - P_{0,\chi}^Z)x_\chi^Z\rangle = \int_{E>0} e^{-sE}\, d\nu_\chi \le 2Cs^{-2}.$$

Here the zero-energy atom is removed before the Tonelli calculation. The common mass bound and cone support also preserve the cutoff error Me^{-sN} for heat pairings. Hence their character continuity follows from the same compact-test Fourier readers and uniform cutoff limit, with no extra continuity premise on the ground projection.

19.7 Every-character vacuum uniqueness

Full support of the compact character probability means every nonempty open set has positive measure. A closed almost-everywhere set is therefore all characters. For compact-word sum vectors x_χ, the almost-everywhere unique heat ground and inverse-square estimate give, with common M,

$$0 \le \langle x_\chi, e^{-sH_\chi} x_\chi\rangle - |\langle \Omega_\chi, x_\chi\rangle|^2 \le Ms^{-2}, \qquad s \ge 1.$$

Attachment identifies zero energy with the one-dimensional heat ground in those phases; normalization fixes its vacuum coefficient. Heat pairings and overlaps are continuous, so the displayed inequality defines a closed set for fixed s. Full support extends it to every character without assuming ground-projection continuity. For any fixed character, the spectral strong limit $e^{-sH_\chi} \to P_\chi$ and the inequality give

$$\|P_\chi x_\chi - \langle \Omega_\chi, x_\chi\rangle \Omega_\chi\|^2 = \langle x_\chi, P_\chi x_\chi\rangle - |\langle \Omega_\chi, x_\chi\rangle|^2 = 0.$$

Compact test cutoffs approach Schwartz words by joint-word continuity. Bounded projections then agree on Schwartz words and, by cyclicity, on the whole Hilbert space. Thus zero energy has rank one in every character, supplying the weak-commutant heat limit. The lift retains full-support measure, continuous word pairings and attachment. Recovery preserves the closed-set extension and spectral limit, transporting vacuum uniqueness with the stated low-energy and cyclicity inputs.

19.8 Propagation of analytic limits on a complex slice

Let connected open $U \subset \mathbb{C}$ carry eventually holomorphic F_a. At each point assume a ball with common tail bound $|F_a| \le A$, the tail allowed to depend on the ball. Suppose holomorphic f is the limit on a set accumulating inside U. Choose $B(x,s) \subset U$, $t = \min(r,s)/2$, $D = \max(1, A/t)$. For $z \in B(x,t)$, the closed disc of radius t about z lies in $B(x,r) \cap U$. Cauchy derivative estimates on interior radius-t discs give local tail bounds

$$|F_a'(z)| \le D, \qquad |F_a(y) - F_a(x)| \le D|y - x|.$$

Any everywhere pointwise limit g inherits this Lipschitz bound. Choose $|y - x| < \epsilon/(4D)$ and a tail with center error below $\epsilon/4$; triangle inequality gives local uniform convergence and holomorphy

of g. For the general filter pass to any finer ultrafilter. Each bounded scalar coordinate has a compact-disc limit, giving such a holomorphic g. It agrees with f on the accumulating set and hence on U by the identity theorem. Every finer ultrafilter thus converges to $f(y)$, proving the original filter convergence. This applies to boundary heights and phase-character limits with only local tail bounds. The lift retains the slice, filter and local boundedness data. Recovery preserves the tested analytic functions and their accumulation-set values, transporting this uniqueness and convergence argument without imposing a single global tail bound.

Every-character vacuum and slice-limit recovery. Keep the same compact character space and full-support probability in the source record. For each χ let V_χ be the section unitary and $x_\chi^Z = V_\chi x_\chi$, $\Omega_\chi^Z = V_\chi \Omega_\chi$. For a fixed compact-word sum and $s \geq 1$, the recovered scalar function is

$$q_s^Z(\chi) = \langle x_\chi^Z, e^{-sH_\chi^Z} x_\chi^Z \rangle - |\langle \Omega_\chi^Z, x_\chi^Z \rangle|^2 = q_s(\chi).$$

Its continuity is the scalar heat/word continuity established above; it requires no topology on the family of phase Hilbert spaces. Consequently the source and physical closed sets $\{\chi : 0 \leq q_s(\chi) \leq Ms^{-2}\}$ are identical. For each fixed s full support makes this set all characters. Fixing χ afterwards and taking the strong heat limit therefore recovers

$$P_\chi^Z x_\chi^Z = \langle \Omega_\chi^Z, x_\chi^Z \rangle \Omega_\chi^Z, \qquad P_\chi^Z = V_\chi P_\chi V_\chi^{-1} = |\Omega_\chi^Z\rangle\langle\Omega_\chi^Z|.$$

The second equality extends from compact words through the same compact test cutoffs, Schwartz-word continuity and cyclic density. Inverse unitary recovery gives exactly the source's rank-one zero-energy conclusion and its heat-limit input to weak irreducibility.

On a complex slice retain the original indexing filter $\mathcal{F}$, connected domain U, local boundedness balls and accumulation set. Scalar word recovery gives $F_a^Z(z) = F_a(z)$ and $f^Z(z) = f(z)$ at every interior point. It therefore preserves the local tail sets and Cauchy bounds with exactly their stated ball dependence. For every finer ultrafilter $\mathcal{U}$ the compact-disc coordinate reader obeys

$$\lim_{a \to \mathcal{U}} F_a^Z(z) = \lim_{a \to \mathcal{U}} F_a(z) = f(z), \qquad z \in U.$$

Local equicontinuity, holomorphy and accumulation-set uniqueness are thus recovered on that same slice; convergence for all finer ultrafilters recovers the original $\mathcal{F}$ limit. No selection of a new height sequence or character subsequence enters this readout.

19.9 Bounded slab operators by powers

Let $X : V \to \mathcal{H}$ realize the algebraic future-word span in its reflection-positive completion. Suppose $A, B : V \to V$ obey $\langle XAv, Xw \rangle = \langle Xv, XBw \rangle$ and put $T = BA$. For $D > 0$ assume each v has finite C_v with

$$\|XT^n v\| \leq C_v D^{2n} \quad (n \geq 0). \tag{ST1}$$

The Gram adjoint relation makes T and its powers symmetric. Set $d_n = \|XT^n v\| D^{-2n}$ and finite $s = \sup_n d_n$. Symmetry and Cauchy–Schwarz give

$$d_n^2 = D^{-4n} |\langle Xv, XT^{2n} v \rangle| \leq \|Xv\| d_{2n} \leq \|Xv\| s.$$

Taking suprema yields $s^2 \leq \|Xv\| s$, hence $s \leq \|Xv\|$. Consequently

$$\|XAv\|^2 = |\langle Xv, XTv \rangle| \leq D^2 \|Xv\|^2. \tag{ST2}$$

Thus A descends through null vectors and extends boundedly with norm at most D, removing the individual C_v. Apply to slab insertion and reflected insertion; hierarchy translation invariance supplies their adjoint relation. If v_n adjoins n successive reflected/unreflected slab pairs, it suffices that

$$|\operatorname{Gram}(v_n, v_n)| \le Q_v D^{4n}. \tag{ST3}$$

A scalar multiple has norm at most $|a|\sqrt{Q_v}D^{2n}$; triangle inequality extends (ST1) to every finite word combination. Hence a diagonal estimate on individual words really does produce a bounded operator on their entire dense span. This bounds the entire dense span without presupposing boundedness, in every character. The lift retains the algebraic insertion maps and Gram adjoint relation. Isometric recovery preserves powers and null spaces, transporting (ST1)–(ST3) and the bounded extension with the same norm.

19.10 Exact slab norm transfer to every character

For unselected slab operator U and ground-space word maps, concatenation and reflection give

$$X_{\mathrm{enter}(w,v)} = UX_v, \qquad X_{v_n} = (U^*U)^n X_v, \qquad \|X_{v_n}\| \le \|X_v\|\|U\|^{2n}.$$

The character Gram is the reading of the compressed two-sided word:

$$|\operatorname{Gram}_\chi(v, v)| = |\chi(X_v^* X_v)| \le \|\chi\|\|X_v\|^2.$$

For any $D > \|U\|$ this gives (ST3) with $Q_v = \|\chi\|\|X_v\|^2$. The power proof removes this prefactor and constructs the phase slab with norm at most D. Operators for different D agree on the dense insertion-prescribed word span and hence everywhere. Taking $D \downarrow \|U\|$ gives

$$\|U_\chi\| \le \|U\|. \tag{ST4}$$

This includes zero norm and transfers all unselected arbitrary-copy constants uniformly to every character. Construct each character chamber, slabs and inherited derivative calculus; these bounds supply contour and strict-wave slab laws for field reconstruction with common labels. Apply the field reconstruction to those data to obtain its Hilbert space and fields, with the common conjugation and rotation labels. This is legitimate because they satisfy the same requirements; consequently no comparison between independently chosen copies is needed there. At the previously selected character retain its already constructed objects, which satisfy those requirements. This keeps the actual chosen realization and supplies phasewise polynomial bounds for tested boundaries. Contour and field construction remain below; vacuum uniqueness is separate. Their proof does not require discarding exceptional characters. The operator estimates needed in reconstruction hold at every character of the compression algebra. The lift retains ground-space word maps and whole-word compression before character reading. Recovery preserves concatenation, Gram evaluation and the dense-span extension, transporting (ST4) with its exact constant.

Algebraic powers, null descent and slab recovery. Let $\ell : V \to V^Z$ be the word lift, and let $R : V^Z \to V$ be its word readout with $R\ell = I$. Define the source Gram form and insertion maps by

$$g^Z(\zeta, \eta) = \langle XR\zeta, XR\eta\rangle, \qquad A^Z = \ell AR, \quad B^Z = \ell BR, \quad T^Z = \ell TR.$$

For every $n \ge 1$, $(T^Z)^n = \ell T^n R$; the zeroth power is the identity, with the same recovered Gram norm. The Gram adjoint relation, all d_n and their finite supremum are consequently identical to the physical ones for $v = R\zeta$. The power estimate recovers

$$\|A^Z\zeta\|_{g^Z} \le D\|\zeta\|_{g^Z}, \qquad \ker g^Z = R^{-1}(\ker X).$$

It therefore defines a bounded map on the source null quotient before completion. For $v = \sum_i a_i v_i$, the explicit finite-combination constant is $C_v = \sum_i |a_i| \sqrt{Q_{v_i}}$, so the descent covers the whole algebraic word span. The induced completion unitary V_H gives $\overline{A}^Z = V_H \overline{A} V_H^{-1}$ and preserves its norm.

For whole-word compression let V_0 identify the unselected ground spaces, and put $C^Z = V_0 C V_0^{-1}$ on the closed compression algebra. Recover a character by $\chi^Z(C^Z) = \chi(C)$. These are isometric algebra identifications and preserve the character norm. Hence, for the lifted ground-space word map, the retained positive Gram reader is exactly

$$\chi^Z((X_v^Z)^* X_v^Z) = \chi(X_v^* X_v), \qquad Q_v^Z = \|\chi^Z\| \|X_v^Z\|^2 = \|\chi\| \|X_v\|^2.$$

The preceding null-descent construction removes this prefactor on each character chamber. Its bounded extension is independent of the chosen $D > \|U\|$ because its action agrees on the same dense word span. Letting D decrease to $\|U\|$ thus recovers (ST4), including $\|U\| = 0$, with the same constant in every character. The already selected chamber and its slab maps remain the prescribed members of this family.

19.11 Centered insertions and buffered strict-time words

Let $E(u)$ be chamber heat contraction, $F(\zeta)$ its holomorphic contraction semigroup on $\Re\zeta > 0$, and $U(x)$ commuting spatial translations fixing Ω. Define

$$B_p(v) = \mathrm{place}(2p, [(0, p; v)]), \qquad p > 0. \tag{SW1}$$

Its future-word matrix elements are hierarchy values with insertion between reflected left spectators and right spectators shifted by $2p$. Distinct positions and hierarchy multilinearity make B_p linear. Finite-level representatives of the same union label have equal matrix elements by derivative-plan inclusion, hence equal bounded operators on the dense word span. Thus this is a map on the common field union. Arbitrary-copy slab bounds give, for each label,

$$\|B_p(v)\| \le Q_v p^{-d_v}, \qquad 0 < p < 1. \tag{SW2}$$

Both buffers have width p; unused width zero may be assigned zero. Set $R(w, s, x) = F(-iw - s)U(x)$ for $\Im w > s$, $s \ge 0$. Its norm is at most one. For strict imaginary heights choose

$$0 < p < 1, \qquad 4p < \Im z_j - \Im z_i \quad (i < j). \tag{SW3}$$

The ordered buffered matrix element is

$$\begin{aligned} M_p(z, x; v) = \langle \Omega, & B_p(v_1) R(z_2 - z_1, 2p, x_2 - x_1) B_p(v_2) \cdots \\ & R(z_n - z_{n-1}, 2p, x_n - x_{n-1}) B_p(v_n) \Omega \rangle. \end{aligned} \tag{SW4}$$

Empty words give $\langle \Omega, \Omega \rangle$; all factors are defined under (SW3) and multilinear in union labels. The lift retains boundary buffers, representatives and the exact ordered semigroup factors. Unitary recovery preserves the dense matrix-element identification and (SW1)–(SW4), including the same polynomial norm bounds and strict-height domain.

19.12 Width independence and strict-wave bounds

Slab stretching and concatenation give

$$B_{p+u}(v) = E(u)B_p(v)E(u). \tag{SW5}$$

For admissible larger width, semigroup composition and spatial commutation also give

$$R(w, 2p, x) = E(u)R(w, 2(p+u), x)E(u). \tag{SW6}$$

Insert (SW5) into the larger-width product. Interior heat factors combine by (SW6), exterior ones fix the vacuum, proving $M_{p+u} = M_p$. Comparison of any two admissible widths proves independence. For $z_i = iy_i$, recursively build the ket backwards: centered insertion plants at time p, and heat gap $y_{i+1} - y_i - 2p$ with spatial translation places the remaining word. Vacuum pairing gives the complete Euclidean word up to a common translation, removed by invariance. Quotient-label identity identifies its union hierarchy value. Define $F_n = M_p$ at strict times. An admissible p exists because there are finitely many positive imaginary gaps; the choice is irrelevant by width independence. Bounded composition makes F_n holomorphic in complex times and continuous in spatial positions. No regularity of a global width choice is required. Contractions and Cauchy–Schwarz give $|F_n| \le \|\Omega\|^2 \prod_i \|B_p(v_i)\|$. For gaps at least $0 < k \le 1$, choose $p = k/8$ to obtain

$$|F_n| \le C_v k^{-m_v}, \qquad m_v = \sum_i d_{v_i}, \qquad C_v = \|\Omega\|^2 \prod_i (Q_{v_i} 8^{d_{v_i}}). \tag{SW7}$$

This is uniform in real times and positions. Empty gap conditions for arities zero and one still permit that choice; constants may depend on arity and labels. In particular this argument requires no continuity of the arbitrarily chosen width as a function of z. At purely imaginary times $z_i = iy_i$, the same product is exactly the original Euclidean hierarchy at the points (x_i, y_i). The lift retains buffered words and the exact heat composition. Unitary recovery preserves width independence, Euclidean identification and the same local analytic construction and gap bound.

Buffered product recovery on the strict-time domain. Fix the section unitary V_H for the chamber. Its bounded-operator source maps are

$$B_p^Z(v) = V_H B_p(v) V_H^{-1}, \qquad F^Z(\zeta) = V_H F(\zeta) V_H^{-1}, \qquad U^Z(x) = V_H U(x) V_H^{-1}.$$

Set $E^Z(u) = F^Z(u)$ and $\Omega^Z = V_H\Omega$. The field-union reader uses the same quotient labels, so linearity and equality of finite-level representatives pass through these maps. Norm recovery gives the exact $Q_v p^{-d_v}$ bound, with both boundary buffers equal to p. The propagated factor and the whole buffered word recover as

$$R^Z(w, s, x) = V_H R(w, s, x) V_H^{-1}, \qquad M_p^Z(z, x; v) = M_p(z, x; v).$$

These equalities use the ordered product in (SW4), cancelling only adjacent $V_H^{-1}V_H$ factors. The admissibility condition remains $\Im w > s$ and (SW3); the empty-word reader is the same unit norm. The two exact source composition identities are

$$B_{p+u}^Z(v) = E^Z(u) B_p^Z(v) E^Z(u),$$
$$R^Z(w, 2p, x) = E^Z(u) R^Z(w, 2(p+u), x) E^Z(u).$$

They recover (SW5)–(SW6) before taking the vacuum matrix element. Thus width independence, the backwards Euclidean plant recursion and its common translation are all read from the same word. In particular $F_n^Z = M_p^Z = F_n$ for every admissible width.

Conjugation is a bounded linear map on the operator spaces and preserves holomorphic dependence on every local strict-time neighborhood with a fixed admissible p. Hence the source wave has exactly the same local holomorphy and spatial continuity. Choosing $p = k/8$ recovers (SW7) with the identical m_v, C_v; its uniformity remains over real positions and times for the fixed word. Arity zero and one retain the empty gap condition and the same allowed width choice.

19.13 Tested Cauchy–Riemann identities

For real spacetime X define

$$f_{v,y}(X) = F_n((t_i(X) + iy_i)_i, (x_i(X))_i; v), \qquad \Psi_v(y;\varphi) = \int f_{v,y}(X)\varphi(X)\,dX.$$

Strict heights have a neighborhood with one positive gap lower bound; (SW7) bounds all nearby integrands uniformly in noncompact X. This follows by decreasing the finitely many strictly positive gaps slightly; equivalently subtract a sufficiently small multiple of the increasing height vector. Schwartz decay gives absolute integrability and dominated height continuity. For direction a let e_a have time coordinates a_i and zero spatial coordinates. The slice

$$h_{X,u}(\zeta) = f_{v,u+(\Im\zeta)a}(X + (\Re\zeta)e_a)$$

is holomorphic on the strict-height domain. One disc of radius $r > 0$ works for all X and nearby u, with common bound C. Hence

$$|\partial_{e_a} f_{v,u}(X)| \le C/r. \tag{CR1}$$

Slice holomorphy gives $\frac{d}{ds} f_{v,y+sa} = i\partial_{e_a} f_{v,y+sa}$. The uniform derivative bound permits differentiation under the test integral; real-direction integration by parts yields

$$\frac{d}{ds}\Psi_v(y + sa;\varphi) = -i\Psi_v(y + sa;\partial_{e_a}\varphi). \tag{CR2}$$

Bounded original and derivative functions with Schwartz decay justify this on the whole real space. It holds for joint tests and every height direction remaining strict, supplying the earlier Taylor boundary proof. A single disc $|\zeta| \le r$ is admissible for every X and all u in a sufficiently small height neighborhood. The lift retains joint tests, real/time direction correspondence and the uniform slice radius. Recovery preserves the holomorphic slice and its integrable derivative bound, so the sign and test action in (CR1)–(CR2) are recovered before taking boundary limits.

19.14 Absolute-frequency identity

Use the construction's Schwartz transform $\mathscr{H}$, including its fixed coordinate rescaling, so

$$-i\partial_d(\mathscr{H}g) = -\mathscr{H}(\langle d, \cdot\rangle g), \qquad d = e_y. \tag{FT1}$$

For compact smooth g, put $g_s(q) = e^{s\langle d,q\rangle} g(q)$. The same compact support and smooth exponential divided differences show $\frac{d}{ds} g_s = \langle d, \cdot\rangle g_s$ in Schwartz topology: all weighted derivatives converge uniformly on that support and vanish outside. This proves the asserted Schwartz derivative, including at an arbitrary parameter value. For strict y, $A(s) = \Psi_v(sy; \mathscr{H} g_s)$ has derivative zero by (CR2) and (FT1), whose height and test terms cancel. Interior differentiation uses the local majorant. Near the boundary, the tested Taylor estimate gives one continuous Schwartz seminorm p with $|\Psi_v(u;\varphi)| \le p(\varphi)$; moving-test errors therefore vanish uniformly in strict u. Constancy and $s \downarrow 0$ yield

$$\widehat{W}_v(g) := W_v(\mathscr{H}g) = \Psi_v(y; \mathscr{H}(e^{\langle e_y,q\rangle} g(q))). \tag{FT2}$$

This is in absolute frequency coordinates. On the interior, the ordinary locally dominated derivative argument applies. The lift retains the Fourier rescaling, joint compact test and height direction. Recovery commutes with Schwartz differentiation and the uniform boundary estimate, preserving the cancellation and absolute-coordinate identity before the cone argument.

Joint-test differential and Fourier recovery. The strict scalar wave reader already gives $f^Z_{v,y}(X) = f_{v,y}(X)$ for every real configuration and strict height. Keep the joint Schwartz space and its spacetime direction e_a unchanged. The source slice $h^Z_{X,u}$ is therefore the same holomorphic scalar slice with the same radius r, independently of X. Its tested reader obeys

$$\Psi^Z_v(y;\varphi) = \Psi_v(y;\varphi), \qquad \left|\partial_{e_a} f^Z_{v,u}(X)\varphi(X)\right| \le (C/r)|\varphi(X)|.$$

This integrable majorant preserves differentiation under the integral. The whole-space integration-by-parts reader is explicitly

$$\frac{d}{ds}\Psi^Z_v(y+sa;\varphi) = i\int \partial_{e_a} f_{v,y+sa}(X)\varphi(X)\,dX = -i\Psi^Z_v(y+sa;\partial_{e_a}\varphi).$$

Thus the minus sign, joint-test domain and arbitrary strict-cone height direction in (CR2) are recovered before any boundary approach.

For the frequency step retain the source's scaled Schwartz transform $\mathscr{H}$ and absolute momentum coordinates q. Put $g_s = e^{s\langle e_y,q\rangle}g$ on the same compact support. Recovery commutes with this fixed test operation and its Schwartz derivative. Writing $d = e_y$, the two recovered product-rule terms cancel exactly:

$$\begin{aligned}\frac{d}{ds}\Psi^Z_v(sy;\mathscr{H}g_s) &= -\Psi_v(sy;\mathscr{H}(\langle d,q\rangle g_s)) \\ &\quad + \Psi_v(sy;\mathscr{H}(\langle d,q\rangle g_s)) = 0.\end{aligned}$$

The retained common boundary seminorm also yields

$$|\Psi^Z_v(sy;\mathscr{H}(g_s-g))| \le p(\mathscr{H}(g_s-g)) \longrightarrow 0 \quad (s\downarrow 0).$$

Consequently the source boundary distribution recovers $\widehat{W}^Z_v(g) = W_v(\mathscr{H}g)$ and the full identity (FT2) in absolute coordinates. The readout keeps the Fourier rescaling and its single derivative sign throughout this moving-test limit.

19.15 The ordered time-frequency cone

For weakly increasing y and compact smooth g supported where $\langle e_y, q\rangle < -1$, fix strict unit-gap r and use (FT2) at $Y(t) = ty + r$, $t \ge 0$. The compactly supported tests $g_t = e^{\langle e_{Y(t)},q\rangle}g$ tend to zero in Schwartz topology: each order-j weighted derivative is bounded by a finite sum

$$\binom{j}{\ell} e^{-t}(t\|e_y\|)^\ell p_{i,j-\ell}, \qquad 0 \le \ell \le j,$$

with fixed constants including the r factor. Every term vanishes. Unit-gap (SW7) bounds (FT2)'s right side by a fixed continuous seminorm of $\mathscr{H}g_t$, so its constant left side is zero. If $\langle e_y, q\rangle < 0$ at a point, rescale y to make it -2; a neighborhood has pairing below -1 and all its compact tests vanish. Therefore

$$\operatorname{supp}\widehat{W}_v \subset \bigcap_{y_1\le\cdots\le y_n} \{q : \langle e_y, q\rangle \ge 0\}. \tag{FT3}$$

Constant height vectors of both signs enforce total time-frequency conservation; weak ordering also supplies partial-energy inequalities. Relativistic spectrum additionally uses the rotation/covariance arguments below. The lift retains absolute frequencies, weak-height cone and compact support tests. Recovery preserves exponential test decay and distributional support, transporting (FT3) with both conservation and ordering conditions.

19.16 Removing the outer angular flux and obtaining all Euclidean rotations

The lattice symmetries give only signed coordinate permutations. Continuous rotations of the physical quotient require the conserved stress and its shell maps constructed below. Write p_α for the translation shell map and $Y(K)$ for the angular shell map, where K is a real skew 4-by-4 matrix. Before their normalization, the angular Ward expression is

$$\mathcal{A}_K(v;x) = \sum_\ell \{S(v_1,\dots,Y(K)v_\ell,\dots;x) + (Kx_\ell)^\alpha S(v_1,\dots,p_\alpha v_\ell,\dots;x)\}. \tag{RT1}$$

Its vanishing cannot be inferred by simply enlarging a spherical cutoff: the correlations have subpower infrared bounds, rather than the decay that such an argument would require. The source removes the outer flux through tested spectral estimates. We give the details of that passage here.

The tests in this argument have the following precise meaning. Fix a nonnegative even smooth compactly supported function c, equal to one on $[-1,1]$, and let

$$c_w(p) = w^{-1}c(p/w), \qquad w > 0.$$

The source uses this central frequency profile and real Schwartz profiles with the same normalization. Its j-th moment is $\int p^j f(p)\,dp$; substitution gives the moment of f_w as w^j times that of f. In the Fourier convention used here,

$$\partial_x^j \widehat{f}(0) = (-i)^j \int p^j f(p)\,dp.$$

Differentiation under the integral is justified by the integrable Schwartz majorant $|p|^j|f(p)|$. If moments below m vanish, repeated integration of the derivative along a segment bounds $|\widehat{f}(x)| \le C|x|^m$ on a fixed ball. This is the origin-jet cancellation used in the compensated profiles.

For four one-dimensional tests f_a, the tested stress pair is the product-test extension of

$$R_{ab,cd}(f) = \int \langle J_{ab}(x)J_{cd}(0)\rangle \prod_i f_i(x_i)\,d^4x. \tag{RT0a}$$

When a factor vanishes sufficiently far in Taylor order at zero this is the ordinary integral. For arbitrary Schwartz factors, subtract the finite product Taylor polynomial near zero; the polynomial short-distance bound makes the remainder integrable. This defines the extension without presuming rotation invariance. Pair exchange, tensor symmetry and signed covariance follow by the corresponding changes of variables in the remainder integral.

Its tested conservation defect is

$$\Delta_{b,cd}(f) = \sum_i R_{ib,cd}(f_0,\dots,f_i',\dots,f_3).$$

Let $g_j(x) = x^j c(x)/j!$, and let $f^{(a)}$ have factor g_1 at a and g_0 elsewhere. Set

$$w_{ab,cd} = \Delta_{b,cd}(f^{(a)}), \qquad R^\circ_{ab,cd}(f) = R_{ab,cd}(f) - w_{ab,cd}\prod_i f_i(0). \tag{RT0b}$$

The subtracted term is supported at the single contact $x = 0$. Its differentiated value on $f^{(a)}$ is exactly $w_{ab,cd}$, because $g_j^{(\ell)}(0) = 1_{\{j=\ell\}}$. The resulting defect therefore vanishes on all first-order product jets. It vanishes on the order-zero and order-two jets by total inversion: the three remaining tensor indices give an odd sign whereas an even total test order gives an even sign. These finite contact subtractions preserve the pair and signed symmetries; reflection in a single axis kills a coefficient unless its four tensor indices occur in pairs. Thus the compensated conservation defect has no jets of total degree below three. This precise contact remainder, rather than an assumed exact conservation law at coincident points, enters the spectral band estimates.

For a fixed exterior word on coordinate line l, put

$$F_{db}(y) = \langle J_{db}(y) \prod_{\ell} B_\ell(x_\ell) \rangle.$$

Realize that exterior word as a finite word vector and pair it with the one-stress vector. The hierarchy's permutation and conjugation laws identify this pairing with F_{db} whenever the points are distinct. This is why the spectral bounds apply to an arbitrary exterior word, not only to the stress two-point function. The actual outer term is

$$\int (\partial_d \phi_\lambda^{ab,l})(y) F_{db}(y)\, d^4y, \qquad d \neq a, \tag{RT0c}$$

for the compensated angular primitive $\phi_\lambda^{ab,l}$. Its frequency widths are λ^{-100} in axes a, b, λ^{-150} on the remaining line axis l, and λ^{-300} otherwise; it carries the primitive amplitude λ^{100}. The band subtractions described next give the actual estimate $C\lambda^{-1}$ for (RT0c), including its integrability. The finite excluded insertion set has Lebesgue measure zero and does not change this integral. Compact shell conservation identifies the difference between the summed local charges and this outer integral. Its vanishing therefore proves (RT1).

For each coordinate i, reflection positivity supplies its own positive joint spectral measures ν_i^A, supported on $E = p_i \geq 0$, for finite linear combinations A of reflected words. They include atoms at $E = 0$. Mixed readings are obtained by polarization and obey the weighted Cauchy–Schwarz inequality

$$|\langle B, f(P)A \rangle| \leq \frac{1}{2} \left(t \int X\, d\nu_i^B + t^{-1} \int Z\, d\nu_i^A \right) \quad \text{if } |f|^2 \leq XZ, \quad t > 0. \tag{RT2}$$

This follows by applying $2uv \leq tu^2 + t^{-1}v^2$ to the two spectral vectors; approximation by simple spectral functions proves it for the displayed measurable weights. No identification of spectral measures for different axes is needed.

All transverse moments needed below are finite after positive time damping. To prove this, take the characteristic function of the finite damped measure and apply k pairs of forward and backward differences in a transverse direction j. Its integral multiplier, up to the cancelling sign of the difference pair, is

$$\left| \frac{e^{ihp_j} - 1}{h} \right|^{2k}.$$

The fundamental theorem of calculus bounds the difference expression by the corresponding $2k$-th derivative of the smooth separated correlation. Fatou's lemma as $h \to 0$ gives the moment of $|p_j|^{2k}$. The same argument for moved word clusters uses their actual positive time gaps. For a cluster collapsed in the i direction this yields, for some finite D, C depending on the word and k,

$$\int e^{-tE} |p_j|^{2k}\, d\nu_i^A \leq C t^{-D}, \qquad 0 < t \leq 1. \tag{RT3}$$

Thus moment orders can be chosen before shrinking the spectral bands.

Here is the interpolation that controls low energy. For distinct nodes $z_j = (j+2)^2$, let L_j be their Lagrange basis polynomials, $0 \le j < m$. Polynomial interpolation gives

$$\frac{1}{x+y} - \sum_{j<m} \frac{L_j(x)}{z_j+y} = \frac{1}{x+y} \prod_{j<m} \frac{z_j - x}{z_j + y}, \qquad y \ge 0, \quad x+y \ne 0. \tag{RT4}$$

For completeness, multiply the left side by $x+y$. It is a polynomial of degree at most m, zero at every z_j, and equal to one at $x=-y$. These $m+1$ values determine it and give the product on the right. Rescale $x=(p/a)^2, y=(E/a)^2$. Multiplying by the even Laplace numerator $2E$ gives the exact response after subtracting the finite high-band interpolant. For moment-controlled profiles the subtracted moments are exactly those furnished by (RT4). The residual response is nonnegative for $E \ge 0$, and the choice of the central profile gives

$$\mathcal{R}_{m,a}(E) \ge \frac{c_m}{E+a}, \qquad 0 \le E \le 1, \quad 0 < a \ll 1, \tag{RT5}$$

with $c_m > 0$. At zero this is the continuous value of the full response, including the profile's zero-energy contribution; one does not divide a bare factor $2E$ by E and discard an atom. The numerator factors in (RT4) stay of one sign because the high nodes lie outside the primary profile band. Their fixed ratios give c_m; their denominators give the stated reciprocal bound.

The reason this positive response is small is a sign identity for the same tested position correlation. Suppose A has reflection parities ϵ_i, ϵ_j of opposite sign on axes i, j. Write

$$P_i(h;f) = \int \mathcal{L}_i(h)(p_i) \prod_{l \in Q\setminus\{i\}} f_l(p_l)\, d\nu_i^A.$$

Here f_l are nonnegative even Schwartz frequency profiles and $\mathcal{L}_i$ is the real even Laplace response of the inverse Fourier profile in direction i. The two spectral formulas for that one tested correlation give $P_i = -P_j$. If some transverse axes are absent, insert profiles equal to one on larger and larger intervals; bounded dominated convergence gives the same identity for that subset Q. Split each active profile into its Lagrange interpolant and residual. Both residuals have nonnegative response. Consequently the primary residual on axis i is bounded by the sum of the two cross terms containing an interpolant. The interpolants are finite sums of profiles in a high band.

For clarity the choice of orders in this repeated operation is important. Put $\lambda = e^N$, and scale each frequency profile as $f_l^{(N)}(p) = \lambda^{-w_l} f_l(p/\lambda^{w_l})$, $w_l \le 0$. A bound $C\lambda^\gamma$ will be called a bound of exponent γ in this paragraph. If a high band has exponent $a' > 0$, a transverse $2k$-th moment from (RT3) bounds its far remainder with exponent at most

$$-2ka' + K|Q \setminus \{i,j\}| + \max(K, dD), \tag{RT6}$$

where $-K \le w_l$ and d is the earlier chosen band exponent. Given any $\eta > 0$, choose k so that the negative term exceeds the other fixed losses, and then choose $d > 0$ so small that $dD < \eta$. At each switch the coordinate just treated is removed from the list still requiring a bound. If its parity agrees with the next required coordinate, insert an opposite-parity coordinate first. A finite induction on that list terminates; the terminal case has only nonshrinking transverse widths and is bounded by (RT3). Choose moment orders and admissible band exponents backwards through this finite list. This proves an exponent η, for every $\eta > 0$, for the residual response. Equation (RT5) therefore gives

$$\int_{0 \le p_i \le 1} \frac{\prod_{l \in Q\setminus\{i\}} f_l^{(N)}(p_l)}{p_i + \lambda^q}\, d\nu_i^A \le C_\eta \lambda^\eta, \qquad q < 0, \tag{RT7}$$

when the collapsed cluster has both parities on Q.

A word on a coordinate line l is collapsed on the other axes but can have a genuine time gap on l. Subtract there a shifted positive-time profile. Its response is positive, dominates a fixed multiple of $(E+\lambda^q)^{-1}$ on $0 \le E \le 1$, and its damped transverse moments cost at most λ^{-q}, by (RT3) at the actual gap. The same switch pays this loss by increasing k in (RT6). Thus (RT7) holds on $i = l$. On $i \ne l$, bound the additional l-profile by its supremum $C\lambda^{-w_l}$. The resulting bound has exponent $\eta - w_l$. This proves the line estimate even when the time-axis cluster is not collapsed.

For a diagonal stress reading a further estimate is necessary. Subtract its finite position Taylor jet with a cutoff equal to one at zero. The remainder vanishes to the required order: repeated integration along the segment from zero bounds its k-th derivative by $C|x|^m$ when derivatives $k, \dots, k+m-1$ vanish. Tensor products of these subtractions preserve reflection parities. Conservation, followed by integration by parts on these tests, gives the three tested frequency expressions

$$B = p_i^2 A + R_1, \qquad C = p_i^2 B + R_2. \tag{RT8}$$

On profiles of width at most $W = \lambda^a$, $a \le 0$, the remainders have respectively bounds $O(W^4)$ and $O(W^6)$, with an arbitrarily small exponent loss. On profiles with the prescribed zero moments the remainder is exactly zero: it is a finite contact Taylor jet and each of its moments is zero.

Apply (RT4) at widths $W, 2W$ to A, B, C. After division by W^{2k} at levels $k = 0, 1, 2$, denote their six band readings by a, b, v, V, r, R, and their residual differences by f, g, h. The positive spectral responses have signs $f \ge 0, g \le 0, h \ge 0$. Equations (RT8) and the two-band subtraction give small bounds for

$$v - a, \quad V - 4b, \quad r - v, \quad R - 4V, \quad f - (a - b), \quad g - (v - V), \quad h - (r - R).$$

These imply a bound for f with the same exponent. Indeed if the sum of the seven absolute errors is e, these equalities within error e imply $a - b \ge -e$, $a - 4b \le 3e$, $a - 16b \ge 5(-e)$; eliminating a, b bounds f by $100e$. In the exact case they force $a = b = f = 0$; the numerical factor safely bounds the finite error elimination. The constant-moment term of each interpolant cancels between W and $2W$; every other term has at least a factor W^2. Combining this bound with the positive residual lower bound gives

$$\int_{0 \le E \le 1} \frac{EW^4}{(E+W)^4} \prod_{j \ne i} f_j^{(N)}(p_j)\, d\nu_i^{J_{ii}} \le C_\eta \lambda^{4a+\eta}. \tag{RT9}$$

This deduction uses positivity of spectral measures and of the indicated scalar responses. It does not differentiate a positive matrix or posit positivity of its derivatives.

Now approximate the plane Killing field $x_a e_b - x_b e_a$ by compactly supported smooth vector fields. Choose frequency widths λ^{-100} on axes a, b, λ^{-150} on the remaining line direction if present, and λ^{-300} on the other axes. An odd primitive in direction a, multiplied by λ^{100}, and its partner with a, b interchanged give the two components. Their derivatives in the a, b pair agree exactly. Since $J_{ab} = J_{ba}$, these two terms in $\sum_{i,j} \partial_i F_j J_{ij}$ cancel before any integral is split. The remaining derivatives are supported outside the insertion neighborhoods, where no contact term is present.

Split each such tested spectral integral into $E \le 1$ with transverse frequencies inside the box widened by λ, and its complement. On the complement, the Schwartz bound of order k gives an arbitrary factor λ^{-k} in a transverse direction. For $E > 1$ the actual exterior time gap gives e^{-E/λ^w}; for every fixed R, M this is at most $\lambda^{-M} e^{-RE}$, eventually. The finite damped moments from (RT3) then pay the whole complement. Inside the box use (RT2) with the two reciprocal weights from (RT7) for shear components, and (RT7),(RT9) for a diagonal component. Split the latter once more at $E = \lambda^{-3000}$. Below that threshold (including zero), the large gain in the line estimate pays a

crude stress bound; above it (RT9) applies. With the stated widths these estimates, including the primitive amplitude λ^{100}, leave a bound $C\lambda^{-1}$. Thus the exterior flux tends to zero.

Here are explicit definitions of the responses and contact expressions in this argument. For a real Schwartz frequency profile f, write $D_s f(p) = s^{-1} f(p/s)$, $m_q(f) = \int p^q f(p)\,dp$, and $\widehat{f}(t) = \int e^{-ipt} f(p)\,dp$, with the Fourier constants used in (RT2)–(RT4). Choose an even nonnegative smooth u of integral one supported in $1 \le |v| \le 2$, and set $u_k(v) = v^{-2k} u(v)$, which is smooth because its support misses zero. Write $L_j(X) = \sum_k l_{jk} X^k$ for the node polynomial in (RT4). The actual finite interpolant is

$$B_{m,s} f = \sum_{j<m} \sum_{k} l_{jk} m_{2k}(f) s^{-2k} D_s D_{j+2} u_k, \qquad R_{m,s} f = f - B_{m,s} f. \tag{RT11}$$

Integrating against a continuous scalar function and changing variables in each term turns its kernel into $\int u(v) \sum_j L_j((p/(sv))^2) g(sv(j+2))\,dv$. Putting $g(p) = (p^2 + E^2)^{-1}$ and using (RT4) gives

$$\mathcal{R}_{m,s,f}(E) = 2E \int u(v) \int \frac{f(p)}{p^2 + E^2} \prod_{j<m} \frac{(sv(j+2))^2 - p^2}{(sv(j+2))^2 + E^2}\,dp\,dv \quad (E > 0). \tag{RT12}$$

It is the real part of $\int e^{-|t|E} \widehat{R_{m,s} f}(t)\,dt$, not a rational expression assigned independently at $E = 0$. The latter integral gives its continuous value there, namely pref $f(0)$, since the interpolant vanishes near zero. The odd-parity response uses $\int \operatorname{sgn}(t) e^{-|t|E} \widehat{f}(t)\,dt = -2i \int pf(p)/(p^2 + E^2)\,dp$ for $E > 0$. Its subtraction is obtained by applying the even interpolant to $pf(p)$ and dividing the result by p; division is on support away from zero. Even and odd residuals have the required first $2m$ moments zero.

For $f \ge 0$ supported in $[-s, s]$, every product factor in (RT12) lies between zero and one, because $|v| \ge 1$ and $j + 2 \ge 2$. Each is at most $(2(j+2))^2 (s/E)^2$; if $s \ge 1$, it is at least $\frac{1}{2}(1+E)^{-2}$. The lower estimate follows from $(sv(j+2))^2 \ge 4s^2$ and $p^2 \le s^2$. For the central profile $f = D_w \chi$, where $\chi = 1$ on $[-1, 1]$, integration just over $0 < p < wE/(E+w)$ gives

$$2E \int \frac{D_w \chi(p)}{p^2 + E^2}\,dp \ge (E + w)^{-1}.$$

Consequently

$$\mathcal{R}_{m,s,D_w\chi}(E) \ge 2^{-m} (E + w)^{-1} (1 + E)^{-2m}. \tag{RT13}$$

Continuity proves the same inequality at zero; on $0 \le E \le 1$, (RT5) follows with $c_m = 2^{-3m}$. The fixed subtraction band $s \ge 1$ and the shrinking primary width w are distinct. At high energy the product upper bound gives, for $q \le 2m$, a bound $C_{m,q}(w^{-1} + s^q)(1 + E)^{-q}$. These are the precise positive majorants used in the parity switch. For an arbitrary profile of either parity, a nonnegative even compact majorant and the odd division construction bound its subtracted response by a constant times an even response of this form.

To specify the contacts in (RT8), let $R_{ab,cd}$ denote the compensated product-test distribution (RT0b), let $T_{ab,cd}$ be its value after Fourier transforming all four test slots, and let $Q_{bcd} = i\,\text{Fourier}(\sum_j \partial_j R_{jb,cd})$. Writing P_j for multiplication of frequency slot j by p_j, the source identity is $Q_{bcd} = \sum_j P_j T_{jb,cd}$. Thus the quantities called A, B, C in (RT8) are

$$A = T_{ii,ii}, \qquad B = \sum_{j,a \ne i} P_j P_a T_{ai,ji}, \qquad C = \sum_{j,a,b,c \ne i} P_j P_a P_b P_c T_{cj,ab}. \tag{RT14}$$

Their contact remainders are exactly

$$R_1 = \sum_{j \ne i} P_j Q_{iji} - P_i Q_{iii}, \qquad R_2 = \sum_{j,a,b \ne i} P_j P_a P_b Q_{jab} - P_i \sum_{j,a \ne i} P_j P_a Q_{aji}. \tag{RT15}$$

Expand $Q = \sum PT$, use the exchange symmetries of the two tensor pairs, and commute the scalar multipliers P_j; this gives $B = P_i^2 A + R_1$ and $C = P_i^2 B + R_2$ exactly. The compensated Q has no contact jet of total order below three by (RT0b). On widths $\le \lambda^a$ this costs $3a$ with arbitrarily small exponent slack. Multiplication by one frequency costs a and by three costs $3a$; hence (RT15) costs $4a$ and $6a$, respectively. This proves the powers asserted after (RT8). Vanishing on profiles with the required moments follows directly by evaluating the finite contact polynomial on those moments.

The exterior-flux powers can also be made explicit. Put $E = p_d$, $H_N(p) = \prod_{j\neq d} D_{\lambda^{1+w_j}\chi}(p_j)$, and

$$D_q(p) = \mathbf{1}_{E\le 1}\frac{H_N(p)}{E+\lambda^q}, \qquad Z_z(p) = \mathbf{1}_{E\le 1}\frac{E\lambda^{4z}}{(E+\lambda^z)^4}H_N(p). \tag{RT16}$$

The exterior-word integral of D_{-3000} has exponent $u \le 150$: $u = 1$ if $d = l$, and $u = -(1+w_l)+1$ otherwise, by the line estimate. For a shear stress, the stress integral of D_{w_d} has exponent one. The squared response is at most $C^2\lambda^{2(3+w_d)}D_{-3000}D_{w_d}$, by $e^{-E/\lambda^{w_d}} \le \lambda^{w_d}/(E+\lambda^{w_d})$. Weighted Cauchy–Schwarz therefore gives exponent $3 + w_d + (u+1)/2$. If $d = l$, then $u = 1$ and $w_d \le -150$, giving at most -146. If $d \neq l, a, b$, then $w_d = -300$ and $u \le 150$, giving at most -221.5. These exhaust the shear case $d \neq a, b$; both bounds are stronger than exponent -101.

If $d = b$, split instead at $E = \lambda^{-3000}$. On the upper part use Z_{-99}. Its stress integral has exponent $1 + 4(-99) = -395$, by (RT9) with slack one, and the squared response is at most $C^2\lambda^6 D_{-3000}Z_{-99}$. The resulting exponent is at most $3+(150-395)/2 < -101$. On the lower part use the crude stress weight D_0, whose integral has exponent at most $3\cdot 300 = 900$. The squared response is at most $C^2\lambda^{2(3-1500)}D_{-3000}D_0$, giving exponent at most $3-1500+(150+900)/2 < -101$. These pointwise estimates are valid on the nonnegative spectrum and include $E = 0$. The complement of the low box has arbitrary negative exponent by the damped moments and Schwartz tails discussed after (RT9). Thus the spectral reading has exponent -101. Multiplying by the angular primitive's λ^{100} leaves exponent -1, as claimed in (RT0c). Decomposition into the finitely many reflection parities preserves the bound; parity-mismatched terms integrate to zero by reflecting their odd axis.

The compact cutoff flux identity is therefore (RT1) equal to zero for every separated tuple on a coordinate line. Translation along that line changes Kx_ℓ by the fixed vector Ke_l and leaves the correlation unchanged. Subtracting the two angular identities gives the corresponding sum of the translation shell maps equal to zero. Empty tuples give empty sums throughout.

19.17 The order of choices in the spectral sign walk

We record the quantitative induction that supplies the small mixed-band bound in (RT6). Let Q be the finite set of axes being tested, with two signs assigned so that each axis has an opposite-sign partner. Let $U \subseteq Q$ be the axes whose width exponent is still allowed to be negative, and let i be the active axis. A finite route alternates signs and removes the currently visited axis from U. Such a route exists by induction on $|U|$: if the active axis is in U, remove it and move to an opposite partner. If it is not, choose an axis still in U; reach it directly if its sign is opposite, and otherwise take one opposite-sign detour. This reduces the nonempty set after at most two such moves.

The induction statement is uniform in a width vector w with $-K \le w_j \le a$, $w_i \le d$, and $w_j \ge 0$ outside U, for every $0 \le d \le a$. It asserts a residual budget $\delta > 0$ after sufficiently high moment order, for nonnegative even compact profiles and all fixed support factors. If U is empty, the direct spectral moment bound of order zero has a finite price exponent D. Choose $a > 0$ with $aD < \delta$. The nonnegative widths then pay its residual by that budget.

At an induction step from axis i to an opposite-sign axis j, first obtain the future bandwidth $a' > 0$ from the induction hypothesis for $U \setminus \{i\}$. Put $p = K|Q \setminus \{i, j\}|$ and $z = \max(K, \delta)$. Choose

an integer

$$k > (p + z)/a'. \tag{RSW1}$$

Only now purchase the moment estimate of order k, with its finite price exponent D, and choose $0 < a < a'$ with $aD < \delta$. The positive profile majorant constructed by the finite interpolant bounds the baked profile and keeps the remainder nonnegative on the spectral half-line. The far part of the two-axis comparison has exponent at most

$$-2ka' + p + \max(-w_i, dD) \le -2ka' + p + z < 0. \tag{RSW2}$$

The near part is paid by the future hypothesis after replacing the active profile by that majorant and its exponent by the nonnegative d. The two residuals have nonnegative spectral integrands. The signed comparison inequality from (RT5) therefore bounds their sum by the two paid positive quantities; no spectrum is transferred from one axis to another. Moment orders and thresholds can be enlarged to their maximum. Induction along the finite route gives the bound for arbitrary initial nonpositive widths, with one sufficiently small positive a. This explains why the future bandwidth is chosen before the present moment order and why the contact-price exponent is chosen before the present final bandwidth. It supplies the precise quantifiers behind the mixed-band estimate used to derive the collinear Ward identities.

19.18 Normalized two-band diagonal estimate

For clarity, the six normalized readings used in (RT8)–(RT9) have the following common normalization. Let $W = \lambda^a$, $a \le 0$, and let $q_k^{(pW)}$ denote the order-k profile from (RT4) at width pW, with all other profiles fixed. Put

$$a_0 = \operatorname{Re} A(q_0^{(W)}), \quad b_0 = \operatorname{Re} A(q_0^{(2W)}),$$

$$v = W^{-2} \operatorname{Re} B(q_1^{(W)}), \quad V = W^{-2} \operatorname{Re} B(q_1^{(2W)}), \quad r = W^{-4} \operatorname{Re} C(q_2^{(W)}), \quad R = W^{-4} \operatorname{Re} C(q_2^{(2W)}). \tag{DB1}$$

The normalization on the second band is still by W, not by $2W$. Multiplication of the order-$k+1$ profile by the frequency square gives $(pW)^2$ times its order-k predecessor. Consequently (RT8) gives small errors for $v - a_0, V - 4b_0, r - v, R - 4V$. The first contact error has unnormalized order W^4, the second order W^6; their respective normalizations leave order W^2, including the arbitrary small exponent slack. The zero-moment contact subtraction has the same bound: its constant interpolation coefficient cancels between the two bands, and every nonconstant even coefficient has at least a factor W^2.

Let f, g, h be the real spectral gap readings at these three orders, normalized by $1, W^{-2}, W^{-4}$. Their signs are $f \ge 0, g \le 0, h \ge 0$. The difference between a gap profile and its two-band difference is exactly the finite interpolation debris just bounded. Hence the additional errors $f - (a_0 - b_0), g - (v - V), h - (r - R)$ have the same small budget. Let K be the sum of the absolute values of these seven errors. From $g \le 0$ we get $a_0 - 4b_0 \le 3K$; from $h \ge 0$ we get $a_0 - 16b_0 \ge -8K$. Therefore $12b_0 \le 11K$, and

$$0 \le f \le a_0 - b_0 + K \le 3b_0 + 4K \le 7K \le 100K. \tag{DB2}$$

Finite sums and fixed scalar multiples preserve the budget exponent, so (DB2) proves the small diagonal gap bound. This is the algebraic step in (RT9); it needs positivity of the three response signs and the contact bounds, and makes no positivity claim for the six individual band readings.

19.19 Ward kernel transfer and the rotation action

We explain how these identities extend to general tuples and to a group action. Differentiating a shell integral is allowed because its fixed compact annulus remains separated under small movements of the distinguished point. Differentiation acts both on the correlation and on the centered angular coefficient. The latter derivative is the relevant constant translation coefficient. Hence every position derivative of a line Ward test is again a finite sum of line Ward tests for successor-degree labels. The all-order scaling comparison (WS7) transports these tests to the free quotient, where the translation and angular actions are explicit derivatives of polynomials. Finitely many separated reading maps suffice at each finite degree, by the detector argument in (WS4)–(WS5). Their matrices converge to injective free matrices. This transfers the free rank statements for the spaces of possible internal defects and forces any such defect to vanish. Thus the translation shell map is one positive scalar c times the already constructed derivative, and the angular rows, divided by c, satisfy the Ward identity at every separated tuple.

The finite-dimensional rank arguments just used have two different domains. For translation rows, the domain at cap d is the real subspace of $\prod_{i=1}^{4}\operatorname{Hom}_{\mathbb{C}}(\mathcal{F}_d,\mathcal{F}_{d+1})$ consisting of rows that kill the unit, preserve each lower filtered image, commute with conjugation, and transform as a vector under signed permutations. A reading is a position derivative at a fixed separated collinear tuple of the Ward sum $\sum_\ell S(v_1,\dots,T_iv_\ell,\dots,v_n)$. The free readings have common kernel precisely the real line spanned by the derivative row. Select finitely many such readings by minimizing the dimension of their common kernel. Adjoining a further reading cannot lower that minimal dimension, so this finite kernel already equals the full kernel. Pullback by U_k turns the physical readings into matrices on this one fixed real space; their convergence is (WS7). Eventual rank and the two known rows in their kernels then give the translation normalization stated above.

For the internal angular defect, the domain is instead the real subspace

$$\mathcal{I}_d\subset\operatorname{Hom}_{\mathbb{R}}(\mathfrak{so}(4),\operatorname{Hom}_{\mathbb{R}}(\mathfrak{so}(4),\operatorname{End}_{\mathbb{C}}\mathcal{F}_d))\,. \tag{RT17}$$

Its elements $T(K,L)$ are alternating in K,L, kill the unit, commute with conjugation, preserve every lower filtered image, and intertwine simultaneous conjugation of K,L by orientation-preserving signed permutations with their action on labels. The reading for a separated tuple, labels v_ℓ, and K,L, is $\sum_\ell S(v_1,\dots,T(K,L)v_\ell,\dots,v_n)$. The free internal Ward separation says the common kernel of these readings is zero at the required sufficiently large caps; lower caps follow by inclusion. Select finitely many readings by the same decreasing-kernel-dimension argument. Their joint map is injective. Transport by $T\mapsto U_kTU_k^{-1}$, retaining both Lie algebra slots, identifies the actual and free domains; reality, signed covariance, unit preservation and lower-image preservation follow from (WS1). On fixed free basis inputs the actual reading matrices converge to the free ones by order-zero joint comparison. Linear independence of finitely many matrix columns is open, so they are eventually injective. This permits testing the unknown, scale-dependent pulled-back defect without assuming it is bounded. Its readings vanish by the Ward identity; injectivity makes the defect itself zero.

The free internal separation used here has a concrete angular argument. Identify fields with their Wick polynomials as in (FM3)–(FM7). For a fixed pair K,L, the assumed zero Ward sums say that the corresponding polynomial endomorphism T is silent in every moment; in particular its two-point pairings obey

$$\langle Tp(x),q(0)\rangle+\langle p(x),Tq(0)\rangle=0. \tag{RT18}$$

It preserves the degree-four subspace by the lower-image condition. The explicit curvature contraction (FM2), expanded on its degree-four invariant polynomials, gives at equal radii $x,y\neq 0$ the

identity

$$\langle p(x), q(0)\rangle = \langle (\rho(H_y H_x)p)(y), q(0)\rangle, \tag{RT19}$$

where $H_z = I - 2zz^{\mathsf{T}}/|z|^2$ is reflection in the hyperplane normal to z, and ρ acts on all tensor legs. This follows by substituting the orthogonal factor $I - 2zz^{\mathsf{T}}/|z|^2$ in the degree-four Wick pairings; the radial factors agree, and contractions use the invariant color inner product. The explicit low-degree evaluations and the extension to every filtered cap are (FS6)–(FS14), with (WE1)–(WE4) and (HC1)–(HC4) supplying their concrete evaluation and collision steps. Apply (RT18) at x and y, use (RT19) in both terms, and subtract. Positive definiteness of the reflected polynomial pairing at $x = -re_i$ forces T to commute on degree four with $\rho(H_x H_y)$. Taking $y = e_i + e_j$, with the matching radius, makes this product a signed quarter-turn in the ij-plane.

Signed covariance of the two Lie algebra slots now makes $T(K, L)p$, for each degree-four p, invariant under simultaneous quarter-turn conjugation of K, L. An alternating bilinear map with this invariance is zero. To check this without representation theory, use the plane basis E_{ij} of $\mathfrak{so}(4)$. Equal arguments vanish by alternation. For two planes sharing one axis, the square of a quarter-turn in the remaining two axes preserves one plane and negates the other. For disjoint planes, two quarter-turn substitutions exchange the arguments with the sign which, combined with alternation, negates the value. Every coefficient is therefore its own negative. This proves that the internal endomorphism kills the whole degree-four subspace, including the free stress.

The local angular Ward identity for the free stress then propagates this to higher degrees: insert the stress in the silent moment sum, apply the compact angular shell integral, and use that its T-term is zero. The remaining identity is the moment pairing of the commutator of T with the free angular generator. Nondegeneracy of the Wick polynomial tests, proved by the reflected Gram argument following (FM7), makes this commutator zero. Finite-dimensional ODE uniqueness exponentiates that commutation to each quarter-turn at the higher cap. Signed covariance and the same plane-basis calculation again force $T = 0$. This proves the separation of $\mathcal{I}_d$ for every $d \geq 4$ used in (RT17), rather than assuming an additional internal symmetry classification.

For the translation detector there are two further points behind its free one-dimensional kernel. A separated target tuple can be joined to a tuple on any prescribed coordinate line by a segment staying in configuration space: order the line coordinates by the target coordinates on that axis, break ties by distinct arbitrarily small increments, and interpolate. Strict inequalities prevent a collision along the segment, and tied target coordinates remain separated until the endpoint. All position derivatives of the Ward sum vanish at the line tuple when all the scene readings vanish. Analyticity of the explicit Wick kernels gives zero in its neighborhood and then along this segment. Thus scene vanishing is the full silent-moment identity. The color-vector, reality and filtration restrictions of (RT17)'s translation analogue then give precisely the derivative line: the free Wick coefficient calculation identifies its degree-four charge as a real scalar translation charge; subtract that scalar derivative. The remaining silent row kills the free stress. The radial shell identity makes its commutator with the engineering-weight derivation a ghost. The fixed exterior-cap separation (FS13) kills that commutator, and the inversion-parity calculation (FS14) kills the row on every higher cap. The complete charge and polynomial-separation calculations are (FS1)–(FS14), built on the hierarchy (FM1)–(FM7) and the invariant spaces (FC1)–(FC15). The zero scalar case is included.

Apply two orbital derivatives to that identity and subtract in opposite orders. The vector fields $x \mapsto Kx$ have bracket with the opposite sign to the matrix bracket, while the substitutions in a multilinear correlation have the same opposite sign. The resulting defect is $[Y(K), Y(L)] - cY([K, L])$, tested in the internal defect space just eliminated. It is zero. This proves the Lie relations on the actual quotient.

There is a useful additional detail in fitting this representation to the free one: the deformed matrices need not be normal. Put $D = c^{-1}Y$, and work at one finite degree in the space of linear maps between the two quotients. Its relative generators are $R_K(T) = D(K)T - TY_{\mathrm{free}}(K)$. Their sum of squares commutes with every R_K: expand the commutator, use the plane matrix brackets, and pair the two opposite ordered-plane terms. On the free side this sum is negative semidefinite in the invariant inner product. A real polynomial in it projects onto the zero eigenspace, by interpolation on its finite real spectrum. Evaluate that same polynomial on the nearby deformed sum of squares. The resulting matrices P_k converge to the free projection P, commute with all the deformed generators, and obey $P_k^2 - P_k \to 0$; they are not yet projections.

Correct them by iteration of $\phi(Z) = 3Z^2 - 2Z^3$. If $e(Z) = Z^2 - Z$, direct polynomial multiplication gives

$$\phi(Z) - Z = (1 - 2Z)e(Z), \qquad e(\phi(Z)) = (4e(Z) - 3)e(Z)^2. \tag{RT10}$$

For bounded Z and sufficiently small initial defect, the defect decreases at least geometrically by $1/2$, and the increments are bounded by a fixed constant times that defect. The iterates are Cauchy and converge to an exact idempotent Q_k, with $Q_k - P_k \to 0$. Polynomial iteration preserves every commutation and every closed real symmetry constraint.

It remains to show that the relative generators vanish on its range. Each plane generator is the bracket of two other plane generators. Since Q_k commutes with them and is idempotent, the restricted matrices R_iQ_k satisfy those same perfect bracket identities. They tend to zero because the free generators kill P. If M is the largest of their finitely many norms, the bracket identities give $M \le 2M^2$. Eventually $M < 1/2$, hence $M = 0$. Apply Q_k to the comparison map corresponding to the identity: its limit is the identity, so the result is invertible for large k and is an exact intertwiner. Reality and signed permutations preserve it, because signed conjugation permutes the plane generators with signs whose squares are one. This proves the required compatibility without assuming a spectral theorem for the deformed matrices.

Transport the free orthogonal representation by this intertwiner. Its exponential derivative is $c^{-1}Y(K)$. Along $x_\ell(t) = e^{tK}x_\ell$ and the simultaneously moved labels, the ordinary product rule and the Ward identity show that the derivative of the correlation is zero. The product rule is valid jointly in positions and labels: expansion in fixed finite bases is a finite sum of products of smooth position functions and polynomial coefficient functions. Thus the correlation is constant along every plane rotation. Plane rotations align any unit vector with a coordinate axis; every hyperplane reflection is a conjugate of a coordinate reflection by such rotations. Reflections generate the orthogonal group, so covariance holds for all of $O(4)$.

All actions preserve the inclusions. This is first true for their generators by the quotient derivative and Ward identities; the two sides of an inclusion intertwining equation then solve the same finite-dimensional linear differential equation along every exponential, with the same initial value. Uniqueness proves it on the exponentials and the group generators. Finally average the normalized stress $c^{-1}J$, with its two tensor indices moved as well, over normalized Haar measure on compact $O(4)$. Conservation, reality and each shell identity are finite-dimensional linear identities, so they pass through the integral. This produces a symmetric conserved covariant stress with the normalized translation and angular charges on the same hierarchy. The next subsection uses precisely this Euclidean action.

Source–readout formulation. The four subsections above establish a symmetry of the recovered hierarchy that is not a symmetry of the realized lattice source. Three statements should be kept distinct.

1. *The realized lattice source has only the signed permutation symmetry*; the frame realization intertwines exactly that group (Section 9). No continuous rotation acts on $\mathcal{Z}$ or on $\mathcal{U}$.

2. *The readout* (the continuum hierarchy on the physical quotient) acquires the full $O(4)$ covariance. It is therefore a symmetry of the recovered object that is not the image of a symmetry of the source: it is derived, through the stress shell maps, the vanishing of the outer flux and the rank transfer from the free comparison sector.

3. The representation on labels is obtained by *transport along an intertwiner* from the free sector, constructed by an exact idempotent correction; the choice of intertwiner is fixed up to the commutant, and the resulting action on the quotient is determined by its generators $c^{-1}Y(K)$, which are read from the hierarchy.

The quantitative heart of the flux estimate, and the conclusion, are the following.

Lemma 19.2 (Two-band elimination)**.** *With the six band readings* (DB1) *and the gap readings $f \geq 0$, $g \leq 0$, $h \geq 0$, let K be the sum of the absolute values of the seven errors $v - a_0$, $V - 4b_0$, $r - v$, $R - 4V$, $f - (a_0 - b_0)$, $g - (v - V)$, $h - (r - R)$. Then*

$$0 \leq f \leq a_0 - b_0 + K \leq 3b_0 + 4K \leq 7K.$$

Proof. $g \leq 0$ gives $a_0 - 4b_0 \leq 3K$ (errors $g - (v - V)$, $v - a_0$, $V - 4b_0$); $h \geq 0$ gives $a_0 - 16b_0 \geq -8K$ (errors $h - (r - R)$, $r - v$, $R - 4V$, $v - a_0$, and $V - 4b_0$ four times). Subtracting, $12b_0 \leq 11K$. Then $a_0 - b_0 = (a_0 - 4b_0) + 3b_0 \leq 3K + 3b_0 \leq 3K + \frac{11}{4}K$, and adding the error K of f gives at most $\frac{27}{4}K \leq 7K$. □

Theorem 19.3 (Euclidean covariance of the hierarchy)**.** *The free orthogonal representation transported by the intertwiner has exponential derivative $c^{-1}Y(K)$; along $x_\ell(t) = e^{tK}x_\ell$ with simultaneously moved labels the correlation is constant; hence the hierarchy is covariant under all of $O(4)$, the actions preserve the inclusions, and averaging the normalized stress $c^{-1}J$ (with its tensor indices moved) over Haar measure of $O(4)$ gives a symmetric conserved covariant stress with normalized translation and angular charges on the same hierarchy.*

Proof. The product rule holds jointly in positions and labels (finite basis expansions), and with $\mathcal{A}_K = 0$ the derivative along a plane rotation vanishes. Plane rotations align any unit vector with a coordinate axis, every hyperplane reflection is conjugate to a coordinate reflection, and reflections generate $O(4)$. Both sides of an inclusion-intertwining equation solve the same finite-dimensional linear differential equation with the same initial value. Conservation, reality and the shell identities are finite-dimensional linear identities and pass through the Haar integral. □

The constant $c > 0$ is a normalization of the stress, fixed by the translation Ward identity; its positivity is part of the rank-transfer conclusion. The contact subtraction (RT0b) is a retained residual: the unsubtracted stress pair is not asserted to be conserved at coincident points. The order of choices in the spectral sign walk (RSW1)–(RSW2) is part of the realized record: the bands, the window and the tolerance are fixed before the limit in which the outer flux is removed, and the seven error terms of Lemma 19.2 are each retained with their own bound until they are summed. The passage from the Lie algebra to the group uses only the exponential of the generators on the finite-dimensional filtered pieces and the signed permutations already present, so that all of $O(4)$, and not only its identity component, is obtained without assuming normality of a subgroup.

19.20 Euclidean rotations and tested complex continuation

Let Euclidean generator Ω_a have vector field $x_a\partial_t - t\partial_a$, label action L_a and boost action $K_a = -iL_a$. Differentiating covariance on distinct configurations gives

$$D_{(\Omega_a X_i)_i} S_v(X) = -\sum_i S_{v[i\leftarrow L_a v_i]}(X). \tag{BC1}$$

This follows by inverse rotation on finite-dimensional labels and smooth positional differentiation. Each label belongs to a finite-dimensional field space, so the multilinear scalar reading is differentiable by its finite basis expansion. Continue only after joint compact testing. Set $\zeta_i(z, X) = zt_i(X) + iy_i$. The parameters giving strict imaginary order for every support point form an open convex U, containing the real axis, by compactness and affine gap inequalities. At $z = iw$, folded Euclidean points are $Q_w(X)_i = (x_i, y_i + wt_i)$. The divergence-free vector field

$$V_w = \sum_i [x_{i,a}\partial_{t_i} - w(y_i + wt_i)\partial_{x_{i,a}}]$$

maps under the linear part of Q_w to w times its rotation field, including $w = 0$ without division. Chain rule, (BC1) and compact integration by parts give

$$\begin{aligned} 0 = \int \Big[& F_v(\zeta(iw, X), x) \sum_i \{x_{i,a}\partial_{t_i} + iw\zeta_i(iw, X)\partial_{x_{i,a}}\}\varphi \\ & - iw \sum_i F_{v[i\leftarrow K_a v_i]}(\zeta(iw, X), x)\varphi \Big] dX. \end{aligned} \tag{BC2}$$

Folded strict heights make the compactly supported product smooth; zero divergence makes its derivative integral vanish. Replace iw by $z \in U$. Pointwise holomorphy and joint continuity give uniform local Cauchy formulas on the compact support, so bounded integration is holomorphic in z. Imaginary zeros accumulate at zero; the identity theorem makes the integral zero at one. The lift retains the compact test, folded map and both label generators. Recovery preserves the finite-dimensional chain rule and tested holomorphic integral, including signs and the nonsingular $w = 0$ case.

Absolute cone and tested rotation recovery. Write $\tau_i = q_i^0$ for the absolute time-frequency coordinates. Every weakly increasing height vector has the unique description $y_i = c + \sum_{j<i} \delta_j$, with $c \in \mathbb{R}$, $\delta_j \geq 0$. Consequently its physical pairing is

$$\langle e_y, q\rangle = c\sum_{i=1}^{n} \tau_i + \sum_{j=1}^{n-1} \delta_j \sum_{i=j+1}^{n} \tau_i.$$

Thus the cone on the right of (FT3) is exactly

$$\sum_{i=1}^{n} \tau_i = 0, \qquad \sum_{i=j+1}^{n} \tau_i \geq 0 \quad (1 \leq j < n).$$

This calculation preserves the absolute coordinates and makes explicit both conclusions of the weak-height reader. Scalar source recovery of $\widehat{W}_v$ on every compact smooth test is equality with the physical distribution, so the local vanishing argument and its distributional support are identical. The same exponential tests g_t and their e^{-t} times polynomial seminorm bounds establish that vanishing before any rotation or relativistic-spectrum step.

For each finite label carrier let j_d, p_d be its section and recovery. The source label generators are $L_a^Z = j_d L_a p_d$ and $K_a^Z = j_d K_a p_d = -iL_a^Z$ on the horizontal image. For lifted labels,

$$p_d L_a^Z j_d = L_a, \qquad p_d K_a^Z j_d = K_a.$$

Finite label replacement and scalar table reading therefore commute with (BC1). Keep the physical folded configuration Q_w and compact test unchanged. Its linear derivative satisfies exactly

$$DQ_w(X)V_w(X) = w(\Omega_a Q_w(X)_i)_i, \qquad \operatorname{div} V_w = 0.$$

Indeed its time component is $wx_{i,a}$ and its spatial-a component is $-w(y_i + wt_i)$, also at $w = 0$. Hence the source tested integral in (BC2) equals the physical tested integral, with identical label signs. For arbitrary complex z the compact support determines the same open convex parameter set U. Equality holds pointwise in the integrand before bounded integration on that support; the local uniform Cauchy formula and identity theorem therefore recover its value zero at $z = 1$. This keeps continuation at the tested level and introduces no spatial differentiability premise for strict waves.

19.21 Tested boost identity and Schwartz motion

The continuation at $z = 1$ gives

$$\sum_i \Psi_{v[i\leftarrow K_a v_i]}(y;\varphi) = \Psi_v(y;\mathcal{B}_{a,y}\varphi), \qquad \mathcal{B}_{a,y} = \sum_i [x_{i,a}\partial_{t_i} + (t_i + iy_i)\partial_{x_{i,a}}]. \tag{BC3}$$

Compact cutoffs converge in Schwartz topology. Continuity of this polynomial-coefficient differential operator and the tested functional extends (BC3) to every joint Schwartz test. Poincare test action is inverse placement $(g\cdot\varphi)(X) = \varphi(g^{-1}X)$. For invertible A, translation b and weighted derivative seminorms, chain rule and the coordinate bound give

$$p_{i,k}(\varphi(A(\cdot - b))) \le (\|A^{-1}\| + \|b\|)^i \|A\|^k 2^i \max_{r\le i,\ s\le k} p_{r,s}(\varphi). \tag{BC4}$$

Compact-set derivative continuity as A, b vary and higher weighted seminorm tail control prove strong Schwartz continuity. Boost inverse placement differentiates pointwise as

$$\frac{d}{ds}(g_s\cdot\varphi)\Big|_{s=0} = -\mathcal{B}_{a,0}\varphi. \tag{BC5}$$

The lift retains inverse placement and polynomial differential test operators. Recovery preserves their finite coordinate actions, compact cutoff extension and seminorm bounds, including the inverse-action sign. The strong derivative justification continues with the source argument.

19.22 Strong test derivatives and finite boost covariance

The boost derivative (BC5) holds in Schwartz topology. At parameter s its pointwise derivative is the strongly continuous family $D(s) = g_s\cdot(-\mathcal{B}_{a,0}\varphi)$. Pointwise fundamental theorem writes the quotient error as $\int_0^1 [D(us) - D(0)](X)\,du$. The same holds after every coordinate derivative. Taking each weighted supremum bounds its seminorm by $\sup_{0\le u\le 1} p_{i,k}(D(us) - D(0))$, which tends to zero by continuity on the compact parameter interval. This proves strong differentiation without a Schwartz-valued integral. In (BC3), extra height terms $i\sum_i y_i \Psi_v(y;\partial_{x_{i,a}}\varphi)$ vanish at the boundary by each fixed-test limit. Hence

$$\sum_i W_{v[i\leftarrow K_a v_i]}(\varphi) = W_v(\mathcal{B}_{a,0}\varphi). \tag{BC6}$$

Place all labels in one finite field space and set $H(s) = W_{(\rho(g_s)v_i)_i}(g_s \cdot \varphi)$. Finite-basis expansion justifies the product rule; label derivatives in (BC6) cancel the negative test derivative. Repeating at moved labels/tests and using the group law gives $H'(s) = 0$, so

$$W_{(\rho(g_s)v_i)_i}(g_s \cdot \varphi) = W_v(\varphi). \tag{BC7}$$

Translations and spatial rotations follow by strict-height analytic uniqueness and boundary change of variables with unit-modulus Jacobian. Covariant transformations form a subgroup by representation and test-action laws, including inverses. Coordinate boosts and spatial rotations then give full covariance with the stated group-generation input below. The lift retains finite label spaces, strong test actions and boundary pairings. Recovery preserves finite-basis differentiation and the cancelling height terms, transporting (BC6)–(BC7) with the same group-generation dependency.

Recovery of strong test motion and finite boosts. Use the same joint Schwartz test carrier on both sides of the readout. For $T_{A,b}\varphi(X) = \varphi(A(X-b))$, the retained coordinate estimate is

$$\|X\| \le (\|A^{-1}\| + \|b\|)(1 + \|A(X-b)\|).$$

Together with the derivative chain rule it recovers (BC4) in every weighted seminorm. Thus test recovery commutes with inverse placement, polynomial-coefficient differentiation and compact cutoff approximation. In particular the source tested boost operator is exactly $\mathcal{B}^Z_{a,y}\varphi = \mathcal{B}_{a,y}\varphi$. For the difference-quotient error E_s in (BC5), its source seminorm reader is the same bound

$$p_{i,k}(E^Z_s) = p_{i,k}(E_s) \le \sup_{0\le u\le 1} p_{i,k}(D(us) - D(0)) \longrightarrow 0.$$

This recovers the strong derivative with its inverse-placement minus sign using the original pointwise fundamental-theorem argument.

The finite-dimensional label section intertwines the boost action: $\rho^Z(g_s)j_d = j_d\rho(g_s)$, with generator $K^Z_a = j_d K_a p_d$. Every label in the fixed word lies in one common finite field space. Writing $\widehat{v}_i = j_d v_i$, the scalar boundary reader obeys

$$W^Z_{(\rho^Z(g_s)\widehat{v}_i)_i}(g_s \cdot \varphi) = W_{(\rho(g_s)v_i)_i}(g_s \cdot \varphi).$$

The recovered extra height term is $i\sum_i y_i \Psi_v(y; \partial_{x_{i,a}}\varphi)$; each summand vanishes by its fixed derivative-test boundary limit before finite-boost integration. The label and test derivatives of the displayed scalar pairing then cancel by (BC6) at every s, recovering (BC7). The same reader preserves the representation law, inverse test action and unit-modulus change-of-variables Jacobian, so the covariance subgroup and its subsequent group-generation dependency remain those of the retained construction.

19.23 Dual cone and smooth frequency cutoff

For absolute momenta with total zero and forward-cone proper suffix sums $P_j = \sum_{i=j}^n p_i$, summation by parts gives, for closed dual-cone successive height differences,

$$\langle p, y\rangle = \sum_{j=2}^{n} \langle P_j, y_j - y_{j-1}\rangle \ge 0. \tag{DC1}$$

Endpoints vanish; cone self-duality gives each inequality, including the empty tuple. For interior height choose $B(y,r)$ in the open cone. If $\|u-y\| < r/2$, the point $u - rp/(2\|p\|)$ remains inside it for nonzero cone p. Applying (DC1) gives

$$\langle p, u\rangle \ge (r/2)\|p\|, \tag{DC2}$$

with zero momentum immediate. This is a common neighborhood damping margin. Convolve the indicator of the cone's open distance-two neighborhood with a nonnegative unit-integral smooth bump b supported within radius $1/2$. The cutoff η is one within distance one, supported within distance $5/2$, with closed support inside distance three. All derivatives are globally bounded, since derivatives can be put on the compactly supported bump and the indicator has absolute value at most one: $\|D^k\eta\|_\infty \le \|D^k b\|_{L^1}$. These use full absolute-frequency Euclidean distance, including dimension zero. For closed height y and x in closed support, choose cone z within three. Then

$$-\langle y, x\rangle \le \langle y, z - x\rangle \le 3\|y\|.$$

The lift retains suffix momenta, dual heights and this explicit convolution. Recovery preserves summation by parts and distance margins, transporting the damping and derivative bounds in the same absolute-frequency metric.

19.24 Closed-height multipliers and Fourier conventions

On cutoff support and $\|y\| \le R$, the preceding exponential bound gives $\|D_x^\ell e^{-\langle y,x\rangle}\| \le e^{3R}R^\ell$. Leibniz and cutoff bounds yield, in the source's polynomial-growth form,

$$\|D_x^k(\eta e^{-\langle y,\cdot\rangle})(x)\| \le C_{k,R}(1 + \|x\|)^{N_k}. \tag{DC3}$$

Derivatives vanish off closed support. Thus multiplication acts continuously on Schwartz tests uniformly for bounded closed heights; interior heights also retain (DC2) damping. These are the cutoff and damping estimates needed for the subsequent tube construction. For $N = 4n$ fix inverse radian Schwartz maps

$$\mathscr{H}g(q) = \int e^{-i\langle x,q\rangle} g(x)\,dx, \qquad \mathscr{P}f(q) = (2\pi)^{-N}\int e^{i\langle q,x\rangle} f(x)\,dx.$$

For closed height define $D_y g(q) = \eta(q)e^{-\langle y,q\rangle}g(q)$. (DC3) and Leibniz control its seminorms by finitely many test seminorms, uniformly on bounded height sets. Joint multiplier smoothness gives jet convergence on compact frequencies, and one extra polynomial weight controls tails. Hence $y \mapsto D_y g$ is Schwartz-continuous even at boundary heights. The lift retains these inverse transforms and multiplier seminorm bounds. Recovery preserves finite derivatives and weighted-tail control, transporting closed-height continuity with the exact Fourier constants.

Dual-height and multiplier readout. Retain the full absolute-frequency Euclidean space and its closed cone C as physical reader coordinates. The source record for a cutoff is (C, b, η) with

$$\eta(q) = \int \mathbf{1}_{\{\operatorname{dist}(q-h,C)<2\}} b(h)\,dh, \quad b \ge 0, \quad \int b = 1, \quad \operatorname{supp} b \subset \{\|h\| \le 1/2\}.$$

Its readout is this actual convolution, preserving the distance-one unit region, distance-three closed-support bound and derivative bills. For the suffix-sum linear map $S(p) = (P_j)_{j=2}^n$ and height-difference map $\partial y = (y_j - y_{j-1})_{j=2}^n$, the dual reader is precisely

$$\langle p, y\rangle = \langle S(p), \partial y\rangle \quad \text{when } \sum_i p_i = 0.$$

This retains (DC1), its interior ball margin (DC2), and the empty tuple without changing coordinates or pairings.

Keep the Schwartz test space unchanged under recovery. Writing $m_y(q) = \eta(q)e^{-\langle y,q\rangle}$, its retained derivative bounds and the multiindex Leibniz formula give explicitly

$$\partial^\alpha(D_y g) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\partial^\beta m_y)\partial^{\alpha-\beta} g.$$

If $s_{i,k}(g) = \max_{|\alpha|\le k} \sup_q (1+\|q\|)^i |\partial^\alpha g(q)|$, (DC3) therefore yields

$$s_{i,k}(D_y g) \le C'_{k,R}\, s_{i+N,k}(g), \qquad N = \max_{j\le k} N_j, \quad \|y\| \le R.$$

These are the same test seminorms on the source carrier. On frequency tails one additional weight gives a factor $(1+L)^{-1}$ outside $\|q\| \le L$; on the remaining compact set joint smoothness gives jet convergence as y varies. This explicitly recovers closed-height Schwartz continuity. The Fourier readers retain the same inverse maps $\mathscr{H}$, $\mathscr{P}$ and the factor $(2\pi)^{-4n}$, so composition with D_y recovers exactly the multiplier used in the subsequent tube construction.

19.25 Schwartz-valued tube lens

For open-tube $Z = X + iY$ define $\lambda_Z(q) = \eta(q)e^{i\langle X,q\rangle - \langle Y,q\rangle}$. Interior damping and the support-distance bound give $-\langle Y,q\rangle \le A - c\|q\|$ on support, $c > 0$, with $A = 3(\|Y\|+c)$ sufficient. Frequency derivatives cost powers of the linear functional norm. Leibniz and $(1+t)^k e^{A-at} \le k! a^{-k} e^{A+a}$ prove

$$\sup_q (1+\|q\|)^k e^{(c/2)\|q\|} \|D_q^j \lambda_Z(q)\| < \infty. \tag{TL1}$$

Thus the lens is a Schwartz test with spare exponential margin. For small real-linear complex functional L, $\|L\| \le \min(a,1)$, exponential remainders satisfy

$$\|D^j(e^{L(q)} - 1 - L(q))\| \le \|L\|^2 (1+\|q\|)^2 e^{a\|q\|}. \tag{TL2}$$

Order zero uses the series remainder, order one uses $e^L - 1$, and higher orders use $\|L\|^j e^{\Re L}$ with $\|L\| \le 1$. Multiply by the lens and use (TL1) to bound every seminorm of $\lambda_{Z+H} - \lambda_Z - i\langle H,\cdot\rangle\lambda_Z$ by $C_{k,j}\|H\|^2$. For $j \ge 2$, derivatives of the affine terms vanish and the bound follows from $\|L\|^j e^{\Re L(q)}$, using $\|L\| \le 1$. This proves the joint complex Frechet derivative

$$d\lambda_Z(H)(q) = i\langle H,q\rangle \lambda_Z(q). \tag{TL3}$$

For the tempered frequency functional set

$$\mathcal{F}_v(Z) = (2\pi)^{-N} \widehat{W}_v(\lambda_Z). \tag{TL4}$$

The lift retains the actual Schwartz lens, spare margin and Fourier factor. Recovery commutes with its continuous distribution pairing and seminorm Taylor estimate, preserving the joint derivative and tube definition.

19.26 Holomorphic emission and damped-distribution identity

Continuity of $\widehat{W}_v$ transfers (TL3) to complex differentiability of $\mathcal{F}_v$, hence full-tube holomorphy. At this point no spectral support assertion is needed for holomorphy itself; the support condition will make the result independent of the cutoff. Fix interior Y and Schwartz $g(q) = \eta(q)e^{-\langle Y,q\rangle}$, $g_X = e^{i\langle q,X\rangle} g$. Its transform is $(\mathscr{H}g)(\cdot - X)$ and

$$p_{i,j}(h(\cdot - X)) \le 2^i [p_{i,j}(h) + \|X\|^i p_{0,j}(h)]. \tag{TL5}$$

This follows from $\|q\| \le \|q - X\| + \|X\|$, splitting according to which summand is larger. Finite-seminorm continuity of the tempered functional gives continuous polynomial growth of $\widehat{W}_v(g_X)$, hence absolute integrability after joint Schwartz testing. For compact test φ, uniform jet control permits passing the functional through the compact integral; Fourier inversion yields

$$\widehat{W}_v(D_Y(\mathscr{P}\varphi)) = \int \mathcal{F}_v(X + iY)\varphi(X)\,dX. \tag{TL6}$$

Compact cutoffs extend this to all Schwartz tests: left continuity and right polynomial-growth dominated convergence justify both passages. Thus the emission is the actual damped boundary distribution at every interior height, without a fictitious norm on Schwartz space or an exchange of unbounded integrals. The lift retains the translated Schwartz tests and finite-seminorm estimates. Recovery preserves their continuous pairing and compact-cutoff limit, transporting (TL5)–(TL6) with exact Fourier constants and absolute integral convergence.

Schwartz lens and emission recovery. Retain (η, Z, λ_Z) as a source test record on the same absolute frequency space. Its test readout is the actual λ_Z, and its scalar distribution readout is $\widehat{W}_v^Z(g) = \widehat{W}_v(g)$. The damping margin and (TL2) therefore give in each Schwartz seminorm $s_{i,j}$ the recovered second-order remainder

$$s_{i,j}(\lambda_{Z+H} - \lambda_Z - i\langle H, \cdot\rangle\lambda_Z) \le C_{i,j,Z}\|H\|^2$$

for sufficiently small H in the same open tube. This is a family of seminorm bounds, rather than one norm on the test space. Continuous distribution reading recovers the derivative

$$d\mathcal{F}_v^Z(Z)(H) = (2\pi)^{-N}\widehat{W}_v(i\langle H, \cdot\rangle\lambda_Z) = d\mathcal{F}_v(Z)(H).$$

Thus the source emission is the same jointly holomorphic full-tube function, with the specified Fourier normalization.

For fixed interior Y, the compact-support test integral identity behind (TL6) is, as an equality of Schwartz functions,

$$(2\pi)^{-N}\int g_X(q)\varphi(X)\,dX = D_Y(\mathscr{P}\varphi)(q).$$

Uniform jet control on the compact X support permits the continuous frequency reader to pass through this integral. The two resulting scalar readings are exactly the two sides of (TL6). For a general joint Schwartz test, the same compact cutoffs converge on the left in the Schwartz topology and on the right under the polynomial-growth bound from (TL5). Hence recovery preserves the absolutely convergent smeared emission on every interior height, before cutoff independence is used.

19.27 Tube identification and cutoff-independent covariance

A cone-supported distribution annihilates Schwartz tests vanishing near the cone. For compact smooth g, the tests $\eta e^{-\langle Y,q\rangle}g$ and $e^{-\langle Y,q\rangle}g$ agree near it, hence have equal readings. Compact support is essential to the second test. At strict temporal y, bounded continuous $f_{v,y}$ defines tempered u_y. Apply (FT2) to $e^{-\langle e_y,q\rangle}g$; exponentials cancel and cutoff removal gives $\widehat{W}_v(D_{e_y}g) = \widehat{u}_y(g)$. Schwartz continuity and compact cutoffs extend equality to all tests. Fourier inversion and (TL6) identify their smeared functions; continuity and full-support Lebesgue upgrade almost-everywhere to pointwise equality:

$$\mathcal{F}_v(X + ie_y) = f_{v,y}(X). \tag{TI1}$$

Thus this full tube continues the actual strict waves. For real Lorentz r, let $r^\vee$ satisfy $\langle rX, r^\vee q\rangle = \langle X, q\rangle$. The height and frequency cones are invariant. Lenses $\lambda_Z(q)$ and $\lambda_{rZ}(r^\vee q)$ have identical exponentials, and both cutoffs are one on a neighborhood of each cone point. Their cutoff factors need not agree globally, but both equal one near every cone point p: use the intersection of the unit ball about p and the inverse image of the unit ball about $r^\vee p$. Their difference has closed support disjoint from it; hence the supported distribution gives them the same value. Fourier boundary covariance therefore gives

$$\mathcal{F}_{(\rho(r)v_i)_i}(rZ) = \mathcal{F}_v(Z). \tag{TI2}$$

The same local support comparison proves independence of permitted cutoff choices. The lift retains the cone-supported distribution, dual action and test domains. Recovery preserves these local lens identities and Fourier inversion, transporting (TI1)–(TI2) without an invariant-cutoff assumption or unrestricted exponential multiplication.

Cutoff removal and covariance recovery. Let C be the retained closed frequency cone. The source frequency reader is the same tempered functional $\widehat{W}_v$ with support in C. If $h \in \mathcal{S}$ vanishes on a neighborhood of C, choose smooth compact cutoffs κ_R with $\kappa_R h \to h$ in $\mathcal{S}$. Each compact test $\kappa_R h$ is supported away from C, so

$$\widehat{W}_v^Z(h) = \lim_R \widehat{W}_v(\kappa_R h) = 0.$$

This makes the support-based Schwartz readout explicit. For compact smooth g it applies to $(\eta - 1)e^{-\langle Y,q\rangle}g$; compact support ensures the test belongs to $\mathcal{S}$. At strict temporal y, the Fourier test and damping reader thus recover

$$\widehat{W}_v^Z(D_{e_y}g) = \widehat{u}_y(g), \qquad \int \mathcal{F}_v^Z(X + ie_y)\varphi(X)\,dX = \int f_{v,y}(X)\varphi(X)\,dX.$$

Continuity on both sides then gives exactly (TI1) pointwise, on the original strict-wave configurations.

For a retained Lorentz motion, the dual frequency map is $r^\vee = (r^{-1})^{\mathsf{T}}$ for the stated Euclidean pairing. For each $p \in C$ the open set

$$\{q : \|q - p\| < 1\} \ \cap\ \{q : \|r^\vee q - r^\vee p\| < 1\}$$

contains p and makes both cutoff factors equal to one. Its union over $p \in C$ is the neighborhood on which the two compared lenses agree. Their difference is Schwartz and the preceding support reader therefore annihilates it. Together with the recovered boundary covariance this recovers (TI2) with the same finite label action and dual map. Replacing the two cutoff factors by any two permitted choices gives the identical argument for cutoff-independent emission. The readout retains each individual cutoff and does not require it to be invariant.

19.28 Adjacent spacelike analytic germs

Fix adjacent slots $i, j = i + 1$ and their transposition σ. Choose $y_k^{(m)} = a_k/(m + 1)$ with $a_k = k$ except $a_j = i$. Exactly the selected gap is flat; every other ordered gap is strict. For two arguments this profile is zero. Both orders belong to their closed height cones. At a real spacelike difference $v = X_j - X_i$, put $T = \|\mathbf{v}\| > |v^0|$, $e = \mathbf{v}/T$, and

$$C_e v = (v^0 e, e \cdot \mathbf{v}), \qquad A_e(z)Z = Z + \sinh z\, C_e Z + (\cosh z - 1)S_e Z,$$

where S_e projects onto the spatial-e and time plane. The time component of $C_e v$ is T and its spatial norm is $|v^0|$, so it is strictly future timelike. Under $z = ib$ an adjacent imaginary difference becomes

$$Y + \sin b\, V + (\cos b - 1)W, \qquad V = C_e \operatorname{Re}(\Delta Z), \quad W = S_e Y. \tag{LG1}$$

Strict gaps remain strict in a neighborhood of $b = 0$. At the flat gap,

$$\sin b\, V + (\cos b - 1)W = \sin b \left(V - \frac{\sin b}{1 + \cos b} W \right).$$

The parenthesis remains strictly timelike near $(0, Z_0)$; for small $b > 0$ its addition to a closed-cone Y enters the open cone. Finitely many gaps permit one product neighborhood and one angle, uniformly down to zero. For the transposed order use the opposite imaginary boost: both sign reversals at the flat gap cancel. Thus the original tube and the transposed tube each have a small imaginary-boost path into their interiors, with this path property holding for nearby points of the corresponding tube, not just at the boundary point itself.

Here is the label continuation, retaining the source's finite-dimensional hypothesis. For a continuous one-parameter group $u(t)$ in a unital Banach algebra, set $V(t) = \int_0^t u(s)\, ds$. Then $V'(t) = u(t)$ and $V(t)/t \to 1$. Choose nonzero a with $g = V(a)/a$ invertible. The group law yields

$$u(t)V(a) = V(t + a) - V(t), \qquad u(t) = a^{-1}[V(t + a) - V(t)]g^{-1}.$$

Thus u is differentiable. With $B = u'(0)$ it satisfies $u' = uB$; $d[u(t)e^{-tB}]/dt = 0$ and $u(0) = 1$ give $u(t) = e^{tB}$. On the finite field space containing the chosen labels this supplies the entire extension $u(z) = e^{zB}$.

At the chosen boundary point take the small endpoint it above and put

$$G(Z) = \mathcal{F}_{(u(it)v_k)_k}(A_e(it)Z). \tag{LG2}$$

Its endpoint is interior, hence G is holomorphic on a full neighborhood. For a nearby original-tube point consider

$$Q_Z(z) = \mathcal{F}_{(u(z)v_k)_k}(A_e(z)Z).$$

Expansion of the moving labels in the fixed finite basis makes this holomorphic near the whole boost path. For real z, TI2 makes it constant with value $\mathcal{F}_v(Z)$. The one-variable identity theorem propagates that constant along the connected path to it. Thus G is an actual extension of the original emission. The opposite path, with permuted labels, gives the second germ: permutation is complex linear and commutes with real boosts. Both germs use the same spectator height sequence tending to zero. The next argument identifies the two germs and passes their equality to the boundary exchange relation.

In the lifted object retain the ordered labels, transposition, height profile, finite label space and the entire boost action together with the lens data. The readout LG2 is compatible with the original tube readout by the path argument, and restriction recovers the original emission. The germ is therefore independent of its path presentation on overlaps by analytic uniqueness; the retained order still records which boundary exchange is to be established next.

Recovery of the boost-path germ. The source germ record retains the adjacent transposition σ, physical boundary point Z_0, spectator-height profile, direction e, endpoint angle t, and one finite label carrier containing the word. The configuration reader is the actual complex boost $A_e(z)Z$. Directly taking its imaginary part at $z = ib$ gives (LG1), so the same closed-to-open cone path and

its product neighborhood are retained. The angle and neighborhood are chosen with the quantifiers supplied for that boundary configuration and profile; the record does not replace them by a global uniform angle over all profiles.

On the horizontal image of the finite coefficient section j_d, put $B^Z = j_d B p_d$ and $u^Z(z) = j_d e^{zB} p_d$. Since $p_d j_d = I$, its entire power series and recovery give

$$p_d u^Z(z) j_d = e^{zB}, \qquad u^Z(z) j_d = j_d u(z), \qquad u^Z(z_1) u^Z(z_2) = u^Z(z_1 + z_2).$$

Here the identity at $z = 0$ is the identity on that horizontal image. This preserves the source's finite-dimensional analytic label action. Define the source germ reader using those lifted labels and the same boosted point:

$$G^Z(Z) = \mathcal{F}^Z_{(u^Z(it) j_d v_k)_k}(A_e(it)Z) = \mathcal{F}_{(u(it)v_k)_k}(A_e(it)Z) = G(Z).$$

The same equality holds for $Q_Z(z)$ throughout its retained connected path neighborhood. Real boost covariance and the one-variable identity theorem therefore recover its constant value at the endpoint before restricting to the original tube. Permutation commutes with the componentwise lifted boost action, so the opposite path recovers the transposed germ with the same spectator profile. These explicit readers preserve the local holomorphic extension and its overlap uniqueness; identification of the two orders remains the subsequent germ-comparison step.

19.29 Germ identification and actual partial boundaries

First set all real times to zero and retain the common spectator profile with only the selected adjacent heights tied. The Euclidean configuration $q = ((\mathbf{x}_k, y_k))_k$ has no collisions: untied slots have different heights and the tied spacelike pair has different spatial positions. Define the totally real Wick slice

$$M(q) = (\mathbf{q}_k, i q_k^0)_k.$$

Its image spans the entire complex configuration space: every point is $M(u) + iM(v)$, using the real and imaginary spatial parts and, for time, the imaginary part and negative real part respectively. The germ $G_+ \circ M$ equals the smooth Euclidean hierarchy S_v on the strictly ordered side. On the opposite open side $G_- \circ M$ equals the permuted hierarchy by strict-wave identification, and hence S_v by permutation symmetry. Both open sides accumulate at q, since the tied heights can be separated in either direction. For every m, continuity of derivatives up to the tied-height point gives

$$D^m(G_+ \circ M)(q) = D^m S_v(q) = D^m(G_- \circ M)(q).$$

The chain rule identifies these with restrictions of the complex m-linear derivatives. Complex multilinearity and the spanning property give equality on all complex arguments. Equality of the convergent Taylor series therefore proves

$$G_+ = G_- \quad \text{near } M(q). \tag{GE1}$$

This uses equality of limiting jets, not an assumed common open Euclidean side.

To identify the distributions represented by these germs, let E be an interior tube function and G a holomorphic extension through $X + iY$. Suppose the interior height slices converge weakly to Q. Choose a small real ball about X and a small height neighborhood whose product lies in the germ domain and on which G is bounded by C. For every compact smooth joint test φ supported

in that ball, agreement with E at interior points and dominated convergence with majorant $C|\varphi|$ give

$$Q(\varphi) = \int G(X' + iY)\varphi(X')\,dX'. \tag{GE2}$$

The approach is nonempty since the base point is in the tube closure; the weak limit is the one already supplied by the damped-distribution boundary theorem. Thus (GE2) identifies that particular partial boundary; it does not merely define a new distribution. Apply GE2 to each order. In the transposed case reindex all real variables by σ: it preserves Lebesgue measure, is its own inverse, pulls back the test and permutes the heights. Both partial boundaries are consequently represented by the two germs in the same original variables. GE1 gives their local equality at these zero-real-time spacelike configurations. The extension to general spacelike configurations and passage to zero height remain the subsequent source steps. Apply this argument separately to the original and transposed tubes.

The lifted germ record retains the Wick slice, both ordered domains, all limiting jets, the common smooth hierarchy and the tested boundary functional. Its realization is the pair of analytic restrictions; its observable readout is GE2. The spanning argument proves recovery of the entire germ from the retained jets, while restriction and the specified weak limit recover the original partial distributions. Reindexing is an invertible source operation, so no orientation or test-variable information is lost by comparing the two orders.

Explicit germ recovery from Wick-slice jets. For a complex tangent argument H_a, retain its unique real decomposition $H_a = M(u_a) + iM(v_a)$. If $J_m = D^m(G \circ M)(q)$ is the retained real jet, its full complex derivative is recovered by the finite formula

$$D^m G(M(q))[H_1, \dots, H_m] = \sum_{\epsilon \in \{0,1\}^m} i^{|\epsilon|} J_m[w_{1,\epsilon_1}, \dots, w_{m,\epsilon_m}], \quad w_{a,0} = u_a, \quad w_{a,1} = v_a.$$

The zeroth jet is the scalar value. Thus the equal limiting jets from the two different Euclidean sides recover all equal complex derivatives. The analytic germ readout is their convergent local Taylor series,

$$G(M(q) + H) = \sum_{m=0}^{\infty} \frac{1}{m!} D^m G(M(q))[H, \dots, H].$$

This is the explicit recovery map giving (GE1) from the supplied smooth hierarchy jets and the analytic germs already constructed.

For the partial boundary retain the original weak-limit functional Q_Y, the local germ ball and the admissible interior height approach. Scalar source recovery gives on its compact tests

$$Q_Y^Z(\varphi) = Q_Y(\varphi) = \int G(X + iY)\varphi(X)\,dX.$$

The same local bound $C|\varphi|$ controls both readers, so this is the actual partial limit (GE2). If Q_Y^- is the transposed reader in its own coordinates, its recovered original-variable functional is

$$(\sigma^* Q_Y^-)(\varphi) = Q_Y^-(\varphi \circ \sigma), \qquad |\det \sigma| = 1, \quad \sigma^{-1} = \sigma.$$

The heights and ordered labels are permuted with these same variables. It follows that the two recovered partial boundaries are represented in one coordinate system by G_+ and G_- respectively, and (GE1) gives their local equality with the source's zero-real-time scope. The later passage to general spacelike points and zero height remains attached to its subsequent construction.

19.30 Independent heating, Gram recovery and product-test density

For a positive ordered height profile y, choose p smaller than its initial height and all its gaps. With $z_i = t_i(X) + iy_i$, retain the actual chamber vector

$$\xi_{p,y,v}(X) = R(z_1, p, x_1)B_p(v_1)R(z_2 - z_1, 2p, x_2 - x_1)B_p(v_2)\cdots \\ R(z_n - z_{n-1}, 2p, x_n - x_{n-1})B_p(v_n)\Omega.$$

At arity zero it is Ω. Recursive contractions give

$$\|\xi_{p,y,v}(X)\| \le \|\Omega\| \prod_i \|B_p(v_i)\|. \tag{HV1}$$

These vectors vary strongly continuously in X: heat factors vary in operator norm in the open half-plane, spatial translations vary strongly, and induction applies to the recursively varying ket. At zero real times the vector is the planted future word at (x_i, y_i), by the insertion/translation recursion of SW4 with reference width zero. Its Gram product with an independently chosen q, z, w is the Euclidean hierarchy of the reflected reversed first word followed by the second. For arbitrary real times multiply the first time array by $\bar{c}$ and the second by c. Antilinearity of the first slot makes the Gram product holomorphic in c on an open horizontal strip containing $0, 1$ where both widths satisfy their height inequalities. The hierarchy is holomorphic there too. On purely imaginary c both words are Euclidean planted words with shifted heights. Analytic uniqueness yields

$$\langle \xi_{p,y,v}(X), \xi_{q,z,w}(Y)\rangle = f_{(Jv_n,\dots,Jv_1,w_1,\dots,w_m),(-y_n,\dots,-y_1,z_1,\dots,z_m)} \\ (X_n, \dots, X_1, Y_1, \dots, Y_m). \tag{HV2}$$

The widths need not agree. The first real block is reversed without another geometric reflection; its imaginary heights already changed sign.

For a smeared word $b = ((v_i, \varphi_i))_{i=1}^n$, put $\varphi_b(X) = \prod_i \varphi_i(X_i)$, $y_i(t) = it$ (the integer slot index times the positive real t), $p(t) = \min(t, 1)/8$, and

$$\eta_b(t) = \int \varphi_b(X)\xi_{p(t),y(t),v}(X)\, dX.$$

HV1 and Schwartz integrability make this an absolutely integrable Hilbert-valued integral. Common real translation changes only the initial propagator; all relative ones are unchanged. At arity zero, heat invariance of Ω extends by holomorphic uniqueness to the half-plane and then its real-time edge. Changing integration variables gives

$$\eta_{a\cdot b}(t) = U(a)\eta_b(t). \tag{HV3}$$

Integrating HV2 is justified by the same fixed-height bounds. Reversal of the first block is a measure-preserving real linear isometry; its test is $\overline{\varphi_b(X)}\varphi_c(Y)$, exactly that of b^*c. The joined height profile is strictly ordered for all independent $t, u > 0$ and tends to zero with the pair. The independence of the parameters is stronger than a diagonal limit and is what will make the shot vectors Cauchy. The all-height boundary theorem gives

$$\lim_{\substack{t\downarrow 0\\ u\downarrow 0}} \langle \eta_b(t), \eta_c(u)\rangle = W(b^*c). \tag{HV4}$$

The two-parameter conclusion, rather than only a diagonal limit, is retained for the subsequent Cauchy argument.

Finite sums of product Schwartz tests are dense in joint Schwartz space. For a compactly supported joint test φ, take a product cutoff $\chi = \prod_i \chi_i$ equal to one on its support. Compact frequency cutoffs of its Fourier transform, inverted and multiplied by χ, converge to φ in Schwartz topology. Each is an integral over a compact frequency set of product tests

$$\chi(X)e^{i\langle q,X\rangle} = \prod_i \chi_i(X_i)e^{i\langle q_i,X_i\rangle}.$$

This family is continuous in every Schwartz seminorm: its derivatives vary jointly on common compact spatial support. Quadrature approximates the integral in each seminorm by finite sums of products. A diagonal choice over the countable seminorms gives Schwartz convergence. Finally smooth compact spatial cutoffs approximate any Schwartz test. For arity zero, the constant one on the one-point space already spans it.

The lifted heated-word source retains p, y, q, z independently, the chamber vectors and their joint Gram kernel. Readout HV2 recovers the hierarchy, and HV4 recovers its boundary word functional. The bound HV1 supplies the integration domain, while product-test density recovers joint tested distributions from these word readouts. No diagonal identification of heating parameters enters this recovery.

Independent heated-vector and joint-test recovery maps. Use the chamber section unitary V_H and the already lifted buffered factors. Form the source shot in exactly the displayed order, with the initial factor $R^Z(z_1, p, x_1)$. Cancellation of adjacent inverse unitaries gives

$$\xi^Z_{p,y,v}(X) = V_H\xi_{p,y,v}(X), \qquad \|\xi^Z_{p,y,v}(X)\| = \|\xi_{p,y,v}(X)\|.$$

Thus (HV1) and strong continuity recover on the same positive-height and fixed-width domains. For independent widths and profiles the Gram reader is

$$\langle \xi^Z_{p,y,v}(X), \xi^Z_{q,z,w}(Y)\rangle = \langle \xi_{p,y,v}(X), \xi_{q,z,w}(Y)\rangle.$$

It recovers the complete reversed-adjoint/second-word ordering in (HV2), including the negative reversed first heights and unchanged reversed real coordinates. Antilinearity in the first slot retains the $\bar{c}, c$ continuation used to establish that identity.

Boundedness of V_H permits it to pass through the absolutely integrable Hilbert-valued smearing integral, so

$$\eta^Z_b(t) = V_H\eta_b(t), \qquad \eta^Z_{a\cdot b}(t) = U^Z(a)\eta^Z_b(t), \qquad \lim_{t,u\downarrow 0}\langle \eta^Z_b(t), \eta^Z_c(u)\rangle = W(b^*c).$$

The limit uses the full two-parameter approach and the same joined height profile. No equality of widths or heating parameters is imposed. The measure-preserving reversed-variable map and conjugated first test are retained before this limit.

Finally let φ_r be the finite product-test sums obtained by the source's compact frequency quadrature and spatial cutoff construction, with $\varphi_r \to \varphi$ in all joint Schwartz seminorms. Every retained continuous joint distribution reader Q recovers by

$$Q^Z(\varphi) = \lim_r Q^Z(\varphi_r) = \lim_r Q(\varphi_r) = Q(\varphi).$$

This recovery is independent of the approximating product sums by continuity. It includes the one-point test space at arity zero and preserves the passage from word-vector identities to full joint tests.

19.31 Spacelike exchange with all spectator variables

Two germs anchored to the same tube at a closure point agree: on a small connected ball both equal the tube function on a nonempty open intersection, so analytic uniqueness applies. Along a continuous path in two tube closures, nearby selected germs therefore agree with the restrictions of the germs at the original base point. Equality of the two extensions is locally constant along the path, in either direction, and hence constant on a connected interval. For any designated spacelike pair, contract all real times by

$$X_k(s) = (\mathbf{x}_k, st_k), \qquad 0 \le s \le 1.$$

Spacelikeness persists since $s|t_j - t_i| \le |t_j - t_i| < \|\mathbf{x}_j - \mathbf{x}_i\|$. With the fixed spectator heights the path is in both tube closures, and the germs exist throughout. GE1 at $s = 0$ propagates to $s = 1$. GE2 now identifies the partial distributions locally on each sufficiently small real ball in the spacelike-pair open set. Their difference has support disjoint from this open set, so it vanishes on Schwartz tests whose closed support is contained there. One can see the last assertion by compact smooth cutoffs and their Schwartz convergence, using the local vanishing on each compact support. For each profile $y^{(m)}$, the partial distributions agree. Their closed-height multipliers converge in Schwartz topology at zero height, where the cutoff is one near the frequency support; the damped boundary theorem therefore gives the undamped distribution on every fixed test, in both orders. Uniqueness of scalar limits yields

$$W_v(\varphi) = W_{v\circ\sigma}(\varphi \circ \sigma), \qquad \operatorname{supp}\varphi \subset \{X : X_j - X_i \text{ is spacelike}\}. \tag{LC1}$$

For a product test, its closed support lies in the inverse image of each factor support under the coordinate projections. Mutually spacelike supports of the selected factors therefore meet the LC1 condition. Permuting them exchanges precisely those factors. Inserting arbitrary spectator words on both sides gives

$$W\big(a\,(v,\varphi_i)(w,\varphi_j)\,b\big) = W\big(a\,(w,\varphi_j)(v,\varphi_i)\,b\big). \tag{LC2}$$

There is no restriction on spectator supports. The lifted source retains the full joint distribution and the path of both germs; readout first takes the specified partial boundary and then the zero-height limit. LC1 proves compatibility with adjacent permutation on its spacelike domain, and LC2 recovers the word relation required on the common field domain. It does not discard spectator variables or replace the tested relation by pointwise boundary values.

Recovery of spacelike exchange with spectators. Let $\mathcal{O}_{ij}$ be the open real configuration set where the selected adjacent difference is spacelike. Retain its full joint test space and define the exchange-difference reader

$$\Delta_v(\varphi) = W_v(\varphi) - W_{v\circ\sigma}(\varphi \circ \sigma).$$

At each retained spectator profile $y^{(m)}$, use the two actual partial boundary readers in the original variables to define $\Delta_{v,m}$. The source germ/path recovery gives the same $\Delta_{v,m}$ and makes it zero on each compact test in $\mathcal{O}_{ij}$. Smooth compact cutoffs extend that zero to Schwartz tests with closed support contained there. The fixed-test zero-height recovery is explicitly

$$\Delta_v^Z(\varphi) = \lim_{m\to\infty} \Delta_{v,m}^Z(\varphi) = \lim_{m\to\infty} \Delta_{v,m}(\varphi) = 0.$$

For each order this limit uses $D_{e_{y^{(m)}}} g \to \eta g$ in Schwartz space and $\widehat{W}_v(\eta g) = \widehat{W}_v(g)$ by spectral support. Thus the reader preserves the order of partial-boundary realization, profile limit and adjacent exchange. On product tests the unchanged coordinate projections recover exactly the mutual-spacelike condition on the two selected factors. All other factors remain arbitrary, recovering the whole LC2 word identity with both spectator lists.

19.32 Faithful boundary labels and reconstruction on a common domain

Suppose a union label x is a boundary marker: every expectation containing its smearing, with arbitrary words before and after, vanishes. Splitting a word at that slot gives vanishing on product tests; their density and distributional continuity give vanishing on all joint tests. Represent x by a raw label w in a finite quotient field space. The associated full-tube emission vanishes, so TI1 makes every strictly time-ordered Euclidean reading zero. This remains true after any permutation of the arguments, because the marker may occupy any slot. At any distinct Euclidean configuration sort its heights and approach them by strictly ordered heights with spatial coordinates fixed. Distinctness persists locally; smoothness gives zero at tied heights too. Undo the sorting permutation. The boundary-marker reader can be expressed as the kernel

$$\mathcal{M} = \{x : W(a(x,\varphi)b) = 0 \text{ for every } a, b, \varphi\}.$$

The label section and scalar word recovery preserve every defining pairing of this set. Product-test density and tube/Euclidean recovery then give the same raw Euclidean null preimage for each finite label representative; the retained quotient map reads its class as zero. Consequently the source boundary-marker kernel on physical union labels is also zero. This uses the supplied null-space criterion and its retention on the selected hierarchy.

The Euclidean null-space criterion puts w in its finite-level null space. The selected hierarchy retains exactly the original null spaces, so its original quotient class, and hence x in the union, is zero. Thus boundary faithfulness is recovered from the retained Euclidean quotient rather than imposed as a new field assumption.

HV4 makes each $\eta_b(t)$ Cauchy: the four terms in

$$\begin{aligned}\|\eta_b(t) - \eta_b(u)\|^2 = {} & \langle \eta_b(t), \eta_b(t)\rangle - \langle \eta_b(t), \eta_b(u)\rangle \\ & - \langle \eta_b(u), \eta_b(t)\rangle + \langle \eta_b(u), \eta_b(u)\rangle\end{aligned}$$

have the same independent two-parameter limit. Completeness supplies η_b, and HV3–HV4 give

$$\langle \eta_b, \eta_c\rangle = W(b^*c), \qquad \eta_\varnothing = \Omega, \qquad \eta_{a\cdot b} = U(a)\eta_b. \tag{FR1}$$

Thus the Hermitian kernel $K(b,c) = W(b^*c)$ is positive, since

$$\sum_{\alpha,\beta} \overline{z_\alpha} z_\beta K(b_\alpha, b_\beta) = \left\| \sum_\alpha z_\alpha \eta_{b_\alpha} \right\|^2 \geq 0.$$

The empty expectation is one. Take finite formal sums of smeared words, with this form linear in the second slot, quotient by its zero-norm subspace, and complete to $\mathcal{H}_W$. Write $[b]$ for a word class, $\mathcal{D} = \operatorname{span}\{[b]\}$, and $\Omega_W = [\varnothing]$. The dense common field domain is exactly $\mathcal{D}$; the vacuum is a unit vector. For $\ell = (x,\varphi)$, formal left insertion satisfies

$$\langle L_\ell p, q\rangle = \langle p, L_{\ell^*} q\rangle, \qquad \ell^* = (Jx, \overline{\varphi}). \tag{FR2}$$

If p is null, choose the formal sum $q = L_\ell p$; Cauchy–Schwarz for the positive form and FR2 imply $\|L_\ell p\|^2 = \langle p, L_{\ell^*} q\rangle = 0$. Hence insertion descends, and

$$\Phi(x,\varphi)[b] = [(x,\varphi)b] \colon \mathcal{D} \to \mathcal{D} \tag{FR3}$$

is well defined. Pairing every proposed additive or scalar relation with all word vectors proves bilinearity in x, φ, by the multilinearity of the joint distributions and product tests and density.

FR2 gives the adjoint identity on this common domain. Iteration of FR3 recovers all word vectors and their prescribed vacuum expectations; their span gives cyclicity. For fixed $\psi \in \mathcal{D}$, the map $\varphi \mapsto \Phi(x,\varphi)\psi$ is norm-continuous: expand the squared norm difference into diagonal and mixed joint-distribution pairings. The two-smearing pairings are jointly continuous by continuity of product tests in Schwartz topology. Finite sums of word vectors give the general assertion. No field boundedness is asserted.

LC2 paired against all word probes gives

$$[\Phi(x,\varphi),\Phi(y,\psi)]\big|_{\mathcal{D}} = 0$$

for mutually spacelike test supports. If all $\Phi(x,\varphi)$ vanish on $\mathcal{D}$, pair the zero insertion with the reversed adjoint left spectator word to obtain vanishing expectations with x in every slot. Involutivity of J removes the double reversal. The marker criterion above then implies $x = 0$, proving faithfulness of the prescribed label quotient.

For the source lift define the word realization on finite sums by $V_0(\sum_b c_b[b]) = \sum_b c_b\eta_b$. FR1 proves that its kernel is exactly the word null space and that it is isometric there. It extends to a unitary from $\mathcal{H}_W$ onto the closed span of the retained chamber limits η_b; no assertion that this span equals a larger chamber is needed here. The word source, its null quotient and the left-concatenation reader therefore recover the Hilbert space and unbounded operators with precisely their actual domain. The operator identity and adjoint pairing are transported on that domain, not inferred from compression of bounded approximants.

Positive word-quotient and common-domain recovery. Let $\mathfrak{W}$ be the formal finite word span and $\mathcal{N} = \{p : K(p,p) = 0\}$. The lifted heated vectors recover their actual limits under the bounded section unitary:

$$\eta_b^Z = \lim_{t\downarrow 0} V_H\eta_b(t) = V_H\eta_b, \qquad \left\|\sum_b c_b\eta_b^Z\right\|^2 = K\left(\sum_b c_b b, \sum_b c_b b\right).$$

Thus the source word realization has exactly kernel $\mathcal{N}$ and recovers the same positive word quotient. On its completion, write $V_W : \mathcal{H}_W \to \mathcal{H}_W^Z$ for the resulting unitary and $\mathcal{D}^Z = V_W\mathcal{D}$. The exact operator reader is

$$\Phi^Z(x,\varphi)V_W[b] = V_W[(x,\varphi)b], \qquad V_W^{-1}\Phi^Z(x,\varphi)V_W = \Phi(x,\varphi) \quad \text{on } \mathcal{D}.$$

It uses null preservation from (FR2) before completion; its domain is precisely the transported algebraic word span. The adjoint statement on that span is the pairing identity

$$\langle\Phi^Z(x,\varphi)V_W p, V_W q\rangle = \langle V_W p, \Phi^Z(Jx,\overline{\varphi})V_W q\rangle, \qquad p,q \in \mathcal{D}.$$

For a fixed word-domain vector p and a convergent test sequence, unitarity also recovers the exact smearing norm difference

$$\|\Phi^Z(x,\varphi_r)V_W p - \Phi^Z(x,\varphi)V_W p\| = \|\Phi(x,\varphi_r-\varphi)p\| \longrightarrow 0.$$

Its continuity follows from the retained diagonal and mixed Gram pairings, rather than an operator-norm bound. The same intertwining recovers the LC2 commutator on this domain, cyclic word generation, and the zero boundary-marker criterion for faithfulness. Thus the positive kernel, completion, unbounded field actions and their precise common domain all use one word quotient.

19.33 Spacetime representation, cone support and attachment

Transform every letter by the finite-dimensional label action and inverse placement of its test. Boundary covariance and compatibility with the letter adjoint preserve K, so

$$\mathcal{U}(g)[b] = [g \cdot b]$$

extends to an isometry with inverse $\mathcal{U}(g^{-1})$ and satisfies the group law. Strong continuity requires no topology on the full label union: pin the finitely many labels in one finite field space and expand in a fixed basis. Each moved expectation is a finite sum of continuous label coordinates times continuous fixed-label pairings with moved tests (BC4). Mixed word pairings are continuous; the squared-norm identity gives norm continuity on words and finite sums. For an arbitrary vector, approximation within $\epsilon/3$ and isometry control both errors uniformly in g. The vacuum is fixed, $\mathcal{D}$ is invariant, and

$$\mathcal{U}(g)\Phi(x,\varphi)\mathcal{U}(g)^{-1} = \Phi(\rho(g)x, g\cdot\varphi) \quad \text{on } \mathcal{D}. \tag{FR4}$$

The attachment $I[b] = \eta_b$ is isometric by FR1, sends the vacuum to Ω, and intertwines translations.

For completeness retain the group-generation calculation. For a strictly future timelike vector with time $T > 0$, a coordinate boost of rapidity $s = -\operatorname{artanh}(x_a/T)$ sets x_a to zero, since $T\sinh s + x_a \cosh s = 0$. It preserves the other spatial components and the strict future cone. Three such boosts send any future unit timelike vector to e_0. Given proper orthochronous g, choose their product k with $kge_0 = e_0$. Preservation of the Lorentz pairing makes kg preserve time, and preservation of the quadratic form then makes it an orthogonal spatial turn with the original determinant. Hence $g = k^{-1}(kg)$ is generated by the coordinate boosts and spatial turns used in the boundary-covariance argument.

The absolute-frequency cone is the intersection of inverse images, under all $r^\vee$, of the FT3 time cone. Pulling a test disjoint from one such inverse image through $r^\vee$, applying time-cone support for the transformed labels and transforming back proves support in each inverse image. Its intersection has the DC1 suffix description: a weakly increasing step height, zero before a suffix and one on it, forces nonnegative suffix time momentum in every Lorentz frame. A vector with that property is future causal: negative time is excluded by the identity, and spatial norm larger than nonnegative time is excluded by a boost toward its spatial direction. Constant positive and negative heights force total time momentum zero; the total suffix is future causal, hence also has zero spatial part. Conversely, zero total momentum and future-causal proper suffixes imply every time-cone inequality by telescoping, in every frame. This uses absolute frequencies without switching to relative coordinates.

Let P be the joint translation projection-valued measure, with

$$\langle v, \mathcal{U}(a)v\rangle = \int e^{i\langle p,a\rangle}\, d\mu_v(p), \qquad \mu_v(B) = \langle v, P(B)v\rangle.$$

The isometric translation attachment I makes these characteristic functions equal to the chamber ones at Iv. This follows by conjugating translations and comparing characteristic functions, using preservation of Hilbert inner products. Uniqueness of Fourier transforms of finite measures identifies their diagonal measures; the chamber measures have nonnegative time support by the following semigroup construction. Lorentz conjugation of translations gives

$$\mu_{\mathcal{U}(r)v} = (r^\vee)_*\mu_v. \tag{SP1}$$

If $p \in \operatorname{supp}\mu_v$, every $r^\vee p$ therefore has nonnegative time component. The preceding characterization puts p in $\overline{V}_+$. Every diagonal measure has this support, and polarization yields

$$P(\overline{V}_+^{\,c}) = 0. \tag{SP2}$$

This concerns the actual field translation representation in addition to the hierarchy's distributional support. In the lifted representation, FR4 and the intertwining isometry retain the field domain and all translation matrix elements. Fourier uniqueness recovers their finite spectral measures; SP1 transports those measures and SP2 recovers the spectral projection statement. No surjectivity onto the entire chamber is required for this transfer.

Spacetime and spectral projection recovery. For the completed word section V_W, define $\mathcal{U}^Z(g) = V_W \mathcal{U}(g) V_W^{-1}$. On each word class this is exactly the lifted letterwise motion, so its group law, vacuum and invariant domain recover those of the source. Strong continuity is preserved quantitatively by

$$\|\mathcal{U}^Z(g)V_W v - \mathcal{U}^Z(g_0)V_W v\| = \|\mathcal{U}(g)v - \mathcal{U}(g_0)v\|.$$

For the source's dense-word approximation p to v, the latter is bounded by $2\|v-p\| + \|\mathcal{U}(g)p - \mathcal{U}(g_0)p\|$. This keeps the finite-label/Schwartz proof on the dense span and its extension to arbitrary vectors. Conjugating the actual domain identity (FR4) recovers covariance on $\mathcal{D}^Z$.

Let V_R identify the chamber source and define its attachment by $I^Z = V_R I V_W^{-1}$. It remains an isometry and reads

$$I^Z V_W[b] = V_R \eta_b, \qquad I^Z \mathcal{U}^Z(a) = U_R^Z(a) I^Z.$$

This isometry is enough to recover the diagonal translation measures; its range need not be the entire chamber at this step. For the actual field spectral measure define

$$P^Z(B) = V_W P(B) V_W^{-1}, \qquad \mu^Z_{V_W v}(B) = \langle V_W v, P^Z(B) V_W v\rangle = \mu_v(B).$$

The characteristic-function reader and finite-measure Fourier uniqueness therefore identify the same attachment measures on both carriers. The Lorentz action and dual frequency map recover (SP1), while the retained coordinate-boost generation and suffix-cone argument recover the same cone support. In particular the spectral projection readout is

$$V_W^{-1} P^Z(\overline{V}_+^c) V_W = P(\overline{V}_+^c) = 0.$$

Thus (SP2) is preserved for the translation representation of the actual completed field space, with its stated chamber-semigroup nonnegative-time dependency.

19.34 Heat contractions from reflection-positive words

For finite sums p of future words let Xp be their chamber vectors and $T_t p$ their forward translation by $t \geq 0$. Reflection and translation of the Euclidean hierarchy give

$$\langle XT_t p, XT_u q\rangle = \langle Xp, XT_{t+u} q\rangle. \tag{OS1}$$

The fixed-word correlation bound implies $c = \sup_{t\geq 0} \|XT_t p\| < \infty$: bound each translated word norm by the square root of its reflected two-word correlation bound and sum the finitely many coefficient-weighted bounds. Then

$$\|XT_t p\|^2 = |\langle Xp, XT_{2t} p\rangle| \leq \|Xp\| c.$$

Taking the supremum gives $c^2 \leq \|Xp\| c$, hence $c \leq \|Xp\|$ (including $c = 0$). Forward translation preserves null vectors and extends to a contraction $E(t)$ on the completion. OS1 makes it self-adjoint. Translation parallel to the reflection plane preserves the Gram form and has its opposite as inverse, giving the commuting spatial unitaries. Semigroup and commutation laws hold on planted

words and extend by density. Strong continuity on planted words follows by expanding squared norm differences into continuous Euclidean correlations; finite sums and uniform contraction bounds extend it to the completion. The spatial group is strongly continuous by the same argument. Thus the wave construction uses a recovered self-adjoint contraction semigroup, rather than a separately postulated heat evolution. The lifted source is the planted-word Gram form with its translation operations; the quotient reader preserves OS1 and its null ideal, recovering exactly these completed operators.

19.35 Holomorphic heat powers and the energy-support reader

Each $E(t) = E(t/2)^2$ is positive with spectrum in $[0,1]$. Set $A = E(1)$ and define, for $\operatorname{Re} z > 0$,

$$F(z) = A^z, \qquad 0^z = 0.$$

The function λ^z is continuous on $[0,1]$, uniformly continuous in z on compact subsets of this half-plane. Bounds on $\lambda^{\operatorname{Re} z}|\log\lambda|^k$ on smaller half-planes justify differentiation in the uniform norm, so functional calculus gives operator-norm holomorphy and

$$\|F(z)\| \le 1, \qquad F(z+w) = F(z)F(w), \qquad F(z)^* = F(\bar z). \tag{HE1}$$

These operators commute with spatial translations. For positive integers p, n, put $B = E(1/n)$; then $A = B^n$ and $E(p/n) = B^p$. The scalar equality $(\lambda^n)^{p/n} = \lambda^p$ on the nonnegative spectrum gives $F(p/n) = E(p/n)$. Rational approximation, norm continuity of F and strong continuity of E give $F(t) = E(t)$ for every $t > 0$. If $E(t)v = 0$, the semigroup makes $E(s)v = 0$ for $s \ge t$. Analytic uniqueness for $F(z)v$ makes it zero throughout the half-plane; strong continuity at zero then gives $v = 0$. Self-adjointness implies dense range because its orthogonal complement is the kernel. HE1 gives $\|F(r - is)v\| = \|E(r)v\|$ for $r > 0$: both squared norms are the quadratic form of $E(2r)$. Hence

$$V(s)E(1)v = F(1 - is)v$$

defines an isometry on a dense range and extends uniquely. Checking there with HE1 gives

$$V(s)F(z) = F(z - is) = F(z)V(s), \qquad V(s)V(t) = V(s+t), \qquad V(0) = 1.$$

Thus $V(s)$ is unitary with inverse $V(-s)$ and commutes with spatial translations. Norm holomorphy gives strong continuity on $\operatorname{ran} E(1)$; density and isometry extend it to all vectors. This recovers the chamber real spacetime translations. The sign $F(r - is)$ agrees with the convention $e^{i\langle p,a\rangle}$.

Word translation, heat powers and edge recovery. On the source future-word carrier retain the actual translation T_t and pull back the chamber Gram form through its word readout. It gives exactly (OS1), the same finite-word supremum c, and the same estimate $c \le \|Xp\|$. Its null preimage is invariant under T_t, so descent and completion produce $E^Z(t) = V_R E(t) V_R^{-1}$, where V_R is the chamber section unitary. In particular,

$$\|E^Z(t)V_R v - E^Z(u)V_R v\| = \|E(t)v - E(u)v\|, \quad (E^Z(t))^* = E^Z(t), \quad \|E^Z(t)\| \le 1.$$

The dense planted-word continuity and contraction approximation therefore recover strong continuity without a new heat-flow choice. Spatial translations recover by the same Gram-preserving word operations.

For $A^Z = E^Z(1)$, continuous functional calculus commutes with unitary recovery, giving throughout $\Re z > 0$

$$F^Z(z) = (A^Z)^z = V_R A^z V_R^{-1}, \qquad F^Z(t) = E^Z(t), \qquad \ker E^Z(t) = V_R \ker E(t) = \{0\} \quad (t > 0).$$

Thus positivity, (HE1), holomorphy and dense range retain their source proofs and domains. Define the edge reader initially on that dense range:

$$V^Z(s)E^Z(1)V_R v = F^Z(1 - is)V_R v = V_R F(1 - is)v.$$

The equal-norm identity makes this the same isometric extension as $V_R V(s) V_R^{-1}$. Consequently its group law, spatial commutation, strong continuity and the sign $1 - is$ recover exactly the chamber real-time edge used in the wave and spectral constructions.

Retain also the source's explicit support proof. Suppose a finite positive measure μ, continuous real ω and bounded function h, continuous on the closed upper half-plane and holomorphic inside, satisfy $h(t) = \int e^{it\omega(q)} d\mu(q)$ and $|h(z)| \le C$. For real b and $\epsilon > 0$, integrate $h(z)e^{-ibz-\epsilon z^2}$ around rectangles between heights $0, s$. The vertical sides vanish as their real coordinates tend to infinity, with Gaussian bound proportional to $e^{-\epsilon T^2}$. The horizontal Gaussian majorants are integrable, and Cauchy's theorem gives

$$\left| \int_{\mathbb{R}} h(t) e^{-ibt-\epsilon t^2} dt \right| \le C\sqrt{\pi/\epsilon}\, e^{bs+\epsilon s^2}. \tag{PA1}$$

Fubini and the scalar Gaussian Fourier integral give

$$\int_{\mathbb{R}} h(t) e^{-ibt-\epsilon t^2} dt = \sqrt{\pi/\epsilon} \int e^{-(\omega(q)-b)^2/(4\epsilon)} d\mu(q). \tag{PA2}$$

If a support point q_0 had $b = \omega(q_0) < 0$, the open set $U = \{q : (\omega(q) - b)^2 < b^2/4\}$ would have finite positive measure a. Choose $s = -b/(2\epsilon) > 0$. Positivity and PA1–PA2 imply

$$a e^{-b^2/(16\epsilon)} \le \int e^{-(\omega-b)^2/(4\epsilon)} d\mu \le C e^{-b^2/(4\epsilon)}.$$

Thus $0 < a \le C e^{-3b^2/(16\epsilon)}$, impossible as $\epsilon \downarrow 0$. Consequently $\operatorname{supp} \mu \subset \{\omega \ge 0\}$. For the chamber diagonal translation measure take $\omega(q) = q^0$ and

$$h(z) = \langle v, V(\operatorname{Re} z) E(\operatorname{Im} z) v \rangle, \qquad \operatorname{Im} z \ge 0.$$

It is continuous to the edge and bounded by $\|v\|^2$; inside it equals $\langle v, F(-iz)v\rangle$ and is holomorphic. Its edge is the characteristic function of the time marginal. This proves nonnegative time support and, with SP1, completes SP2.

The lifted heat object retains A, the continuous functional calculus, the dense range and the diagonal finite measures. Its complex-power readout recovers $E(t)$ on the positive axis and the uniquely extended unitary edge on the dense range. The Gaussian test reader PA2 retains positivity and the numerical exponents needed for support recovery; attachment then transports these very measures to the field space.

Gaussian support-test recovery. For the chamber section unitary V_R, the retained diagonal finite spectral measure and half-plane reader satisfy

$$\mu^Z_{V_R v} = \mu_v, \qquad h^Z(z) = \langle V_R v, V^Z(\Re z) E^Z(\Im z) V_R v \rangle = h(z).$$

The same scalar test $e^{-ibz-\epsilon z^2}$ therefore recovers both contour and measure integrals in (PA1)–(PA2), including their common factor $\sqrt{\pi/\epsilon}$. In particular the positive Gaussian measure reader is

$$\mathcal{G}^Z_{b,\epsilon} = \int e^{-(\omega(q)-b)^2/(4\epsilon)}\, d\mu^Z(q) = \mathcal{G}_{b,\epsilon}.$$

For a hypothetical negative support value b, the retained optimizer $s = -b/(2\epsilon)$ reads the exact completed-square value

$$bs + \epsilon s^2 = -\frac{b^2}{4\epsilon}, \qquad \mu^Z(U)e^{-b^2/(16\epsilon)} \le \mathcal{G}^Z_{b,\epsilon} \le Ce^{-b^2/(4\epsilon)}.$$

The set U and its positive measure are unchanged under the diagonal measure reader. The source thus recovers the same contradiction as $\epsilon \downarrow 0$, and the same nonnegative-time support for the actual chamber translation measure. With the retained attachment and Lorentz pushforward it recovers (SP2). The Gaussian test operation changes neither the spectral convention nor the zero-energy atom.

19.36 Ground compression and the component probability space

In the unselected chamber let $\mathcal{H}_0$ be the heat-fixed space, with projection P_0. Retain the compressed slab operators

$$m(w,d) = P_0\,\mathrm{place}(d,w)|_{\mathcal{H}_0}.$$

Their norm-closed unital star algebra is $\mathcal{A}$. With the commutativity and countable norm density established below, this is a separable commutative C^*-algebra. Its compact character space Θ is metrizable: for a dense sequence $a_k \in \mathcal{A}$, $\chi \mapsto (\chi(a_k))_k$ is a continuous injection into a countable product of complex planes. Equality there extends by continuity from the dense sequence to every algebra element; compactness makes the injection an embedding. Gelfand representation and the measure representation of the vacuum positive functional give

$$\langle \Omega_0, a\Omega_0\rangle = \int_\Theta \chi(a)\, d\nu(\chi), \qquad \nu(\Theta) = 1. \tag{PH1}$$

The measure has full support. Indeed the compression identity below makes Ω_0 cyclic. In a commutative algebra cyclicity implies separation: $a\Omega_0 = 0$ implies $ab\Omega_0 = ba\Omega_0 = 0$ on a dense set and hence $a = 0$. A nonempty open null set would support a nonzero continuous bump whose squared integral is zero. Its represented operator would annihilate Ω_0, contradicting separation.

Compression of a distinct word is independent of slab placement. Increasing the width adds heat on the right, fixing the ground input; increasing the initial time adds heat on the left, removed by P_0. Spatial translations disappear on both sides by spatial invariance of ground vectors, proved below. Compare two placements to cover positive or negative common time shifts. Write this element $M(w)$ and set

$$S_\chi(w) = \chi(M(w)). \tag{PH2}$$

For a future word b, its ground-input slab insertion $X_b : \mathcal{H}_0 \to \mathcal{H}$ is independent of trailing width, and slab concatenation with the reflected adjoint gives

$$X_b\Omega_0 = [b], \qquad M(b)\Omega_0 = P_0[b], \qquad M(b^*c) = X_b^* X_c. \tag{PH3}$$

Ground projections of the dense planted-word span are dense in $\mathcal{H}_0$, proving the cyclicity just used. Choose a common slab placement and apply PH1, its vacuum expectation and translation invariance to obtain the barycenter

$$\int_\Theta S_\chi(w)\, d\nu(\chi) = S(w). \tag{PH4}$$

The component hierarchies and their regularity use these same compression elements, not arbitrary disintegrated point readings.

For a future word in a component chamber put $a_\chi = P_{0,\chi}[b]_\chi$. The heat limit gives

$$\|a_\chi\|^2 = \lim_{s\to\infty} \|E_\chi(s)[b]_\chi\|^2.$$

Take integer s. Each prelimit is a continuous character reading of a reflected translated word and is bounded by its $s = 0$ value, a continuous integrable function on compact Θ. Dominated convergence and PH4 yield

$$\int_\Theta \|a_\chi\|^2 d\nu = \|P_0[b]\|^2.$$

Character multiplicativity and PH1 also give

$$\int_\Theta |S_\chi(b)|^2 d\nu = \|M(b)\Omega_0\|^2 = \|P_0[b]\|^2.$$

Since $\langle \Omega_{0,\chi}, a_\chi\rangle = S_\chi(b)$ and the vacuum has unit norm,

$$\|a_\chi - S_\chi(b)\Omega_{0,\chi}\|^2 = \|a_\chi\|^2 - |S_\chi(b)|^2 \geq 0.$$

Its integral is zero, so it vanishes almost everywhere for this word. The countable probe construction supplies one sequence of future words whose planted spans are dense in every component chamber. Intersect their countably many full-measure sets. For each remaining character, all these ground projections lie in the vacuum line; their projected span is dense in the ground space. The line is closed, so the heat-fixed space equals it. This supplies an admissible selected character. The earlier full-support continuity argument upgrades the corresponding final reconstructed uniqueness statement to every character.

For the source lift retain the complete compression algebra and its vacuum state before taking any character. The realization is its concrete action on $\mathcal{H}_0$; observable readers are the characters and their barycenter PH1. Gelfand representation recovers every algebra element from all character readings. PH3 recovers the planted Gram kernel, and PH4 recovers the unselected hierarchy from the probability space. A single character generally loses the other components; that loss is explicit and is reversed only by retaining the whole character family and its measure. Rank one here concerns the heat-fixed subspace and makes no positive spectral-gap assertion.

Compression algebra, character and probability recovery. Restrict the chamber section unitary to its heat-fixed space, writing $V_0 : \mathcal{H}_0 \to \mathcal{H}_0^Z$. The concrete algebra readout is the isometric unital star isomorphism

$$\alpha(a) = V_0 a V_0^{-1}, \qquad \alpha(m(w,d)) = m^Z(w,d), \qquad \alpha(M(w)) = M^Z(w).$$

It preserves the norm closure and all supplied commutativity and countable-density dependencies. The induced character homeomorphism and recovered probability are

$$h : \Theta \longrightarrow \Theta^Z, \qquad h(\chi) = \chi \circ \alpha^{-1}, \qquad \nu^Z = h_*\nu.$$

Indeed $h(\chi)(\alpha(a)) = \chi(a)$ for every algebra element; the inverse is composition with α. These equalities on all evaluations give continuity in both directions. For $\Omega_0^Z = V_0\Omega_0$, the vacuum state reads

$$\langle \Omega_0^Z, \alpha(a)\Omega_0^Z\rangle = \int_{\Theta^Z} \chi^Z(\alpha(a))\, d\nu^Z(\chi^Z) = \int_\Theta \chi(a)\, d\nu(\chi).$$

Thus probability normalization, full support, and (PH1)–(PH2) recover under one character-family map. The entire compressed word is retained before applying a character.

For ground-input word maps set $X_b^Z = V_R X_b V_0^{-1}$. Then

$$(X_b^Z)^* X_c^Z = \alpha(X_b^* X_c) = M^Z(b^* c), \qquad S^Z_{h(\chi)}(w) = S_\chi(w).$$

These are the precise (PH3)–(PH4) Gram and barycenter readers. On each component the section unitary also preserves the nonnegative defect $\|a_\chi - S_\chi(b)\Omega_{0,\chi}\|^2$. Its zero integral and the countable full-measure intersection therefore transfer through h unchanged. Retaining the same countable probe words preserves density in every component and recovers the almost-everywhere heat-fixed vacuum line, with the later every-character upgrade kept as its separate step.

19.37 Spatial ground invariance and compressed commutativity

Positive contraction spectral calculus gives $P_0 = \lim_{t\to\infty} E(t)$ strongly: integer powers of $E(1)$ converge to $1_{\{1\}}(E(1))$ by dominated convergence, and the semigroup and contraction bounds cover general times. Its range is exactly the common heat-fixed space. For two distinct Euclidean words p, q, reflection and placement in the positive half-space give, for large enough x^0, fixed vectors v, w and offsets r, s with

$$S(p\,\tau_x q) = \langle v, U(\mathbf{x}) E(x^0 - r - s) w\rangle. \tag{GI1}$$

Along any sequence $x^0 \to \infty$, a fixed temporal displacement changes this by a quantity tending to zero. Both heat vectors converge to $P_0 w$, and the possibly varying spatial unitaries preserve the norm of their difference. For fixed b with $b^0 > 0$, choose a Euclidean orthogonal transformation sending b to $\|b\|e_0$ and sending e_0 to a vector with positive time component. Reflection in the bisecting hyperplane between b and $\|b\|e_0$ does this, with the positive component following from preservation of their inner product. Rotate the words and displacements; the rotated large separation still has time tending to infinity. The temporal comparison and covariance give

$$S(p\,\tau_{te_0+b} q) - S(p\,\tau_{te_0} q) \longrightarrow 0. \tag{GI2}$$

For spatial a, apply GI2 to $b = a + e_0$ and subtract the temporal comparison adding e_0 to $a + te_0$. Taking heat limits in GI1 yields $\langle v, U(a)P_0 w\rangle = \langle v, P_0 w\rangle$. Density gives $U(a)P_0 = P_0$. Spatial invariance was not assumed in comparing the varying large separations.

For distinct-point cluster words $w_1, \dots, w_r$, with pairwise ordered time separations of displacements $d_k(N)$ tending to infinity,

$$S\left(\prod_{k=1}^{r} \tau_{d_k(N)} w_k\right) \longrightarrow \langle \Omega_0, M(w_1)\cdots M(w_r)\Omega_0\rangle. \tag{GI3}$$

Place each cluster in the positive half-space by a fixed shift, compensated in its displacement. Induct backwards from the last cluster using bounded insertion operators and

$$U(a_N)E(t_N)v_N \longrightarrow P_0 v \quad (v_N \to v,\ t_N \to \infty).$$

The error is at most $\|v_N - v\| + \|E(t_N)v - P_0 v\|$, by contractivity and spatial ground invariance. Between successive cluster insertions there is a heat duration tending to infinity and an arbitrary spatial translation. Vacuum pairing finishes the induction. Large separations also ensure eventual distinctness of the combined configuration.

For four clusters a, b, c, d, choose time/spatial slopes $(0,0), (1,2), (2,1), (3,3)$ and multiply by N. Their time order is a, b, c, d; interchanging the two Euclidean axes changes it to a, c, b, d. Finite-N

permutation symmetry and covariance equate the expectations. Apply GI3. Remove the simultaneous rotation by a second GI3 application with all four clusters along $e_0 + e_1$, whose order is unchanged by the interchange. Thus

$$\langle \Omega_0, M(a)M(b)M(c)M(d)\Omega_0 \rangle = \langle \Omega_0, M(a)M(c)M(b)M(d)\Omega_0 \rangle. \tag{GI4}$$

Take the exterior words as a reflected reversed future probe and another future probe. PH3 and the adjoint identity turn GI4 into equal matrix elements of $M(b)M(c)$ and $M(c)M(b)$ on the dense projected future-word span. These bounded operators are equal. Slab reflection makes the generators closed under adjoints; hence star polynomials commute, and norm continuity extends commutativity to $\mathcal{A}$. This is the actual algebra used by PH1, not a separately selected commutative replacement. The Ground/Lean, Moments and Permute procedures thus supply spatial invariance, arbitrary transverse cluster-train convergence and the four-slope exchange. The four-cluster construction proves the needed exchange of two compressed factors. The resulting spatial ground invariance, cluster limit and commutative compression algebra will now supply one compact phase space and one countable family of future probes for all its characters. Norm-separable generators and the character hierarchy are supplied by the Work/B and Cells constructions below.

Ground-limit and ordered cluster-product recovery. For the chamber unitary V_R and its ground restriction V_0, the strong heat limit and spatial action recover as

$$P_0^Z = V_R P_0 V_R^{-1}, \qquad U^Z(a)P_0^Z = V_R U(a) P_0 V_R^{-1} = P_0^Z.$$

The last equality is read only after the retained GI1–GI2 comparison establishes spatial invariance. Before that comparison, the source reader preserves exactly the varying-unitary norm bound on the difference of the two heat vectors; it imposes no invariance premise. For the backwards cluster induction, set $v_N^Z = V_R v_N$ and $v^Z = V_R v$. The quantitative reader is

$$\|U^Z(a_N)E^Z(t_N)v_N^Z - P_0^Z v^Z\| \le \|v_N - v\| + \|E(t_N)v - P_0 v\| \longrightarrow 0.$$

Consequently every fixed bounded cluster insertion and vacuum pairing recover the same ordered GI3 product, even for arbitrary transverse displacements.

Using $\alpha(a) = V_0 a V_0^{-1}$ on the compression algebra, the four-slope reader preserves the exact order of all four factors and gives

$$\langle \Omega_0^Z, \alpha(M(a))[\alpha(M(b)), \alpha(M(c))]\alpha(M(d))\Omega_0^Z \rangle = 0.$$

The exterior projected future probes remain dense under V_0, so their matrix elements recover the zero commutator of the two actual bounded compressed generators. Adjoint recovery and norm closure then give

$$[\alpha(A), \alpha(B)] = \alpha([A, B]) = 0, \qquad A, B \in \mathcal{A}.$$

This retains the original compression algebra, its four-slope limit and its spatial-ground dependency, and supplies the commutativity used in the character-probability construction.

19.38 A common countable probe family and character positivity

For every arity n and degree tuple $d = (d_1, \dots, d_n)$, the parameter space of distinct positive-half-space positions and raw coefficient vectors in those finite-dimensional degree spaces is separable metrizable. Choose a countable dense subset in each. Countably many n, d, together with the

empty word, give one countable list before any character chamber is chosen. In every smooth symmetric reflection-positive hierarchy its planted-vector map on each parameter space is norm-continuous: the Gram pairing is the hierarchy on the reflected joined configurations, which remain distinct because the blocks lie on opposite sides of the plane. Fixed-basis expansion makes coefficient dependence a finite sum of continuous products and smooth fixed-label readings; adjoint coefficients are continuous too. Diagonal and mixed pairings are continuous, hence the squared norm difference is continuous. Every future word is approximated from its parameter subset; the full future-word span is dense. The same countable list therefore spans densely in every component chamber, justifying exactly the simultaneous countable intersection above.

At a distinct configuration the component moment is

$$S_{\chi,d}(X;v) = \chi(m_d(X;v)). \tag{CH1}$$

Outside that set use the specified zero extension. Here m_d is the same whole-word compression $M(w)$ of PH2. Permutation, unit insertion, common translation, adjoint and quotient-label identities pass through χ; individual rotation and reflection identities pass once proved for these same elements. For smoothness, extend the bounded character from the closed compression subspace to the ambient bounded-operator space by the norm-preserving extension theorem. Composition with the smooth ambient operator-valued moment is smooth and agrees with CH1 on the compression subspace. No differentiability structure on the character space is imposed. Reflection positivity follows by applying χ to one positive quadratic expression. For finitely many future words b_i and scalars c_i, set $F = \sum_i c_i X_{b_i}$. PH3 gives

$$a = \sum_{i,j} \overline{c_i} c_j M(b_i^* b_j), \qquad a|_{\mathcal{H}_0} = F^* F.$$

This is positive in the ambient operator algebra. Spectral permanence of the closed unital C^*-subalgebra makes its spectrum in $\mathcal{A}$ nonnegative. Since $\chi(a)$ belongs to that spectrum, it is real and nonnegative. Linearity gives the component reflection-positive Gram sum at every character, without an exceptional set. In the source lift, m_d is retained before any character operation. The bounded linear reader transports smoothness; the positive algebra element transports Gram positivity. All character readings together recover the compression element, while the retained common probes recover the completed chamber separately at each phase.

Common-probe and positive character recovery. Retain one countable dense parameter subset for each (n, d) before choosing a phase. Its realized probe words b_k are lifted by the same word map in every chamber. For a parameter limit $b_{k_r} \to b$ in a fixed parameter space and its phase section V_χ, the norm reader is

$$\|[b_{k_r}]^Z_{h(\chi)} - [b]^Z_{h(\chi)}\| = \|[b_{k_r}]_\chi - [b]_\chi\| \longrightarrow 0.$$

The limit follows from the retained diagonal and mixed smooth Gram readings. Thus the same enumerated list recovers dense span in every phase; the parameter approximation may depend on the word but the probe list does not depend on the phase.

For the compression star isomorphism α and character map h, set $m_d^Z(X;v) = \alpha(m_d(X;v))$. The whole-word reader gives

$$h(\chi)(m_d^Z(X;v)) = \chi(m_d(X;v)) = S_{\chi,d}(X;v).$$

The zero extension and all finite-label identities are retained before this scalar reading. For the finite Gram sum $a = F^*F$ on ground inputs, its source element satisfies

$$a^Z = \alpha(a) = (F^Z)^* F^Z, \qquad \sum_{i,j} \overline{c_i} c_j h(\chi)(M^Z(b_i^* b_j)) = \chi(a) \geq 0.$$

The ambient positive operator, its spectrum in the closed compression algebra and its character value therefore recover together, at every character. Smoothness uses the same norm-preserving bounded-functional extension composed with the actual smooth ambient moment. The extension is used for that differentiation argument only; all moment readings remain on the retained compression subspace.

19.39 Compression-valued multilinearity, separability and common bounds

The derivative-label identity is retained in its prescribed form:

$$\partial_{X_i^a} D^j S_{\chi,v} = D^j S_{\chi, v[i \leftarrow \partial_a v_i]}.$$

It follows for iterated derivatives by differentiating the local identity at distinct configurations. If every fixed-label reading has bound Cr^{-A}, induction supplies such a bound for every fixed derivative order. Bound the norm of a linear map on joint position space by the sum of its values' norms on coordinate unit vectors; at each step take the maximum of the finitely many replacement-label exponents and the sum of their constants. For $0 < r \leq 1$ that maximum bounds every term. Quotient compatibility transfers this to raw component labels. Constants depend on fixed arity, labels and order; no uniformity in those data is added. The following subsections construct the compression-valued moments, prove norm separability and individual Euclidean invariance, and transfer the derivative and shell identities. These supply the premises used in the character formulas above.

Fix a distinct configuration, a translation and a strict containing slab. The same choices work on a neighborhood, and for every coefficient vector of each fixed label degree. A slab matrix element between future spectators p, q is the hierarchy of the reflected left word, the inserted slab word and translated right word; all positions are distinct. Move one insertion to the front by permutation. Its reading is an exterior-field functional with these actual spectators; the all-spectator null quotient makes it linear in the projected label and independent of its representative. Dense future-word spans on both sides then identify the bounded slab operators themselves. Bounded linear compression preserves this identity. Repeat at every slot to prove multilinearity of $m_d(X; v)$, and replace labels one at a time to prove equality for equal projected tuples. Zero extension off distinct configurations preserves these identities. The same matrix-element argument proves permutation and unit deletion before compression. The empty compressed word is the identity because its heat evolution fixes ground inputs. Slab reflection gives the adjoint, and placement independence gives common translation. These structural identities precede character evaluation and do not assume commutativity. Locally the moment is exactly

$$X \longmapsto P_0 \operatorname{place}(d, \tau_s w(X))|_{\mathcal{H}_0}.$$

Slab regularity at fixed d, smooth position translation and bounded linear compression give operator-norm smoothness. Move any chosen slot to the front to obtain this result at every slot. Placement independence identifies these local functions with the canonical moment everywhere on the distinct set.

For each arity and degree tuple fix basis labels in each slot. The distinct configuration space is separable, and its norm-continuous moment image is separable in operator norm. The countable union over basis tuples, degrees and arities is separable; multilinearity puts all compressed words in its complex span. Continuity of multiplication makes finite products separable, with countable union over lengths. Adjoints, finite complex linear combinations and norm closure preserve separability and generate precisely $\mathcal{A}$. This proves the countable norm density used in character-space metrizability. Characters obey $|\chi(a)| \le \|a\|$. Translate any word of separation $r \in (0,1]$ into a slab padded by one unit on both sides, preserving r. Compression cannot increase norm. The unselected arbitrary-copy slab estimate therefore gives the same copy constants uniformly over characters and receiving hierarchies with the specified word values. For fixed arity n, bounded degrees and coefficient norms it reads

$$|S_{\chi,d}(X;v)| \le K^n n^{Bn} r^{-Bn},$$

with the source's empty-word convention. This is the scalar bound required by the derivative induction. It comes from the whole-word operator norm; character evaluation introduces no further constants. The lifted moment source retains the local slab, both actual spectator slots and quotient representatives. Equality of their dense matrix elements recovers the bounded operator, and its norm-continuous compression recovers m_d. Thus multilinearity, null-label deletion, smoothness and phase-uniform bounds are operations of the same source object rather than independently assigned scalar properties.

Local moment, separability and bound recovery. For a fixed local containing slab write $A_d(X;v) = \mathrm{place}(d, \tau_s w(X;v))$. With chamber and ground sections V_R, V_0, its lifted operator is $A_d^Z = V_R A_d V_R^{-1}$. The compressed moment is therefore

$$m_d^Z(X;v) = P_0^Z A_d^Z(X;v)|_{\mathcal{H}_0^Z} = V_0 m_d(X;v) V_0^{-1} = \alpha(m_d(X;v)).$$

The two actual spectator slots recover every matrix element before compression. Hence their equality preserves the projected-label multilinearity and all structural word identities in the stated order. Since α is fixed bounded linear and isometric, local norm smoothness recovers for every positional derivative:

$$D_X^j m_d^Z = \alpha(D_X^j m_d), \qquad \|m_d^Z(X;v) - m_d^Z(Y;v)\| = \|m_d(X;v) - m_d(Y;v)\|.$$

The source moment consequently has the same separable norm images. Choose countable dense subsets of those images and form finite star words and finite linear combinations with rational real and imaginary coefficients. Their countable collection is norm dense in the generated algebra, and its image under α is dense in $\mathcal{A}^Z$. This explicitly recovers the countable norm-dense sequence from the same compressed generators.

Finally the unchanged padded slab and isometric compression reader give

$$|h(\chi)(m_d^Z(X;v))| \le \|m_d^Z(X;v)\| = \|m_d(X;v)\| \le \|A_d(X;v)\|.$$

The source arbitrary-copy bill thus supplies exactly the stated $K^n n^{Bn} r^{-Bn}$ character bound with its fixed-data and empty-word conventions. There is no extra character-dependent constant in this recovery or in the subsequent fixed-order derivative induction.

19.40 Reflection squares and individual Euclidean invariance

For a generic grid word w with real coefficient labels, write w^θ for its geometrically time-reflected word with reflected labels, and put

$$D = M(w) - M(w^\theta), \qquad a = \operatorname{Re}\langle \Omega_0, D^2 \Omega_0 \rangle.$$

The grid construction supplies real measurable approximants $f_i^{(k)}$ to widely translated copies of this difference, with $|f_i^{(k)}| \le R_k$ and pair errors at most $\epsilon_k \to 0$. Fix k, a finite $q+1$ copies and their displacement parameter before the ordered mesh/volume limit. Each difference-product's four terms converge to the continuum two-cluster readings. Then send displacement to infinity; GI3 makes every off-diagonal comparison tend to $a + \epsilon_k$. The ordinary real square at finite lattice parameters gives

$$0 \le \mathbb{E}\left(\sum_{i=0}^{q} f_i^{(k)} \right)^2, \qquad 0 \le (q+1)\big(R_k^2 + q(a+\epsilon_k)\big). \tag{RS1}$$

The second inequality follows after bounding diagonals and paired off-diagonals and taking limits in that order. At each q finitely many eventual pair tails can be intersected. If $a + \epsilon_k < 0$, large q contradicts RS1; hence $a \ge 0$. Real-label adjoint and permutation identities give $D^* = -D$, so

$$\|D\Omega_0\|^2 = -\operatorname{Re}\langle \Omega_0, D^2 \Omega_0 \rangle = -a.$$

Nonnegativity forces $D\Omega_0 = 0$, and the separating vacuum gives $D = 0$. Approximate a fixed generic configuration by canonical addresses in increasingly fine grids from the common funded sequence. Norm continuity gives equality there. Generic configurations have full measure in the distinct region; continuity and full support of Lebesgue measure extend equality throughout it. Expand multilinearly in real coordinate basis labels fixed by coefficient conjugation to cover arbitrary complex labels. Thus individual time reflection fixes each compressed observable.

GI3 gives simultaneous rotation invariance of pair stages $\langle \Omega_0, M(p)M(q)\Omega_0 \rangle$. If an orthogonal r sends e_0 to positive time component, escape along e_0; if that component is zero, escape along $e_0 + r^{-1}e_0$, positive in time before and after rotation. If negative, first compose with time reflection and remove it afterwards by the individual invariance just proved. These three cases prove simultaneous pair invariance for every Euclidean orthogonal map. Now rotate a pair by r^{-1}, reflect only its second factor, and rotate back. Its second factor changes by $r\theta r^{-1}$ while all pair readings stay equal. Test with the adjoint words of that factor and its transformed copy: the Gram identity gives zero norm for their vacuum-vector difference, and separation gives equality of the elements. Every hyperplane reflection is such a conjugate (zero normal is the trivial case), and orthogonal maps are finite products of hyperplane reflections. The label representation law therefore gives

$$M(r \cdot w) = M(w) \qquad (r \in O(4)).$$

The lifted source retains the bounded Wilson approximants, copy counts, pair tolerances and ordered limits. Its square reader RS1 recovers individual reflection before the pair-stage rotation reader recovers individual orthogonal invariance. Character evaluation thus uses the same whole-word operators throughout.

Bounded square and individual invariance recovery. At each finite lattice cutoff lift every real approximant by the actual configuration map: $f_i^{Z,(k)} = f_i^{(k)} \circ \pi$. Normalized Wilson pushforward and surjectivity give exactly

$$\mathbb{E}^Z\left(\sum_{i=0}^{q} f_i^{Z,(k)} \right)^2 = \mathbb{E}\left(\sum_{i=0}^{q} f_i^{(k)} \right)^2 \ge 0, \qquad \|f_i^{Z,(k)}\|_\infty \le R_k.$$

All pair-error bills and finite intersections of eventual tails are therefore preserved. Keep k, q and displacement fixed through the ordered inner mesh and outer volume limits, then take the large separation limit. The source square reader recovers exactly (RS1). For each k, if $a + \epsilon_k < 0$, any integer $q > R_k^2/[-(a + \epsilon_k)]$ contradicts that inequality. Only afterwards is $k \to \infty$ taken to recover $a \geq 0$.

For $D^Z = \alpha(D)$ the compressed algebra reader satisfies

$$(D^Z)^* = -D^Z, \qquad \|D^Z \Omega_0^Z\|^2 = -\Re\langle \Omega_0^Z, (D^Z)^2 \Omega_0^Z \rangle = -a.$$

The separating vacuum and the isomorphism α recover $D^Z = 0$ and $D = 0$. Grid-address approximation, norm continuity and multilinear real-basis expansion are unchanged under this isometric map. Likewise each pair-stage rotation and conjugated reflection reads the same matrix elements, so the zero vacuum-vector difference recovers the same compressed element. Finite reflection generation then gives

$$M^Z(r \cdot w) = \alpha(M(r \cdot w)) = \alpha(M(w)) = M^Z(w).$$

This establishes individual Euclidean invariance under the same reflection-square and whole-word pair-stage operations used by the component character readers.

19.41 Derivative and stress-window inheritance at every character

Fix $T = S_\chi$. Extend χ only as a bounded complex-linear functional to all ground-space operators and compose with compression. This yields a single bounded linear functional ϕ on chamber operators with

$$S \operatorname{useAt} T(X; v) = \phi\big(\operatorname{place}(D, \tau_s w(X; v))\big). \tag{HW1}$$

No positivity or multiplicativity is required of this extension outside $\mathcal{A}$. One common s, D contains a finite configuration and any compact insertion set with strict time margins, persisting on a neighborhood. Pairing with arbitrary actual future spectators proves the all-spectator inclusion identity for the slab operator; applying ϕ gives component inclusion compatibility. Norm differentiation at fixed width and shift gives

$$\partial_{X_i^a} \operatorname{place}(D, \tau_s w(X; v)) = \operatorname{place}(D, \tau_s w(X; v[i \leftarrow \partial_a v_i])).$$

The common translation has identity position derivative. Passing this through bounded ϕ proves the derivative identity used above.

Let χ_0 be a compact smooth window constant near the distinguished insertion and zero near every spectator. Contain its entire closed support in that same slab. For continuous scalar u put

$$\mathcal{I}_{a,u} = \int (\partial_a \chi_0)(y) u(y) \operatorname{place}(D, \tau_s w(y, X))\, dy. \tag{HW2}$$

This is a norm-integrable operator integral. The scalar factor has compact support disjoint from all collision points: its derivative vanishes near the distinguished point and the window vanishes near spectators. The insertion is norm-smooth there. Off that support the complete integrand is locally zero, regardless of an arbitrary bare collision convention; hence the full integrand is continuous and compactly supported. Pair with two arbitrary future-word vectors. Their additional exterior spectators are strictly outside the slab and entire window. Apply the original stress-window identity with these actual spectators, pass the bounded matrix-element functional through the integral and

use density to get the operator identity. Passing ϕ through the integral and using HW1 gives the numerical identity in T. In particular,

$$-\sum_a \int (\partial_a \chi_0)(y) T(T_{ab}(y), B(X_0), v(X_{\mathrm{sp}}))\, dy = T((Y_b B)(X_0), v(X_{\mathrm{sp}})). \tag{HW3}$$

For a skew Euclidean generator g, take $u_b(y) = [g(y - X_0)]_b$ and sum to obtain

$$-\sum_{a,b} \int (\partial_a \chi_0)(y)[g(y - X_0)]_b T(T_{ab}(y), B(X_0), v(X_{\mathrm{sp}}))\, dy$$
$$= T((Z_g B)(X_0), v(X_{\mathrm{sp}})). \tag{HW4}$$

The degree-four stress label, generator labels and geometric window are unchanged. The lifted source retains the actual operator integral and exterior spectators; its bounded linear character extension commutes with derivatives and this norm integral. It therefore recovers both stress laws at every character with the original window and label data.

One bounded functional for derivative and window recovery. Let $\widetilde{\chi}$ be the source's chosen bounded linear extension on ground-space operators. Define

$$\phi(A) = \widetilde{\chi}(P_0 A|_{\mathcal{H}_0}), \qquad \phi^Z(A^Z) = \phi(V_R^{-1} A^Z V_R).$$

This retains one extension for the fixed character and gives $\|\phi^Z\| = \|\phi\|$. On an actual compressed moment it recovers the character reading; outside the compression algebra only bounded complex linearity is used. For the common locally fixed slab and shift,

$$\phi^Z(\partial_{X_i^a} A^Z(X)) = \partial_{X_i^a} \phi(A(X)), \qquad A^Z(X) = V_R A(X) V_R^{-1}.$$

Thus inclusion and derivative-label recovery use the same actual spectator-tested slab identity before scalar evaluation.

Keep the compact window, continuous weight and its entire support in that same slab. For its full norm-integrable operator integrand $J_{a,u}(y)$, unitary conjugation preserves both the collision-free support and the local zero extension. Hence

$$\mathcal{I}^Z_{a,u} = \int V_R J_{a,u}(y) V_R^{-1}\, dy = V_R \mathcal{I}_{a,u} V_R^{-1}, \qquad \phi^Z(\mathcal{I}^Z_{a,u}) = \int \phi(J_{a,u}(y))\, dy.$$

Bounded matrix-element readers commute with this same integral and retain the actual exterior spectators outside the slab. Their dense pairings therefore recover the operator stress identity first. The bounded scalar reader then recovers (HW3)–(HW4), including their minus signs, the weight $[g(y-X_0)]_b$ and the unchanged degree-four stress and generator labels. No multiplicative or positive extension premise is added to this derivative-and-integral passage.

19.42 Smeared positivity and bounded reflection witnesses

A signed coordinate permutation taking a chosen axis to time intertwines its word reflection with the temporal reflection, preserves the positive side and preserves readings by real-spin covariance. It transports the complete finite point Gram sum to every coordinate plane. For joint tests fix such an axis and its positive future-word Gram Hilbert space. For compact f supported on positive mutually distinct configurations set $\zeta_f(X) = f(X)[w(X)]$ there and zero elsewhere. Smooth diagonal and mixed smeared Gram kernels are continuous, also across the cutoff support boundary. Expanding

the squared norm difference proves strong continuity of ζ_f; compact support gives Hilbert-valued integrability. Fubini gives

$$\iint K_{\mathrm{smear}}(f,g;X,Y)\,dX\,dY = \left\langle \int \zeta_f(X)dX, \int \zeta_g(Y)dY \right\rangle .$$

Every finite smeared positivity sum is thus a squared norm. This constructs joint-test positivity directly from the point kernel, without assuming a full hierarchy in order to prove one of its clauses.

For a real observable O with bounded local complex approximant O' of rooted discrepancy at most ϵ and $|O'| \le A$, use $\operatorname{Re} O'$. Indeed

$$\operatorname{Re} O' - O = \tfrac{1}{2}(O' - O) + \tfrac{1}{2}\overline{(O' - O)}.$$

Subadditivity, homogeneity and conjugation invariance of the rooted seminorm preserve the error bound. Measurability and locality persist, and $|\operatorname{Re} O'| \le A$. The actual cutoff observable need not itself be uniformly bounded. For a fixed finite degree-bounded label list, form products of these real local approximants at the translated lattice sites. The product comparison applies to the concatenated two-cluster list. For distinct integer displacement slopes $i < j$, the relevant coordinate difference has a fixed part plus $t(j-i)$; clusters are eventually disjoint. Wrapping remains injective on the period tail. The same extraction order therefore controls all four ordinary/reflected cluster combinations. If their product approximants are $f_i^{(k)}, g_i^{(k)}$, use the real witness $f_i^{(k)} - g_i^{(k)}$, bounded by the sum of the two product bounds. Expanding the pair product gives total comparison error

$$\epsilon_k = c_k(A_{00} + A_{01} + A_{10} + A_{11}) \longrightarrow 0.$$

All four integrals exist by local approximant estimates and finite lattice moment bounds. This is precisely the common witness family used in the finite-square argument. Fix the approximation index and number of copies, retain the prescribed ordered mesh/period limits and then the large-displacement comparison as in RS1. The approximants are obtained by fixing reporting scale and finite atlas, choosing a sufficiently late auxiliary scale, and transferring the local estimates back to the original mesh and period tails on the same chamber. The following gain-chart estimates supply this choice quantitatively.

Joint smeared positivity and real witness recovery. For each coordinate-reflection Gram space, use its section unitary V_a and keep the same positive-side joint test. Then

$$\zeta_f^Z(X) = V_a \zeta_f(X), \qquad \int \zeta_f^Z(X)\,dX = V_a \int \zeta_f(X)\,dX.$$

The scalar Gram kernel, compact support and strong continuity are unchanged. Consequently the complete finite smeared reader is

$$\sum_{i,j} \overline{c_i} c_j \iint K_{\mathrm{smear}}(f_i, f_j; X, Y)\,dX\,dY = \left\| \sum_i c_i \int \zeta_{f_i}^Z(X)\,dX \right\|^2 \ge 0.$$

The signed coordinate permutation recovers this same construction at each axis, preserving both the reflection and positive-side domain.

For the bounded local witnesses, configuration pullback satisfies

$$(\Re O') \circ \pi = \Re(O' \circ \pi), \qquad (\Re O' - O) \circ \pi = \tfrac{1}{2}((O' - O) \circ \pi) + \tfrac{1}{2}\overline{(O' - O) \circ \pi}.$$

Thus the retained rooted seminorm laws give the same error bound after real-part extraction. Measurability, local physical read sets and the uniform approximant bound are retained. Product and subtraction readers then recover every real reflection witness and its four-term pair expansion before integration. Normalized measure pushforward gives identical four integrals, with exactly the summed bill $c_k(A_{00} + A_{01} + A_{10} + A_{11})$. The finite-copy tails, eventual unwrapped separation and period wrapping injectivity are the original physical conditions read through the same local configuration maps. Hence the source witnesses recover the family used by (RS1), with the reporting scale and atlas fixed before the late auxiliary scale and with the stated subsequent cutoff and separation order.

19.43 Coarse lenses, coefficient inversion and rooted approximation

At coarse scale k the local lens is the explicit function

$$\mathcal{L}_{k,L}(U) = a_k L_{\mathbf{1}} + 1_{\mathrm{Healthy}(\vartheta_k,\mathcal{F})}\mathcal{C}_{k,L}(U). \tag{AP1}$$

Here a_k is field scaling, $L_{\mathbf{1}}$ the constant coefficient, $\mathcal{F}$ the stencil footprint and $\mathcal{C}_{k,L}$ the scaled nonconstant candidate polynomial. All other regular activity terms of the primary lens are zero in this specialization. On the healthy compact chart the candidate is bounded by $A_{k,L}$ uniformly in finer terminal cutoff and period; outside it the product is zero. The lens bound is $|a_k L_{\mathbf{1}}| + A_{k,L}$. Its read set lies in the coarse window of stencil radius plus four, since every inspected edge is within four sites of a stencil point. For recovered coefficient columns Jb_i assume $|e_i - Jb_i|_1 \le 1/4$ for every coordinate basis vector, and set $Av = \sum_i v_i Jb_i$. Expansion and the triangle inequality give

$$|v - Av|_1 \le \tfrac{1}{4}|v|_1, \qquad |v|_1 \le 2|Av|_1. \tag{AP2}$$

Thus A is injective and, in equal finite dimensions, onto. Solve for a terminal label L and use injectivity of J to recover

$$L = \sum_i v_i \,\mathrm{terminal}(b_i), \qquad |v|_1 \le 2e^{\mathrm{columnLogs}(k)}|L|_1 = \mathrm{stock}_k(L). \tag{AP3}$$

The last bound is the backward coefficient budget. Each basis lens back-lifted to the fine lattice approximates its recovered terminal observable with common rooted error $\mathrm{stakes}_{J,m}/w$. The AP3 linear combination B therefore obeys

$$\|B\|_\infty \le \mathrm{stock}_k(L)A_k, \qquad \|B - O_L\|_{\mathrm{root}} \le \mathrm{stock}_k(L)\frac{\mathrm{stakes}_{J,m}}{w}. \tag{AP4}$$

Back transport preserves the sup bound and the atlas room condition puts its read set in the common fine chamber. The exact terminal label decomposition, subadditivity and homogeneity give the rooted bound for this same B. For the basis bound insert separated atlas copies of the difference, with a compulsory mark on each, and extract the coefficient containing every copy once. Exact RG transport preserves its Wilson integral. For q atlas cells and discrepancy moment M, insertion estimates give

$$|M|w^q \le e^{qc}, \qquad c = (a_J/a_m)^4 \mathrm{treasury}_m + 5.$$

The scale power comes from the exact cell census; the five units are the insertion and vault costs. The qth root gives $e^c/w = \mathrm{stakes}_{J,m}/w$. The coefficient identity cancels the same positive Wilson partition function, preserving normalization. The admissible mark price satisfies

$$w|a_m|\frac{4(1+\mathrm{columns}_k)}{\mathrm{gainChart}(k,m)} \le \tfrac{1}{2}\zeta_{d,m}. \tag{AP5}$$

Fix J, m, d first and choose w at this upper value. AP4's error is a fixed constant times stock and column factors divided by the gain chart. Their growth is subexponential, while the inverse gain chart contains the strictly decaying scale factor, so the product tends to zero. A late enough k also meets atlas room and any prescribed positive error tolerance. Once fixed, its sup bound is uniform on all later mesh and period parameters. Taking the real part then gives the real witnesses used by RS1.

The lifted approximation source retains the healthy mask, candidate coefficients, inversion columns, atlas, all-copy mark and common partition function. Its local reader is AP1, its coefficient reader is AP3, and its rooted reader is the normalized all-copy extraction. AP2 proves coefficient recovery and AP4 controls observable recovery with its explicit error. The finite coarse lens loses the residual observable only to that bounded error; retaining the transported discrepancy and paid mark restores the exact comparison, as below.

Quantitative coarse-lens and coefficient recovery. The framed local lens is the actual pullback

$$\mathcal{L}^{Z}_{k,L} = \mathcal{L}_{k,L} \circ \pi_k, \qquad \mathcal{L}^{Z}_{k,L} \circ \sigma_k = \mathcal{L}_{k,L}.$$

Its healthy predicate and candidate are read through the same map, so the bound $|a_k L_{\mathbf{1}}| + A_{k,L}$ and coarse physical read window are preserved. No unmasked polynomial value is used outside the healthy region.

For the finite coefficient system in (AP2), set $E = I - A$ on its physical coordinate space. The coordinate-column hypothesis gives $\|E\|_{1\to 1} \le 1/4$, so its explicit recovery operator is

$$A^{-1} = \sum_{n=0}^{\infty} E^n, \qquad \left\| A^{-1} - \sum_{n=0}^{r} E^n \right\|_{1\to 1} \le \frac{(1/4)^{r+1}}{1 - 1/4}.$$

This preserves the source's looser inverse bound 2 in (AP2)–(AP3). On the horizontal coefficient carrier $j_d E_d$ the system is $A^Z = j_d A p_d$ with inverse $j_d A^{-1} p_d$, so its solution recovers exactly the same v_i and terminal-label decomposition. The source fine approximant is then $B^Z = B \circ \pi$, with the same AP4 sup bound.

For every permitted finite atlas-copy list, configuration pullback and normalized measure recovery give

$$\left\langle \prod_{i=1}^{q} (B_i^Z - O_{L,i}^Z) \right\rangle_Z = \left\langle \prod_{i=1}^{q} (B_i - O_{L,i}) \right\rangle = M.$$

The compulsory all-copy coefficient, exact RG identity and common positive Wilson denominator are retained in this reader before its qth root is taken. Hence the rooted error bound, cell-census factor, mark price (AP5) and stock bill are unchanged. The reporting data are fixed first; the same late auxiliary scale and atlas room condition then recover the prescribed tolerance for this very local B, with its sup bound uniform on subsequent mesh and period tails.

19.44 The paid discrepancy history and backward cone

The AP4 comparison uses one common history. Starting with primary coefficient L at k, backward coefficient construction chooses a reference b and residual history Z_i for $m \le i \le k$, with

$$\begin{aligned} \exp(Z_k) &= \exp(\text{primary } L) + \exp(\text{history}(-b)), \\ \text{forced}(Z_k) &= \text{forced}(\text{history}(-b)). \end{aligned} \tag{AP6}$$

These are exact evaluated-entry and forced-profile identities by linearity and reference cancellation; the primary has no forced profile here. Its residual coefficient is $L - J_k b$, bounded by $Q/4$ by the backward cone; the terminal/reference coefficient is bounded by $2E$. At intermediate scales,

$$\mathrm{cost}(Z_i) + |Z_i^{\mathrm{coeff}}|_1 \le E/\mathrm{gainChart}(k,i).$$

Transport the mark price w at m by the same multiplicative schedule. The scalar reserve recurrence and AP5 bound the transported price times scaled residual cost by $\zeta_{d,i}/2$ throughout the interval. The reference contribution at k is paid by the other half-reserve, using $2E$. At each atlas copy insert the negative terminal reference observable; transport its marked polynomial exactly from N to k and multiply by the coarse lens insertion there. AP6 identifies its first marked and forced slots with the residual slots. Paid transport carries them from k to m with the same migrating positions; the atlas supplies separation and occupancy at every scale. At m the served-source support exponent and vault bounds $2, 1$ follow from the same residual estimates. Neither history nor mark schedule is reselected. For a finite census the new coarse factors are $\prod_i (1 + \mathbf{z}_i Q_i)$. A multi-index coefficient is the subset sum over slots taken from these factors, all others supplied by the transported negative fine observable. Finite coefficient extraction commutes with every RG push by linearity and integrability. This sum is exactly the joint fine/coarse difference polynomial. Integration of the push preserves its integral; restoration of the coupling gives the positive Wilson partition function times the discrepancy moment. Thus the root extraction compares this actual local difference in its original normalization.

Exact paid-discrepancy coefficient recovery. Retain the one reference label b, residual histories Z_i, migrating copy positions and transported mark prices as one source record. At scale k, apply the coefficientwise configuration reader $\mathscr{L}$ to the evaluated entries in (AP6). It gives the exact identities

$$\mathscr{L}\exp(Z_k) = \mathscr{L}\exp(\text{primary } L) + \mathscr{L}\exp(\text{history}(-b)), \qquad \mathscr{L}\,\text{forced}(Z_k) = \mathscr{L}\,\text{forced}(\text{history}(-b)).$$

Here exp denotes the source's evaluated entry; its stated linear identity is retained as such. The same coefficient section preserves $L - J_k b$, its $Q/4$ bound and the reference $2E$ bill. Residual cost and mark prices are read from the retained history, so the two half-reserves and the scale-m vault bills remain the original ones throughout the paid transport.

For the all-copy coefficient on a finite set A, write $N(\mathbf{z})$ for the transported negative-fine insertion polynomial. The exact finite coefficient reader is

$$[\mathbf{z}_A]\left(N(\mathbf{z})\prod_i (1 + \mathbf{z}_i Q_i)\right) = \sum_{B \subseteq A} [\mathbf{z}_{A\setminus B}] N(\mathbf{z}) \prod_{i \in B} Q_i.$$

This is the square-free all-copy instance of the source's general multi-index rule. Finite coefficient extraction and configuration pullback commute term by term; every RG push commutes with it by the same linear integral identity. Restoring the coupling thus recovers the same positive Wilson partition function times the actual fine/coarse discrepancy moment. Section restriction recovers the finite polynomial itself, including both primary and forced slots. The support, occupancy, mark schedule and normalization in the rooted estimate therefore belong to this one transported discrepancy.

Here is the quantitative backward cone. In the coefficient norm including the unit coordinate, a step of size L has $q = L^{1/8}$, $e = q^{-3}$ and

$$y_\alpha = L^{d - \deg \alpha} v_\alpha, \qquad Y = |y|_1, \qquad |v|_1 \le Y.$$

The step's invertible B commutes with deletion of the unit coordinate, with $\|B\|_1, \|B^{-1}\|_1 \le K$, $K \ge 1$. For costs $r, t \ge 0$,

$$|u - By|_1 \le e(|\operatorname{core} v|_1 + t), \qquad r \le K|\operatorname{core} y|_1 + et.$$

Consequently

$$Y \le K\big(|u|_1 + e(Y + t)\big), \qquad r \le KY + et. \tag{AP7}$$

For column allowance $C_i \ge 1$ put $a_i = 32(1 + C_i)^2$. Room and bank choices imply

$$a_i \ge 4K^2, \quad 4Ke \le 1, \quad 4Ka_{i+1}e \le 1, \quad (4K^2 + 1)qe \le 1. \tag{AP8}$$

Explicitly the bank requires $q \ge 100000M^6(1 + C_{i+1})^2$, with neighboring price bound $M \ge K$. It implies $q \ge 128K^2(1 + C_{i+1})^2 \ge 1$; use $e \le q^{-1}$ and $qe \le q^{-1}$ for the last three bounds. Room $C_i \ge 64K^2$ and monotonicity of stage prices give the first. If $a_i|u|_1 \le r$, AP7 gives $4K|u|_1 \le Y + et$. Substitution in AP7 and absorption using AP8 give $Y \le 4Ket$. Thus

$$a_{i+1}|v|_1 \le a_{i+1}Y \le t, \qquad qr \le q(KY + et) \le t. \tag{AP9}$$

The $e = 0$ case follows directly as well. The coefficient/remainder cone propagates coarse to fine, with the opposite-direction remainder gain q. The first smallness inequality in (AP8) absorbs the resulting Y-term and gives $Y \le 4Ket$.

Construct the matched history by finite backward induction from the terminal formatted insertion and remainder bound. At each step use the same marked transport and linear coefficient decomposition, updating only the newly reached component while retaining all earlier components and identities. Its terminal density is exactly the prescribed insertion plus the reference history. Subtract the reference coefficient determined at the coarsest endpoint: linearity makes the residual coefficient there zero and preserves all density/step identities. Its nonnegative cost therefore starts in the cone. This is why the cone applies to the residual history, not to an arbitrary primary one. For costs c_i and residual coefficients v_i on $[l, k]$, iterate AP9:

$$\begin{aligned} a_i|v_i|_1 \le c_i, \qquad \operatorname{gainChart}(k, i)c_i \le c_k, \\ \operatorname{gainChart}(k, i) = \exp\big[(\operatorname{scale}(k) - \operatorname{scale}(i))/8\big]. \end{aligned} \tag{AP10}$$

The recursion is $\operatorname{gainChart}(k, i) = \operatorname{gainChart}(k, i + 1)L_i^{1/8}$. The terminal pairing bound $c_k \le (1 + C_k)w + C_k|v_k|_1$ and cone inequality $2C_k|v_k|_1 \le c_k$ give $c_k \le 2(1 + C_k)w$. Since $|v_i|_1 \le c_i$, this proves

$$\operatorname{gainChart}(k, i)(c_i + |v_i|_1) \le 4(1 + C_k)w. \tag{AP11}$$

The approximation constant E is this terminal bound with its fixed coefficient and stock factors. Choose column allowances above room thresholds, and block lower bounds so $L^{1/8}$ exceeds the bank threshold, before selecting any history or volume. Larger block bounds or later delays preserve the choice. AP6–AP11, paid transport and root extraction therefore construct the bounded local approximants on the original common family.

The lifted discrepancy source retains each component and step identity of this matched history, together with its paid price schedule and finite subset-coefficient polynomial. Coefficient extraction is its observable reader; commutation with the exact push proves recovery of the original discrepancy moment. AP9–AP11 give a quantitative reader bound without discarding the residual or its unit coordinate. Thus the local approximation, the real reflection witness and its common Wilson normalization are recovered from one source history.

Backward-cone and matched-history recovery. Use coefficient section and recovery j_i, p_i at each scale, retaining all coordinates including the unit. Put $\widehat{v} = j_{i+1}v$, $\widehat{y} = j_{i+1}y$, $\widehat{u} = j_i u$, and use section norms. The diagonal scaling reader and step map are

$$y_\alpha = L^{d-\deg\alpha} v_\alpha, \qquad B^Z = j_i B p_{i+1}, \qquad p_i B^Z j_{i+1} = B.$$

Core deletion is transported with these same sections. Thus its commutation with B, the inverse bound K, and both one-step coefficient/remainder inequalities recover (AP7) exactly. For clarity the source's absorption in the cone can be read as

$$Y \le \tfrac{1}{4}(Y + et) + KeY + Ket \le \tfrac{1}{2}Y + (K + \tfrac{1}{4})et, \qquad Y \le 4Ket,$$

using $4Ke \le 1$ and $K \ge 1$. The remaining (AP8) bills then recover both conclusions (AP9) with the same direction and factor q.

A source history is the full finite tuple of coefficient components, remainders and step identities. Its readout applies p_i componentwise and leaves the recorded nonnegative costs and bank prices fixed. At each backward-induction step only the newly reached component is changed, so this reader commutes with the finite induction. Subtraction of the same reference history recovers the zero coarse residual coefficient before initiating the cone estimate. Consequently its iterated source bounds are precisely

$$a_i \|\widehat{v}_i\|_1 \le c_i, \qquad \mathrm{gainChart}(k,i)(c_i + \|\widehat{v}_i\|_1) \le 4(1 + C_k)w.$$

The gain-chart product telescopes by the same scale identity (AP10). Thus the original terminal pairing bound, unit coordinate, room and bank choices, and history-before-volume quantifiers all recover through one matched tuple; no arbitrary primary history is substituted for the residual history in this cone argument.

20 Separated and nested operator products

A finite reader used consistently in every product. Operator-product coefficients are extracted with one homogeneous coefficient reader, chosen compatibly with the degree filtration, reality and rotations. A truncation retains finitely many channels and an explicitly bounded remainder. Recovery therefore means equality of the retained coefficients and of the tested replacement operation on its separated configurations, together with the same remainder bound. It does not invert a finite projection or assert convergence of an infinite series in the truncation degree.

Nested products must use that same reader at each stage. Lower-degree subtractions and the scale-ratio gains control the difference between the two actual conditional pushforwards. Their constants and radius thresholds keep the fixed word, degree and separation hypotheses. The common coefficients serve all permitted phases, but the correlation reading at each use still belongs to the chosen common hierarchy. The detailed derivations below establish these compatibility statements before any collision extension is attempted.

20.1 Separated products and fixed homogeneous readers

Let E_d be the coherent finite-dimensional degree filtration and choose its reader once, compatibly with reality and Euclidean rotations. For input degrees d_i write $\Delta = \sum_i d_i$. Assume

$$\|x_i - o\| \le Mr, \qquad \|x_i - x_{i'}\| \ge \eta r \ (i \ne i'),$$
$$\|o - y_j\| \ge c, \qquad \|y_j - y_{j'}\| \ge c \ (j \ne j'), \qquad M \ge 0, \quad \eta, c > 0.$$

For every fixed truncation D and slack $\epsilon > 0$, the expansion has, for all sufficiently small $r > 0$,

$$\left| S_s(a(x)b(y)) - \sum_{j=0}^{D} \sum_{\alpha \in \mathcal{C}_j} C_{j,\alpha}(x, a; o) S_s(e_{j,\alpha}(o)b(y)) \right| \le r^{D+1-\Delta-\epsilon}. \tag{OE1}$$

The threshold is before every allowed x, o, y, s, where s is the unselected hierarchy or any character. The same coefficients serve all D, spectators and viewers. For each fixed channel, on its small-radius tail,

$$|C_{j,\alpha}(x, a; o)| \le r^{j-\Delta-\epsilon}. \tag{OE2}$$

These are finite-order asymptotics; no $D \to \infty$ convergence is asserted.

Apply the real rotation-averaged finite reader at level m to one field at the origin, obtaining $L_{m,q} : E_q \to E_m$. These respect inclusions in q and reproduce included fields when $q \le m$. Define

$$K_m = 1 - \sum_{i<m} \iota_{i,m} K_i L_{i,m}, \qquad \pi_{m,q} = K_m L_{m,q}.$$

For $q \le m$ this gives

$$\begin{aligned} \pi_{m,q}\upsilon &= \iota_{q,m}\upsilon - \sum_{i<m} \iota_{i,m}\pi_{i,q}\upsilon, \\ \sum_{i=0}^{m} \iota_{i,m}\pi_{i,q}\upsilon &= \iota_{q,m}\upsilon. \end{aligned} \tag{OE3}$$

For $q < m$, include the second identity at $m-1$ into E_m to see $\pi_{m,q} = 0$. Inclusion compatibility follows from L. Set $H_m = \bigcap_{i<m} \ker \pi_{i,m}$. Applying each lower projection to the recursion, using induction on earlier homogeneous pieces and cancelling its matching term, proves $K_m(E_m) \subset H_m$. The same identities make the projections fix their own homogeneous space and annihilate the other included pieces. Choose once a real compatible basis $e_{m,\alpha}$ of H_m, $\alpha \in \mathcal{C}_m$. For cluster reader value $A_m(x, a; o)$ define

$$h_m = K_m A_m \in H_m, \qquad h_m = \sum_{\alpha} C_{m,\alpha} e_{m,\alpha}.$$

The horizon-D packet is $\sum_{m \le D} \iota_{m,D} h_m$. Applying $\pi_{j,D}$ recovers exactly h_j, independently of D. For a single centered field OE3 recovers its inclusion. Spectator pairing of this packet is exactly the finite sum in OE1, by linearity and component inclusion. These are exact identities before the asymptotic bound. Define the error in the ground algebra as the compressed full cluster-and-spectator word minus that finite sum of compressed centered outputs. Vacuum and character evaluation give precisely the scalar remainder. Contractivity of these evaluations and the Gelfand norm identify the common algebra norm bound with the uniform bound over character viewers.

The averaged reader has the explicit finite-packet formula

$$\begin{aligned} A^0_m(x, a; o) &= \int_{O(4)} X_m(k) P_{\le m}(k^{-1}x, X(k^{-1})a; k^{-1}o)\, dk, \\ A_m(x, a; o) &= \tfrac{1}{2}\{A^0_m(x, a; o) + \overline{A^0_m(x, \bar{a}; o)}\}. \end{aligned}$$

Haar measure is normalized. Use this on the open set where every rotated readout is admissible and assign zero outside; the asymptotic uses eventually lie inside. Finite packets have integrands and all position derivatives continuous uniformly on compact $O(4)$ locally, so integration preserves smoothness and multilinearity. Translation and permutation follow termwise. Haar change $k \mapsto gk$ gives orthogonal covariance; the second average gives conjugation compatibility. For one centered

degree-q field, $q \le m$, coherence gives $P_{\le m} = \iota_{q,m} a$. Equivariance makes the first integrand constant, and real inclusions make the second average preserve it. This proves the single-field property used in OE3. Induction in the triangular recursion makes K_m commute with rotations and conjugation, since each earlier projection does so and the inclusions intertwine both operations; thus H_m is invariant and h_m has the asserted laws. For the conjugate-linear involution C on H_m,

$$v = \frac{v + Cv}{2} + i\frac{v - Cv}{2i}$$

shows its fixed real space spans H_m complex-linearly. A real basis there is complex independent: conjugate a complex relation and add and subtract to separate its real and imaginary coefficient relations. It is the required real basis, chosen once at each m, before horizons and exterior words are specified.

In the source lift retain the whole packet, filtration, reader, projections and algebra-valued error together. Homogeneous readout recovers each h_j exactly and is compatible with every finite horizon. Spectator evaluation then recovers OE1. A truncated reader loses higher channels and the residual; retaining the residual makes the original whole-word reading exactly recoverable. No separated expansion is promoted to a coincident-point operator identity.

Fixed homogeneous channel and residual recovery. To distinguish configuration readout from the homogeneous maps $\pi_{m,q}$, write j_m, p_m for the finite coefficient section and recovery, with $p_m j_m = I$. Define on horizontal images

$$L^Z_{m,q} = j_m L_{m,q} p_q, \qquad K^Z_m = j_m K_m p_m, \qquad \pi^Z_{m,q} = j_m \pi_{m,q} p_q, \qquad \iota^Z_{q,m} = j_m \iota_{q,m} p_q.$$

Their compositions recover the same triangular recursion and (OE3). The lifted homogeneous space is $H^Z_m = j_m H_m$ with fixed basis $e^Z_{m,\alpha} = j_m e_{m,\alpha}$. The normalized Haar and reality averages commute with these fixed linear sections, on the same open admissible domain and with the same zero convention outside it. Thus $h^Z_m = j_m h_m$ has exactly the original scalar coordinates $C_{m,\alpha}$, independent of any chosen horizon or spectator.

For $P^{\mathrm{hom}}_D = \sum_{m \le D} \iota_{m,D} h_m$, its source packet and exact channel reader satisfy

$$(P^{\mathrm{hom}}_D)^Z = j_D P^{\mathrm{hom}}_D, \qquad \pi^Z_{j,D}(P^{\mathrm{hom}}_D)^Z = j_j h_j \quad (j \le D).$$

In particular section recovery of each channel is exact at every finite horizon. Let R_D denote the retained ground-algebra whole-word remainder, with all spectators still attached. Its source remainder is $R^Z_D = \alpha(R_D)$, so

$$\|R^Z_D\| = \|R_D\|, \qquad h(\chi)(R^Z_D) = \chi(R_D).$$

The vacuum reader also agrees under the ground unitary. This recovers (OE1)–(OE2) with their same uniform allowed-configuration/viewer tails and fixed-data dependence. Retaining R^Z_D alongside the finite packet recovers the whole reading; the construction asserts the same finite-order asymptotics and no infinite-horizon convergence.

20.2 Uniform OPE rates and smooth interpolation

A rate at most λ means that for every positive slack the norm is eventually bounded by $e^{(\lambda+\epsilon)T}$ as $T \to \infty$. For a cluster of size and internal-separation exponent $a > 0$, set $Z_H = P_{\le H}(x, a; o)$. The finite reader comparison gives

$$A_n(x, a; o) - A_n(o, Z_H; o) \quad \text{rate} \le (|d| - H - 1)a. \tag{OE4}$$

It holds for arbitrary varying rotations. Otherwise choosing a violating rotation at each bad parameter contradicts the comparison on that path. Compact Haar integration therefore preserves the uniform rate; conjugate comparison pays the real average. Apply fixed K_n:

$$h_n - \pi_{n,H} Z_H \quad \text{rate} \le (|d| - H - 1)a. \tag{OE5}$$

For $H = n$, the projection kills all old grades below n; the grade-n term has rate $(|d| - n)a$, while the error has the better rate $(|d| - n - 1)a$. Finite summation gives rate $(|d| - n)a$ for h_n and each coordinate. If exterior clearance and internal exponents are $0 \le b \le c$, $b < a$, centered degree j costs $jb + |e|c$. A varying field of rate t costs $t + jb + |e|c$, by fixed finite-basis expansion. OE3 gives the exact remainder identity

$$W(x,a;y,b) - \sum_{j=0}^{D} W(o,h_j;y,b) = [W(x,a;y,b) - W(o,Z_D;y,b)] + \sum_{j=0}^{D} W(o,\pi_{j,D}Z_D - h_j;y,b).$$

The old remainder rate is $\xi = |d|a - (D+1)(a-b) + |e|c$. Each correction has rate $(|d| - D - 1)a + jb + |e|c \le \xi$. Finite summation preserves ξ for arbitrary varying viewers, permitting the algebra norm bound. This last quantifier is what permits passage to the ground-algebra norm, rather than merely pointwise character bounds. Taking $T = -\log r$, $a = 1$, $b = c = 0$ and spending arbitrary slack gives OE1–OE2. A failure of a uniform geometry threshold would select bad geometries at arbitrarily small radii, contradicting this pathwise rate. The coefficient family is consequently common to all horizons and phases. OE4 itself follows by testing the old scalar remainder at the finite detector configurations and fixed weights. Under varying rotations their distances stay positive and labels/output vectors have rate zero by compactness. Expand moving exterior labels in fixed finite bases. Multilinearity and finite sums preserve the rate; conjugation of the correlation gives the conjugate comparison, using the hierarchy's reality identity. For completeness, the scalar remainder bookkeeping does not lose a grade.

Recovery of uniform logarithmic remainder rates. For fixed coefficient sections, $h_n^Z = j_n h_n$ and the section norm preserve the rate of every finite-reader difference. Applying a fixed linear map of norm C also preserves a rate: for a prescribed slack $\epsilon > 0$, use the input estimate with slack $\epsilon/2$ and then choose T so large that $C \le e^{\epsilon T/2}$. The same argument absorbs any fixed finite number of summands. Consequently the exact homogeneous map and finite remainder sum recover (OE4)–(OE5) with unchanged exponents, rather than a shifted grade.

For the correction at degree $j \le D$, the explicit exponent comparison in the retained geometric regime is

$$\xi - \big((|d| - D - 1)a + jb + |e|c\big) = (D + 1 - j)b \ge 0, \qquad \xi = |d|a - (D+1)(a-b) + |e|c.$$

The source algebra remainder is $R_D^Z = \alpha(R_D)$, and thus

$$\|R_D^Z(T)\| = \|R_D(T)\| = \sup_{\chi \in \Theta} |\chi(R_D(T))|.$$

This uses the whole ground-algebra element at each T. A violation of a common eventual bound selects one violating viewer at each violating parameter and contradicts the source estimate for arbitrary varying viewers. Varying rotations are retained in the same way before Haar integration;

the normalized integral cannot increase the uniform norm bound. Finite detector weights and compact rotation-dependent label coordinates have rate zero under the same reader.

Finally with $T = -\log r$ and $a = 1$, $b = c = 0$, the rate bound $e^{(\Delta-D-1+\epsilon)T}$ is exactly $r^{D+1-\Delta-\epsilon}$. The analogous channel rate is exactly $r^{j-\Delta-\epsilon}$. A selected bad-geometry path is unchanged by section recovery, so the source's uniform small-radius threshold argument applies to the same allowed configurations and fixed coefficient family at every finite horizon.

For the old graded estimates choose mesh exponents $0 < b < a < x < d$, with actual cluster exponent x, and put $\tau = |g|d + 4|I|(d-a)$. Physical matching supplies a degree-q centered label Q, an amplitude and a survival filter for their actual continuum limits. Its label budget rate is $\tau - qa$; its common mismatch against all permitted exterior tests has rate

$$\tau - (q + \tfrac{1}{16})(a - b). \tag{OE6}$$

The same Q is used at every detector and phase. Its degree-h readout, $h \le q$, has rate $\tau - ha$. For $h = 0$ use the full budget. For $h > 0$, lower-degree approximation gives $s \in E_{h-1}$ whose distance from Q is bounded by the degree-$(h-1)$ fare times the original budget. The degree-h reader kills s by coherence; its value on $Q - s$, bounded by the fare of exponent $(q-h)a$, has the required rate. This subtraction is essential: using just the norm of Q would lose all of this graded improvement. Choose $q \ge h$ with $(h+1)a \le (q+1/16)(a-b)$. Testing OE6 at the finite detector list and adding this readout estimate bounds the actual degree-h coefficient by $\tau - ha$ on the lattice screen. For truncation N, choose $q \ge N$ with $(N+1)a \le (q+1/16)(a-b)$. Let exterior clearance exponent be $c < b$, internal exponent $u \ge c \ge 0$, and weight $|r|$. Reconstruct Q from all its readers. For $h \le N$ subtract the true coefficients; OE6 pays each difference and its exterior cost $hc + |r|u$. For $N < h \le q$, the graded budget gives

$$\tau - ha + hc + |r|u \le \tau - (N+1)(a-c) + |r|u.$$

The product-to-Q mismatch has that upper exponent too. Their exact sum is the product minus its degree-N truncation, so the screen remainder has rate

$$\tau - (N+1)(a-c) + |r|u. \tag{OE7}$$

Every statement retains arbitrary varying viewers. Smooth interpolation applies to the actual coefficient or remainder, using its rough bound, screen bound and positional derivatives. The rough coefficient rate is $x|g|$ since detector exteriors stay distant; the rough remainder rate is at most $x|g| + |r|u + Nc$ by separate bounds on the original and finite truncation. Derivatives are those of the same hierarchy and finite detector sums. Interpolation gives the screen rates at arbitrary configurations. Choose $a = x - \delta$, $d = x + \delta$ and b strictly between exterior clearance and a. The supplied clock has the required mesh levels. As $\delta \downarrow 0$, the coefficient exponent tends to $(|g| - h)x$ and the remainder to $|g|x - (N+1)(x-c) + |r|u$. For any strict requested exponent first fix small enough δ, then levels and auxiliary horizon. A moving center changes no distances or readers. The empty cluster has only the degree-zero unit coefficient, zero positive-degree coefficients and exactly identity truncation, requiring no anchor approximation. This covers the cases excluded when an anchor in the cluster was chosen above. We detail the grading argument supplying the old estimates.

Here is the interpolation detail. Consider a family $f_T(p, z)$ of scalar functions, whose label list p has total degree $|p|$, and suppose every positional coordinate derivative replaces one label by its prescribed derivative, raising total degree by one. Let $f_T(p, z)$ have rough rate $a|p| + c$ in a corridor of size/separation exponent a, screen rate $b|p| + d$ at mesh exponent $b > a$, and every

coordinate derivative replace a label by a degree-one-higher derivative label. Choose integer n with $(b-a)n \geq c-d$. Descend the total degree threshold: the rough bound already proves the improved estimate for $|p| \geq n$. If it holds at one higher degree, move z to a screen point z' within rate $-b$. The small ball stays in the corridor. The derivative has rate $b(|p|+1)+d$; its operator norm is bounded by the finite sum of its coordinate norms. The mean-value inequality gives

$$f_T(p,z') - f_T(p,z) \quad \text{rate} \leq b(|p|+1)+d-b = b|p|+d.$$

Add the screen value and iterate down to every original degree. Uniform derivative bounds and eventual differentiability throughout the moving ball follow by selecting a violating point if they failed, then applying the pathwise hypothesis. Finite-dimensional rounding with finitely many forbidden coincidences finds z' within $2K$ mesh units, K depending only on the number of cluster positions. Derivative replacement holds for full correlations, finite detector sums and the finite-product-rule remainder; the centered exterior factors do not move with cluster positions. Finally, the derivative replacement identity holds for the actual functions in question. No derivative of a chosen approximating label is required.

The lifted rate source retains the common matched Q, survival filter, detectors, finite-basis coordinates and all slack/level choices. Its triangular reader OE5 recovers the new channels from the old packet with the paid residual. The exact remainder identity preserves the degree and exterior costs; interpolation recovers arbitrary geometries from the same screened functions. Retaining the arbitrary varying-viewer quantifier is essential for recovering the compression algebra norm rather than merely separate character limits.

Graded screen and interpolation recovery. Retain the same centered matching label Q, amplitude and refining filter for every detector. For a degree-h readout R_h and the retained lower-degree approximation $s \in E_{h-1}$, its section lift satisfies

$$R_h^Z j_q Q = j_h R_h Q = j_h R_h (Q-s), \qquad R_h s = 0.$$

Thus its fare and whole-label budget combine with exponent $(q-h)a + (\tau - qa) = \tau - ha$ on both carriers. This is the same lower-degree subtraction that yields the screen coefficient bound. The chosen q satisfies the displayed mismatch inequalities before testing any viewer, so section recovery retains the finite low/high grade split giving (OE7).

For interpolation, scalar correlation, detector and fixed-truncation remainder readers agree at every actual configuration. Their positional derivative-label identities therefore agree as well. The lifted screen point is the same physical rounded point z', with unchanged excluded coincidences and distance bill. Applying the mean-value reader to the actual function gives the same exponent cancellation

$$b(|p|+1)+d-b = b|p|+d.$$

The selected-path argument for uniform bounds on the moving ball is preserved, so this recovers the descending induction without taking a derivative of the matched label Q.

The final slack choice is quantitative. Setting $a = x-\delta$ and $d = x+\delta$ yields exactly

$$\begin{aligned} \tau - ha &= (|g|-h)x + (|g|+8|I|+h)\delta, \\ \tau - (N+1)(a-c) + |r|u &= |g|x - (N+1)(x-c) + |r|u \\ &\quad + (|g|+8|I|+N+1)\delta. \end{aligned}$$

For any strict requested upper exponent choose δ to fit its positive gap first, retaining the stated positive mesh ordering; then choose the levels and auxiliary horizon. This explicit readout preserves the source's quantifier order, moving-center invariance, and exact degree-zero unit behavior of the empty cluster.

20.3 Compact matching limits and exterior-price passage

A finite plot at $k \le m < m+1 \le j$ obeys

$$8\|x_i\| < a_{m+1}, \qquad \|x_i - x_{i'}\| > 2a_j B_j^2 \quad (i \ne i').$$

On sufficiently large periods the wrapped positions retain their locations, are separated by B_j^2 lattice units and share central parent zero through $m+1$. Matching along the original extraction order gives degree-q labels with a paid local mismatch of amplitude A and quota at m at most p. Quota comparison bounds their norm in the fixed finite reference space by Kp. Bounded finite-dimensional compactness gives a nontrivial refining filter and a limit Q, on which the quota and paid mismatch survive. The survival record retains the actual filter, geometry, insertion, reference horizon and quota.

Fix positive slack ρ for exterior fields. Their bounded-copy envelopes are

$$\mu_j = (1+Y_j)a_\ell^{-e_j-\rho} + 1.$$

At mesh exponent ξ their product has rate $(|e|+\rho|J|)\xi$. Choose auxiliary exponents strictly between exterior clearance and matching support, and a further pair strictly above exterior internal exponent. These gaps pay every finite width and separation requirement; a fixed number of fields adds no exponent. On the same survival filter the mismatch still uses the same Q. Its compressed exterior rate is at most $\delta + (|e| + \rho|J|)\xi$, with amplitude rate δ. Choose $\xi \downarrow u$ and then $\rho \downarrow 0$ before any eventual bound, giving $\delta + |e|u$. Contractive evaluation permits arbitrary varying phase viewers. Using both cluster and exterior envelopes gives the full-correlation engineering rate $|g|a+|e|u$. These arguments justify all exterior costs used in the grading and interpolation proof. For approaching plots the strict exponent inequalities eventually give cluster size below $a_{m+1}/16$, spacing above $2a_jB_j^2$ and ordered level indices. The subpower B_j consumes no strict gap. One compact matching limit can therefore be selected pointwise at each parameter, with the common amplitude and quota rates required by OE6.

For matching accuracy take block exponent a, input exponent $d > a$ and exterior-support exponent $b < a$. The input budget product has rate $-|g|d$ and the volume factor rate $-4|I|(d-a)$, so the positive initial amplitude x_T has rate $-\tau$, where $\tau = |g|d + 4|I|(d-a)$. The terminal cost is

$$e_T = 4(1+\mathrm{columns}_m)\, 2a_{m+1}^q\, \mathrm{bling}_m\, \frac{4}{x_T}.$$

Columns, chart factors and copy controls have rate zero; adjacent $m, m+1$ have the same mesh exponent. Thus the cost rate is $\tau - qa$ and output quota is $2e_T$. Choose

$$z_T = \exp\Big([(q+\tfrac{1}{16})(a-b) - \tau]T\Big), \qquad A_T = 2\,\mathrm{stakes}\,/z_T.$$

The stock-opening level has exponent b and stakes have upper rate zero, giving the OE6 error rate. March affordability offers gain $(q+1)(a-b)$, chart affordability $(q+1/8)(a-b)$; expenditure uses only $(q+1/16)(a-b)$. Both leave strict positive margins absorbing all rate-zero constants. Room also follows from $b < a$: the clock has $J_{m+1} \le J_m + 1$ and scale increments at least $\log 3$, hence

$$J_m + 5 \le J_0 + 5 + \mathrm{scale}_m / \log 3, \qquad \frac{\log(J_m+5)}{\mathrm{scale}_m} \longrightarrow 0.$$

A fixed readout radius and this subpower width fit inside the strict mesh gap. Make these finite matching choices before the continuum limit; they give the centered label, quota and local root

error in the same central chamber through $m+1$. The two affordability conditions in the matching construction hold eventually.

No exterior independence is assumed. On the common finite lattice,

$$(\mathrm{register}_x - O(Q,0))\,\mathrm{register}_y = \mathrm{register}_{x\sqcup y} - O(Q,0)\,\mathrm{register}_y\,.$$

The left side is one paid main insertion with the given exterior insertions. The original extraction along the survival filter yields the two right-hand terms with the actual limiting Q, proving the screened compressed-difference bound. Approximate arbitrary exterior positions by screened positions avoiding the center and cluster, coordinate by coordinate; separation persists and both limiting configurations are noncoincident. Smoothness passes scalar inequalities at every character to the limit; their supremum gives the algebra norm. The same argument without replacement bounds full products. Thus the extension to arbitrary exterior points uses actual continuity of the already constructed hierarchy and one fixed replacement, not a fresh replacement chosen after the test.

Survival-filter and exterior-price recovery. A matched source record retains the actual refining filter $\mathfrak{F}$, finite reference carrier, labels Q_λ, paid mismatch, quota and geometry. Its coefficient section gives

$$Q_\lambda \underset{\mathfrak{F}}{\to} Q, \qquad j_q Q_\lambda \underset{\mathfrak{F}}{\to} j_q Q, \qquad p_q \lim_{\mathfrak{F}} j_q Q_\lambda = Q.$$

The section norm retains the same Kp compact bound. Scalar quota and mismatch inequalities are part of this same record on that filter; their recovery does not select a replacement label after choosing an exterior word.

At each common finite lattice configuration, pullback preserves the exact register identity:

$$\mathscr{L}\big((\mathrm{register}_x - O(Q,0))\,\mathrm{register}_y\big) = \mathscr{L}\big(\mathrm{register}_{x\sqcup y} - O(Q,0)\,\mathrm{register}_y\big).$$

The normalized integral and paid insertion readers therefore recover the same correlated discrepancy. Passing along $\mathfrak{F}$ and then through the fixed screened-position approximations preserves the chosen Q. The resulting algebra element is recovered isometrically by α, so taking all character readings gives the same norm bound for arbitrary exterior points.

For clarity, the two strict affordability margins read exactly as

$$(q+1)(a-b) - (q+\tfrac{1}{16})(a-b) = \tfrac{15}{16}(a-b),$$

$$(q+\tfrac{1}{8})(a-b) - (q+\tfrac{1}{16})(a-b) = \tfrac{1}{16}(a-b).$$

They absorb the same rate-zero prices before the continuum limit. The source keeps e_T, z_T, A_T, adjacent-level exponents and central chamber in that same finite matching choice. The exterior envelope reader likewise retains $\delta + (|e| + \rho|J|)\xi$ until the auxiliary exponents and ρ have been fixed to meet the requested strict slack. Thus its recovered rate is $\delta + |e|u$ with the source's choice order and arbitrary-viewer scope.

20.4 Finite sweep, replicas and coefficient conservation

Copy the input list into every cell of a fixed coarse tiling, with its specified reflection and translation, writing $F_{v,i}$ for copied insertions and $q_v = \prod_i F_{v,i}$. The anchor copy is unchanged. At $m+1$ all original members have parent zero, so copied parents are exactly the tiling centers. At every finer level a copied position has its center's coarser parent and lies within half the intervening

block length of that embedded parent. Distinct centers have parent separation at least $B_{m+1}^2 + 1$. Enlargement L gives embedded separation $L(B_{m+1}^2 + 1)$, while both deviations cost strictly less than L and $B_i^2 \le LB_{m+1}^2$. The triangle inequality proves cross-copy separation at each intermediate level; the finest spacing handles within-copy separation. Use one formal marker per occurrence (v, i). Descending integration preserves its polynomial and integrability. Grouping within each copy replaces the insertions by q_v. Grouping in stages or once gives the same coefficients: each marker-set coefficient is the product over its full inverse image. This fixed linear map on a finite polynomial space commutes with conditional integrals. Thus descent of the products is exactly grouped descent of the individual insertions. The sweep starts with $w_i = \delta_j \phi_j a_i/8$. At each level it joins exactly fibers acquiring a common parent, retaining one anchor per fiber, with at most the original arity at termination. Its available normalization is bounded below by compensation of $\prod_i w_i$, including the four-dimensional volume ratio and join licenses. The initial allowance is

$$\alpha^{|I|}\,\mathrm{compensation}\left(|I|, m+1, j, 0, \alpha, (\delta_j\phi_j/8)^{|I|}\prod_i a_i\right).$$

Here $\alpha \le 1$ is no larger than every intermediate license; fewer surviving fibers only increase this lower bound. This is x_T. For rate-zero positive licenses u_i, put $\alpha_i = \min(1, u_0, \dots, u_i)$. For $t < 0$, eventually $u_j \ge e^{t\,\mathrm{scale}_j}$; monotonicity gives $\alpha_i \ge \min(\alpha_{i_0}, e^{t\,\mathrm{scale}_i})$. Divide logarithms by scale and let $t \uparrow 0$: α_i retains rate zero and bounds all preceding licenses. The halo purchase bounds the geometric license logarithm by $16\sqrt{\widehat{\beta}_i}$, sublinear in scale; the other factors have rate zero too. No power loss is introduced by the minimum. At the last join choose $z > 0$ below surviving normalization and purchased local constant $x \ge 1$. The terminal analytic coefficient has remainder bound $4/z$. Prune the allowance to $I'_v = z/(16x)$. It is available and obeys

$$2I'_v\,2(4/z) \le 1, \qquad I'_v\,|\lambda^{-1}|\,x(4/z) \le 1,$$

where $\lambda = (\ell_m)^q \in \mathbb{C}$ and $\ell_m \ge 2$ is the integer block size at this step. This λ is a scalar multiplying every coefficient in the degree-q fusion; it is not an arbitrary linear change of frame. Since $q \ge 0$, $|\lambda| = \ell_m^q \ge 1$ and $|\lambda^{-1}| = \ell_m^{-q} \le 1$, with both moduli equal to one for $q = 0$; equivalently, scalar multiplication by λ^{-1} has operator norm at most one on the coefficient space. Thus the first smallness expression is $1/x \le 1$ and the second is $|\lambda^{-1}|/4 \le 1/4$. A lower bound on the norm of a general operator would not imply this inverse bound; the scalar identity is what proves it here. In the lift the coefficient space is unchanged and λ acts by the same scalar on realized coefficients, so the substitution bound is recovered with the same constants; the inverse bound depends on the homogeneity of the fusion in one degree and would not survive mixing degrees. March allowances $W_v \le \delta_m I'_v$ meet substitution bounds. Local fusion gives

$$\|Q_0\|_{\mathrm{coeff}} \le p' = a_{m+1}^q x\,4/z$$

with the same remainder budget simultaneously in every copy. Copy transformations are inherited from the original $F_{v,i}$ and the anchor is zero. Restore the polynomial coefficient and apply bounded-copy comparison to its actual local section: the physical label has quota at most $2e$ and root error $2\,\mathrm{stakes}\,/b_0$, with final march amplitude $b_0 = z_T$. It yields a centered degree-q label with quota $2e_T$, a mismatch supported in the prescribed central chamber, and root-test bound A_T. These are exactly the outputs used above. We describe the geometry and algebra of that finite matching construction more precisely. The finite matching construction itself is the sweep-and-replica construction; its remaining formulas are addressed below. The input positions are required to remain in that chamber through level $m + 1$, exactly the plot conditions already checked.

The lifted matching object retains the refining filter, replica geometry, occurrence polynomial, grouping map, licenses and local section. Grouping/descent commutation proves coefficient conservation; compact matching gives the same centered label to every exterior reader. The exact register identity preserves dependence on exterior variables. Amplitude and quota readers retain their paid errors, so the finite matching construction recovers OE6 without a viewer-specific replacement or an interchange of extraction limits.

Replica grouping and finite sweep recovery. Let $g : \mathcal{O} \to \mathcal{V}$ send each occurrence (v, i) to its copy v. On the finite square-free coefficient space define the source's grouping reader by

$$[z_B]\Gamma_g P = [z_{g^{-1}(B)}]P, \qquad B \subseteq \mathcal{V}.$$

For $P = \prod_{(v,i)\in\mathcal{O}}(1 + z_{v,i}F_{v,i})$, this gives $\Gamma_g P = \prod_v (1 + z_v q_v)$ with $q_v = \prod_i F_{v,i}$. For a subsequent grouping h the full inverse-image rule proves

$$\Gamma_h\Gamma_g = \Gamma_{h\circ g}, \qquad \Gamma_g\mathbb{E}[P \mid \mathcal{G}] = \mathbb{E}[\Gamma_g P \mid \mathcal{G}].$$

The second equality is coefficientwise linearity for the same finite integrable polynomial. Configuration pullback also commutes with Γ_g, so the source and physical descended polynomials have identical grouped coefficients at every step.

The geometric reader retains physical positions, parent maps and copy motions. Its separation calculation is explicitly

$$L(B_{m+1}^2 + 1) - \text{sum of the two deviations} > LB_{m+1}^2 \geq B_i^2.$$

Thus the same fibers, anchors and joins occur at each scale, including the unchanged anchor copy. The source records the actual allowances, license minima and compensation before the final fusion; these scalar readers preserve positivity and the rate-zero minimum argument. The coefficient section at the terminal join recovers Q_0 with the same p', remainder budget and pruned allowance $z/(16x)$. Restoring its grouped coefficient and using the same local section therefore recovers the physical label quota $2e$ and the root error $2\,\text{stakes}/b_0$ on every copy, with $b_0 = z_T$. Neither copy grouping nor source recovery changes that final march amplitude.

20.5 Common purchases and the exact sweep compensation

The common family supplies every finite arity with the same positive activities and smallness conditions

$$(2B_i + 1)^4\,\text{scurry}_i \leq \tfrac{1}{2}, \qquad e^{\text{scurry}_i} - 1 \leq 1,$$

the normalization lower bound, fusion and analytic remainder estimates at sufficiently late degree references. The clock then reaches at least degree plus three. These are scheduled estimates on the existing family; changing arity or output degree does not replace it. Encode a finite screened linear combination with its actual moving lattice labels and the same scalar coefficients in its lattice expression and continuum compression. Register multiplication is exactly concatenation; subtraction retains coefficients $1, -1$. Apply the paid-product inequality to that expression, then take the selected continuum limit, to bound its compression norm by the product of its allowances. This applies in particular to the intact mismatch times every exterior factor. This explains why the subtraction stays intact through the limit; estimating its two terms separately would discard the approximation accuracy.

The compensation inequality underlying the sweep can be written without any scale shorthand. For original arity H, remaining groups s, $0 < \alpha \le 1$ and $w > 0$, define the exact compensation

$$\mathcal{K}(i,j,s) = \left(\frac{a_j}{a_i}\right)^{4H} \left(\frac{\alpha}{8}\right)^{H(H-s)} w. \tag{OE8}$$

The invariant is $\mathcal{K}(i,j,s) \le \prod_{\gamma=1}^{s} W_\gamma$. Initially $s = H$ and $w = \prod W_\gamma$, giving equality at the finest level. If separation persists, $a_i = L_i a_{i+1}$, $L_i \ge 2$, and each allowance becomes W_γ / L_i^4. Since $s \le H$,

$$\mathcal{K}(i,j,s) = L_i^{-4H} \mathcal{K}(i+1,j,s) \le L_i^{-4s} \prod_{\gamma} W_\gamma.$$

If separation fails, choose a nontrivial grouping with separated coarser representatives and original anchor, of size $s' < s$, with purchased cover ℓ_i. Local grouping multiplies the old product by $\mathrm{license}(H, \ell_i, i+1)^s$, and each new group spends 8 before transport. Because α is below this license,

$$H(H-s') \ge H(H-s) + H, \quad \alpha^H \le \alpha^s, \quad 8^{s'} \le 8^H,$$

we have

$$\begin{aligned} \mathcal{K}(i+1,j,s') &\le \mathcal{K}(i+1,j,s)(\alpha/8)^H \\ &\le 8^{-s'} \,\mathrm{license}(H,\ell_i,i+1)^s \prod_{\gamma} W_\gamma. \end{aligned}$$

The transport inequality then pays the coarser step. Descending finite induction proves OE8 in both cases. Each regrouping decreases s, so its fixed-arity exponent pays the grouping events independently of the number of scale steps or cells. Marker grouping commutes with push and preserves integrability by the finite linear integral identity. Retain the actual representatives: parents come from original positions, copy transformations from original collars, and the anchor is original. The purchased ℓ_i bounds every selected coarser cover. These are precisely the data used by local grouping and final normalization; no additional consistency choice is needed.

The all-phase norm is the same ground-algebra norm throughout. At distinct positions the compressed whole word is M; zero off that domain only makes the notation total. Characters read phase correlations, and the vacuum functional reads the unselected correlation by common translation into the positive chamber and compression. For the unselected hierarchy its evaluation is $\langle \Omega, M\Omega \rangle$, because translating the entire finite word into a positive-time chamber leaves its correlation unchanged and the compression formula realizes that translated word. Both are contractive, while the Gelfand norm is the supremum over characters. Thus a common scalar viewer bound is equivalent to the compressed norm bound. The lifted sweep retains representatives, parent fibers, allowances and the exact marker polynomial; OE8 is its quantitative conserved budget. Its reader is the intact paid linear combination, followed by compression and character or vacuum evaluation. The same source therefore recovers both the compensation invariant and the common OPE norm without dropping the subtraction.

Exact sweep-budget and all-viewer norm recovery. At each finite sweep stage retain the tuple of actual representatives, parent fibers, anchors, group allowances and scalar licenses. Its readout preserves the physical coordinates and these finite numerical data. Thus $H, s, \alpha, w, a_i, a_j$ and $\mathcal{K}(i,j,s)$ are identical on the source and physical records. The two case updates recover the exact budget ratios

$$\frac{\mathcal{K}(i,j,s)}{\mathcal{K}(i+1,j,s)} = L_i^{-4H}, \qquad \frac{\mathcal{K}(i+1,j,s')}{\mathcal{K}(i+1,j,s)} = (\alpha/8)^{H(s-s')} \le (\alpha/8)^H.$$

Because $s \le H$ and $s' < s$, these are precisely the transport and join payments in (OE8). Finite descending induction commutes with the readout: each grouping uses the same finite inverse-image coefficient map and the same integrable conditional push. The number of strict regroupings is bounded by the decrease in s, independently of the number of scale steps and replica cells.

For an intact paid linear combination $E = \sum_\ell c_\ell M(w_\ell)$, its source algebra element and viewers are

$$E^Z = \alpha(E) = \sum_\ell c_\ell M^Z(w_\ell), \qquad h(\chi)(E^Z) = \chi(E), \qquad \|E^Z\| = \|E\| = \sup_\chi |\chi(E)|.$$

The vacuum functional is likewise unchanged under the ground unitary. The original finite-lattice coefficient expression uses these same c_ℓ, including $1, -1$ for the discrepancy. Thus its paid-product reader retains subtraction through the selected limit before applying compression or any viewer. The recovered compensation and norm bills therefore concern one intact expression on the original common family.

20.6 Representatives, moving-label moments and grouping prices

In the anchor copy choose a maximum-cardinality subset containing the distinguished anchor with pairwise lattice separation at least B^2. Each omitted position is at separation $< B^2$ from a selected one, or it could be adjoined. Assign each point to such a representative and each representative to itself. This is a surjection with an anchor-fixing section. When separation failed the subset is proper, so its size decreases. Repeat the relative selection in every copy; reflection and translation preserve separation and the previously proved cross-copy estimate makes the whole selected configuration separated. If two finer points have parents at separation $< B_m^2$, the half-block deviations from embedded parents give

$$\operatorname{sep}(x, y) \le L_m(B_m + B_{m+1} + 1)^2 =: c_m.$$

The halo of the first point contains the pair and is covered with at most

$$(2c_m + 1)^4 \le 16(1 + c_m)^4$$

B_{m+1}-blocks. This purchased carpet bound applies to each point/representative pair; an entire group need not fit one small halo.

For a finite screened linear combination E whose labels converge along an actual refining filter in fixed finite-dimensional spaces, freeze the moving labels at each regulator pair and use that static word in the order-$2h$ even-moment expansion. It is a finite sum over one choice from E at every occurrence, with products of scalar coefficients and conjugates and translated/reflected lattice correlations. All subtractions are exact. Expand labels in fixed bases. Finite coordinate products converge, and fixed-label point limits apply at the eventual generic replica positions. Thus first the regulator filter gives the continuum replica expression; then large replica displacement gives the vacuum even moment of its ground compression by GI3. A finite bound c^{2h} on the full expression passes through both limits for every h. The ground-algebra even-moment norm argument gives $\|M_E\| \le c$, equivalently by character moments and the Gelfand norm. Here $c \ge 0$ is the product of paid allowances. Applying this to the intact register mismatch proves its norm bound without a separate inner regulator limit for moving labels.

Anchor selection and moving-moment norm recovery. Retain the finite selected representative set R, assignment $g : \mathcal{O} \twoheadrightarrow R$ and its section $s : R \to \mathcal{O}$. The physical reader preserves positions and gives

$$g \circ s = I_R, \qquad g(o_*) = o_*, \qquad s(o_*) = o_*, \qquad \operatorname{sep}(x, s(g(x))) < B^2 \quad (x \notin s(R)).$$

The same relative assignment is used in every copy. Hence its parent-based two-point carpet bill remains $(2c_m+1)^4 \le 16(1+c_m)^4$ for each point/representative pair. It is not replaced by a single cover assumption for an entire group.

For each regulator parameter, retain the actual finite-basis coordinates of every moving label. The source section fixes these coordinates, so every finite even-moment expansion, with all scalar coefficients and conjugates, is identical before taking limits. Coefficient products and fixed-label correlations converge on the same refining filter. Its continuum replica expression and subsequent large-displacement limit therefore recover the same ground-algebra moments. In the commutative algebra the resulting norm test is

$$\int_\Theta |\chi(M_E)|^{2h}\, d\nu(\chi) = \langle \Omega_0, (M_E^* M_E)^h \Omega_0\rangle \le c^{2h}, \qquad h \ge 1.$$

For $c > 0$, if a character value exceeded c, continuity would give an open set of positive measure a on which it exceeded some $d > c$. The inequalities $ad^{2h} \le c^{2h}$ contradict $h \to \infty$. For $c = 0$, already the second moment and full support force zero. Thus the Gelfand reader gives $\|M_E\| \le c$. The algebra isomorphism α preserves these moments and norms exactly. This recovers the paid bound for the intact mismatch with its moving labels and original filter, without inserting a separate inner limit for those labels.

The actual local grouping allowances are

$$\lambda_i(H,\ell) = \frac{(\delta_i/H)\delta_i}{\exp((e_i+\frac{1}{2})\ell)}, \qquad W'_\gamma = \prod_{j\in\gamma} \lambda_i(H,\ell) W_j, \quad e_i = \text{supportExponent}_i\,. \tag{OE9}$$

Each occurrence receives $\delta_i W_j$, allocation among at most H costs δ_i/H, and moving support costs the displayed exponential. The fibers partition the old occurrences, so their product is exactly $\lambda_i(H,\ell)^s \prod_j W_j$, as used by OE8. Apply finite marker fusion with these fibers, the cover and occupancy bounds and both smallness inequalities. The original allowances bound the actual local analytic norms and $\delta_i \le 1$ preserves them. Positive background normalization is the actual stage density by the stage representation. Fusion gives local group functions and dominated profiles representing the exactly grouped polynomial, with remainder $4/W'_\gamma$. Pruning to $W'_\gamma/8$ gives $4/W'_\gamma = 1/[2(W'_\gamma/8)]$, the next-step bound. Newly grouped leading counterterms are zero. Translations and reflections of functions/profiles persist because all copies use the same grouping. At the terminal join there is one group per copy and no $1/8$ pruning before normalization: its allowance is $\prod_j \lambda_i(H,\ell)W_j$, with remainder four divided by it.

20.7 Nested chessboards and the intact paid even moment

For two field groups of separation scales a_b, a_ℓ, choose

$$b \le f \le \ell \le n, \qquad f \le Q, \qquad |I| 10 a_f < a_b, \quad |J| 6 a_n < a_\ell.$$

The seating lemma gives the required interior margins in translated reflection chambers. Replica itineraries preserve those margins; within a replica original spacing persists and between replicas distances diverge. At sufficiently large displacement and then fine regulator all lattice separations exceed their chamber scales. At the $2h$ occurrences use adjoints on alternating even positions and unchanged fields on odd positions. Adjoints preserve allowances. The mixed nested-chamber chessboard inequality bounds the integral by

$$\left[\left(\prod_{i\in I} B_i\right)\left(\prod_{j\in J} D_j\right)\right]^{2h}.$$

Register translation and finite-product conjugation identify this with the even moment of the whole registered insertion Z. Finally, the paid even-moment bound can be seen directly from its lattice layout. For a mismatch, Z is that mismatch times the exterior register; the main group has one occurrence per replica with price A. It is never split into two independently estimated fields.

To prove the nested inequality, a paid interior field with chamber root bound A retains both price and support under permitted diagonal reflection/conjugation. A margin of three original chamber scales inside a larger chamber makes its placement a tiling copy of an interior function with the same bound, by the offset/reflection identity. Collect finer fields in each larger cell and copy that product to all larger cells. Their finer cells are distinct: different larger cells cannot meet in one finer cell, and finer placements in a common larger cell are farther apart than its finer cell extent. Reflection preserves this. Apply finer chessboard, then take the root of order the positive number of larger cells. This root extraction is legitimate because the prices and root tests are nonnegative and the number of cells is positive. The collected larger root bound is the product of its nonnegative individual prices. Coarser main fields occupy distinct larger cells, none occupied by finer exterior fields, since otherwise the cell extent contradicts separation. Apply larger chessboard to those main fields and collected products. This gives $\prod B_i \prod D_j$ without replacing the two scales by one. The common seating translation exists by two finite avoidance steps. First translate main points five buffer scales from coarse boundaries, possible by $|I|10a_f < a_b$. Then shift at most one intermediate scale to put exterior points three fine scales from boundaries, possible by $|J|6a_n < a_\ell$. This consumes at most two main buffer scales and leaves three. Nested cell periods divide the larger ones and replica displacements are their integer multiples, so residues and margins persist. For large periods Euclidean distances are at most twice mesh times wrapped lattice separations; subsequent embedding multiplies lattice separation by mesh ratio. Thus all finer cutoff inequalities follow.

One-step transport uses $W'_\gamma = W_\gamma/L_i^4$. Its analytic remainder cost $6L_i$ fits $6L_i \le L_i^4$ for $L_i \ge 2$:

$$(6L_i)/(2W_\gamma) \le 1/(2W'_\gamma).$$

Marker tolls are finite sums of nonnegative allowance products, hence remain bounded after decreasing allowances. Separated representatives give support substitution, occupancy holds before and after parents, and transition density gives the exact descended polynomial. Counterterms preserve their copied scalar transformations and $W'_\gamma|c_\gamma| \le 1$ since $W'_\gamma \le W_\gamma$. Initially the solved representation gives $4/W_\gamma$; choosing $8W'_\gamma \le W_\gamma$ converts it to $1/(2W'_\gamma)$. These are the initialization and transport constants of OE8.

The lifted source retains the anchor-fixing surjection and section, individual connecting covers, convergent label coordinates, regulator filter and replica itinerary. Its moment reader preserves every signed coefficient before both limits, and its nested root reader preserves the two chamber scales and actual mismatch. OE9 and the pruning identity recover the exact grouped polynomial and its next-step bound. These retained data prevent loss of approximation accuracy through representative selection, grouping, or compressed-norm recovery.

Grouping allowances and nested-root recovery. Retain the actual local allowance W'_γ, cover, occupancy, normalized stage density and dominated profiles from (OE9). Their configuration reader is the same local grouped function and its coefficient polynomial. Conditional-integration recovery preserves that positive normalized density before the remainder is read. Pruning is the scalar operation $\widehat{W}_\gamma = W'_\gamma/8$, so the source recovers exactly

$$\frac{4}{W'_\gamma} = \frac{1}{2\widehat{W}_\gamma}.$$

The terminal group retains its stated unpruned allowance instead. Zero leading counterterms and the common copied transformations are part of this same record.

For a finite paid expression Z let $\mathcal{E}_{2h}(Z)$ denote the actual alternating-adjoint replica integrand. Configuration pullback commutes with every finite product and conjugation, and normalized Wilson recovery gives

$$\langle \mathscr{L}\mathcal{E}_{2h}(Z)\rangle_Z = \langle \mathcal{E}_{2h}(Z)\rangle, \qquad |\langle \mathcal{E}_{2h}(Z)\rangle| \le \left(\prod_{i\in I} B_i \prod_{j\in J} D_j\right)^{2h}.$$

The source root reader keeps the two chamber scales separately: first the finer copied-product inequality, then its positive larger-cell root, then the coarser mixed inequality. Each support, reflection, price and seating residue is the same physical datum under readout. For the mismatch Z its main occurrence is retained as one paid expression in every replica.

Finally the next-step allowance W_γ/L_i^4 is unchanged under source recovery. The original estimates $6L_i \le L_i^4$ and $8W_\gamma^{\mathrm{init}} \le W_\gamma$ for the initial allowance recover the transport and initialization bounds respectively. Monotonicity of the finite nonnegative toll sums preserves all subsequent decreases of those allowances. Thus the grouped remainder, nested even moment and paid sweep use one normalized stage and the original numerical constants.

20.8 Quantitative marker fusion and exact restoration

Let Δ map old occurrences onto groups, with an injective representative section. Extend old supports by the chosen connecting covers, each of cost at most ℓ, and set

$$\widetilde{W}_i = (\eta/H) W_i e^{-\alpha\ell}, \qquad \alpha = e + c.$$

An old coefficient on occurrence set A gains $(\eta/H)^{|A|}$ after its cover costs are paid. A group's allowance is the product of $\widetilde{W}_i$ over its full fiber. Distinct occurrences keep distinct nilpotent markers. Eligible products have disjoint old marker sets, so their toll product is exactly the toll of their union and, for complete fibers, the corresponding product of group allowances. With occupancy at most s, scalar weighted bound a, activity bound b and $W_i|c_i| \le T$, the extended inventory envelope is

$$\gamma = 2(1+s)\eta(sTe^{\alpha} + a) + (1+s)b. \tag{OE10}$$

For nonempty A, $(\eta/H)^{|A|} \le \eta/H$. At most H old siblings charge a representative, cancelling H^{-1}. The injective section places representatives at old locations, retaining occupancy s. Reallocation yields the incidence factor $1+s$; two scalar inventories yield the factor 2, and the activity inventory is charged once. At an original site singleton scalars contribute at most sTe^{α}, because singleton cover number is at most one; ordinary nonempty coefficients contribute a, activities b. This proves OE10 without dropping the unit contribution. Finite eligible products are bounded by products of entry envelopes, using the disjoint-marker toll identity. The connected-set forest estimate at excess exponent c applies when $(2B+1)^4\gamma \le c$. At exponent e it bounds both grouped ordinary and activity coefficients by $e^{\gamma}-1$. This is the same connected-set estimate already proved for the general insertion calculus, now with the extended supports and group tolls. Choose $e^{\gamma}-1 \le Z$. Singleton restoration with the normalized nonzero background gives the new function and dominated activity profile, with remainder $2Z/W'_\gamma + 2Z/W'_\gamma = 4Z/W'_\gamma$. The actual empty background activity coefficient is retained, so the profile is admissibly dominated, not merely numerically bounded. Higher coefficients give the simultaneous served representation. The sweep uses $c = 1/2$, $Z = 1$, recovering OE9.

Weighted fusion and singleton recovery. Retain the occurrence-to-group surjection Δ, its representative section and the actual connecting covers. For an old roster A write $t(A) = \prod_{i\in A} \widetilde{W}_i$. The source toll reader has the exact identities

$$t(A)t(B) = t(A \cup B) \quad (A \cap B = \varnothing), \qquad t(\Delta^{-1}(J)) = \prod_{\gamma\in J} W'_\gamma.$$

Thus eligible full-fiber products retain their weighted coefficient price. Configuration pullback fixes the cover costs, occupancy and all scalar allowances; it preserves coefficientwise sup bounds and the same sitewise nonnegative envelope (OE10). The forest reader therefore has identical inputs $(2B+1)^4\gamma \le c$ and $e^\gamma - 1 \le Z$ on both carriers.

Let $P^{\mathrm{ord}}, P^{\mathrm{act}}$ denote the two grouped polynomials. Coefficientwise pullback commutes with their singleton readers:

$$[\zeta_\gamma]\mathscr{L}P^{\mathrm{ord}} = \mathscr{L}[\zeta_\gamma]P^{\mathrm{ord}}, \qquad [\zeta_\gamma]\mathscr{L}P^{\mathrm{act}} = \mathscr{L}[\zeta_\gamma]P^{\mathrm{act}}.$$

The empty-marker activity and actual normalized background are retained by this same map. Consequently restoration recovers both the local function and its admissibly dominated profile, with remainder $4Z/W'_\gamma$, and all higher served coefficients. A prescribed copy motion is a bijection of the retained inventories and covers; reindexing the same finite coefficient products commutes with these singleton and empty-marker readers. Thus copy covariance and the hypotheses for the next grouping survive exactly, including the sweep values $c = 1/2$ and $Z = 1$.

Here is the full inventory identity. For scalar markers c_i, ordinary polynomials A_S and flagged activity polynomials $K_{S,F}$, take one singleton entry for each c_i and one entry for every nonempty coefficient of A_S. Extend each support by representatives of all mentioned groups. Split each scalar value into its value on healthy extended support and its value on unhealthy extended support, zero on the complementary configurations. Their sum is exactly the value. Retain every allowed activity coefficient, extended support and original flagged data. These finite inventories recover $\sum_i c_i z_i + \sum_S A_S$ and each $K_{S,F}$ coefficientwise; locality places all mentioned markers inside their support. Connected eligible collections receive union supports and markers, with their prescribed flagged intersections. Expansion of the finite nilpotent exponential and gas, followed by the unique connected component partition, yields

$$\mathcal{R}_\Delta\left[\exp_z\left(\sum_i c_i z_i + \sum_S A_S\right)\mathrm{Gas}_z(K)\right] = \exp_\zeta\left(\sum_S \widetilde{A}_S\right)\mathrm{Gas}_\zeta(\widetilde{K}). \tag{OE11}$$

Here the full-fiber reader is

$$\mathcal{R}_\Delta(z_A a) = \begin{cases} \zeta_{\Delta(A)} a, & A = \Delta^{-1}(\Delta(A)), \\ 0, & \text{otherwise.} \end{cases}$$

For retained monomials, new rosters are disjoint exactly when old rosters are: a shared group would place its entire nonempty fiber in both. Thus the reader commutes with eligible products, while it is not asserted to preserve arbitrary products involving partial fibers.

Two entries connect if their group rosters meet, or their extended supports meet and both are activity-collecting types (unhealthy scalar or old activity). Ordinary grouped entries are nonempty connected eligible collections with no such type. Grouped activity entries are connected eligible collections containing an old activity, with their flagged intersection equal to the union of the old flagged subsets; the coefficient is the product if this condition holds, zero otherwise. Old activity supports inside each collection must be pairwise disjoint, as in the original gas. An unhealthy scalar

component without any activity contributes zero under a selected old flag pattern: its required flag belongs to an old activity support, which would connect that activity into the component. Hence each contributing component is one of the two grouped types. Conversely selected grouped entries have disjoint group rosters and disjoint activity supports, so their expanded components have no further connections. Union and component partition are inverse operations, preserving coefficients and flagged unions. The separate flag condition holds for each grouped activity: a flag assigned to another would violate disjoint grouped supports. This proves OE11 at every configuration, independently of entry zeros. With no new markers no scalar entry can occur. Two empty-marker activities cannot connect, since their prescribed flagged supports are disjoint and there is no group marker. Thus empty connected collections are exactly single old empty-marker activity entries, preserving both value and original support-and-flag label. Each nonzero coefficient has its marker locations inside its support by locality, so every term required for either recovery is present. This proves preservation of the empty activity coefficient before any bound. Restoring the nonzero background makes OE11 precisely the served-polynomial identity.

All grouped supports contain their representatives. Entry dependence is confined to variables and flags in the extended support, and product dependence to its union. Gauge invariance and measurability persist, as do flag indicators. Nonzero ordinary groups are healthy on their whole support; activities have exactly their recorded flag intersection. Thus the same conditions hold for the next grouping. Copy covariance holds coefficientwise: a focused marker set uses its copy's inventory; the copy permutation is a bijection taking extended supports and coefficients to the prescribed transformed ones. Reindex finite sums and products to obtain covariance of both polynomials, including empty markers and internal interactions, and hence their singleton functions and profiles after repeated regrouping.

Finally extract each fine served singleton. Restoration identifies its normalized local insertion, including the dominated activity correction. Anchor analytic transport gives a coarse function and dominated profile with unchanged leading counterterm and remainder multiplied by the driving constant. Define other copies by the prescribed reflection and translation and reindex activity labels by the actual motion bijection. Conditional integration intertwines these transformations; integrability permits coefficient extraction to commute with it. The marked transport theorem gives the full coarse representation at δI, and support substitution with target weights J, occupancy and duty inequalities restores the served representation at J. All estimates hold for the arbitrary finite occurrence set $V \times I$; no two-occurrence restriction is used. The lifted fusion source is this complete finite inventory, flag pattern and occurrence-fiber map. Its realization is the connected collection polynomial and its observable reader is full-fiber extraction followed by normalized singleton restoration. OE11 proves exact recovery of the grouped served polynomial; retaining the old inventory exposes the partial-fiber information discarded by the reader. OE10 pays the support transformation quantitatively, while empty-coefficient preservation recovers the same background and makes iteration compatible with the sweep.

Inventory bijection and exact served-polynomial recovery. For each retained scalar inventory value a on extended support S, its two routed entries recover pointwise as

$$a_{\mathrm{h}} = 1_{\mathrm{Healthy(S)}} a, \qquad a_{\mathrm{u}} = (1 - 1_{\mathrm{Healthy(S)}}) a, \qquad a_{\mathrm{h}} + a_{\mathrm{u}} = a.$$

Configuration pullback preserves this split and every original support-and-flag label. Let $\mathscr{L}$ act coefficientwise on these finite polynomial inventories. It commutes with the full-fiber reader $\mathcal{R}_{\Delta}$, because neither operation changes the other's coefficient indices. The source realization of (OE11) is therefore exactly its coefficientwise pullback, with the same finite nilpotent exponential and flagged gas.

More explicitly, retain an eligible old collection together with its unique connected-component partition. The forward reader replaces that collection by its ordinary or activity components; the inverse reader takes their union with their recorded old entries. On the selected flag pattern these operations are inverse, preserve the product of scalar coefficients, and preserve the union of flags. The zero contribution of an unhealthy scalar-only component is retained before summing. This gives recovery of every coefficient of (OE11), including the empty coefficient's bijection with single old empty-marker activities. Individual zeros require no division.

Normalized singleton restoration thus uses the same nonzero background on both carriers. At the final one-step transport, coefficientwise pullback and extraction commute with the integrable conditional push; copy reindexing preserves the same activity-motion bijection. The retained marked transport at δI and support substitution at J recover the entire served polynomial, with its dominated profiles and original duty bounds. The occurrence set remains the arbitrary finite set $V \times I$, so this recovery applies to every census used in the sweep.

20.9 Nested expansions on arbitrary finite trees

Let t be a finite rooted tree, with fields of degrees w_ℓ at its leaves, and centers z_v and radii $s_v > 0$ at internal vertices. Zero and one child are allowed. All subtree positions at v lie in radius Ms_v; different immediate child subtrees have separation at least ηs_v. Internal child radii do not exceed parent radii. The root radius is $r \downarrow 0$; no positive lower bound is imposed on child/parent ratios. Bounds on immediate centers imply these full subtree conditions after a small uniform upper ratio choice: if child displacement constants are C_i, set $C' = \sum_i C_i$ and choose $0 < \beta \le 1$, $\beta \le \eta/[4(C'+1)]$, below all previous child bounds. Descendant deviations are at most $C'\beta s_v$, so sibling separation loses at most $2C'\beta s_v \le \eta s_v/2$ and the parent displacement is at most $(M + C')s_v$. Induction gives the geometry. Put $W_v = \sum_{\ell \text{ below } v} w_\ell$ and $q_v = s_v/s_{\mathrm{parent}(v)}$, with virtual root-parent radius 1, so $q_{\mathrm{root}} = r$. For fixed output degree m define

$$\mathcal{T}_{m,\epsilon}(s) = r^{m-\sum_\ell w_\ell - \epsilon} \prod_{\substack{v \text{ internal} \\ v \neq \mathrm{root}}} q_v^{-W_v-\epsilon}. \tag{NE1}$$

Fix every internal truncation D_v before taking the limit. The direct coefficient reads the full leaf cluster at the root; the iterated coefficient literally multiplies and sums the same homogeneous OPE channels up the tree through the assigned child horizons, ending in the fixed root channel (m, α). Uniformly over admissible geometry and positive radii, for small enough r,

$$\begin{aligned} |C_{\mathrm{direct}}| &\le \mathcal{T}_{m,\epsilon}, \\ |C_{\mathrm{direct}} - C_{\mathrm{iterated}}| &\le \mathcal{T}_{m,\epsilon} \sum_{\substack{v \text{ internal} \\ v \neq \mathrm{root}}} q_v^{D_v+1}. \end{aligned} \tag{NE2}$$

With no nonroot internal vertices this sum is zero and equality is exact. The root horizon does not enter the sum, since its output is already fixed. With exterior positions separated from all tree positions and each other by $\delta > 0$, the common ground-algebra norm of the full leaf product minus its direct root expansion through m is at most

$$\mathcal{T}_{m+1,\epsilon}(s). \tag{NE3}$$

This serves both vacuum and every character. All finite expressions are smooth on their common open reader domain; for fixed requested degrees the tree geometry eventually belongs to it, by shrinking diameters and sibling distinctness recursively.

Suppose first the radii have rates $-a_v$, nondecreasing down the tree, and set $\Delta_v = a_v - a_{\mathrm{parent}(v)} \geq 0$, with virtual exponent zero. The engineering cost is

$$\mathcal{E} = \sum_{v \text{ internal}} W_v \Delta_v. \tag{NE4}$$

Telescoping along each leaf path also writes this as its degree times its nearest internal parent's exponent, summed over leaves. Direct output rate is $\mathcal{E} - m a_{\mathrm{root}}$, tested remainder rate $\mathcal{E} - (m+1)a_{\mathrm{root}}$. NE1 adds slack $\epsilon \sum_v \Delta_v$ to this engineering rate. When $\sum_v a_v = 1$, this slack is positive: some $a_v > 0$ and each a_v is bounded by the sum of all nonnegative advances. It absorbs fixed constants even when individual gaps vanish.

Finite-tree channel and price recovery. Retain the complete finite rooted tree, all positive radii and centers, leaf labels, fixed internal horizons and the already chosen homogeneous channel bases. Lift each finite coefficient carrier by its fixed section. Every node coefficient is unchanged because the homogeneous coordinate reader recovers the same scalar C. Thus each term in the finite iterated coefficient, indexed by the retained intermediate degrees and channels, has the identical product on both carriers. Summing those same indices gives

$$C^Z_{\mathrm{direct}} = C_{\mathrm{direct}}, \qquad C^Z_{\mathrm{iterated}} = C_{\mathrm{iterated}}, \qquad \mathcal{T}^Z_{m,\epsilon}(s) = \mathcal{T}_{m,\epsilon}(s).$$

The tree reader keeps every child-to-parent ratio as an independent positive geometric datum. It introduces no lower ratio bound, permits zero and one child, and retains the fixed root output outside the nonroot error sum. The tested root remainder is lifted by α, so its norm and all vacuum/character evaluations recover (NE3).

The engineering identity behind this recovery is the exact finite path interchange

$$\sum_v W_v \Delta_v = \sum_\ell w_\ell \sum_{v \text{ on the internal path to } \ell} \Delta_v = \sum_\ell w_\ell a_{\mathrm{parent}(\ell)}.$$

Substitution in (NE1) gives its stated engineering rate plus $\epsilon \sum_v \Delta_v$. If the tree has $n_{\mathrm{int}} > 0$ internal vertices and $\sum_v a_v = 1$, then

$$1 = \sum_v a_v \leq n_{\mathrm{int}} \sum_v \Delta_v, \qquad \sum_v \Delta_v \geq 1/n_{\mathrm{int}} > 0.$$

This explicit retained slack absorbs fixed finite channel-count and map constants even when individual advances vanish. The same physical geometry and finite admissible reader domains are read at each node, so the resulting bounds preserve all of (NE1)–(NE4)'s stated finite-tree and finite-horizon scope.

The finite recursive proof retains leaves as actual labels of their actual degrees, with coefficient one; a bud is not a degree-zero projection, a convention that includes arbitrary input labels correctly. Child alternatives through D are its homogeneous fields of degrees $0, \ldots, D$, weighted by direct coefficients. Parent alternatives multiply these coefficients by the OPE coefficient of those resulting fields. Inductively every raw subtree has (i) a finite comparable family at its parent's exponent, to any prescribed tested accuracy, whose amplitudes pay engineering cost minus resulting degree times that exponent, and (ii) its direct coefficient and full tested remainder estimates at its own exponent. Leaves satisfy both. Products of child families add costs as in NE4 and sibling separation makes their positions admissible. If a child has the same exponent as its parent retain its comparable family. If it is strictly larger, use its second assertion to replace it through a sufficiently high

auxiliary finite horizon H. For child exponent $a > k$ with cost Ω and later exterior bound Wb, replacing through auxiliary finite H has error rate $\Omega-(H+1)(a-k)+Wb$. For requested ρ, choose $H > (\Omega+Wb-\rho)/(a-k)$, above all earlier requested horizons. Replace children in order, choosing each new horizon after the earlier finite degrees and their exterior costs are fixed. The exact telescoping sum has one error at a time, avoiding circular horizon choices. Finite readers transfer these tested errors to coefficients. Apply the single-center estimate to the resulting comparable family to complete both parent assertions. For prescribed truncations choose gain $\gamma \geq 0$ satisfying

$$\gamma \leq (D_v + 1)\Delta_v \quad \text{at every nonroot internal } v. \tag{NE5}$$

The exact product difference

$$\prod_j x_j - \prod_j y_j = \sum_j (x_j - y_j) \prod_{i<j} y_i \prod_{i>j} x_i$$

transfers one gain γ, not its multiple. For $\gamma > 0$ all nontrivial immediate child gaps are positive. Replace through auxiliary $H_j \geq D_j$ to arbitrarily good accuracy. The auxiliary sum minus the desired sum is the finite omitted-channel sum; every term has a child degree at least $D_j + 1$, gaining $(D_j+1)\Delta_j \geq \gamma$ between its coefficient cost and the parent's returned degree. Other degree-gap contributions are nonnegative. Recursive replacement of retained direct child coefficients gains the same γ by the product identity. The three exact differences therefore give

$$\text{direct-to-iterated error rate} \leq \mathcal{E} - ma_{\text{root}} - \gamma. \tag{NE6}$$

For $\gamma = 0$, bound direct and iterated terms by the common raw rate; no high-accuracy substitution is needed at zero gap. With no nonroot internal vertices every child choice has coefficient one and permutation invariance gives exact equality. Take γ as the minimum of $(D_v + 1)\Delta_v$; an attaining vertex contributes ratio power of rate $-\gamma$. Positive NE1 slack absorbs constants; its one term is bounded by the NE2 sum. Direct/remainder rates give the remaining claims, and contractivity plus the character supremum gives NE3.

Recursive finite-family and truncation-gain recovery. A source alternative retains its actual ordered word, positions and scalar coefficient. Lift its field labels by the fixed coefficient sections and leave that scalar unchanged. Products and finite sums of alternatives then recover term by term under the same correlation reader. In particular a leaf retains its original degree and coefficient one, and a parent uses exactly the same intermediate channel indices. For a strict child exponent gap $a - k > 0$, one explicit allowed auxiliary choice is any integer

$$H \geq \max\left\{H_{\text{previous}}, 0, \left\lfloor \frac{\Omega + Wb - \rho}{a - k} \right\rfloor + 1\right\}.$$

It recovers the required strict error rate and is chosen after the preceding finite alternatives and exterior costs. Equal-gap children keep their comparable families and require no such division.

Let x_j, y_j be direct and recursively composed child scalar coefficients. The source product reader preserves the exact finite identity

$$\prod_j x_j - \prod_j y_j = \sum_j (x_j - y_j) \prod_{i<j} y_i \prod_{i>j} x_i.$$

Every term contains one changed coefficient, so it retains one gain γ under (NE5). For an omitted channel of degree $h_j \geq D_j + 1$, the finer coefficient cost and parent return have net gain $h_j(a_j -$

$a_{\mathrm{parent}}) \geq (D_j + 1)\Delta_j$. All other advances are nonnegative. Thus the same three finite differences recover (NE6), including its $\gamma = 0$ branch and the exact case with no nonroot internal vertices. Choosing an attaining minimum vertex and then the positive NE1 slack recovers the original sum of ratio powers in (NE2). The corresponding algebra remainder reader is isometric, giving the same all-phase (NE3) bound.

Here are the finite-family quantifiers used above. An alternative is a finite word at selected subtree positions times a scalar of finite upper rate. Eligibility at b means positions separated at that exponent. Agreement to rate ρ up to exterior weight W means the full correlation difference has that rate against every fixed exterior list of weight at most W, arbitrary admissible position paths and viewer paths. A supply at parent exponent k and cost M provides such comparable families for sufficiently large exterior exponent b and every W, ρ, in the exact continuum hierarchy. To replace one branch, fix the other branches' alternative, of degree d and amplitude rate ζ. Request weight $W + d$ and accuracy $\rho - \zeta$ for the changed branch; multiplication restores rate ρ. Finite maxima over all other alternatives suffice before the replacement. Sibling separation ensures admissible augmented exteriors. Successive replacements use only previous choices and unchanged later branches; transitivity adds complete differences. For coefficient transfer choose W covering every finite detector word through m. Rotating detector words retain degree and bounded coordinates and remain a fixed positive distance apart and from the shrinking cluster. Finite detector sums, compact rotation averaging, real averaging, K_m and a coordinate reader preserve agreement rate ρ. Uniformity under arbitrary rotation paths follows as in OE4. For complete remainder transfer an extra debit is needed, which is why agreement of products alone would not suffice: if desired pre-exterior rate is $\rho_0 = M - (m + 1)(k - u)$, $0 \leq u \leq k$, request replacement agreement $\rho_0 - mu$, covering actual exterior and detector weights. The product difference has rate at most ρ_0. Degree-j coefficient differences cost at most $\rho_0 - mu + ju + |e|b \leq \rho_0 + |e|b$. Add the replacement remainder and subtract all these coefficient differences to recover exactly the original full remainder. For a comparable alternative of degree d, amplitude rate $M - dk$ and geometry exponent $k > 0$, its degree-m coefficient costs $(d - m)k$, giving $M - mk$. Include its fields in the exterior test of the changed branch. At $k = 0$, finite detector words cost zero, so its amplitude times their correlations has rate M; comparison with the empty family gives the same coefficient bound. Write each rotated reader as its finite sum of these tested correlations times its fixed weights and rotated loaded vectors. Actual shrinking ensures eventual admissibility even at zero exponent. For $u < k$ multiply the single-center remainder by the amplitude, giving $M-(m+1)(k-u)+|e|b$. At $u = k$, separately bound the full correlation and each finite centered term; engineering degrees cancel to $M + |e|b$. Finite summation covers both boundary cases.

Finite-family agreement and whole-remainder recovery. Retain each alternative as its ordered word, scalar amplitude and geometric path. Its lift preserves every complete correlation against the same exterior labels and viewer. Hence agreement to rate ρ through weight W is preserved with all its original universal path quantifiers. For a fixed other-branch alternative of degree d and amplitude rate ζ, requesting $(W+d, \rho-\zeta)$ before the replacement gives exactly rate ρ after multiplication. Taking the finite maxima first recovers the triangular branch procedure. The coefficient reader uses the same finite detector words, normalized rotation average, real average and homogeneous coordinate maps; its recovery therefore preserves that agreement rate as well.

Write $\mathcal{R}_m(F)$ for the complete tested remainder of a family F, and $C_j(F)$ for its centered degree-j coefficient field, with the fixed homogeneous basis sum understood. For a comparable

replacement G the exact readout identity is

$$\mathcal{R}_m(F) - \mathcal{R}_m(G) = \mathrm{Corr}(F - G) - \sum_{j=0}^{m} \mathrm{Corr}(C_j(F) - C_j(G);\ \text{exterior}).$$

Both source and physical readers retain this subtraction before any estimate. With requested agreement $\rho_0 - mu$, its jth centered term has upper rate

$$\rho_0 - mu + ju + |e|b = \rho_0 - (m - j)u + |e|b \le \rho_0 + |e|b.$$

This recovers the precise debit needed for the full remainder, not only the product. For the boundary cases the same scalar rate identities give $M - mk = M$ at $k = 0$ and $M - (m + 1)(k - u) + |e|b = M + |e|b$ at $u = k$. The source keeps actual shrinking and eventual admissibility in the zero-exponent case, so no positive gap or degree loss is inserted in these cases.

No prior existence of logarithmic rates is required. For any approaching admissible family, every internal radius tends to zero; set

$$T = \sum_{v \text{ internal}} \log(1/s_v).$$

Eventually $T > 0$ and tends to infinity. The normalized log-radius vector lies in a finite compact cube and has a limit a on a nontrivial refining filter, with $\sum_v a_v = 1$. Radius monotonicity forces $a_v \ge a_{\mathrm{parent}(v)}$ by exponential-rate comparison. Geometry gives the size rates and actual shrinking; sibling separation gives separation at each strictly larger exponent. The proof applies even when some $a_v = 0$, with positive total slack. If a uniform small-radius threshold failed, select violating arrays at arbitrarily small radii. Extracting a refining filter of those same arrays yields eventual validity, a contradiction. Include exterior geometry and a violating viewer in the selection when necessary. Thresholds thus depend only on the fixed tree, labels, horizons, $M, \eta, \epsilon, \delta$, without lower ratio bounds.

For any measure space (K, μ) and complex $u \in L^1$ with $\int |u| d\mu \le B$, take an almost everywhere measurable admissible plot map at one fixed radius array. Smooth coefficients on the common open domain, zero extended outside, are measurable. If $|g| \le C$ where $u \ne 0$, then

$$|ug| \le C|u|, \qquad \left| \int ug \, d\mu \right| \le BC. \tag{NE7}$$

Apply to the direct coefficient and its direct-minus-iterated difference with NE2's two bounds. Both are integrable, hence so is the iterated coefficient, and subtraction proves the weighted NE2. No product measure or absolute continuity assumption is used. Jointly varying exterior positions and characters are allowed for the tested remainder: smooth position dependence and continuous character evaluation on the compact metrizable character space give joint measurability on the guard domain. Apply NE7 with $C = \mathcal{T}_{m+1,\epsilon}$. The raw correlation has bound $\mathcal{T}_{0,\epsilon}$ by adding its degree-zero terms and remainder with their engineering rates, so it too is integrable. The truncation is a difference of integrable expressions, its finite terms can be integrated, and the integrated full discrepancy is bounded by $B\mathcal{T}_{m+1,\epsilon}$. Probability phase mixtures use $u = 1$, $B = 1$, with identical coefficients outside the phase integral. Covariance concerns the full homogeneous vectors, not invariant scalar basis channels. Leaves emit their actual labels. Internal vertices include the child emissions into their fixed horizon spaces, apply the multilinear h_j at their centers, and sum through their own horizon. Multilinearity gives exactly the finite channel formula. Induction using the laws of h_j and inclusions gives translation, conjugation and orthogonal covariance; applying only h_m at the root gives the degree-m law. The basis can transform nontrivially within H_m; no assertion that

its vectors are individually rotation invariant is needed. The full tested discrepancy inherits these laws.

The lifted nested object retains the finite rooted tree, all radii, centers, actual leaf degrees, fixed horizons and full homogeneous vectors. Its iterated reader is their finite multilinear substitution; its direct reader retains the full cluster with the same fixed basis. The exact telescoping comparison retains the residual between readers, with NE2 quantifying recovery at arbitrary positive ratios. Keeping that residual recovers the original coefficient exactly; NE3 similarly retains the tested whole-word remainder. Filter normalization proves uniform reader bounds without a scale-gap assumption. NE7 transports these same readers under joint weighted integration and phase mixture.

Normalized radii, weighted readers and vector recovery. For the nested source record z, retain its radii unchanged under the physical readout p and section j. In particular the finite vector

$$a_v(T) = \frac{\log(1/s_v)}{T}, \qquad \sum_v a_v(T) = 1, \qquad a_{\text{virtual parent}} = 0$$

is literally the same function of z, pz and jpz. Its compact filter limit therefore needs no new choice of physical scales. For v below its parent, $s_v \le s_{\text{parent}(v)}$ gives $a_v(T) \ge a_{\text{parent}(v)}(T)$ eventually, and hence the same inequality in the limit. Each radius has rate $-a_v$ relative to T; even at $a_v = 0$ its actual convergence to zero remains part of the retained geometry. The geometric size estimate Ms_v gives the required envelope rate at vertex v, and also actual convergence of every descendant displacement to zero. The bounds Ms_v and ηs_v retain, respectively, the descendant envelope and separation at every exponent strictly above a_v. Induction over the tree thus gives all the regularity hypotheses of the preceding rate proof, including vertices with $a_v = 0$. With $\Delta_v = a_v - a_{\text{parent}(v)}$, the preceding finite-tree argument retains $\epsilon \sum_v \Delta_v > 0$. Selecting a violation before compact extraction retains that very position, exterior and viewer record; its refined eventual bound contradicts the selected violation. This is the recovery of the common threshold, including all-viewer quantifiers and arbitrarily small positive ratios.

Write C_m^{dir} and C_m^{it} for the direct and iterated readers. Their lifted versions satisfy $C^Z = C \circ p$ and $C^Z \circ j = C$. For any measurable plot $z(i)$ the exact identities

$$\int_K u(i) C^Z(jz(i))\, d\mu(i) = \int_K u(i) C(z(i))\, d\mu(i), \qquad \left| \int_K u (C_m^{\text{dir}} - C_m^{\text{it}})\, d\mu \right| \le B\mathcal{T}_{m,\epsilon} \sum_{v \ne \text{root}} q_v^{D_v+1}$$

use the same fixed radii and the same zero extension on the common open domain. Domination by $C|u|$ supplies integrability before subtraction. Joint position and character readouts similarly give $B\mathcal{T}_{m+1,\epsilon}$ for the complete tested discrepancy; the raw bound $\mathcal{T}_{0,\epsilon}$ then makes its finite truncation integrable. Thus finite coefficient sums and probability phase integrals retain their order and coefficients.

For a subtree rooted at v, denote its emitted vector by e_v and its fixed horizon by D_v. With the prescribed inclusions understood, the recursive reader is exactly

$$e_v = \sum_{j=0}^{D_v} \iota_j\, h_j\big((x_w)_{w \text{ child of } v}; (\iota_w e_w)_{w \text{ child of } v}\big), \qquad e_\ell = \text{the actual leaf label.}$$

The lifted recursion uses $h_j^Z = j_j h_j(p\,\cdot)$ and the same inclusions, so induction gives $pe_v^Z = e_v$ and $e_v^Z = je_v$ for section inputs. Multilinearity recovers every finite channel term. If $\rho_j(R)$ is the actual orthogonal action on H_j, use $\rho_j^Z(R) = j_j \rho_j(R) p_j$ on its horizontal image. The same induction intertwines these actions, translations and conjugation. At the root retain only h_m to recover the specified degree-m output. This transports the full homogeneous vector and its tested discrepancy while allowing basis channels to mix under $\rho_m(R)$.

21 Extension across collision sets

Why collision data must be retained. Separated correlations alone do not specify a distribution on a collision diagonal: two extensions can differ by a distribution supported there. The raw record consequently includes the subtraction windows, finite Taylor orders, collision charts, normal directions and forest choices used by the construction. The readout is the resulting Taylor-subtracted pairing for those devices.

Cancellation is established inside each combined shell expression before its integral norm is bounded. Absolute L^1 summability then justifies Fubini and passage through the finite forest overlaps. Integrating separately divergent pieces would destroy that argument. The finite cutoffs and compatible projections below show how to handle arbitrary labels using the same prescribed extension. What is recovered is thus a complete extension with declared devices and finite horizons, not independence from arbitrary alternative subtraction prescriptions.

21.1 Collision extensions with specified subtraction data

The contact conclusion is a Euclidean distributional extension using the same reader and a specified subtraction system. Let $P : E \to (\mathbb{R}^4)^k$ be a real linear isometry, with Haar volume on finite-dimensional E. An admissible forest records compatible collision groups, local continuation charts, support windows, finite Taylor orders and projection choices. For each such forest and compact smooth chart test the construction gives an integrable Taylor-subtracted pairing, equal to its prescribed annular expansion with absolute summability in integral norm. Finite ordered iterations of centered collision groups have the same property. Away from total collapse, annuli recover the punctured correlation. On permitted isometric collision planes containing the collision directions the endpoint has the stated finite total test order. Devices carry the hierarchy's actual rotation action; projection bases are the OPE reader prescription. A real system basis and compatible positive thresholds exist simultaneously. Contact values are therefore attached to these devices and forest conditions, not asserted independent of all subtraction choices.

For a nonempty group g, define

$$N_g^{\mathrm{full}} = \{x : x_j = 0 \ (j \notin g),\ \sum_{j\in g} x_j = 0\}, \qquad N_g = P^{-1} N_g^{\mathrm{full}}.$$

Admissibility requires $N_g^{\mathrm{full}} \subset \operatorname{ran} P$, so P is an isometry of these normals. Their orthogonal complements contain collapsed configurations and the centroid. Omitting one member identifies the full normal space linearly with $(\mathbb{R}^4)^{|g|-1}$, the omitted coordinate being minus their sum; hence

$$\dim N_g = 4(|g| - 1) =: \nu_g. \tag{CE1}$$

Disjoint groups have orthogonal normals. Centroids of disjoint nonempty proper groups differ almost everywhere in the normal variables: equality is the kernel of a nonzero linear map, nonzero by opposite vectors at one member of each group. Its proper linear kernel has Haar measure zero. This supports almost-everywhere shell identities without assigning punctured readings on a coincidence plane. With orthogonal projection π_g, normal Taylor terms are

$$T_{g,j} f(x) = \frac{1}{j!} D^j f(x - \pi_g x)[\pi_g x, \dots, \pi_g x].$$

For fixed $s > 0$, let the prescribed radial smooth χ_s equal one near zero and vanish for normal norm at least $2s$. Put

$$\mathsf{T}_{g,s,m} f = \chi_s(\pi_g x) \sum_{j=0}^{m} T_{g,j} f, \qquad \mathsf{R}_{g,s,m} = 1 - \mathsf{T}_{g,s,m}. \tag{CE2}$$

Tangential variables remain parameters. This depends only on normal jets through m along $N_g^\perp$ and vanishes for tests supported away from that subspace. Its support lies in $\{x : x - \pi_g x \in \operatorname{supp} f,\ \|\pi_g x\| \le 2s\}$. For orthogonal normals projections annihilate each other, derivatives commute and cutoffs use separate variables; Taylor and remainder operators therefore commute for disjoint groups. These facts follow directly from the displayed derivative formula and the cutoff.

For input degrees d_j, define $\omega_g = \sum_{j\in g} d_j$ and $L_g = \max(0, \omega_g + 1 - \nu_g)$. Resolve the expression into degrees $h = 0, \dots, L_g - 1$ and its remaining component using the fixed field reader. Pair the remaining component with f, and the projected degree-h term with the normal remainder through $L_g - 1 - h$. For these finite choices v, gains are $G_v = L_g$ and $G_v = h$ respectively; after normal integration,

$$e_v = \nu_g + G_v - \omega_g. \tag{CE3}$$

The remaining branch has $e_v \ge 1$, including $L_g = 0$. A projected branch has $e_v + (L_g - 1 - h) = 0$. At shell radius $a \le 1$, Taylor remainder supplies the additional order $a^{L_g - h}$, giving

$$a^{e_v} a^{L_g - h} = a. \tag{CE4}$$

For $a > 1$ the cutoff bound uses only $a^{L_g - 1 - h}$ and the product is exactly one. The remaining branch meets only shells up to the fixed geometric reserve, where its a^{e_v} is bounded. This is the near/far integrability gain.

A uniform local chart retains a partition Q into disjoint nonempty parts, length $s > 0$, collapsed offset, linear resizing frame A and a closed stage in a fixed bounded resized set. Part radii $R_i \in (0, 1]$ scale their normal directions. A common detector collar $B \in (0, 1]$ contains the normal extent and subtraction halo, with $2s \le B$; nonsingleton radii are at most a fixed multiple of s. Require $B \le \text{clearance}(\sum_j d_j)$, decreasing with degree and hence covering all projection degrees. The continuation of other groups, centers and fields is constant along each current part's normal directions. Ancestor-subtracted expressions are thus retained as exterior continuations with their cancellations intact. Refine a nonsingleton i by proper partition U on an angular annular patch, with new radius a and other radii fixed. Compose A with normal multiplication by a/R_i, identity on $N_i^\perp$. Then

$$|\det A'| = |\det A| (a/R_i)^{\nu_i}. \tag{CE5}$$

Engineering fields contribute $(a/R_i)^{-\omega_i}$, and continuation gain contributes $(a/R_i)^{G_v}$; together these are exactly $(a/R_i)^{e_v}$. Keep both the old stage and its normally collapsed, cutoff-bounded part, since Taylor supports may enter the halo. Orthogonality preserves other normals. The continuation remains locally continuous: Taylor acts within the detector collar and the angular patch separates distinct part palettes.

The lifted contact source is the isometric plane together with its normal spaces, Haar volume, ordered forest, fixed reader projections, cutoff jets and resizing frames. CE2 is its test readout, paired with the matching projected field component; CE5 transports its volume and CE3–CE4 preserve its scale price. Retaining subtraction jets exposes the contact data lost by restricting to punctured configurations. Restriction of the eventual extension recovers the punctured reading; contact recovery additionally requires these actual devices, not only the punctured observable.

Collision-plane and normal-jet recovery. The contact record includes the actual isometry P, Haar volume dx_E, admissible forest, fixed homogeneous field projections and all cutoff and continuation choices. For its horizontal coordinate copy write $J_E : E \to E^Z$ and $Q_E : E^Z \to E$, with $Q_E J_E = 1$ and $E^Z = J_E E$. Equip this copy with the transported inner product and volume

$(J_E)_* dx_E$. These are horizontal coordinates, not an integration over additional redundant directions. Define

$$P^Z = PQ_E, \quad N_g^Z = J_E N_g, \quad \pi_g^Z = J_E \pi_g Q_E, \quad f^Z = fQ_E, \quad \chi_s^Z = \chi_s Q_E.$$

Then $(\pi_g^Z)^2 = \pi_g^Z$, $\pi_g^Z J_E = J_E \pi_g$ and $\dim N_g^Z = \nu_g$. Orthogonality and the zero-measure coincidence kernels are transported by the same isometry. In particular the omitted-coordinate parametrization used for CE1 is only linear; its coordinate Lebesgue measure is not silently substituted for the induced Haar volume.

The chain rule gives, for horizontal arguments $J_E u_i$,

$$D^j f^Z(J_E x)[J_E u_1, \dots, J_E u_j] = D^j f(x)[u_1, \dots, u_j], \qquad T_{g,j}^Z f^Z(J_E x) = T_{g,j} f(x).$$

Thus $\mathsf{T}_{g,s,m}^Z f^Z = (\mathsf{T}_{g,s,m} f) Q_E$ and likewise for R. The collapsed base point, normal jet order, cutoff halo and tangential parameters are all recovered. For disjoint groups the conjugated projections still annihilate one another, so the commuting Taylor and remainder identities remain exact. The same identities transport the finite ordered forest operations with their prescribed order.

If F_g is the current group field expression, let $\Pi_h F_g$ denote its fixed degree-h reader component and put $F_{\mathrm{rem}} = F_g - \sum_{h<L_g} \Pi_h F_g$. With the exterior continuation held fixed, the contact pairing has the precise form

$$\mathcal{C}_g(F_g, f) = \langle F_{\mathrm{rem}}, f \rangle + \sum_{h=0}^{L_g - 1} \langle \Pi_h F_g, \mathsf{R}_{g,s,L_g-1-h} f \rangle.$$

An empty sum at $L_g = 0$ is retained. Transporting each component, test and horizontal volume gives $\mathcal{C}_g^Z(F_g^Z, f^Z) = \mathcal{C}_g(F_g, f)$ on the source's admissible charts. In the projected branch, $L_g = \omega_g + 1 - \nu_g$, so explicitly

$$(\nu_g + h - \omega_g) + (L_g - h) = 1, \qquad (\nu_g + h - \omega_g) + (L_g - 1 - h) = 0.$$

These recover respectively the near-shell power a and far-shell power 1, while the remaining branch has exponent at least one. For resizing $S_i = (a/R_i)\pi_i + (1 - \pi_i)$, its lifted frame is $J_E A S_i Q_E$; its determinant in transported orthonormal coordinates is $\det A\,(a/R_i)^{\nu_i}$. The unchanged engineering and continuation factors then recover CE5's total $(a/R_i)^{e_v}$, including the old stage and collapsed cutoff halo. Consequently this lift preserves the stated contact devices and scope before the subsequent shell summability argument.

21.2 Uniform contact budgets and shell refinement

For a current partition Q of g choose

$$\tau_Q = 3^{|g|-|Q|}, \qquad \varepsilon_Q = \tau_Q \gamma, \qquad \gamma > 0, \quad (\tau_Q + 1)\gamma \le 1.$$

A proper refinement Q' increases the part count, so

$$2\varepsilon_{Q'} + \gamma \le \varepsilon_Q. \tag{CE6}$$

For chart continuation bound B_w, frame A_w, and all recorded radius registers $R_j(w)$, including unchanged ones, put

$$\mathcal{B}_w(\varepsilon) = |\det A_w| B_w \prod_{i \in Q} R_i(w)^{-\omega_i} \prod_j R_j(w)^{-\varepsilon}. \tag{CE7}$$

A dyadic restriction of part i to shell number n_i supplies $2^{-n_i\gamma}$; an unrestricted part supplies one. With $N_Q = (2+\sum_j d_j)(|g|-|Q|)$, every permitted projected/remaining choice and shell restriction has an integrable complete integrand F:

$$\int_E \|F(x)\|dx \le C\|f\circ A_w\|_{C^{N_Q}}\mathcal{B}_w(\varepsilon_Q)\prod_{\substack{i\in Q\\ |i|>1}} 2^{-n_i\gamma}. \tag{CE8}$$

The constant is uniform after geometric reserves, degree bounds and continuation estimates are fixed. The norm is taken after combining the finite subtraction expressions, preserving their cancellations. Under refinement the child's budget is the parent's at $\varepsilon_{Q'}$, multiplied by $(a/R_i)^{e_v}a^{-\varepsilon_{Q'}}$. CE4 and its bounded far-shell counterpart make the Taylor-adjusted factor at most a reserve constant times $(a/R_i)^t$ for every $0\le t\le 1$. Choose $t=\gamma+\varepsilon_{Q'}\le 1$:

$$(a/R_i)^{\gamma+\varepsilon_{Q'}}a^{-\varepsilon_{Q'}} = a^\gamma R_i^{-(\gamma+\varepsilon_{Q'})}.$$

CE6 absorbs the radius loss in the parent's surcharge. Other shell factors stay unchanged, and each new annulus contributes the summable a^γ. Proper partitions strictly decrease $|g|-|Q|$, giving finite induction and the derivative count N_Q.

The shells are

$$\psi_{g,a} = \chi_a(\pi_g x) - \chi_{a/2}(\pi_g x), \qquad a_n = 2^{-n}, \qquad \operatorname{supp}\psi_{g,a}\subset\{a/2\le\|\pi_g x\|\le 2a\}.$$

Their telescoping identity is

$$\sum_{n<N}\psi_{g,a_n} = \chi_1 - \chi_{a_N}. \tag{CE9}$$

For tests supported at normal norm at most one and any nonzero normal point, multiplying by this right side is eventually equality in a whole neighborhood, including every derivative used by the finite Taylor operations. On the compact unit normal annulus group labels by equal positions. This is a proper partition since total collapse is absent. Subcentroids in a part coincide; those in different parts are distinct. The latter is open, and the finitely many proper partitions/subgroups give an open cover. Choose a smooth finite partition of unity subordinate to it. Its rescaled angular factors times $\psi_{g,a}$ give $\psi_{g,a,U}$ summing to $\psi_{g,a}$. On each compact mask support different-part subcentroids have a positive minimum distance at radius one, hence distance at least $c_U a$ after rescaling. This supplies reader separation also for all subgroups used in subsequent refinement. The lifted shell record retains the full radius register, finite angular atlas and combined Taylor expression. The annular reader CE9 is eventually exact off the collapsed normal plane; CE8 pays its integral norm before summing. CE6 proves that refinement preserves that paid budget, so iterated readout recovers the same tested pairing rather than separately bounding terms whose cancellation supplies integrability.

Exact budget and annular-mask recovery. Retain each radius register as a separately named entry in the contact source, even when its value is unchanged by a refinement. The section and readout keep these entries, B_w, the frame determinant and all input degrees fixed. Thus $\mathcal{B}^Z_w(\varepsilon) = \mathcal{B}_w(\varepsilon)$, and the transported horizontal norm gives

$$\int_{E^Z}\|F^Z\|\,d(J_E)_*x_E = \int_E\|F\|\,dx_E, \qquad \|f^Z\circ(J_E A_w Q_E)\|_{C^{N_Q}} = \|f\circ A_w\|_{C^{N_Q}}.$$

Here derivatives and operator norms on the horizontal copy use the transported orthonormal frame. This recovers CE8 with its complete combined integrand; individual unsubtracted terms are not substituted for F in this equality.

For a proper refinement set $d = |g| - |Q|$. Then $\tau_{Q'} \le \tau_Q/3$ and $\tau_Q \ge 3$, whence

$$2\varepsilon_{Q'} + \gamma \le (2\tau_Q/3 + 1)\gamma \le \tau_Q\gamma, \quad t = \gamma + \varepsilon_{Q'} \le 1, \quad \varepsilon_Q - \varepsilon_{Q'} \ge t.$$

Consequently for $0 < R_i \le 1$ the parent's extra surcharge obeys $R_i^{-(\varepsilon_Q - \varepsilon_{Q'})} \ge R_i^{-t}$. Every other retained radius register contributes a factor at least one to this surcharge ratio. Combined with the exact identity

$$(a/R_i)^t a^{-\varepsilon_{Q'}} = a^\gamma R_i^{-t}, \qquad \sum_{n\ge 0} (2^{-n})^\gamma = \frac{1}{1 - 2^{-\gamma}},$$

this explicitly recovers the radius absorption and convergent shell price used in the finite partition induction. No unchanged register is removed to obtain this estimate.

For each fixed finite angular partition let its normalized masks be θ_U, so $\psi_{g,a,U} = \theta_U \psi_{g,a}$ with the stated rescaling understood. Pull back the actual masks along Q_E. Then, as identities of smooth horizontal functions,

$$\sum_U \psi^Z_{g,a,U} = \psi^Z_{g,a}, \qquad \sum_{n<N} \psi^Z_{g,2^{-n}} = (\chi_1 - \chi_{2^{-N}})Q_E.$$

The normal distances are preserved by J_E, so the same positive $c_U a$ separates all prescribed different-part subcentroids. The source retains the eventual equality of germs off $N_g^\perp$, including every finite jet entering later operations. This retains the actual angular partition and shell readers, rather than only their support bounds.

21.3 Uniform Taylor bounds and absolute contact resolution

On a fixed compact unit annulus, with derivatives of f bounded through $n+m+1$, normal rescaling by a gives

$$\|\mathsf{R}_{g,s,m} f\|_{C^n \text{ after rescaling}} \le C\|f\|_{C^{n+m+1}} \begin{cases} a^{m+1}, & 0 < a \le 1, \\ a^m, & a > 1. \end{cases} \tag{CE10}$$

The parent scale R obeys $R \le Bs$, $Ra \le Bs$, with the fixed normalized support bound. All derivatives through n of $\chi_1(u\pi_g x)$, $0 \le u \le B$, have a common bound by compact smoothness. The rescaled normal Taylor polynomial is bounded by $C\max(1, a)^m$; the rescaled original function has uniform derivatives on its compact support and is zero as a germ beyond it. Leibniz gives the far bound. For $a \le 1$, if $3Ra/s \le 1$ the cutoff is identically one near the annulus, so integral Taylor remainder gives a^{m+1}. Otherwise $1 \le 3Ba$, and multiplying the rough bound by $(3Ba)^{m+1}$ proves the same near bound. Fixed normalized angular mask derivatives add only a constant. This proves uniformity in s.

The normal Taylor estimate also intertwines exactly with the horizontal copy: its rescaled C^n norm and its parameter family $\chi_1(u\pi_g x)$, $0 \le u \le B$, are unchanged. In the near branch the normalized integral Taylor remainder is

$$f(b + ay) - \sum_{j=0}^{m} \frac{a^j}{j!} D^j f(b)[y^j] = \frac{a^{m+1}}{m!} \int_0^1 (1-t)^m D^{m+1} f(b + tay)[y^{m+1}]\, dt,$$

with y normal and b tangential. Differentiating this formula in the normalized coordinates uses at most $n+m+1$ derivatives. When the cutoff is not identically one, the retained alternative $1 \le 3Ba$ converts the rough near bound to the same power a^{m+1}; the far branch retains a^m. Hence CE10 transports with exactly its two regimes and a constant uniform in the subtraction length and fixed normalized angular masks.

For the explicit CE8 induction, the singleton partition has no normal direction left: apply continuity, the permitted continuation bound and bounded resized support. Otherwise choose nonsingleton i, decompose its election into the finite projected/remaining choices v, and apply CE9 after normal Taylor operation and the angular partition. Each resulting term equals the refined forest expression with the continuation that regroups its children into i and applies its fixed projection or remainder. This is a linear-expression identity before hierarchy evaluation or integration. The refined partition has smaller $|g| - |Q'|$ and retains offsets, length and normal constancy. Its subcentroid palette is admissible by angular separation and the common collar. The inductive estimate and CE10, with CE3–CE7, give

$$C_v \|f \circ A_w\|_{C^{N_Q}} \mathcal{B}_w(\varepsilon_Q) a_n^\gamma \prod_{\substack{j \in Q \\ j \neq i}} 2^{-n_j \gamma}. \tag{CE11}$$

Empty annular stages give zero; nonempty ones have exactly the refined geometry. Finitely many sectors and components preserve uniformity. Because $\sum_{n \geq 0} a_n^\gamma = (1 - 2^{-\gamma})^{-1}$, their complete terms are absolutely summable in integral norm. CE9's eventual germ equality and proper-normal Haar-null exceptions identify the sum almost everywhere with the unsliced expression. More precisely the terms sum in the Banach space $L^1(E; X)$, since the sum of norms is finite. Absolute integral-norm summability also supplies the almost everywhere pointwise sum; uniqueness identifies the same integrable function, without an unproved dominating function. Continuous integration gives

$$\int F_v = \sum_{n=0}^{\infty} \int F_{v,n}, \qquad \sum_n \int \|F_{v,n}\| < \infty. \tag{CE12}$$

Use CE11 directly for a specified shell; sum the finite choices for CE8. If all choices of the election occur, the exact root formula is

$$\int F = \sum_v \sum_{n=0}^{\infty} \int F_{v,n}. \tag{CE13}$$

For any ordered list of distinct unresolved nonsingleton groups, successive shell resolutions have bound proportional to $\prod_j 2^{-n_j \gamma}$. Absolute Fubini gives the multiple sum and its equality with the original pairing in every permitted order. Only the combined subtraction expressions are asserted absolutely summable; this does not separately assert it for their Taylor monomials. For desired loss $\epsilon > 0$, choose $\gamma = \min(1, \epsilon)/(\tau_Q + 1)$. Then the economy condition holds, $\varepsilon_Q \leq \epsilon$, and on unrestricted shells monotonicity of $R^{-\epsilon}$ for $0 < R \leq 1$ gives the requested loss using the same devices and projections.

The continuation hypothesis is a tested inequality on every allowed logarithmic chart path: price rate b and affordable group rates c_i give reading rate at most $b + \sum_i c_i$. It also retains local continuity on palettes and constancy in already assigned normals. The actual hierarchy with separated exterior fields has these properties by the signed-expression estimates below. Refine i with radius rate $-s$ and parent recorded rate $-t$. Angular separation gives between-child palette lower bounds at rate $-s$. Affordable-expression product rules give child product rate $\sum_j c_j$ at exponent s. Projection or remaining-component continuation at the parent gives

$$\sum_j c_j + (t - s) G_v. \tag{CE14}$$

For the remaining component support forces $t \leq s$; the full finite projection remainder through $L_i - 1$ gives $G_v = L_i$. If $L_i = 0$, use coarsening with zero gain. The projected degree-h reader

gives $G_v = h$. The common detector collar makes these operations admissible on the actual moving palettes.

The lifted resolution source retains the complete combined integrand as an L^1 object, its finite election, shell indices and inherited continuation. CE10 transports its test norm; CE11 pays every refined term; CE12–CE13 recover the unsliced pairing by continuous integration. Absolute Fubini proves compatibility of permitted resolution orders. The reader does not split the cancellation-bearing source into independently integrated monomials. CE14 records the exact continuation gain that subsequent refinements must retain.

Absolute-resolution recovery in the tested norm. Let $U_X : X \to X^Z$ be the isometric horizontal output copy (the identity when the tested output is scalar). The lift on integrands is

$$(\mathcal{U}F)(J_E x) = U_X F(x), \qquad \|\mathcal{U}F\|_{L^1(E^Z;X^Z)} = \|F\|_{L^1(E;X)}, \qquad \int_{E^Z} \mathcal{U}F = U_X \int_E F.$$

The finite election, normal Taylor operation, angular multiplication and prescribed regrouping all intertwine with this lift. Hence each complete annular term, before taking its norm, satisfies $F^Z_{v,n} = \mathcal{U}F_{v,n}$. The same holds for an empty stage, whose term is zero. Set

$$K_v = C_v \|f \circ A_w\|_{C^{N_Q}} \mathcal{B}_w(\varepsilon_Q) \prod_{\substack{j \in Q \\ j \neq i}} 2^{-n_j \gamma}, \qquad r = 2^{-\gamma} \in (0,1).$$

CE11 and the isometry give the quantitative common tail estimate

$$\left\| F^Z_v - \sum_{n=0}^{N-1} F^Z_{v,n} \right\|_{L^1} \leq K_v \frac{r^N}{1-r}, \qquad Q_X \int F^Z_v = \sum_{n \geq 0} \int F_{v,n},$$

where $Q_X U_X = 1$. The identification of the limit with F^Z_v uses the transported eventual germ identity and Haar-null exceptional sets. This proves recovery of CE12 in norm as well as after integration. Summing the finite v gives precisely CE13.

For k successively resolved groups, retain the full multi-index $(n_1, \dots, n_k)$ and its complete combined term. Its norm bound is $K r^{n_1 + \cdots + n_k}$, with the fixed common constant supplied by the successive estimates. In particular

$$\sum_{n_1, \dots, n_k \geq 0} K r^{n_1 + \cdots + n_k} = \frac{K}{(1-r)^k}, \qquad \sum_{\max n_j \geq N} K r^{n_1 + \cdots + n_k} \leq \frac{k K r^N}{(1-r)^k}.$$

The rectangular sums therefore recover the original pairing, and absolute Fubini permits every resolution order allowed by the recorded forest. The bound concerns complete subtraction terms only. For the empty list the recovery is the original integral without a shell sum. Finally the choice $\gamma = \min(1, \epsilon)/(\tau_Q + 1)$ leaves $\varepsilon_Q \leq \epsilon$, so $\mathcal{B}_w(\varepsilon_Q) \leq \mathcal{B}_w(\epsilon)$ for the same full radius register and the same devices.

21.4 Continuation inheritance and the global contact distribution

The refined scalar price is exactly $B'_w = B_w (a/R_i)^{G_v}$. Its rate $b + (t-s)G_v$ matches CE14. On a subtraction halo choose an original-stage point differing only in N_i, as guaranteed by the halo definition; for the remaining branch take the same point. Normal constancy permits this evaluation. Orthogonality leaves every other palette and expression unchanged. Substitution of

the child product with its fixed component treatment into the old tagged product is the refined reading. The old path inequality gives $b + (t - s)G_v + \sum_{\text{refined}} c_j$, proving the inheritance in CE11. Continuity follows on the original-stage and normally collapsed-stage closed pieces and glues on their agreeing restrictions, including continuations with previous Taylor cancellations; the auxiliary continuation is thus justified at each step.

For explicit recovery of that inheritance, denote the old continuation by C_w and the finite child-regrouping/component map by M_v. The refined continuation is the actual composition $C_{w'} = C_w \circ M_v$, with all other tagged arguments fixed. Its lifted version is the same composition of the intertwined maps, so $Q_X C^Z_{w'}(J\,\cdot) = C_w(M_v(\,\cdot))$. Here J denotes the product of the relevant horizontal sections. For a path with radius rates $-s, -t$, the scalar price identity is

$$B_{w'} = B_w(a/R_i)^{G_v}, \qquad \operatorname{rate} B_{w'} = b + (t - s)G_v, \qquad \operatorname{rate} C_{w'} \le b + (t - s)G_v + \sum_{\text{refined parts}} c_j.$$

All three are unchanged by section/readout. In the remaining branch retain the support condition $t \le s$ and the complete finite remainder; $L_i = 0$ retains coarsening with gain zero. In the projected branch retain its actual degree h and $G_v = h$. A halo evaluation is transported from the same original-stage point differing only in N_i. Since J_E preserves orthogonality, this move leaves every other palette unchanged. The two closed-piece continuation maps agree on their intersection before and after transport, so their glued continuous map is the same refined reading. Thus subsequent refinements inherit the actual cancellations as well as their rate bound.

At each $x \in E$, partition labels by equality of positions $Px + o$. Each part has zero normal displacement and distinct-part subcentroids are distinct. On a sufficiently small closed ball normals stay in the common collar and these separations have positive lower bounds. The same positive s, $2s \le B$, works on these balls. The local forest estimate gives the test pairing for the equality partition, and the expression overlap identity below identifies it with the single root prescription, without assigning new overlap values. For compact test support K, take a finite such cover and smooth subordinate θ_j. With $N = (2 + \sum_i d_i)|g|$ we have $N_Q \le N$ and $\|\theta_j f\|_{C^N} \le C_{K,j}\|f\|_{C^N}$. Linearity and finite addition of local bounds give

$$\int_E |K_s(f)(x)|dx \le C_K\|f\|_{C^N}, \qquad \operatorname{supp} f \subset K. \tag{CE15}$$

Thus $\mathfrak{S}_s(f) = \int_E K_s(f)(x)dx$ is continuous on every fixed-support smooth test space, hence is a distribution of finite order N. Its forest restrictions are the same test expression readings. This holds for the unselected and every character reader, retaining s, finite projections and normal Taylor convention. We next obtain a distribution on the whole allowed position space.

A score is a finite sum of field expressions tensored with linear test instructions; at x multiply the field value by the instructed test. Products compose instructions in their prescribed order, commuting for disjoint groups by orthogonal normals. Let

$$\mathcal{P}_g = \sum_{\substack{Q \text{ partition of } g \\ Q \ne \{g\}}} \prod_{i \in Q} C_i.$$

For singleton $g = \{j\}$, C_g is the actual field tensored with identity. For $|g| \ge 2$ define

$$C_g = -\sum_{h=0}^{L_g - 1} \operatorname{reshape}(\text{reader}_{g,h})\mathcal{P}_g\, \mathsf{T}_{g,s,L_g-1-h}, \qquad \mathcal{S}_g = \mathcal{P}_g + C_g. \tag{CE16}$$

This finite recursion uses only smaller groups. Its fixed centroid reader acts on fields, and its right Taylor factor on tests; for $L_g = 0$ the correction is empty. Rearrangement is precisely the

component decomposition used by CE3. Each C_g depends only on the test germ at $x - \pi_g x$ by CE2; singleton normals are zero. For partitions Q, R both separated on a test support, $\prod_{i\in Q} \mathcal{S}_i$ expands into corrections indexed by partitions refining Q, since choices of partitions inside its parts are exactly such refinements. A term not refining R has a group crossing two R parts. At its collapse those labels coincide, whereas they remain separated on the test support. Its germ-dependent factor annihilates the test, which is zero near that collapsed locus. All such crossing terms vanish as scores after multiplication by the test mask. Both expansions therefore reduce to common refinements, as masked scores before evaluation and integration. This proves the overlap identity even for nonnested Q, R used in CE15.

Finite score recursion and global distribution recovery. For a test instruction D, define its horizontal transport by $D^Z f^Z = (Df)Q_E$. A score summand $F \otimes D$ is transported to $F^Z \otimes D^Z$, using the corresponding horizontal field section and the same tagged product. Denote this finite linear transport by $\mathcal{L}$. Its evaluation satisfies

$$\mathrm{ev}^Z(\mathcal{L}A, fQ_E)(J_E x) = U_X \,\mathrm{ev}(A, f)(x), \qquad (D_1 D_2)^Z = D_1^Z D_2^Z.$$

Thus the order of test instructions is preserved. For disjoint groups the transported normal projections and Taylor operations commute, exactly as in the original score.

The centroid field reader and its reshape are transported with the same fixed homogeneous output spaces and basis. Induction on $|g|$ then gives

$$C_g^Z = \mathcal{L}C_g, \quad \mathcal{P}_g^Z = \mathcal{L}\mathcal{P}_g, \quad \mathcal{S}_g^Z = \mathcal{L}\mathcal{S}_g.$$

The base is the actual singleton field and identity instruction. Every proper partition contains only smaller groups, and CE16's finite sum uses the already intertwined centroid reader and normal Taylor instruction. This proves the inductive step, including the empty correction when $L_g = 0$. It also recovers the minus sign and normal jet order $L_g - 1 - h$ separately for every component.

For separated partitions Q, R, let $T \preceq Q$ mean that T refines Q. On their common test support the exact masked-score identity is

$$\left(\prod_{i\in Q} \mathcal{S}_i\right)[f] = \left(\sum_{\substack{T\preceq Q\\ T\preceq R}} \prod_{j\in T} C_j\right)[f] = \left(\prod_{i\in R} \mathcal{S}_i\right)[f].$$

Here brackets denote action on the test before hierarchy evaluation. The bijection between choices of partitions inside each part and refinements accounts for every term once. A crossing correction annihilates the vanished germ at its normal collapse. Transport preserves that germ and the support separation, so this same identity holds for $\mathcal{L}$ of each score. No nesting relation between Q and R is required.

On a compact support K, transport the actual finite chart cover and its smooth masks by J_E. Let A_j be the local integral-norm constant for chart j, after fixing its frames and devices. The common test order $N = (2 + \sum_i d_i)|g|$ gives explicitly

$$\|K_s(f)\|_{L^1} \le \sum_j A_j \|\theta_j f\|_{C^N} \le \left(\sum_j A_j C_{K,j}\right) \|f\|_{C^N}.$$

The first inequality uses linearity of the single root prescription; the overlap identity identifies each chart computation with it. Consequently, defining the horizontal root integrand by the lifted score,

one has

$$K_s^Z(fQ_E)(J_Ex) = U_X K_s(f)(x), \qquad \mathfrak{S}_s^Z(fQ_E) = U_X\mathfrak{S}_s(f), \qquad Q_X\mathfrak{S}_s^Z(fQ_E) = \mathfrak{S}_s(f).$$

The same fixed-support seminorm bound proves continuity of the lifted functional on each test space and hence on the compactly supported smooth test-function space. Its order remains at most N. Forest restriction commutes with these identities since it uses the same test score. This recovers the unselected reading and each character reading with the fixed length s, projections and normal Taylor convention retained in the contact source.

For the initial continuation, retain a finite signed-expression proof tree. Leaves are bounded degree-d labels priced R^{-d} and the unit priced one. Products require palette separation by a positive multiple of common radius and multiply prices. Carrying to R' is allowed for diameter $O(R)$ around the center and $R = O(R')$. Subtracting full projection through h costs $(R/R')^{h+1}$ with that size condition and fixed reader clearance. Degree-h projection costs $(R/R')^h$, collapses its palette to the center and requires no ratio bound. Palette enlargement is permitted. Along logarithmic paths the full field, separated product, coarsening and tested projection/remainder estimates prove each price by finite induction, retaining a remainder as its signed difference. Compact normalization of all log radii and the strict surcharge margin give the uniform bound

$$|\langle E\rangle_\chi| \le C_e p \prod_j R_j^{-\epsilon}.$$

Output testing at degree h adds R^h. Separated exterior fields have fixed clearance and bounded norms with radius one, so these bounds and hierarchy continuity supply the initial continuation. An additional root projection or remainder is allowed when its whole palette fits the root detector collar; its price is R^h or R^{h+1} respectively. Apply it after forming the root expression and before testing the lower forest; CE14 preserves that price under shell refinement.

The lifted global contact source retains its finite score recursion, test germ instructions and local equality partitions. Its overlap reader is equality of masked scores, so partition-of-unity assembly recovers one root prescription. CE15 proves continuity of its integral readout with the original finite derivative order. The signed proof tree and exact price update preserve previous cancellations and root operations; the contact distribution is recovered with these chosen devices, rather than from punctured values alone.

Signed-expression price recovery. Retain the finite expression tree together with each radius register, center, palette and price. The horizontal field section intertwines the actual fixed projections: for a degree reader Π_h and its finite sum $\Pi_{\le h}$,

$$\Pi_h^Z J = J\Pi_h, \qquad (1 - \Pi_{\le h}^Z)JE = J(E - \Pi_{\le h}E).$$

Products use the recorded separated palettes; carrying and palette enlargement retain the same physical positions and centers. Thus induction over the finite tree recovers its expression and price at every node. In particular the price updates remain exactly

$$p(E_1E_2) = p(E_1)p(E_2), \quad p(\mathrm{rem}_{\le h}\, E) = p(E)(R/R')^{h+1}, \quad p(\Pi_h E) = p(E)(R/R')^h.$$

These are the assigned prices on the permitted operations, not norm identities. The remaining component is the displayed signed difference; its two pieces are not priced separately. The remainder operation retains $R = O(R')$ and clearance, whereas the degree projection retains its original absence of a ratio bound and its collapsed center palette. The viewer map is transported by

the same ground-algebra isomorphism, so its scalar evaluation is unchanged. On any allowed logarithmic path, both the quantity and assigned bill therefore have the same rates before and after lift. Normalized logarithmic compactness and the same full surcharge $\prod_j R_j^{-\epsilon}$ recover the stated uniform all-viewer bound. Exterior words at radius one introduce no new radius loss. A root output or remainder is retained in its original position in the tree, after forming the root field and before testing the lower forest; its R^h or R^{h+1} price is preserved by CE14.

21.5 Normalized contact scaling and compatible finite cutoffs

For normal scaling R retain the normalized test

$$f_R(x) = R^{-\nu_g} f(x - \pi_g x + R^{-1}\pi_g x). \tag{CE17}$$

The frame multiplying N_g by R and fixing its complement has determinant R^{ν_g}, while f_R composed with it is $R^{-\nu_g} f$. These cancel exactly. The one-root budget becomes $R^{u-\omega_g-\epsilon}$, where $u = 0$ for the full reading, $u = h$ for degree-h output and $u = h + 1$ for remainder through h. Thus

$$\int \|K_s(f_R)(x)\| dx \le C\|f\|_{C^{N_{\{g\}}}} R^{u-\omega_g-\epsilon}. \tag{CE18}$$

It is uniform on permitted normalized supports and chart parameters. The subtraction length can be fixed or vary under the same halo, support and clearance conditions; it need not equal R. If f vanishes near the total diagonal $N_g^{\perp}$, its normal jets annihilate C_g in CE16. Hence full and punctured scores agree on these tests even if proper-subcluster collision sets are met. Every positive normal annulus qualifies, also after scaling, with no uniform lower bound on its hole. CE18 therefore bounds full, output and remainder punctured pairings with the same constants.

Normalized normal-test recovery. Set $S_R = (1 - \pi_g) + R\pi_g$, so $S_R^{-1} = (1 - \pi_g) + R^{-1}\pi_g$ and $f_R = R^{-\nu_g} f \circ S_R^{-1}$. On the horizontal copy use $S_R^Z = J_E S_R Q_E$. Then

$$(f^Z)_R = f_R Q_E, \qquad (f^Z)_R \circ S_R^Z = R^{-\nu_g} f^Z, \qquad |\det S_R^Z|\, \|(f^Z)_R \circ S_R^Z\|_{C^N} = \|f\|_{C^N}.$$

The determinant is taken in the transported orthonormal frame; it is R^{ν_g}. Thus the one-root volume, normalized test and scalar continuation price combine as

$$R^{\nu_g}\, R^{-\nu_g}\, R^u R^{-\omega_g} R^{-\epsilon} = R^{u-\omega_g-\epsilon}.$$

This recovers CE18 for each of its three root readers with the same finite derivative count, permissible support and subtraction length. Moreover S_R preserves $N_g^{\perp}$ and is invertible for each $R > 0$. A test vanishing in a neighborhood of that subspace retains this property under normalization. Its normal jets vanish, so CE16's root correction vanishes exactly and the punctured and full pairings coincide. Proper-subcluster collisions may remain in the support; they are treated by the retained lower corrections. The recovery uses only a positive annulus hole at each test, with no uniform lower bound on it and no requirement $s = R$.

Choose real bases simultaneously in homogeneous contact grades. This is a coordinate basis for the strata of the already fixed splitting; it does not replace the OPE prescription or its projection functionals. The strata are invariant under conjugation and rotations since their lower-grade maps intertwine them. Conjugation-fixed vectors span each stratum by real/imaginary decomposition; a maximal independent real subset is a finite complex basis. In degree zero start with the nonzero real unit and extend it. Expand any field into its finite homogeneous parts and basis coordinates.

Multilinearity extends the score from these basis inputs, retaining each actual homogeneous degree in CE16. Higher ambient horizons change none of these parts, so this linear extension is well defined. For positive requested bounds f_n choose

$$g_n = \min\left(f_n, 2^{-n}, \frac{\text{clearance}(n)}{2(n+1)}\right), \quad t_0 = g_0, \quad t_{n+1} = \min(t_n/8, g_{n+1}).$$

Then $t_n > 0$, $4t_{n+1} < t_n$, $t_n \to 0$ and $2(n+1)t_n \le \text{clearance}(n)$. Take $b_0(s) = 1$, and the prescribed smooth b_n, $n > 0$, equal one for $s \le t_n$ and zero for $s \ge 2t_n$. They are locally finite for $s > 0$; if $b_k(s) \ne 0$ with $k > n$, strict threshold spacing forces $b_n(s) = 1$. The last active index $N(s)$ exists and tends to infinity as $s \downarrow 0$. Active $k > 0$ obey $ks \le \text{clearance}(k)$, since $s < 2t_k$. For fixed C this gives eventually $Cs \le \text{clearance}(N(s))$. Define

$$\mathcal{D}_s(x)E = \sum_{j=0}^{N(s)} b_j(s)\,\text{insertRow}_{x,j}\,E. \tag{CE19}$$

Local finiteness gives smoothness for positive s, also when the last index changes. Biorthogonality gives exactly

$$\text{row}_{x,n}\,\mathcal{D}_s(x)E = b_n(s)\,\text{row}_{x,n}\,E.$$

For $s \le t_n$ all rows through n have weight one, so projection through n is precisely the original finite reader. Slowing thresholds to meet any further positive f_n preserves these identities. No convergence in the growing horizon is asserted. The affordable-expression rules, logarithmic compactness and mixed-label inclusions used here are the UC1–UC3 constructions below.

The lifted normalized-test source includes the normal volume and scaling frame; its integral reader cancels the Jacobian against CE17 exactly. Restriction away from total collapse recovers the punctured score while retaining proper-subcluster corrections. The basis and threshold source retains every finite row and its biorthogonal insertion. CE19 recovers each fixed finite reader exactly below its threshold, so a variable cutoff introduces no replacement of the finite reader assumed in the contact estimates.

Real contact bases and fixed-row cutoff recovery. Write κ for the conjugation on a homogeneous stratum. Its horizontal transport is $\kappa^Z = J\kappa Q$. For every v in that stratum,

$$v = v_{\mathbb{R}} + iv_{\mathbb{I}}, \qquad v_{\mathbb{R}} = \frac{v + \kappa v}{2}, \quad v_{\mathbb{I}} = \frac{v - \kappa v}{2i}, \qquad \kappa v_{\mathbb{R}} = v_{\mathbb{R}}, \quad \kappa v_{\mathbb{I}} = v_{\mathbb{I}}.$$

The same formulas on the horizontal copy give $Jv_{\mathbb{R}}$ and $Jv_{\mathbb{I}}$. A maximal complex-linearly independent subset of these fixed vectors spans, since otherwise the displayed decomposition supplies another independent fixed vector. Transport therefore gives the identical finite real system basis, including the unit in degree zero. The rotation representation transports on the whole stratum; individual basis vectors may mix. For a finite homogeneous expansion in each input, multilinearity gives the exact finite basis sum of scores. Applying Q to that sum recovers the original contact score term by term, with the assigned homogeneous degrees unchanged when its ambient horizon is raised.

The positive sequence t_n is part of the retained source data. Inductively $0 < t_n \le g_n \le f_n$ and $t_{n+1} \le t_n/8$, proving all its stated spacing, clearance and decay conditions. If $k > n > 0$ and $b_k(s) \ne 0$, then

$$s < 2t_k \le 2\,8^{-(k-n)}t_n < t_n,$$

so $b_n(s) = 1$; for $n = 0$ this follows from $b_0 = 1$. On a compact positive interval $s \geq a > 0$, all indices with $2t_n \leq a$ vanish identically, which proves local finiteness and smoothness without differentiating the last-active-index function. For each fixed m, $s \leq t_m$ activates every index through m; hence $N(s) \to \infty$. An active $k > 0$ satisfies more precisely

$$ks < 2kt_k \leq \frac{k}{k+1}\,\mathrm{clearance}(k) \leq \mathrm{clearance}(k).$$

For any fixed positive C, eventually $N(s) \geq C$ and $N(s) > 0$, so $Cs \leq N(s)s \leq \mathrm{clearance}(N(s))$.

Let $r_{x,n}$ denote the fixed degree-n row and $I_{x,j}$ its biorthogonal row insertion used in CE19. Transport both maps with the same horizontal sections of their domain and output. Their composition identity remains $r_{x,n}I_{x,j} = 0$ for $n \neq j$ and the prescribed degree-n row for $n = j$. Because the cutoff sum is locally finite, exact map transport gives

$$\mathcal{D}_s^Z(x)J = J\mathcal{D}_s(x), \qquad r_{x,n}^Z\mathcal{D}_s^Z(x)J = J_n b_n(s) r_{x,n}, \qquad \Pi_{\leq n}^Z\mathcal{D}_s^Z(x)J = J\Pi_{\leq n} \quad (0 < s \leq t_n).$$

Here J_n is the degree-n output section and the last identity uses the fixed finite projection on its indicated domain. Thus every finite reader is recovered at an actual threshold, with no limit in the growing horizon required. Any additional positive requested bounds are imposed in g_n before the same recursion; they retain all these exact row identities.

21.6 Uniform subpower bounds and arbitrary-label contact scores

Let $R_1, \ldots, R_m \in (0, 1]$ be the finite radius registers on a configuration set U, with positive bill q and normed reading F. Assume that on every nontrivial filtered sequence with increasing $T \to \infty$, $\log R_a/T \to -s_a$, $0 \leq s_a \leq 1$, the bill has an exact rate c and F has upper rate at most c. Then for every $\varepsilon > 0$ one finite constant gives

$$\|F(w)\| \leq Cq(w) \prod_{a=1}^{m} R_a(w)^{-\varepsilon} \quad (w \in U). \tag{UC1}$$

If not, select violations with $C = e^n$. Put $p_{n,a} = -\log R_a(w_n)$, $S_n = \sum_a p_{n,a}$ and $T_n = n + S_n$. On an ultrafilter refining the sequence tail, compactness of finitely many coordinates gives $p_{n,a}/T_n \to s_a$ and $n/T_n \to k$, with $k + \sum_a s_a = 1$. The path hypothesis gives upper rate c for F, whereas the violating comparison quantity $e^n q(w_n) \prod_a R_a(w_n)^{-\varepsilon}$ has exact rate $c+k+\varepsilon \sum_a s_a > c$; the increment is positive because k and all s_a are nonnegative and their sum is one. A quantity of strictly larger exact rate eventually exceeds one of smaller upper rate, contradicting every chosen violation. The added n covers radii bounded away from zero. All bills here are generated by constants, positive radii, products and fixed real powers and thus have exact rates. No uniform intermediate OPE horizon or threshold across paths is required to obtain this ordinary uniform layer bound.

Let $M_a^{(d)} : E_{\leq d} \to E_{\leq a}$ be the fixed homogeneous splittings. At fixed chart, partition and route the homogeneous contact score is multilinear before integration: its grade-transport identity, with other inputs fixed, is a linear map. Hence addition and scalar multiplication hold in that slot for the full expression-valued score. For arbitrary labels set

$$\mathcal{B}_d(f_1, \ldots, f_k) = \sum_{0 \leq a_i \leq d_i} \mathcal{A}_{(a_i)}(M_{a_1}^{(d_1)} f_1, \ldots, M_{a_k}^{(d_k)} f_k). \tag{UC2}$$

For homogeneous inputs only their own grade tuple survives, recovering the original score. Raising caps embeds every original tuple (a_i) in the enlarged finite index set and preserves it by splitting/

inclusion compatibility; each added tuple has some $a_i > d_i$, whose component is zero. Multilinearity kills that term. Thus cap independence holds at expression level. On distinct-configuration test supports, each homogeneous contact image is its ordinary field product times the test. The exact identity

$$\sum_{a=0}^{d} \mathrm{insert}_a(M_a^{(d)} f) = \mathrm{insert}_d(f)$$

and finite product expansion recover the mixed-label word. Allowed spectator evaluation gives the separated correlation times the test. Per-grade integral bounds permit termwise integration of the finite sum, giving the full field-space extension with its prescribed separated restriction. The lifted uniform-bound source retains the actual radius-power bill, clock and all path hypotheses; the compact violation argument recovers one finite UC1 constant without selecting a different reader for each radius. Its mixed-label source retains the coherent splitting and the finite score tuple. UC2 commutes exactly with cap inclusion and separated readout; finite integration preserves these identities. Consequently extension to arbitrary labels loses no homogeneous component or assigned degree.

Uniform-layer and mixed-label recovery identities. For a radius-power bill $q = C_0 \prod_a R_a^{\beta_a}$ with $C_0 > 0$, its exact path rate is

$$\frac{\log q}{T} \longrightarrow -\sum_a \beta_a s_a = c.$$

The section retains every named radius and this positive bill. The horizontal norm of the lifted reading is the original norm, so both sides of UC1, as well as any selected violation, recover exactly. For the normalized violating array of the preceding argument the strict increment satisfies the explicit lower bound

$$k + \varepsilon \sum_a s_a = \varepsilon + (1-\varepsilon)k \geq \min(1, \varepsilon) > 0.$$

Thus the contradiction retains a strictly positive margin even when all radius rates vanish, in which case $k = 1$. Taking the viewer as part of the selected configuration retains the same argument for varying characters and the unselected reading, under the original path hypotheses. This transports the common constant without assuming that intermediate path-dependent horizons or thresholds are uniform.

For clarity write $J_{\leq d}, Q_{\leq d}$ for the horizontal maps on a capped label space and J_a, Q_a on its homogeneous output. Use the transported splitting and inclusion maps

$$(M_a^{(d)})^Z = J_a M_a^{(d)} Q_{\leq d}, \qquad (\iota_d^{d'})^Z = J_{\leq d'} \iota_d^{d'} Q_{\leq d}.$$

Their coherent composition identities follow by $QJ = 1$. For fixed chart, partition and route, let the homogeneous score lift satisfy $\mathcal{A}_a^Z(J_{a_1} v_1, \ldots, J_{a_k} v_k) = \mathcal{L}\mathcal{A}_a(v_1, \ldots, v_k)$, where $\mathcal{L}$ is the score transport already defined. UC2 then gives exactly

$$\mathcal{B}_d^Z(J_{\leq d_1} f_1, \ldots, J_{\leq d_k} f_k) = \mathcal{L}\mathcal{B}_d(f_1, \ldots, f_k).$$

Every sum here is finite before score evaluation or integration. For larger caps $d_i' \geq d_i$, splitting/inclusion coherence makes each old summand identical and each new summand zero in a slot with $a_i > d_i$. Consequently both sides are independent of that cap enlargement as expression-valued scores, not only after testing. For homogeneous inputs only their own tuple remains.

On distinct configurations the recovered field expression is

$$\prod_{i=1}^{k}\left(\sum_{a_i=0}^{d_i}\operatorname{insert}_{a_i}(M_{a_i}^{(d_i)}f_i)\right)=\prod_{i=1}^{k}\operatorname{insert}_{d_i}(f_i).$$

The product retains its original order. The same finite expansion holds for transported insertions, scores and any allowed spectator. The finite sum of the per-grade integral bounds supplies integrability of the mixed score and permits termwise integration. Applying its horizontal readout therefore recovers the separated mixed-label correlation with the original test function, while retaining every actual homogeneous degree used in its contact correction.

21.7 Projection gain, tested funds and safe probes

At clock T, an expression is affordable at layer s, cost c, if it is a finite sum of separated words whose coefficient of total weight D has upper rate $c-sD$. Along a logarithmic clock T, an upper rate c means a bound by $e^{(c+\delta)T}$ eventually for every $\delta>0$. This is closed under finite sums, scalar rate addition and eventual equality, with its finite position palette. A precision-q tested error, multiplied by any separated affordable spectator of terminal layer one and cost b, has expectation upper rate $q+b$, for the unselected and every possibly varying character. An expression has funds (s,c) if it has a finite terminal affordability bound and, at every requested precision, agrees to that precision with an affordable expression of layer s, cost c. For palette radius rate e^{-sT} and $0\le u<s\le 1$, one-scale OPE gives a centered finite approximation at layer u to arbitrary precision: choose horizon n so $n(s-u)$ pays the precision deficit. Retained output of weight d has coefficient rate $c-ds$. This horizon may depend on path and precision; uniformity is obtained later by UC1. Finite sums and an existing tested error give this replacement for all expressions with funds. A centered finite sum with these rates has weight-w row rate $c-ws$: lower weights vanish and higher-weight splitting uses fixed linear maps. For a general funded expression, test its error against the row's finite detectors, rotations and conjugates, outside its fixed aperture. A field of weight below w gives zero, while splitting a field of weight at least w applies a fixed linear map and cannot worsen its coefficient rate. Their distances are bounded below and their layer is zero. If their fixed weight sum is $\operatorname{probeFee}(w)$, the row error rate is at most $q+\operatorname{probeFee}(w)$; choose precision to pay that fee. The row bound and finite-projection funds follow. Arbitrary varying rotations preserve rates by finite-dimensional covariance and compactness; selecting a violating rotation would contradict this pathwise bound. Hence normalized Haar averaging, conjugation and finite decoding preserve the same rate without interchanging nonuniform limits; this proves the detector-row estimate. The projection gain underlying (UC1) can now be made precise.

For the prescribed finite projection the exact gain rule is

$$\operatorname{Funds}(s,c;e)\Longrightarrow\operatorname{Funds}\big(u,c-(w+1)(s-u);e-\Pi_{\le w}e\big),\quad u\le s. \tag{UC3}$$

For centered sums all weights through w cancel and every remaining $d\ge w+1$ obeys $c-ds\le c-(w+1)(s-u)-du$. For general e, approximate at precision $\min(q,q-\operatorname{maxFee}(w)-wu)$, with $\operatorname{maxFee}(w)=\sum_{j=0}^{w}\operatorname{probeFee}(j)$. Projection of the error costs at most $\operatorname{maxFee}(w)+wu$, so both errors fit precision q. Subtract them and apply the finite sum result. At $u=s$ simply subtract the projected affordable expression with zero gain. Repeated UC3 supplies each signed ancestor move's full $w+1$ gain in the contact forest. Funds multiply across separated layer-s palettes, $s\le 1$, by adding costs. For e,f with funds $(s,c),(s,d)$ and finite terminal costs a,b, first approximate e by g at precision $q-b$. Then enlarge g to terminal layer one and record its actual finite cost c'.

Approximate f by h at precision $q - c'$. The identity

$$ef - gh = (e - g)f + (f - h)g$$

puts both errors at precision q, while concatenation makes gh affordable at cost $c + d$. Induct over finite separated families. This ordered choice is essential: terminal cost can greatly exceed the outgoing layer cost, so it is recorded before requesting the next approximation; no uniform bound over horizons is imposed. The arbitrary-precision definition is exactly what permits this successive choice. For expectation against a spectator of layer $v \geq s$, cost q, first record its terminal cost b, then approximate e at precision $c + q - b$. Direct separated bounds and error testing both give rate $c + q$. The empty spectator gives expectation rate c, uniformly over characters. This closes the multiplication and expectation rules.

Tested-funds and finite-gain recovery. Lift a finite word using the same ordered labels and positions, and retain its scalar coefficient and total weight. Its coefficient rate $c - sD$ is unchanged. For any allowed terminal spectator v and transported viewer χ^Z, the exact scalar identity is

$$\langle e^Z v^Z \rangle_{\chi^Z} = \langle ev \rangle_\chi.$$

The same identity holds for the unselected hierarchy. The horizontal section and readout therefore transport affordability, precision-q tested errors and finite terminal bounds in both directions on the retained expression class. In particular the funds quantifier remains "for every precision q, there exists a finite approximation," with that approximation allowed to depend on the path and q. No common horizon is inserted in this transport.

For a fixed row of weight w, transport its actual detector family, aperture, rotations, conjugates and finite decoding maps. Positions are unchanged, so the fixed positive placement distances and layer zero are unchanged. Its error estimate retains the exact fee $q + \mathrm{probeFee}(w)$. Given any extra rate tolerance $\delta > 0$, choose the precision to satisfy $q + \mathrm{probeFee}(w) \leq c - ws + \delta$. This recovers the row rate $c - ws$ from the affordable approximation. Varying rotations are retained as part of the tested path. The resulting eventual uniform bound over rotations bounds their normalized Haar integral by that same bound; horizontal transport commutes with this finite-dimensional integral. Thus the full averaged reader, including conjugation and decoding, has the recovered row estimate.

For UC3 put $K_w = \mathrm{maxFee}(w) + wu$ and $q_0 = \min(q, q - K_w)$. If a is the chosen finite approximation, the exact residual identity is

$$(e - \Pi_{\leq w} e) - (a - \Pi_{\leq w} a) = (e - a) - \Pi_{\leq w}(e - a), \qquad q_0 \leq q, \quad q_0 + K_w \leq q.$$

Both tested errors therefore meet precision q before and after horizontal transport. For a surviving weight $d \geq w + 1$,

$$[c - (w + 1)(s - u) - du] - (c - ds) = (d - w - 1)(s - u) \geq 0.$$

This recovers precisely the cost gain and its $u = s$ zero-gain case.

For separated products the successive precision requests remain $q - b$ for $e - g$, followed only after recording g's finite terminal cost c' by $q - c'$ for $f - h$. The two errors in $ef - gh = (e - g)f + (f - h)g$ then have rates at most q; their finite sum has the same upper rate. The product approximation retains cost $c + d$ and the original separated palette. For a spectator of layer $v \geq s$, outgoing cost q and terminal cost b, the request $c + q - b$ gives error rate $c + q$, equal to the direct approximation bound. These identities transport the multiplication and expectation rules, including the empty

spectator and varying viewers, without replacing the recorded terminal costs by outgoing costs or suppressing the signed residuals.

For the earlier equation lift, replace any step traversing a forbidden stored positive edge (in either direction) by three steps: one in a different coordinate, the desired step, then its inverse transverse step. Endpoints agree and the forbidden edge is avoided. Replacing all offending steps gives an avoiding cover path between any two sites. An axes-$i < j$ plaquette can use a forbidden edge based at b only if its base lies in $\{b, b - e_i, b - e_j\}$, as listing its four stored positive edges shows. Outside this finite set both an avoiding transport to its base and its boundary avoid the edge, giving the safe probes used by the finite-dimensional span argument. For the finitely many used cover edges, choose torus side larger than twice the largest absolute base coordinate, including the forbidden base, plus one. Centered wrapping then recovers all bases, preserves distinct stored edges and makes the link variation change no path factor. Thus holonomies, curvatures, adjoint transports and finite products are exactly unchanged, uniformly under stencil translation. Take the maximum finite threshold for the marking identity.

The lifted funds source retains finite approximants, their actual terminal costs, detector fees and signed projection residuals. UC3 is its coarsening reader with paid error; ordered multiplication preserves the complete difference rather than only its separate terms. UC1 then recovers a uniform radius bound from these pathwise readers. The safe-probe source retains the avoiding cover paths and finite wrapping threshold, so its torus readout recovers exact independence of the varied link without a limiting approximation.

Avoiding-probe readout recovery. For a step from x to $x+\epsilon e_i$ traversing the forbidden stored edge, choose $j \neq i$ and retain the ordered detour

$$x \longrightarrow x + e_j \longrightarrow x + e_j + \epsilon e_i \longrightarrow x + \epsilon e_i, \qquad \epsilon \in \{1, -1\}.$$

Its transverse stored edges have direction j, and its middle stored edge is the translated direction-i edge, so none equals the forbidden stored edge. Its endpoints are identical to the original step. The detour is a new recorded probe path; no equality with the original path's holonomy is assumed. The ordered positive stored edges in an (i, j) plaquette based at x are based at $x, x + e_i, x + e_j, x$, with directions i, j, i, j, respectively; its last two occurrences are reversed. This recovers the stated finite excluded-base set.

Let $\mathcal{E}$ be the finite used stored-edge set of the resulting probes. Choose M to bound all base coordinates relative to the stencil anchor, including the forbidden base, and take side $N > 2M + 1$. Wrapping is then injective on these bases and preserves directions. After any common translation the same relative-coordinate statement holds. If $V_{e,\tau}$ varies only the forbidden physical link, each probe reading H therefore satisfies exactly

$$H(V_{e,\tau}U) = H(U), \qquad H^Z(z) = H(\pi z), \qquad H^Z(\sigma V_{e,\tau}U) = H^Z(\sigma U).$$

Here π and σ are the configuration realization and section. The finite products and adjoint transports inherit this equality, so every defined link derivative of the probe is zero. The finite maximum of these side thresholds recovers the same marking identity.

22 Gaussian short-distance comparison

The comparison retains its sequence and normalization. The ultraviolet comparator is a free Maxwell polynomial hierarchy on one weak scaling sequence. Simultaneous finite-degree frames

transport labels and derivatives to that comparator, and finite determining position tests fix the prescribed calibration columns. An exact normalization of these finite columns is a statement about those tests; it is not equality of every interacting moment with its Gaussian value at a finite scale.

Private Gaussian marks, corrected bilinear covariances and the occurrence recursion recover the whole separated kernel and its fixed derivatives. The scalar return and annular quadrature estimates retain their Fourier measures, powers of the scale and normalization. The Gaussian prefactor amendment of Appendix I.1 fixes the earlier finite integration convention; it does not authorize a new normalization of these later kernels. Only their stated errors tend to zero, and the weak-sequence comparison by itself supplies no interpolation to every positive small radius. The later trace argument provides the distinct all-distance passage for its specified selected observable.

22.1 Common weak-sequence Gaussian comparison and stress normalization

The mean Euclidean hierarchy uses the same weak sequence $r_k \to 0$, with inverse ultraviolet scale r_k^{-1}; this statement does not interpolate to every positive small scale. For free Maxwell field spaces E_d^0 and simultaneous frames $U_{k,d} : E_d^0 \to E_d$, unit, conjugation, signed rotation and filtration compatibilities hold eventually in each fixed degree, and

$$r_k U_{k,d+1}^{-1} \partial_a U_{k,d} \longrightarrow \partial_a^0. \tag{PT1}$$

Transported infinitesimal rotations also converge; the real conserved stress space is one-dimensional. The free translation-normalized comparator $T^{0,\mathrm{tr}}$ is the negative of the original Wick Maxwell tensor. To fix the sign, if the same T and nonzero derivative family D satisfy translation shell identities with both $cD, c'D$, subtraction and separated exterior testing give $(c - c')D = 0$, so $c = c'$ by field separation. The explicit Maxwell tensor has convention $-D^0$; compare with the constructed tensor to force constant -1. The curvature-label calculation identifies its underlying Maxwell tensor. For any finite free homogeneous list, any stress-slot subset and indices, use $U_{k,d_i} q_i$ in regular slots and $r_k^4 T_{a_i b_i}$ in stress slots. At positions $r_k x$ the mean moments and every fixed position-derivative order converge uniformly on compact distinct sets to Wick moments, with $T^{0,\mathrm{tr}}$ in stress slots. All use the same sequence and frames. Pull every slot back by U^{-1}: regular labels are fixed and stress labels converge by physical comparison and Ward normalization. Fixed-basis expansion reduces to finitely many compact jet comparisons, with converging coefficients and bounded free derivatives. This proves the mixed claim for any number of stress slots on the same table as the selected/reconstructed hierarchy.

Common-sequence frame and mixed-jet recovery. Use the horizontal maps J_d, Q_d on the interacting degree spaces and keep the free spaces as the common comparison coordinates. Put

$$U_{k,d}^Z = J_d U_{k,d}, \qquad (U_{k,d}^Z)^{-1} = U_{k,d}^{-1} Q_d, \qquad \partial_a^Z = J_{d+1} \partial_a Q_d.$$

The inverse is on the horizontal image. The exact finite-level identity

$$r_k (U_{k,d+1}^Z)^{-1} \partial_a^Z U_{k,d}^Z = r_k U_{k,d+1}^{-1} \partial_a U_{k,d}$$

recovers PT1 before taking its limit. The unit, filtration, conjugation, signed rotations and infinitesimal rotations intertwine by the same cancellation $Q_d J_d = 1$.

For a stress slot the pulled-back label is $U_{k,4}^{-1}(r_k^4 T_{ab})$, identical to the lifted pulled-back label $(U_{k,4}^Z)^{-1}(r_k^4 J_4 T_{ab})$. Its comparator is the same $T_{ab}^{0,\mathrm{tr}}$. The nonzero derivative family in the subtracted Ward identities is transported injectively, so $(c - c')D = 0$ and $(c - c')JD = 0$ are equivalent. This

preserves the source's normalization $T^{0,\mathrm{tr}} = -T^{0,\mathrm{Wick}}$ and its one-dimensional real conserved stress space.

For completeness expand the finitely many pulled-back labels in fixed free bases. Write $c_{k,A}$ for the finite product of their coordinates and $W_{k,A}$ for the corresponding mean basis moment at scaled positions; let $c_{0,A}, W_{0,A}$ denote their limits. For every fixed position multi-index α and compact distinct set K, the exact finite expansion gives

$$\begin{aligned}&\left\|\partial^\alpha \sum_A (c_{k,A}W_{k,A} - c_{0,A}W_{0,A})\right\|_K \\ &\quad\le \sum_A |c_{k,A}|\,\|\partial^\alpha(W_{k,A} - W_{0,A})\|_K \\ &\qquad + \sum_A |c_{k,A} - c_{0,A}|\,\|\partial^\alpha W_{0,A}\|_K \longrightarrow 0.\end{aligned}$$

Coefficients are position independent, bounded and convergent; the free derivatives are bounded on K. Every lifted term has the identical scalar readout, so this also recovers arbitrary mixed stress-slot compact jets. The sequence, frames and continuum witness remain the same for all the fixed finite choices in the source; no interpolation in the scale parameter is used.

22.2 Finite position tests and exact frame normalization

At finite horizon d let $\mathcal{B}_d$ be the real space of small endomorphisms b annihilating the unit, preserving each lower filtration space and commuting with conjugation, signed symmetries and multiplication by i. Set

$$M_b = I + b. \tag{PT2}$$

It is complex linear and near zero invertible by geometric series, fixing the unit and the stated structures. The determining tests choose $b_k \to 0$ and frame $U_{k,d}M_{b_k}$. At degree four all lower fields are fixed exactly: for simple gauge algebra the free spaces below four consist of unit multiples, whose degree inclusions remain the unit, annihilated by b. A test i has finite arity n_i, labels $v_{ij} \in E_d^0$ and smooth compact complex profile ψ_i supported on distinct configurations. Let Λ be a finite real linear combination of real readings. In a complex basis e_a, let w contain every profile-integrated basis moment for each test and tuple. Multilinearity gives

$$D_i(w; v_1, \dots, v_{n_i}) = \sum_{a_1,\dots,a_{n_i}} \prod_j (v_j)_{a_j} w_{i,a_1,\dots,a_{n_i}},$$

and the finite real polynomial map is

$$\mathcal{C}_Q(w, b) = \Lambda\big(\mathrm{Re}\, D_i(w; M_b v_{i1}, \dots, M_b v_{in_i})\big)_i. \tag{PT3}$$

For the free integrated array w_Q^0, a square determining collection has invertible derivative

$$\partial_b \mathcal{C}_Q(w_Q^0, 0) : \mathcal{B}_d \longrightarrow \mathbb{R}^{\dim \mathcal{B}_d}. \tag{PT4}$$

At degree four its existence follows from the rigidity calculation below. A test derivative in direction b is the real integrated value of

$$\mathcal{J}_b(x; v) = \sum_j \langle v_1(x_1) \cdots (bv_j)(x_j) \cdots v_n(x_n)\rangle_0. \tag{PT5}$$

If all real derivatives vanish, using $i\psi$ also kills imaginary parts. Smooth compact test separation and continuity give pointwise vanishing on distinct configurations. Rigidity forces $b = 0$, so these real functionals separate $\mathcal{B}_4$. Choose finitely many spanning their dual span; their combined map is injective and has a linear left inverse. Use it and coordinates on $\mathcal{B}_4$ as Λ. Then PT3's derivative is that coordinate isomorphism, proving PT4 with all profiles away from collisions. The finite-dimensional implicit-function theorem gives a locally unique smooth ϕ near w_Q^0 such that

$$\phi(w_Q^0) = 0, \qquad \mathcal{C}_Q(w, \phi(w)) = \mathcal{C}_Q(w_Q^0, 0). \tag{PT6}$$

For nearby (w, b) equality holds exactly when $b = \phi(w)$. Polynomial smoothness and PT4 supply its hypotheses. The actual mean array $w_Q(k)$ uses $U_{k,d}e_a$ at $r_k x$. Compact separated convergence gives $w_Q(k) \to w_Q^0$: each integrated error is bounded by its supremum on the fixed compact profile support times $\|\psi_i\|_{L^1}$. The compact-profile convergence needed in (PT7) requires no distribution extension through collisions. The lifted comparison source retains the common sequence, frames and Ward-normalized tensor, rather than separately selecting stress and regular limits. Its finite-test reader is the polynomial PT3 with the complete basis moment array. PT4–PT6 recover the unique small frame correction from that array; preserving the profile and coordinate data makes this calibration explicit. The rigidity dependency is retained for the ensuing degree-four calculation.

22.3 Exact frame correction and finite numerical jets

Set

$$b_k = \phi(w_Q(k)). \tag{PT7}$$

Then $b_k \to 0$, M_{b_k} is eventually invertible, and PT6 gives exact determining equations in the interacting mean hierarchy. This identifies the correction chosen by the local map with the correction appearing in the endpoint. Finite-basis expansion and interchange of finite sums with integrable profiles identify these equations with PT3. For any other finite separated test collection R, continuity gives

$$\mathcal{C}_R(w_R(k), b_k) \longrightarrow \mathcal{C}_R(w_R^0, 0). \tag{PT8}$$

Thus exact normalization preserves every other leading position test.

Finite-test calibration and frame recovery. The calibration source records the finite basis, every basis-tuple moment, the actual profiles, input labels and real map Λ. The lifted mean moments have identical scalar readouts, so their profile arrays satisfy $w_Q^Z(k) = w_Q(k)$ exactly in these recorded coordinates. Hence the full polynomial, not only its derivative, is recovered:

$$\mathcal{C}_Q^Z(w, b) = \mathcal{C}_Q(w, b), \qquad \partial_b \mathcal{C}_Q^Z(w_Q^0, 0) = \partial_b \mathcal{C}_Q(w_Q^0, 0).$$

Here the free calibration coordinates $b \in \mathcal{B}_d$ are kept unchanged; the interacting frame carries the horizontal lift. The real derivative in a direction $\dot{b}$ is explicitly

$$\partial_b \mathcal{C}_Q(w, 0)[\dot{b}] = \Lambda \left(\operatorname{Re} \sum_{j=1}^{n_i} D_i(w; v_{i1}, \dots, \dot{b}v_{ij}, \dots, v_{in_i}) \right)_i .$$

Thus PT5 is recovered by the same finite multilinear expansion and profile integral. Replacing a profile by $i\psi$ recovers its imaginary functional as well. The source's ensuing rigidity result makes their common kernel zero at degree four. Thus the degree-four adjustment is compatible with the

already chosen frames in degrees zero through three. Choosing an injective finite combined derivative L and a left inverse A, the prescribed real coordinate map can be written $\Lambda = \mathrm{coord} \circ A$; then $\Lambda L = \mathrm{coord}$. The horizontal translation retains L, A and this coordinate isomorphism, recovering PT4 with the same profiles and the same rigidity dependency.

Since the entire polynomial and base point agree, local uniqueness in PT6 gives the identical correction function ϕ in these coordinates. In particular

$$b_k^Z = \phi(w_Q^Z(k)) = b_k, \qquad \widehat{U}_{k,d}^Z = J_d U_{k,d}(I + b_k), \qquad Q_d \widehat{U}_{k,d}^Z = U_{k,d}(I + b_k).$$

For $\|b_k\| < 1$ its inverse on the horizontal image is $(I + b_k)^{-1} U_{k,d}^{-1} Q_d$, with

$$(I + b_k)^{-1} = \sum_{\ell=0}^{\infty} (-b_k)^\ell, \qquad \left\| (I + b_k)^{-1} - \sum_{\ell=0}^{m} (-b_k)^\ell \right\| \le \frac{\|b_k\|^{m+1}}{1 - \|b_k\|}.$$

The finite filtration, unit, conjugation and signed symmetry laws are retained by this inverse and the correction. At horizon four all included degree-zero through degree-three fields are fixed exactly, since their unit multiples are annihilated by b_k.

For each profile the unchanged integral error estimate is $\|\psi_i\|_{L^1}$ times the corresponding compact moment error. It proves $w_Q^Z(k) \to w_Q^0$, hence $b_k \to 0$, while the determining equations hold exactly for all sufficiently large k. Any other fixed finite profile collection has its same transported array $w_R(k)$ and therefore the same PT8 continuity limit. Thus calibration preserves the other leading position tests on the original common sequence and adjusts the frame by the exact local solution, with no independent selection of moments or stress data.

For a fixed finite torus of side greater than one, let $w_{Q,\ell}(m)$ be order-ℓ coefficients of its regulated profile series; order zero tends to w_Q^0 as $m \to \infty$. At fixed finite formal horizon p, the polynomial $c_{Q,p,m}(t) = \sum_{\ell=0}^{p} t^\ell w_{Q,\ell}(m)$ is used only for numerical derivatives, without convergence of the full series. For sufficiently large m its base point, which tends to w_Q^0, is in the PT6 neighborhood, so $\mathcal{C}_Q(c(t), \phi(c(t)))$ is identically the prescribed constant near zero. All positive derivatives vanish exactly and the zeroth one has that value. Another query has order-zero numerical reading

$$\mathcal{C}_R(w_{R,0}(m), \phi(w_{Q,0}(m))) \longrightarrow \mathcal{C}_R(w_R^0, 0). \tag{PT9}$$

Subtracting at $m = k$ from PT8 gives difference tending to zero on the actual-theory sequence. Order zero is independent of p. This calibrates each finite determining jet and compares other profiles at order zero; it does not compare every higher coefficient to the interacting theory. PT6 local uniqueness makes every other small correction solving the same equations eventually equal to PT7. Together with PT2 this retains all frame symmetries and exactly fixes raised lower fields through degree three.

Finite numerical-jet recovery. Retain each regulated coefficient array as a separate numerical entry, with its finite formal horizon p and mesh index m. Equality of the scalar coefficient readouts gives

$$c_{Q,p,m}^Z(t) = c_{Q,p,m}(t), \qquad \frac{d^j}{dt^j} c_{Q,p,m}(0) = j! w_{Q,j}(m) \quad (0 \le j \le p).$$

The polynomial curve starts at $w_{Q,0}(m)$, so for each sufficiently large m it lies in the local calibration neighborhood for t near zero. Write $H_Q(w) = \mathcal{C}_Q(w, \phi(w))$ there. PT6 makes H_Q the constant $\mathcal{C}_Q(w_Q^0, 0)$, whence

$$\frac{d^j}{dt^j} H_Q(c_{Q,p,m}(t))\big|_{t=0} = \begin{cases} \mathcal{C}_Q(w_Q^0, 0), & j = 0, \\ 0, & j > 0. \end{cases}$$

The identical polynomial and correction function recover this equality in the lift at every retained finite horizon. No infinite series is evaluated. For a different query put $a_k = \mathcal{C}_R(w_R(k), b_k)$ and $d_k = \mathcal{C}_R(w_{R,0}(k), \phi(w_{Q,0}(k)))$. Both converge to $a_0 = \mathcal{C}_R(w_R^0, 0)$, so explicitly

$$\|a_k - d_k\| \le \|a_k - a_0\| + \|d_k - a_0\| \longrightarrow 0.$$

The second reading uses only order-zero arrays and is independent of p. The same local uniqueness recovers the eventual equality of any other small correction satisfying the determining equations. This retains exactly the finite calibration and order-zero comparison scope of the source.

22.4 Degree-four rigidity of the position tests

The free degree-four basis is the unit together with unordered plane-pair curvature labels

$$e_c = \sum_\alpha F^\alpha_{ab} F^\alpha_{uv}. \tag{PT10}$$

Each plane has increasing-axis orientation; reversing one changes sign and swapping the two planes preserves the product. The free quotient calculation supplies this basis and dual coordinates ℓ_*, ℓ_c. At separation $(e_0, 0)$ curvature pairings are diagonal on six planes with nonzero γ_p. Wick contraction gives

$$\langle v(e_0) e_c(0)\rangle_0 = a_c \ell_c(v), \qquad a_c \neq 0. \tag{PT11}$$

Repeated planes have two equal pairings; distinct planes have their single matching pairing. The coefficient is the product of nonzero curvature factors, $\dim \mathfrak{g}$ and its nonzero multiplicity, absorbed in the dual normalization. Expectation is the unit coordinate. For zero unit coordinate the reversed pairing has the same formula. Basis expansion proves these identities without requiring this bilinear form to be positive definite. For $c = (p, q)$ define parity $p \triangle q$. Axis reflection gives sign minus precisely on axes in that set. Commutation with reflections forces $b_{cd} = \ell_c(be_d) = 0$ for different parities. Equal parities admit an involutive axis permutation exchanging the plane pairs up to orientation and swap. If parity is empty both repeat a plane and a transposition or double disjoint transposition exchanges the planes. If it has two elements, each pair consists of two planes sharing one of the other two axes; exchange those common axes when they differ. For size four, a transposition exchanges the different complementary-plane partitions. Identical pairs use the identity. These lie among identity, six transpositions and three double disjoint transpositions. The induced exchange signs agree, since squaring the involution is identity on nonzero basis vectors. Commutation gives $b_{cd} = b_{dc}$. Also $a_c = a_d$ within parity: multiplicities agree, absolute curvature factors agree and the product of curvature signs at the axis-zero separation depends only on whether zero is in parity. If PT5 vanishes for every tuple, arity one gives $\ell_*(bv) = 0$; arity two gives

$$\langle (bv)(e_0) w(0)\rangle_0 + \langle v(e_0)(bw)(0)\rangle_0 = 0. \tag{PT12}$$

Taking $v = e_c, w = e_d$ yields $a_d b_{dc} + a_c b_{cd} = 0$. Different-parity entries vanish already; equal-parity symmetry gives $2a_c b_{cd} = 0$, so all vanish. The unit column is zero since $b1 = 0$ and its row by arity one. Thus $b = 0$, closing PT4's determining-test existence proof through degree four.

Degree-four basis and rigidity readout. Transport the source's actual free quotient basis and its dual by $e_c^Z = J_4 e_c$, $1^Z = J_4 1$, and $\ell_c^Z = \ell_c Q_4$, $\ell_*^Z = \ell_* Q_4$. For the allowable matrix use $b^Z = J_4 b Q_4$. Then

$$\ell_c^Z(b^Z e_d^Z) = b_{cd}, \qquad \langle (J_4 v)(e_0) e_c^Z(0)\rangle_0^Z = a_c \ell_c(v), \qquad \mathcal{J}_{b^Z}^Z = \mathcal{J}_b.$$

The horizontal free reader here is the transported free Wick reader. The oriented-plane signs, unordered products and fixed multiplicities are retained, so no new positivity convention enters PT11. Reflection characters and the finite signed axis permutations are conjugated by J_4, Q_4. They therefore give the same parity blocks, zero off-block matrix entries and symmetric within-block entries.

On the curvature-label block let $D = \operatorname{diag}(a_c)$. The arity-two identity is exactly

$$Db + b^{\mathsf{T}} D = 0.$$

Within each parity block $D = a_c I$ with $a_c \neq 0$, and the source's involutions give $b^{\mathsf{T}} = b$ on that block. Thus the recovered identity is $2a_c b = 0$ there. Every off-block entry already vanishes; the unit column is zero from $b1 = 0$ and the unit row from arity one. The same full matrix is recovered by the dual coordinates after horizontal transport, so $b^Z = 0$ on its horizontal domain if and only if $b = 0$. This recovers the precise degree-four rigidity used for the determining derivative, with its quotient-basis and free contraction dependencies intact.

For the physical stress limit let the nonzero raw centered conserved tensor satisfy $J_{\mathrm{raw}} = c\,\mathrm{center}(T)$, $c \neq 0$, by the one-dimensional divergence kernel. The centered physical tensor is nonzero because its translation charge differentiates a nonzero field. Shell identities and field separation give $D_{\mathrm{raw}} = cD$. If $\xi_k \neq 0$ is the preliminary comparison scale, weak derivative comparison gives $c/\xi_k \to \kappa_0 \neq 0$. Rescaling the raw limit therefore yields

$$U_{k,4}^{-1}(r_k^4\,\mathrm{center}(T)) \longrightarrow \kappa_0^{-1} T^0. \tag{PT13}$$

The difference from T is a fixed unit tensor whose inverse-scaled contribution is r_k^4 times that unit and tends to zero, with eventual exact unit identifications. Ward normalization above gives $\kappa_0 = -1$. The same factor $(c/\xi_k)^{-1}$ normalizes the preliminary angular-generator limit, giving the final rotation limit. For compact separated profiles no contact extension is used: if $f_k \to f_0$ uniformly on support and $\delta = \varepsilon/(1 + \|\psi\|_1)$, eventually

$$\left| \int \psi (f_k - f_0) \right| \leq \|\psi\|_1 \sup_{\operatorname{supp}\psi} |f_k - f_0| \leq \|\psi\|_1 \delta < \varepsilon.$$

The lifted calibration source retains finite formal curves separately from actual hierarchy arrays. PT7 reads their unique small frame correction, and PT9 recovers only the asserted numerical order-zero comparison. The plane-pair basis, parity action and Wick coefficients supply an explicit injective derivative reader, recovering b from the determining tests. PT13 retains the charge normalization and exact unit removal, recovering the same physical stress sign used in all mixed-slot limits.

Physical stress and compact-profile recovery. With the horizontal maps on degree four, the raw/physical relation and Ward derivative relation recover exactly as

$$J_4 J_{\mathrm{raw}} = c J_4\,\mathrm{center}(T), \qquad J D_{\mathrm{raw}} = cJD.$$

The same nonzero c and comparison scale ξ_k are retained. Before any limit the normalization identity in free coordinates is

$$U_{k,4}^{-1}(r_k^4\,\mathrm{center}(T)) = (c/\xi_k)^{-1}\,\xi_k^{-1} U_{k,4}^{-1}(r_k^4 J_{\mathrm{raw}}).$$

Using $(U^Z_{k,4})^{-1} = U^{-1}_{k,4}Q_4$ makes this identical for lifted tensors. Since $c/\xi_k \to \kappa_0 \neq 0$, it recovers PT13 and the same normalization of the angular generator. Write $T - \mathrm{center}(T) = \lambda\mathbf{1}$ componentwise, with fixed λ. Exact unit identification gives

$$(U^Z_{k,4})^{-1}(r_k^4 J_4(T - \mathrm{center}(T))) = r_k^4\lambda\mathbf{1}^0 \longrightarrow 0.$$

Thus horizontal recovery retains the Ward-normalized sign $\kappa_0 = -1$ and the full physical tensor, including the vanishing center correction. For each separated compact profile the finite moment readout is unchanged; its Bochner integral error remains bounded by $\|\psi\|_1$ times the uniform moment error on its support. This recovers the actual arrays without any contact extension.

22.5 Moving Gaussian rows and normalized profile sums

Smooth separated moments make the preceding profile integrals ordinary Bochner integrals, proving every component of $w_Q(k) \to w^0_Q$. For numerical regulator arrays, order-zero Wilson coefficients are Gaussian Wick contractions of reduced local polynomials. The common field-address identity identifies these with the free labels. With fixed refinement base $L > 1$, denominator L^m, spacing L^{-m} and periods growing faster than each fixed polynomial margin, the Gaussian row limit holds for every refining subsequence and moving grid tuple tending to distinct continuum positions. The profile sum then gives

$$w_{Q,0}(m) \longrightarrow w^0_Q. \tag{PT14}$$

Here are its normalization and convergence details. Let $h_m = D_m^{-1} \to 0$, D_m integer, and torus side N_m satisfy $h_m N_m \to \infty$. Coordinate rounding in arity n gives

$$|\rho_m x - x| \leq 4nh_m. \tag{PT15}$$

Every bounded set eventually embeds without wrapping. Assume the joint moving limit: for every subsequence m_j, allowed x, and grid tuple o_j embedded at positions tending to x, $K_{m_j}(o_j) \to K_0(x)$. This implies an eventual uniform bound on rounded points of any compact allowed set. Otherwise choose values exceeding j at $m_j \geq j$; compactness gives a convergent position subsequence, and PT15 plus nonwrapping contradicts the moving limit. For a continuous compact profile ψ supported in the allowed region put $f_m(x) = \psi(\rho_m x)K_m(\rho_m x)$. It converges pointwise to $\psi(x)K_0(x)$; outside closed support the rounded positions are eventually outside. If support lies in a radius-R ball, these step functions lie in radius $R + 4n$ for $h_m \leq 1$ and have one common bound there. They are integrable, and dominated convergence applies. Every grid cell has volume h_m^{4n}, so the integral is exactly

$$h_m^{4n} \sum_{o \in (\mathbb{Z}/N_m\mathbb{Z})^{4n}} \psi(h_m o)K_m(o). \tag{PT16}$$

No wrapping occurs on the support. Arity zero has a one-point grid and space with cell volume $h^0_m = 1$, giving the same conclusion.

Moving-row and grid-integral recovery. Retain the finite regulator row, mesh, period and rounding cells in the source record. Write K^Z_m for its lifted scalar reading; then $K^Z_m(o) = K_m(o)$ for every retained grid tuple. Consequently its joint moving limit and every selected violation of a uniform compact bound are the identical scalar statements after recovery. A fixed compact profile support lies inside a compact neighborhood of the allowed open region. For sufficiently fine meshes, all rounded points relevant to its nonzero values lie in that neighborhood. The moving-limit bound therefore gives some finite M with

$$|\psi(\rho_m x)K^Z_m(\rho_m x)| \leq M\|\psi\|_\infty \mathbf{1}_{\{|x| \leq R+4n\}}(x).$$

This is an integrable common bound in the finite-dimensional configuration space. The transported step function is literally the same function, so dominated convergence commutes with its readout.

For the actual rounding cell $C_{m,o}$ the constant reading and its volume give the exact finite identity

$$\int_{C_{m,o}} \psi(\rho_m x) K_m^Z(\rho_m x)\, dx = h_m^{4n} \psi(h_m o) K_m(o).$$

Summing the cells intersecting the fixed support recovers PT16 with its full $4n$-dimensional normalization. Eventual nonwrapping uses the unchanged $h_m N_m \to \infty$. In arity zero this identity is the one-point integral with weight one. Through the same field-address identity the reduced order-zero polynomials represent the fixed free labels, so this profile recovery gives PT14 once the corrected covariance and moving Wick-row limit below are supplied. Those remaining dependencies are retained rather than replaced by an independent choice of continuum profile data.

Reduce the row limit to covariances by indexing every occurrence of every generator separately. Write $\tau(a)$ for its insertion slot and ζ_a for its generator label. The order-zero Gaussian product moment on a finite occurrence set A satisfies integration by parts:

$$|A| M_m(A) = \sum_{a \in A} \sum_{b \in A \setminus \{a\}} C_m(a,b) M_m(A \setminus \{a,b\}). \tag{PT17}$$

The empty moment is one by probability normalization. Continuum Wick moments obey the same recursion with C_0. Induction on $|A|$, the covariance limit and finite summation prove convergence after division by the positive integer $|A|$. Distinguishable occurrences preserve multiplicities of repeated generators. Finite polynomial expansion then proves the limit for all allowed sections; the field-address identity makes the limit exactly their free Wick moments, without a new fit. Applying this along every refining subsequence gives PT16's moving-limit hypothesis. The corrected covariance limit, including its finite-regulator subtraction, is supplied next. For $A = \varnothing$, $M_m(A) = 1$ because the Gaussian law is a probability measure. The lifted row source retains occurrence indices, reduced polynomial addresses, grid cells and their actual volume. Its Gaussian recursion reader preserves multiplicities and normalized empty moments; its profile reader is PT16. Joint moving convergence supplies the common bound needed by the integral reader, recovering PT14 without a pointwise-only convergence substitution.

22.6 Private Gaussian marks and corrected bilinear covariance

Fix the maximal jet grade d in the row and its finite coordinate keys a. The regulated real potential Z has precision P and covariance P^{-1}; its symmetrically averaged linear jet coordinate at x is $k_{h,x,a}^T Z$, using the perturbative integral's same density. For sine/cosine waves, axes and color labels c, define the infinite lattice subtraction matrix by their responses $l_{h,a,c}(p)$:

$$S_h(a,b) = (2\pi)^{-4} \int_{[-\pi,\pi]^4} \delta(p)^{-1} \sum_c l_{h,a,c}(p) l_{h,b,c}(p)\, dp. \tag{PT18}$$

Responses are bounded by continuity on the cube and δ^{-1} is integrable in four dimensions. Entries exist, symmetry holds, and

$$v^T S_h v = (2\pi)^{-4} \int \delta(p)^{-1} \sum_c \left(\sum_a v_a l_{h,a,c}(p) \right)^2 dp \geq 0.$$

Thus S_h is the covariance of a possibly degenerate real centered Gaussian vector, requiring no inverse. Give each insertion i an independent such vector Y_i, independent of Z, and set $X_{i,a} = k_{h,x_i,a}^T Z + \mathrm{i} Y_{i,a}$.

Their bilinear covariance is

$$C_h((i,a),(j,b)) = k_{h,x_i,a}^T P^{-1} k_{h,x_j,b} - \mathbf{1}_{i=j} S_h(a,b). \tag{PT19}$$

Mixed terms vanish by independence/centering; private marks are independent across insertions and their imaginary coefficients give the minus sign within one insertion. No conjugation occurs. Same-slot entries need not vanish at finite regulator. The normalized perturbative constant term is exactly the product of reduced insertion polynomials evaluated at these X and then averaged. Weight and vacuum denominator have constant coefficient one. Each linear insertion's constant term is its physical coordinate plus imaginary private mark; finite polynomial expansion commutes with constant extraction. Private integration gives the prescribed within-insertion subtraction; independence combines it with physical integration. Polynomial Gaussian integrability, from $C(1+\|z\|)^q$ and finite Gaussian moments, justifies expansion and Fubini. For a real centered linear Gaussian probe of variance v, its moment generating function $\exp(vt^2/2)$ gives $\mathbb{E}L^{r+1} = rv\mathbb{E}L^{r-1}$. Apply to arbitrary real linear combinations and compare coefficients to obtain by polarization

$$|A|\mathbb{E}\prod_{\alpha\in A} X_\alpha = \sum_{\alpha\in A}\sum_{\beta\in A\setminus\{\alpha\}} \mathbb{E}(X_\alpha X_\beta)\mathbb{E}\prod_{\gamma\in A\setminus\{\alpha,\beta\}} X_\gamma. \tag{PT20}$$

Complex multilinear expansion extends this from real probes, including degenerate laws. Repeated generators retain distinct occurrences; the empty product is one. This is precisely PT17's finite recursion.

Private-mark source and exact polynomial recovery. The source retains the real potential law and a separately named private mark for each insertion. A concrete realization of a possibly degenerate mark is $Y_i = S_h^{1/2}\eta_i$, with independent standard real Gaussian vectors η_i, independent also of Z. Its pushforward covariance is exactly S_h; the positive square root requires no inverse. On the resulting product Gaussian source define the complex linear readout

$$\mathcal{X}_{i,a}(Z,\eta_1,\dots,\eta_n) = k_{h,x_i,a}^T Z + \mathrm{i}(S_h^{1/2}\eta_i)_a.$$

Private coordinates are indexed by insertion, not by distinct physical position or by generator name. Thus repeated occurrences in one slot share its mark, while different slots retain independent marks. The same readout on the horizontal potential copy uses the transported Gaussian law and coefficient maps, giving precisely the original $\mathcal{X}_{i,a}$. Its bilinear second moment is PT19, with $\mathrm{i}^2 = -1$ and no conjugation. This preserves finite-regulator same-slot terms rather than setting them to zero.

For an insertion polynomial Q_i and a fixed physical coordinate vector $z_i = (k_{h,x_i,a}^T Z)_a$, the private integral has the exact finite polynomial form

$$\mathbb{E}_{Y_i} Q_i(z_i + \mathrm{i}Y_i) = \left[\exp\left(-\frac{1}{2}\sum_{a,b} S_h(a,b)\partial_a\partial_b\right) Q_i\right](z_i).$$

The exponential terminates on each polynomial. This follows by finite Taylor expansion and Gaussian moments, or by comparing polynomial coefficients of the Gaussian generating function. Independence of the marks gives, conditional on the same physical Z,

$$\mathbb{E}_{Y_1,\dots,Y_n}\prod_i Q_i(z_i+\mathrm{i}Y_i) = \prod_i \mathbb{E}_{Y_i} Q_i(z_i+\mathrm{i}Y_i).$$

Integrating the physical law recovers exactly the normalized perturbative constant term. The weight and vacuum constant terms are one, so no additional denominator appears at this order.

Polynomial integrability justifies each finite expansion and this product-law integration, even for a degenerate mark law.

For occurrence indices $\alpha \in A$, set $G(t) = \mathbb{E}\exp(\sum_{\alpha\in A} t_\alpha X_\alpha)$. Its Gaussian coefficient identity is

$$G(t) = \exp\left(\frac{1}{2}\sum_{\alpha,\beta\in A} t_\alpha C_h(\alpha,\beta)t_\beta\right).$$

Only its finite derivatives at zero are used. The identity $\partial_{t_\alpha} G = \sum_\beta C_h(\alpha,\beta)t_\beta G$, followed by extraction of the multilinear coefficient and summation over α, recovers PT20 and its factor $|A|$. Distinct occurrence variables preserve all repeated-generator multiplicities. The empty moment remains one. Induction over $|A|$ then transports PT17 exactly and, once the corrected covariance limit is supplied, recovers every finite polynomial moving-row limit with the original field addresses.

The corrected covariance limit has two roles: for distinct limiting positions z_i, z_j it tends to the Maxwell kernel $K(k_a, k_b; z_i - z_j)$ on the corresponding free solution-jet functionals; for the same slot it tends to zero:

$$C_h((i,a),(i,b)) \longrightarrow 0. \tag{PT21}$$

This vanishing follows after regulator removal. For the finite signed coordinate group $\mathcal{S}$, $c_{\mathcal{S}} = |\mathcal{S}|^{-1}$, the averaging identity beginning its proof is

$$A(\mathrm{wideLinear}_{h,x}) = c_{\mathcal{S}} \sum_{s\in\mathcal{S}} A \circ \mathrm{turn}_s \circ \mathrm{radialMap}(\mathrm{chart}_{s,h,x}). \tag{PT22}$$

Write A_s for the inner functional. The next sampling/reconstruction identity identifies its free limit. The lifted Gaussian source retains Z and the separate private Y_i, with the same finite density and explicit subtraction matrix. The polynomial reader is expectation after $X = k^T Z + iY$; PT19 proves compatibility with the prescribed subtraction. Keeping bilinear rather than Hermitian covariance recovers the sign and same-slot remainder exactly. PT20 supplies the finite occurrence reader without a covariance inverse, so degeneracy introduces no lost branch. The limit reader retains both the cross-slot kernel and the corrected diagonal tending to zero rather than imposing Wick subtraction prematurely.

22.7 Averaged reconstruction and corrected covariance limits

Radial reconstruction inverts sampling on free solution jets, so

$$\mathrm{plain}_s(A_s) = A \circ \mathrm{gaugeJet}\,.$$

Independently transformed stencils with inverse turns therefore give the same free two-point form for all s, t, with fixed relative endpoints. The finite covariance is $c_{\mathcal{S}}^2 \sum_{s,t} \mathrm{ordinary}_{s,t}(h; x, y; A_s, B_t)$. Expanding both wave factors in PT18 gives the same finite double average of infinite-lattice shell forms; integrability justifies interchange with the integral. The separated chart limit gives the same free pairing for each summand. At equal endpoints the return chart limit makes each ordinary form minus its shell form tend to zero. Finite averaging proves both PT21 and the separated limit. The reporting schedule is

$$h_m = L^{-m}, \qquad N_m = 4L^{m^2+m+1}, \qquad h_m N_m = 4L^{m^2+1}, \quad L > 1 \text{ integer}. \tag{PT23}$$

Thus $N_m h_m^q \to \infty$ for every fixed integer q, and $4 \mid N_m$. Center positions at the quarter-period point. Convergent centered physical coordinates eventually have integer coordinates smaller than

$N_m/4$, so fixed stencil shifts of differences do not wrap and have the required physical limit. Sums of positions are twice the quarter center plus bounded physical displacement; their lifts have a coordinate at least $N_m/4$ in absolute value. Their physical distance tends to infinity, including at coincident observed positions, supplying the inversion-image return condition. PT20 induction now gives moments with zero within-insertion pairings and Maxwell cross-insertion pairings, exactly the reduced free Wick rule. Finite polynomial expansion and PT15–PT16 give every moving row and profile limit. This completes the Gaussian constant-coefficient step needed in the leading perturbative comparison.

An elementary stencil is a finite signed shifted-edge sum, with axis, offsets and ordered derivative list D_a. Its two-point form consequently has four terms, with signs $+,-,-,+$. Its Fourier polynomial factors into sign, translation character and difference multipliers. Multiplication gives the exact identity

$$\sum_{t,s} a_t b_s \chi_p(z + u_t \pm v_s) = \sigma_{a,b,\pm}\chi_p(z + u_{a,\pm} + u_{b,\pm}) \prod_{\nu \in D_a D_b} q_\nu(p). \tag{PT24}$$

Multiplying by the inverse-Laplacian weight and summing identifies covariance with the differentiated scalar torus kernel, including the same sign and factor $h^{-|D_a|-|D_b|-2}$. On the infinite cover, sum sine/cosine channels using $\cos u \cos v + \sin u \sin v = \cos(u-v)$. Unequal axes or colors give zero; matching ones give the offset-difference cosine. Integration against $(2\pi)^{-4}\delta^{-1}$ is the cover scalar Green function, justified by $|\cos| \le 1$ and integrability. Curvature coordinates are differences of two stencils, giving the same four signs $+,-,-,+$ in torus covariance and cover subtraction. PT24 reduces separated terms to scalar differentiated limits and inversion-image terms to the zero far-distance limit. At coincidence subtract the cover value before rescaling: the return estimate vanishes after any fixed inverse power of h by PT23. Finite array coordinate functionals span each raw jet-response dual; bilinear expansion therefore extends these elementary limits to arbitrary chart functionals in PT22; the responses are arrays of color components, vector-index pairs and derivative-index words, so coordinate projections span the dual of each finite array space. The lifted stencil source retains the exact Fourier factors, turns, relative endpoints and common torus/cover subtraction. Its finite average reader recovers the free solution-jet functional, and PT24 transports covariance to scalar kernel readers without replacing any stencil by its limit. The quarter-period placement and PT23 preserve the far-image and return hypotheses needed for the corrected diagonal.

Finite averaged-covariance recovery. Record the actual signed group and every ordered pair (s,t) in its double average. Denote the corresponding torus and infinite-cover forms by $O_{h;s,t}$ and $H_{h;s,t}$. The exact same-slot identity is

$$C_h((i,a),(i,b)) = |\mathcal{S}|^{-2} \sum_{s,t\in\mathcal{S}} (O_{h;s,t} - H_{h;s,t}).$$

Horizontal transport of each stencil functional, sampling map, turn and radial reconstruction leaves these scalar forms unchanged. On free solution jets the equality $\mathrm{plain}_s(A_s) = A \circ \mathrm{gaugeJet}$ recovers the same output for each signed record. In particular

$$|C_h((i,a),(i,b))| \le \max_{s,t} |O_{h;s,t} - H_{h;s,t}| \longrightarrow 0.$$

The subtraction is inside each summand before this limit. For separated slots no private subtraction occurs; if K is their common free pairing, the corresponding estimate is

$$\left| |\mathcal{S}|^{-2} \sum_{s,t} O_{h;s,t} - K \right| \le \max_{s,t} |O_{h;s,t} - K| \longrightarrow 0.$$

Finite group size makes these implications exact and preserves the independent transformations of both stencils. The subsequent scalar return, separated and far-image estimates supply these entrywise limits with their stated rescalings.

The schedule is retained literally. Its explicit growth is

$$N_m h_m^q = 4L^{m^2+(1-q)m+1} \longrightarrow \infty \quad \text{for every fixed integer } q.$$

For a bounded physical displacement and any fixed finite stencil offset, their ratio to $h_m N_m$ tends to zero. Hence differences have the stated nonwrapping lifts; a sum at twice the quarter-period center has lifted distance at least $h_m N_m/4$ eventually, after allowing the bounded displacement and offsets. The same geometry holds when the observed endpoints coincide. No observation point, stencil shift or period is reselected in the horizontal lift.

For explicit scalar-stencil recovery set $P_a(p) = \sum_t a_t \chi_p(u_t)$. Then the left side of PT24 is $\chi_p(z) P_a(p) P_b(\pm p)$. Multiplication of the actual finite factorizations of P_a, P_b gives exactly its sign, offset and ordered difference factors. The horizontal reader retains these coefficients, so both the Fourier sum and the infinite-cover integral have their original normalization and $h^{-|D_a|-|D_b|-2}$ price. If $L_a = L_{a,1} - L_{a,2}$ and $L_b = L_{b,1} - L_{b,2}$ are curvature readings, their bilinear covariance, and separately its cover term, is exactly

$$B(L_a, L_b) = B(L_{a,1}, L_{b,1}) - B(L_{a,1}, L_{b,2}) - B(L_{a,2}, L_{b,1}) + B(L_{a,2}, L_{b,2}).$$

Thus the common four-term subtraction is recovered before scaling. Finite coordinate expansions then recover arbitrary chart functionals. The occurrence recursion uses these very covariance entries, so its limit retains zero within-insertion pairings and the specified Maxwell cross-insertion pairings. This closes the moving-row/profile recovery conditional on precisely the scalar estimates developed next.

22.8 Scalar return with every fixed jet rescaling

On $Q = [-1/2, 1/2]^4$ put $d(x) = \delta(2\pi x)$ and $f_a(x) = d(x)^{-1} \cos(2\pi a \cdot x)$ for fixed integer displacement a, assigning $f_a(0) = 0$ as in the inverse-Laplacian zero-mode convention. The integral is unaffected and the symbol bound gives $|f_a(x)| \le \|x\|^{-2}$. Outside null cell boundaries, quadrature representatives obey $v_N(x) \in Q$, $\|v_N(x) - x\| \le 4/N$ and $\|v_N(x)\| \ge 1/N$ when nonzero. Thus $\|x\| \le 5\|v_N(x)\|$ and $|f_a(v_N(x)) - f_a(x)| \le 26\|x\|^{-2}$, including the zero cell. For $0 < r \le 1$, $8/N \le r$ and $\|x\| > r$, the representative norm is at least $r/2$. Bounded derivatives of cosine and symbol give

$$|f_a(y) - f_a(x)| \le C_a r^{-4} \|y - x\| \quad (\|x\|, \|y\| \ge r),$$

by the exact quotient subtraction

$$(\cos_a(y) - \cos_a(x)) d(y)^{-1} + (d(x) - d(y)) d(y)^{-1} d(x)^{-1} \cos_a(x).$$

Using $r/2$ gives the outer error $A_a N^{-1} r^{-4}$. The inner singularity is integrable in four dimensions, with

$$\int_{\|x\|<2r} \|x\|^{-2} dx = r^2 \int_{\|u\|<2} \|u\|^{-2} du.$$

Split the error at r to obtain

$$|\operatorname{Re} G_N(a) - G_\infty(a)| \le C_a r^2 + A_a N^{-1} r^{-4}. \tag{PT25}$$

The torus sum is exactly $\int_Q f_a(v_N(x))dx$ since cells have volume N^{-4}, and the cover integral is $\int_Q f_a$ by $p = 2\pi x$. Even N makes symmetric representatives tile this cube. For fixed integer $d \geq 0$ choose $r = h^{d+1}$; PT23 eventually ensures $r \leq 1$, $8/N \leq r$. Then

$$h^{-d}|\operatorname{Re} G_N(a) - G_\infty(a)| \leq C_a h^{d+2} + A_a(Nh^{5d+4})^{-1} \longrightarrow 0. \tag{PT26}$$

This retains every fixed inverse mesh power needed by a finite jet. The lifted return source keeps the zero-mode convention, representative cells, exact quadrature normalization and the singular/regular split. Its scalar reader is the difference of torus and cover values before rescaling; PT25 transports that difference with an explicit bound and PT26 recovers its vanishing at the jet reader's actual scale.

Quadrature subtraction and fixed-power recovery. The scalar return source retains the actual representative cells $C_{N,u}$, zero-mode value and fixed integer displacement a. Its two scalar readers and their exact discrepancy are

$$\begin{aligned} R_N(a) &= N^{-4}\sum_u f_a(u) = \int_Q f_a(v_N(x))\,dx, \\ R_\infty(a) &= \int_Q f_a(x)\,dx, \\ E_N(a) &= \int_Q [f_a(v_N(x)) - f_a(x)]\,dx. \end{aligned}$$

Thus $E_N(a) = \operatorname{Re} G_N(a) - G_\infty(a)$ before multiplication by any inverse mesh power. The normalization $(2\pi)^{-4}dp = dx$ under $p = 2\pi x$ and cell volume N^{-4} are retained entries of these readers. Pulling the source record to its horizontal section changes neither reader nor their difference.

For a nonzero representative the inequalities $\|x\| \leq \|v_N(x)\| + 4/N \leq 5\|v_N(x)\|$ give the integrable bound $26\|x\|^{-2}$ for the discrepancy; in the zero cell $f_a(v_N(x)) = 0$ gives the same bound directly. All these statements are up to the stated null boundaries and zero point. Splitting at r gives explicitly

$$|E_N(a)| \leq 26\int_{Q\cap\{\|x\|\leq r\}} \|x\|^{-2}\,dx + A_a N^{-1} r^{-4}\operatorname{vol}(Q).$$

The first term is at most a fixed constant times r^2 in dimension four and $\operatorname{vol}(Q) = 1$. This recovers PT25 using the combined subtraction rather than separate singular estimates after rescaling.

For fixed $d \geq 0$ the retained choice $r = h^{d+1}$ yields exactly

$$h^{-d}r^2 = h^{d+2}, \qquad h^{-d}N^{-1}r^{-4} = (Nh^{5d+4})^{-1}.$$

Along PT23, $Nh^{d+1} \to \infty$ makes $8/N \leq r$ eventually, and $Nh^{5d+4} \to \infty$ gives the second vanishing term. Consequently the same reader recovers PT26 for each fixed power required by a finite jet. If a fixed finite stencil calculation uses $\sum_t \lambda_t E_N(a_t)$ with constant coefficients and bounded rescaling degree, the finite sum of these bounds proves its return limit with the same schedule, after taking the maximum of finitely many thresholds. This transports the actual torus-minus-cover cancellation used for arbitrary finite chart functionals.

22.9 Whole scalar kernels from annular readers

Suppose $A_m(2s) - A_m(s) \to A(2s) - A(s)$ for every $s > 0$, and with constants independent of s, eventually for each fixed s,

$$|A_m(s)| \leq Cs^p, \quad |F_m - A_m(s)| \leq B/s, \quad |A(s)| \leq Es^p, \quad |F - A(s)| \leq D/s, \quad p \geq 1. \tag{PT27}$$

Then $F_m \to F$. Choose small $s \le 1$ with $(C+E)s^p < \varepsilon/3$, then integer q with $(B+D)/(2^q s) < \varepsilon/3$. Finite annular telescoping gives $A_m(2^q s) - A_m(s) \to A(2^q s) - A(s)$. Take m late enough for its error below $\varepsilon/3$. Add large-cutoff tails and small-cutoff sizes to bound $|F_m - F| < \varepsilon$. The order is s, q, m; cutoff-uniform convergence is not assumed. For differentiated torus kernels at nonzero physical separation use $p = |D| + 2$. Smooth Fourier cutoffs have bound $Cs^{|D|+2}$ and complement B/s away from zero. Their mesh conditions $(2\pi/N)(|D|+4) \le hs$, $hs \le 1/2$ hold eventually for fixed s by $Nh \to \infty$, $h \to 0$. Annular Riemann convergence and PT27 give the separated scalar limit. Its continuum candidate is $\partial_D G(z)$, $G(z) = (4\pi^2\|z\|^2)^{-1}$. With Fourier character $e^{2\pi i k\cdot z}$, an integrable heat regulator $b > 0$ gives inverse transform

$$\partial_D G(z) - \partial_D(G(z)e^{-\pi^2\|z\|^2/b}), \quad z \neq 0.$$

Set $b = v^{-2}$; the second term is $v^{|D|+2}\partial_D(Ge^{-\pi^2\|\cdot\|^2})(vz)$, tending to zero by rapid derivative decay at infinity. For a physical displacement whose norm tends to infinity, the same comparison principle applies with $F = 0$, $A(s) = 0$. At fixed Fourier cutoff, dominated convergence removes heat inside the integral. Annular tail bounds persist and sum as $\sum_j C/(2^j s) \le 2C/s$, supplying the continuum estimates in PT27. For diverging physical separation take $F = A = 0$. Summation by parts gives

$$|\operatorname{annulus}_{h,s,D}(x)| \le \frac{C_D}{s}\left(\frac{2\pi}{\|h\operatorname{lift}(x)\|}\right)^{|D|+3},$$

which vanishes for fixed s. Low-frequency and tail bounds remain valid once distance is at least one. PT27 proves the far-image limit.

Whole-kernel recovery from annular differences. Retain both the annular reader and its small-cutoff anchor in the source. For fixed $s > 0$ and finite integer $q \ge 0$, put $S = 2^q s$. The exact telescoping discrepancy is

$$E_{m;s,q} = \sum_{j=0}^{q-1}[A_m(2^{j+1}s) - A_m(2^j s) - A(2^{j+1}s) + A(2^j s)] \longrightarrow 0.$$

An empty sum at $q = 0$ is zero. The full recovered difference satisfies

$$F_m - F = [F_m - A_m(S)] + E_{m;s,q} + [A_m(s) - A(s)] + [A(S) - F],$$

and therefore, eventually for the fixed cutoffs,

$$|F_m - F| \le \frac{B+D}{S} + |E_{m;s,q}| + (C+E)s^p.$$

The choice of small s, then finite q, then sufficiently large m is preserved by the lift. These identities expose the anchoring term $A_m(s) - A(s)$ which controls the additive constant left undetermined by annular differences alone. Every scalar cutoff and torus reader has its identical horizontal readout, so these finite identities and bounds recover the whole kernel, not merely its annular increments.

For the separated torus application, the retained mesh inequalities are exactly

$$\frac{2\pi(|D|+4)}{Nh} \le s, \qquad hs \le \tfrac{1}{2}.$$

For each fixed cutoff they follow from the same reporting schedule; no cutoff-dependent change of continuum witness is introduced. The limiting inverse transform uses the stated character convention and the actual function $G(z) = (4\pi^2\|z\|^2)^{-1}$. For $z \neq 0$, the heat correction retains the exact scaling

$$\partial_D\big(G(z)e^{-\pi^2 v^2\|z\|^2}\big) = v^{|D|+2}[\partial_D(Ge^{-\pi^2\|\cdot\|^2})](vz).$$

Choose a decay power greater than $|D|+2$ to recover its vanishing at each nonzero z as $v \to \infty$. Heat removal at a fixed Fourier cutoff and the retained dyadic tail $\sum_{j\ge 0} C/(2^j s) = 2C/s$ supply the continuum comparison without asserting absolute integrability of an uncut differentiated symbol. These operations use the same scalar readouts before and after horizontal transport.

For the far-image application retain $F = 0$ and $A(s) = 0$. The source's annulus bound gives annular convergence to zero at each fixed s as physical distance diverges. The preceding exact comparison then recovers $F_m \to 0$ with the same small-cutoff and tail constants once distance is at least one. This is the far-image reader used in the quarter-period covariance construction, including the coincident-observation return geometry.

For the nonoscillatory low-frequency bound set $t = hs$, $q = p/(hs)$. The denominator is $d_t(q) = \sum_{\nu=0}^{3} q_\nu^2 \operatorname{sinc}(tq_\nu/2)^2$. For $\|q\| \le 1$, $0 \le t \le 2$, the sinc bound $2/\pi$ gives $d_t(q) \ge (2/\pi)^2 \|q\|^2$. Each normalized difference numerator is at most $|q_\nu| \le 1$, so the normalized cutoff symbol is bounded by $(\pi/2)^2 \|q\|^{-2}$ regardless of list length. For centered integer momentum lifts in the cube B_m, $|B_m| \le (2m+1)^4$. On $B_m \setminus B_{\lfloor m/2 \rfloor}$ inverse squared norm is at most $4/m^2$. Strong induction gives $\sum_{p \in B_m} \|\operatorname{lift}(p)\|^{-2} \le 432 m^2$, with zero term assigned zero: the inner bound is $108m^2$ and outer bound $(3m)^4(4/m^2) = 324m^2$, starting at $m = 0$. Writing $\eta = 2\pi/N$, $T = \eta/t$, rescaling bounds the unit-ball inverse-square sum by $432T^{-4}$. Multiply by physical normalization $(Nh)^{-4} s^{|D|-2}$ and use $Ts = 2\pi/(Nh)$:

$$|\operatorname{piece}_{h,s,D}(x)| \le \frac{432(\pi/2)^2}{(2\pi)^4} s^{|D|+2}. \tag{PT28}$$

This low-frequency estimate also holds at coincidence. The lifted kernel source retains annular differences together with both small-cutoff size and large-cutoff tail data. Its telescoping reader recovers the whole kernel and fixes the constant not determined by differences alone. Heat removal, the zero momentum convention and physical normalization remain attached to the same reader, preserving the separated and far-image limits used in the corrected covariance.

22.10 Discrete oscillation bounds and moving annular quadrature

Translation of a finite torus momentum sum gives exact integration by parts for every integer v:

$$\sum_p \chi_p(x) \Delta_{-e_\nu}^{v} f(p) = (\chi_{e_\nu}(x) - 1)^v \sum_p \chi_p(x) f(p). \tag{PT29}$$

The normalized annular symbol is smooth, supported at $\|q\| \le 2$, with derivatives of every fixed finite order uniformly bounded for $0 < t \le 1$, by cutoff away from zero and smooth sinc extension at $t = 0$. Momentum differences have normalized step T, so repeated fundamental theorem of calculus bounds order v by $C_{D,v} T^v$. If $vT \le 1$, support expands only to $\|q\| \le 3$, with at most $8^4 T^{-4}$ representatives since $T \le 1$. Boundary crossings cause no loss: the periodic uncut symbol agrees on both lifts and the annular cutoff vanishes on both if $\eta v \le 1/2$. For the high-frequency complement at $t \ge 1/2$ use uniform derivatives on the rescaled Fourier box; all N^4 representatives contribute but $N \le 4\pi/T$, giving the same CT^{v-4} bound. Thus

$$|\chi_{e_\nu}(x) - 1|^v \left| \sum_p \chi_p(x) f(p) \right| \le C_{D,v} T^{v-4}.$$

Choose a coordinate at least half the physical norm; the centered-lift circle chord inequality gives $|\chi_{e_\nu}(x) - 1| \ge \|h \operatorname{lift}(x)\|/(Nh)$. With $v = |D| + 3$, physical normalization and $Ts = 2\pi/(Nh)$ give

$$|\operatorname{annulus}_{h,s,D}(x)| \le \frac{C_{D,v}}{(2\pi)^4 s} \left(\frac{2\pi}{\|h \operatorname{lift}(x)\|} \right)^v. \tag{PT30}$$

The same holds for the complement when $1/2 \le hs \le 1$. Choose the first q with $h2^q s > 1/2$, then $h2^q s \le 1$. The full tail is the finite dyadic annuli through $2^{q-1}s$ and that complement. At physical distance at least $R > 0$, their bounds sum to $2C_{D,R}/s + C_{D,R}/s$, giving

$$|\,\mathrm{joint}_{h,D}(x) - \mathrm{piece}_{h,s,D}(x)| \le 3C_{D,R}/s.$$

This is PT27's tail estimate without requiring an absolute uncut infinite-volume derivative sum.

Finite Fourier bound recovery. The source retains the normalized symbol, centered representatives, zero summand, mesh and difference order. Horizontal readout leaves these scalar entries unchanged. In particular the low-frequency estimate uses the actual inverse-square sum and the exact scaling

$$(Nh)^{-4} s^{|D|-2} T^{-4} = (2\pi)^{-4} s^{|D|+2}, \qquad T = \frac{2\pi}{Nhs}.$$

The retained induction constant 432 and symbol constant $(\pi/2)^2$ therefore recover PT28 with its original prefactor. No oscillatory cancellation or positive separation is required for this reader, so its bound still applies at coincidence.

For a smooth normalized annular symbol F, the finite-difference estimate can be written exactly as

$$\Delta^v_{-Te_\nu} F(q) = (-T)^v \int_{[0,1]^v} \partial^v_\nu F(q - T(t_1 + \cdots + t_v)e_\nu)\, dt_1 \cdots dt_v.$$

The prescribed support and derivative bounds thus give CT^v per representative, and counting the enlarged support gives CT^{v-4}. The recorded boundary and high-frequency alternatives supply the same bound for the periodic symbol when a representative changes. This transports the finite difference of the actual symbol rather than replacing it by a continuum derivative.

Let $\mathscr{F}_x(f) = \sum_p \chi_p(x) f(p)$ be the finite Fourier reader. Index translation is a bijection of the same torus, so

$$\mathscr{F}_x(\Delta^v_{-e_\nu} f) = (\chi_{e_\nu}(x) - 1)^v \mathscr{F}_x(f)$$

is an exact readout identity. It remains so on the horizontal source with the same character and representatives. For nonzero physical distance $\rho = \|h\,\mathrm{lift}(x)\|$, the retained chord bound and normalization yield

$$\begin{aligned}(Nh)^{-4} s^{|D|-2} T^{v-4} (Nh/\rho)^v \\ = (2\pi)^{v-4} s^{|D|+2-v} \rho^{-v} = \frac{1}{(2\pi)^4 s} \left(\frac{2\pi}{\rho}\right)^v, \qquad v = |D| + 3.\end{aligned}$$

This recovers PT30 including its powers and physical normalization. The final complement retains the same estimate at its allowed scale. The finite dyadic decomposition is an identity of readers before estimating, and $\sum_{j=0}^{q-1} (2^j s)^{-1} \le 2/s$ together with the final complement bound gives exactly the stated $3C_{D,R}/s$ tail for $\rho \ge R$. Thus both the low-frequency anchor and oscillatory tail used by PT27 are preserved by the same scalar source lift.

For fixed s, set $c_m = N_m h_m$, integer on the reporting schedule. Physical Fourier spacing is c_m^{-1}. The normalized annular integrand is jointly continuous in $t = h_m s$, displacement and Fourier variable, including $t = 0$, uniformly bounded and supported in radius $2s/(2\pi)$. Choose integer R containing it. Eventually $2Rc_m < N_m$, equivalent to $2Rh_m < 1$, so all supporting integer addresses are their own centered representatives. The torus sum is exactly unrestricted box quadrature. If $r_m(k)$ is its representative, $\|r_m(k) - k\| \le 4/c_m$. The step function $F_m(r_m(k))$ integrates to that quadrature with cell volume c_m^{-4}, converges pointwise by joint continuity and is

bounded by a constant times the radius-$R+5$ ball indicator. Dominated convergence proves its moving-displacement limit. Its phase is exact:

$$e^{2\pi i\langle h_m \operatorname{lift}(x_m),\operatorname{lift}(p)/c_m\rangle} = \chi_p(x_m).$$

The normalized radial variable is also exact since $c_m = N_m h_m$.

For continuum heat-uniform oscillation let $H_{D,a}(q) = e^{-a\|q\|^2} M_D(q)$, $a \ge 0$, where M_D is the normalized smooth annular symbol supported in $1/2 \le \|q\| \le 3$. Every fixed derivative order has a common bound on that annulus, uniform in a. Indeed with $y = \sqrt{a}\, q$, order i derivatives of the heat factor are $a^{i/2}$ times Gaussian derivatives. For $\|y\| \le 1$, $\sqrt{a} \le 2$ gives compact bounds; for $\|y\| \ge 1$, $\sqrt{a} \le 2\|y\|$ and rapid decay give the same bound. Leibniz with M_D finishes it. Choose a coordinate of x at least half its norm and integrate by parts j times; compact support removes boundary terms, and the derivative L^1 bound gives

$$\|x\|^j \left| \int e^{2\pi i k\cdot x} H_{D,a}(k)\, dk \right| \le C_{D,j} \quad (a \ge 0).$$

Rescaling $k = (s/(2\pi))q$ contributes $s^{|D|+2}$; choosing $j = |D| + 3$ leaves s^{-1}. The rescaled heat parameter is nonnegative, so this proves the annular estimate uniformly during heat removal and its summable dyadic tail in PT27. The lifted oscillatory source retains periodic representatives, exact momentum differences, normalized phases, cell volumes and heat parameter. PT29 transports the reader by finite index translation; PT30 bounds its actual tail. The quadrature reader recovers the continuum annulus by dominated convergence with moving displacement, and uniform heat bounds preserve recovery when its regulator is removed.

Moving annular integral and heat-uniform recovery. The annular source records the mesh/period pair and the physical Fourier address $k_p = \operatorname{lift}(p)/c_m$. The phase recovery is an identity before any limit, since

$$\langle h_m \operatorname{lift}(x_m), k_p\rangle = \frac{\langle \operatorname{lift}(x_m), \operatorname{lift}(p)\rangle}{N_m}.$$

Thus its exponential is the retained torus character. The radial argument and cell volume are also retained exactly; on the supporting box the annular reader is

$$c_m^{-4} \sum_{u\in\mathbb{Z}^4} F_m(u/c_m) = \int_{\mathbb{R}^4} F_m(r_m(k))\, dk.$$

Only finitely many summands are nonzero. Eventual centered-address injectivity follows from $2Rh_m < 1$. All scalar ingredients have the same horizontal readout, so the equality also holds for the lifted annular reader. Joint continuity with the retained moving displacement and $t \to 0$ gives pointwise convergence. The common bound $M\mathbf{1}_{\{|k|\le R+5\}}$ is integrable and remains identical under readout. Consequently dominated convergence recovers the continuum annulus with its actual phase and normalization.

For explicit heat-uniform control, for a multi-index α of length i and $a > 0$ the derivative of the heat factor is

$$\partial_q^\alpha e^{-a\|q\|^2} = a^{i/2} (\partial^\alpha e^{-\|\cdot\|^2})(\sqrt{a}\, q).$$

On the recorded annulus $\|q\| \ge 1/2$, putting $y = \sqrt{a}\, q$ bounds $\sqrt{a}$ by 2 when $\|y\| \le 1$, and by $2\|y\|$ otherwise. Thus a finite constant controls this derivative uniformly in $a > 0$. The case $a = 0$ is the

constant heat factor, with all positive derivatives zero. Leibniz's formula therefore gives uniform L^1 bounds for every prescribed finite derivative of $H_{D,a}$ on its fixed compact support.

For $x \neq 0$ choose $|x_\nu| \geq \|x\|/2$. The exact continuum reader identity is

$$\int e^{2\pi i k\cdot x} H_{D,a}(k)\,dk = (-2\pi i x_\nu)^{-j} \int e^{2\pi i k\cdot x} \partial_\nu^j H_{D,a}(k)\,dk.$$

It retains no boundary term and transports unchanged. Its derivative bound recovers the uniform oscillatory estimate. After rescaling, $s^{|D|+2} s^{-j} = s^{-1}$ at $j = |D| + 3$; the new heat parameter still ranges over nonnegative values. Summing the dyadic annular bounds gives $2C/s$ uniformly during heat removal. This retains the precise continuum tail used in PT27 and completes recovery of the moving annular step of the Gaussian comparison.

23 Finite stress, scaling maps and shell charges

Actual finite stress, followed by geometric station selection. The stress and current begin as functions on finite group configurations. Their Ward identity, Haar-Jacobian terms and quiet-loop localization are computed there before extraction. Rooted estimates, then the full-support phase argument, transfer these finite readings to the required characters. A single calibration matrix must work for the three panels and their joint higher-arity suppliers; centering removes the unit direction before the relevant rank assertion is used.

The station-selection equation is geometric: $s = \rho_k(b + u)$. It chooses a station at a requested physical radius. The actual interacting and Gaussian moments retain the IG3 comparison error at that station. Consequently the exact equality of radii and the limiting equality of moment readings have different logical roles. The physical conserved stress is then identified on the same quotient and with the same shell-charge normalization. Flux representability is proved with the actual geometry and faces; a coefficient rank count alone would not recover the ordinary face flux.

23.1 The finite trace candidate, its equation identity and its compensation

The local current and stress entering this chapter are finite configuration functions. For a base site o, let

$$C_{ab}(o, U) = \mathrm{curv}(U_a(o) U_b(o + e_a) U_a(o + e_b)^{-1} U_b(o)^{-1}).$$

The curvature map is the real color projection specified in the startup construction, so this expression is defined on every group configuration. Gauge transformations act by the adjoint at o. Define a link-valued current and a scalar tensor by

$$(Y_b(o, U))_{(x,i)} = 1_{\{x=o\}} C_{bi}(o, U), \qquad J_{ab}(o, U) = \tfrac{1}{4}\delta_{ab} \sum_{i,j} \langle C_{ij}, C_{ij}\rangle - \sum_i \langle C_{ai}, C_{bi}\rangle. \tag{TF0a}$$

Color coordinates are converted back to the actual link-coordinate space by the fixed color isometry. At the identity every C vanishes. Its derivative in a logarithmic link direction A is precisely

$$c_{ab}(o, A) = A_a(o) + A_b(o + e_a) - A_a(o + e_b) - A_b(o).$$

Thus $Y(1) = 0$; $J(1) = 0$ and $DJ(1) = 0$. Its Hessian is obtained by replacing a product $\langle C, C'\rangle$ by $\langle c(A), c'(B)\rangle + \langle c(B), c'(A)\rangle$. These formulas supply the actual quadratic jets, rather than an unspecified continuum stress approximation. Initially J is symmetric and has identically zero trace.

The divergence used for the finite Taylor calculation is the fixed high-order antisymmetric stencil

$$\mathscr{D}_a f(o) = \sum_{\ell=1}^{4} d_\ell \{ f(o+\ell e_a) - f(o-\ell e_a) \}, \qquad (d_1, d_2, d_3, d_4) = (4/5, -1/5, 4/105, -1/280). \tag{TF0b}$$

It has first moment one and its odd moments three, five and seven vanish. Substitution of the four rational coefficients proves these identities. Taylor expansion therefore reads the ordinary derivative exactly on polynomials of degree at most eight. Physical spacing factors are inserted only when the finite tensor is rescaled.

During a stationary descent retain the full tuple (J, Y, E, Q, Ξ), with exact identities

$$\begin{aligned} \sum_a J_{aa} = \sum_a \nabla_a Q_a, \qquad J_{ab} - J_{ba} = \sum_i \nabla_i \Xi_{abi}, \\ \sum_a \mathscr{D}_a J_{ab} - E_b \equiv D_{Y_b} \mathcal{P} \quad \text{through Taylor order four.} \end{aligned} \tag{TF0c}$$

Here ∇ is the unit forward difference, and the last equivalence includes orders zero through four at the identity. Initially $Q = \Xi = 0$, and E is defined as the difference on the last line, so that it is an equality. The Hessian of $\mathcal{P}$ is the retained covariant quadratic form; differentiating $D_Y \mathcal{P}$, using $Y(1) = D\mathcal{P}(1) = 0$, gives its two cross Hessian pairings. The local analytic change of variables preserves these four jets by the chain rule. The scalar rows J, E, Q, Ξ remain analytic, gauge invariant, and zero through order one.

To see why the identities persist under blocking, denote the main averaging stencil by M, the directional flux stencil by F_a, and its correction by T_a. Their finite coefficient identities are

$$F_a = M + \nabla_a T_a, \qquad L^5 M \mathscr{D}_a = \mathscr{D}_a L^4 F_a.$$

Pull each fine scalar function back along the stationary section. Transport J_{ab} with $L^4 F_a$, E_b, Y_b with $L^5 M$, and the difference potentials with their degree-three edge stencils. Add $L^4 T_a J_{aa}$ to the transported Q_a, and add $1_{\{i=a\}} L^4 T_a J_{ab} - 1_{\{i=b\}} L^4 T_b J_{ba}$ to the transported Ξ_{abi}. Substitution proves all three identities in (TF0c); the two added terms cancel precisely the change from the main stencil to the flux stencils. The current is transported by the derivative of the stationary section, so the chain rule also preserves the last identity. No exact symmetry or zero trace of each blocked local representative is presumed.

Localization is performed on the scalar and current functions themselves: gauge fix along the prescribed finite axial paths, discard coordinates beyond the chosen radius, and retain the Taylor polynomial through order four in those local coordinates. Denote the localized tensor by J^{loc}, and localize Q, Ξ by the same operation. Restore its exact trace and its six antisymmetric divergences by defining

$$\begin{aligned} \widetilde{J}_{00} &= \sum_i \nabla_i Q_i^{\mathrm{loc}} - \sum_{i \neq 0} J_{ii}^{\mathrm{loc}}, \\ \widetilde{J}_{ab} &= J_{ab}^{\mathrm{loc}} \quad (a < b), \\ \widetilde{J}_{ba} &= J_{ab}^{\mathrm{loc}} - \sum_i \nabla_i \Xi_{abi}^{\mathrm{loc}} \quad (a < b). \end{aligned} \tag{TF0d}$$

Other diagonal entries are unchanged. The six pairs are $01, 02, 03, 12, 13, 23$; their identities follow individually by subtraction. Summing the four diagonal entries proves the trace identity. The same construction has bounded support, enlarged by one difference step.

The localization error of a differentiated tensor is controlled at the actual stencil sites: (TF0b) uses only shifts of length at most four, so its error is bounded by $\sum_\ell |d_\ell| (A+A)$ if the undifferentiated

error is at most A. Summing the four directions gives four times this allowance. Support and anchor weights are enlarged by four at this step. The corresponding derivative and tail allowances are summed with the same absolute coefficients. Exponential path decay pays the discarded tails including their first moments; the stationary four-jet identity pays the local Taylor difference. These are the separate errors retained when the localized divergence is issued as a degree-five insertion.

Finally average the scalar tensor and its divergence over signed permutations and conjugation, with weight $1/|\mathrm{Signed}\times\{0,1\}|$, multiplying each component by its tensor or vector sign. Reindexing this finite average proves reality and the required signed covariance. The vector average kills the identity-configuration constants: the axis reflection reverses the vector sign and pairs identical constants. It does not increase a common rooted bound, because the sum of the absolute averaging weights is one, every transformed chamber has the same rooted seminorm, and the rooted triangle inequality applies. Transport commutes with the same transformations. Thus the issued vector and tensor have the correct exact finite symmetries before any coefficient limit is taken.

We first specify the finite objects whose limits enter the trace argument. This prevents a formal Taylor coefficient from being confused with a continuum insertion. At a fixed finite representative let Y_b be its rooted infinitesimal link-current vector field, $D_Y f$ the directional derivative of a configuration function along its left logarithmic link increments, and $\mathrm{div}_\nu Y$ its divergence for product Haar measure. In exponential coordinates this divergence includes both the ordinary coordinate divergence and the derivative of the logarithm of the Haar density. Write P_A for a finite partial sum of the classical action, L_Q for a finite partial sum of the ordinary logarithmic corrections, and

$$\mathcal{E}_{A,Q} = -\beta P_A - L_Q, \quad \mathcal{R}_b = \beta(B_b - D_{Y_b} P_A). \tag{TF1}$$

Here B_b is the rooted lattice stress expression, with the same mean and curvature rows as in the finite stress construction. The finite sums are taken from the actual source dictionaries, rather than from new representatives of their limiting functions.

The protected configuration change along Y_b, with step $u > 0$, gives a finite difference identity. If $\mathcal{C}_{u,b}$ denotes the resulting density-difference expression, its exact decomposition on the protected healthy region is

$$\frac{\mathcal{C}_{u,b}}{u} = \mathcal{Z}_b - \beta B_b + (\mathrm{div}_\nu Y_b)(1), \tag{TF2}$$

$$\mathcal{Z}_b = \mathcal{R}_b + \{\mathrm{div}_\nu Y_b - (\mathrm{div}_\nu Y_b)(1)\} - D_{Y_b} L_Q + \mathcal{K}_{u,b} + \sum_{\gamma\in\mathcal{F}} \mathcal{H}_{u,b,\gamma}. \tag{TF3}$$

The constant denoted by evaluation at 1 is evaluated at the identity configuration. The fourth term is the finite Taylor remainder of the changed action and Jacobian after subtracting their first directional terms, divided by u. The last sum is the finite correction for the chosen finite shelves of activity terms: each summand is the corresponding changed activity difference with the coefficient and the factor u^{-1} dictated by that difference. Thus (TF3) defines a literal configuration function. To prove (TF2), expand the change of density, distribute $D_Y(-\beta P_A - L_Q)$, and use (TF1); the $\beta D_Y P_A$ terms cancel. Splitting the activity change into the finite retained shelves and their remainder gives exactly the last sum. This is an algebraic equality before any limit or expectation is taken.

The correction shelves can be enlarged to approximate the full changed expression uniformly on configurations. The estimates for rooted activity sums above give nonnegative numbers g_γ with each summand regular on its actual support P_γ, and

$$\sum_{\gamma\in\mathcal{F}} \mathrm{mass}(P_\gamma) g_\gamma \le \varepsilon_{\mathrm{corr}}. \tag{TF4}$$

The finite four-term part of (TF3) is regular on the common neighborhood and has derivative allowance $2c_{\mathrm{small}} + 2c_{\mathrm{loop}}$. The sum in (TF4) is included with its own supports, not paid as if every term had the largest common support.

The conversion of a regular function f into a coefficient insertion is particularly explicit. Let $\Pi_{\le 5} f$ be its Taylor polynomial along the local analytic section at the identity, projected to the degree-five frame, and weight its homogeneous components with the prescribed threshold powers. Call the resulting coefficient $a(f)$. Choose the primary local section $s(a(f))$, and put the masked difference $f - s(a(f))$ into one regular activity slot, with every other slot zero. The resulting formatted insertion equals f on the protected region, has zero extra scalar term, and retains locality and symmetry. If r_{read} is the norm of the finite Taylor reading map and r_{sect} the norm of this section construction, a bound c for the original section gives the two respective bills

$$\mathrm{mass}(P)c(r_{\mathrm{read}} + r_{\mathrm{sect}}), \qquad \mathrm{mass}(P)c((1 + c_{\mathrm{sect}})r_{\mathrm{read}} + r_{\mathrm{sect}}). \tag{TF5}$$

These are finite-dimensional operator bounds and the ordinary triangle inequality for $f - s(a(f))$. Add this construction for the four terms and the finite correction sum. Taylor reading is linear, so the actual candidate coefficient is the sum of the four weighted Taylor readings plus the weighted readings of $\mathcal{H}_{u,b,\gamma}$. Equation (TF4) bounds the size of the latter by $(r_{\mathrm{read}} + r_{\mathrm{sect}})\varepsilon_{\mathrm{corr}}$. This supplies the coefficient and its paid remainder together.

The equation identity for this candidate is also literal. Differentiate the finite change of variables in the normalized Haar integral, or integrate its density difference before dividing by u. A spectator supported outside the changed links is unchanged. The resulting expectation is zero. Apply (TF2), subtract its identity-configuration constant, and then let the shelves increase and u follow its admitted small-step schedule. Their uniform allowances justify these passages by dominated convergence. The same identity with a spectator derivative gives the finite lattice Ward contact term when that spectator meets the support. In particular equation directions have zero separated readings, precisely the equation cancellation already used in constructing the physical quotient.

There is a possible degree-five vector component in this procedure. The raw-to-free vector map has exactly the kernel $\mathbb{R}E$ proved in (FC9)–(FC10). Choose its real signed-equivariant section by the corrected polynomial lift and subtract the unique remaining multiple of E. This is finite-dimensional subtraction before the coefficient limit. It explains why the trace candidate can be compared in fixed vector coordinates despite its original equation component.

The compensation by the quiet logarithmic part is exact at the coefficient level needed here. In logarithmic link coordinates the vector-current diagnostic is a sum, over $d \neq i$, of the quadratic and cubic Taylor coefficients with two equal fixed-color variations in direction i and opposite lattice directions $d, -d$. Its cubic part is the contraction denoted crest_q; denote the quadratic contraction by $\mathrm{suite}_{d,i}\,\mathrm{floft}_{d,i,q}$. Define

$$H_q(X) = \mathrm{crest}_q(X) - \frac{1}{2}\sum_{d \neq i} \mathrm{floft}_{d,i,q}(X_i). \tag{TF6}$$

The quiet logarithmic contribution is constructed with two moment cancellations; gauge invariance alone would not establish the required zero in (TF6). For each fixed ordered pair of plaquette types and color coordinates, let c_γ be the finite two-slot path coefficient and z_γ the displacement of the second slot from the first. Put

$$C_0 = \sum_\gamma c_\gamma, \qquad C_a = \sum_\gamma c_\gamma (z_\gamma)_a. \tag{TF6a}$$

Append a zero-displacement two-slot term with coefficient $-C_0 + \sum_a C_a$, and four terms of displacements e_a with coefficients $-C_a$. Repeat this for every plaquette and color pair. The augmented

finite family then satisfies exactly

$$\sum_\gamma c_\gamma = 0, \qquad \sum_\gamma c_\gamma (z_\gamma)_a = 0 \quad (a = 0, 1, 2, 3). \tag{TF6b}$$

Its appended coefficients remain color invariant because that condition is a real linear subspace. The quiet logarithm is the signed and translated sum of this augmented family. The difference between it and the retained ordinary loop sum is precisely the finite appended part. Its coefficient sizes are bounded by the zero and first moments of the omitted path tail; exponential path decay bounds these moments, giving the small remainder already paid in (TF4).

Here is why this construction cancels (TF6). Let $T = L_{\mathrm{quiet}}$. Its Hessian at the identity vanishes when one direction has bounded support and the curl of the other is affine on the enlarged support neighborhood. In fact the Hessian of a two-slot term is the sum of the two products of its linearized curl slots. With the first direction fixed, the other slot evaluated at displacement z_γ is $A + F(z_\gamma)$, with F additive. Summing over the family gives $A \sum_\gamma c_\gamma + \sum_a F(e_a) \sum_\gamma c_\gamma (z_\gamma)_a = 0$ by (TF6b). The same calculation applies to the reversed slot order and each signed translate. The nonwrapping and support hypotheses ensure that these displacements are the actual integer displacements; in the application $2(t + 2 + M) + 6 < n$, where t bounds the current support and M the retained path length.

For $F_b = D_{Y_b} T$, differentiate twice at the identity. The current vanishes there, and the first derivative of a gauge-invariant scalar there is zero. The product rule consequently reduces the Hessian to

$$D^2 F_b(0)[X, Z] = D^2 T(0)[DY_b(0)X, Z] + D^2 T(0)[DY_b(0)Z, X]. \tag{TF6c}$$

The directions $DY_b(0)X$ have the bounded support of the current. Choose X, Z to be the polynomial link directions of degrees at most two that occur in the coefficient test. Their curls are affine on that neighborhood. The preceding Hessian cancellation makes both terms in (TF6c) zero. Terms with a constant pure-gauge direction vanish by the differentiated gauge identity. Expanding the cubic coefficient in the two position variables of the test uses just these Hessian pairings, so its centered-link reading is zero.

One must still convert the centered-link reading into the actual link reading. If the two position arguments are z_0, z_1, the change is multiplication by

$$e^{\,i((z_0)_i + (z_1)_i)/2}.$$

The constant and first coefficients of the original reading vanish. The normalized third coefficient of this product is therefore its original third coefficient plus $i((z_0)_i + (z_1)_i)/2$ times its second coefficient. Apply the finite coefficient functional in the d, b variables. The latter term is zero unless $i = b$; when $i = b$, its value is $-\frac{1}{2}\,\mathrm{floft}_{d,i,q}$. This follows by expanding the quadratic coefficient on $(e_d, -e_d)$ and using $i^2 = -1$. Summing over b and $d \neq i$ gives exactly (TF6). Its centered-link version has just been shown zero, so $H_q(D_Y L_{\mathrm{quiet}}) = 0$. The factor $1/2$ thus comes from this link-centering phase and the normalized Taylor coefficients. Subtracting the paid appended part gives the stated bound for the ordinary loop contribution. No quiet contribution is left undetermined.

The needed vanishing of the quadratic part can be derived without setting a mass regulator equal to zero inside an integral. For $0 < \mu \leq 1$, the local resolvent estimates below give, for the massive comparison expression L_μ,

$$|\,\mathrm{cob}_{d,i,q}(L_\mu)| \leq C\mu^{-1}. \tag{TF7}$$

The proof differentiates the Fourier-resolvent expression twice; its pointwise integrand is bounded by $C(|p| + \mu)^{-5}$. Counting lattice shells and multiplying by the momentum cell volume gives $C\mu^{-1}$

in four dimensions. This applies to both the physical and comparison resolvents, hence to their sum. At each fixed positive μ, the resolvent identity identifies the unregularized current coefficient with $\mu^2 \operatorname{cob}_{d,i,q}(L_\mu)$, up to an error tending to zero in the retained large-volume limit. Given $\epsilon > 0$, choose $\mu = \min(1, \epsilon/[2(C+1)])$ first. The massive term has magnitude at most $C\mu \le \epsilon/2$; take the volume large enough to make the remaining error below $\epsilon/2$. The unregularized quadratic coefficient therefore tends to zero. The Taylor remainder bounds give the same limit for its finite-difference reading. Summing the finitely many d, i shows that (TF6) has the same limit as its cubic part.

That cubic limit is $-12\, s_G$, where s_G is the one-loop slope constant in the coupling drift. Here is how the logarithmic slope fixes it. For the regulated determinant expression $A(\log \mu)$, the determinant estimates give

$$|A(x) + (-12 s_G)x| \le C, \qquad x \le 0. \tag{TF8}$$

The regulated response is $-A'(x)$, with $x = \log \mu$. The real mean value theorem on an interval $[a, b] \subset (-\infty, 0]$ supplies an intermediate regulator at which this response differs from $-12 s_G$ by at most $2C/(b-a)$. First choose $b - a$ large, then use the local resolvent difference estimate to compare that response with the unregularized cubic current coefficient; this error tends to zero when the upper endpoint b is sufficiently negative and the retained volume is sufficiently large. The intermediate regulators stay in the fixed positive interval $[e^a, e^b]$, so the large-volume estimates apply uniformly there. The triangle inequality proves the limit. Together with (TF7), it proves $H_q \to -12 s_G$ for every color q. These are finite rooted Haar-Jacobian coefficient tests, not an assumption about a mollified continuum field.

Finally the passage through many adjacent coefficient transports retains a relative error. Put

$$\omega_i = \exp\{-\tfrac{2}{5}\, \mathrm{RLog}(\widehat{\beta}_i)\} \longrightarrow 0, \qquad k = k(i) > i. \tag{TF9}$$

The chosen calendar ensures every adjacent transport error between i and k is at most $\omega_i/[2(k-i)]$ times the next coefficient size. If v_j denotes that coefficient, backward induction proves

$$\|v_j - v_k\| \le \frac{\omega_i (k-j)}{k-i} \|v_k\| \le \omega_i \|v_k\|, \qquad i \le j \le k. \tag{TF10}$$

Indeed the induction hypothesis and $\omega_i \le 1$ imply $\|v_{j+1}\| \le 2\|v_k\|$; adding the next adjacent error gives the asserted bound at j. The calendar inequality follows from the positive reserve $-2/5 + 1/2 = 1/10$: logarithms and the number of intervening steps are eventually dominated by the exponential with exponent $1/10$. Thus this accumulated relative error vanishes. The finite candidate, its equation identity, the quiet compensation and its scalar diagnostic consequently survive on the same retained field and enter the following trace comparison.

Source–readout formulation. The retained tuple (J, Y, E, Q, Ξ) is the realized source record of the trace; its three identities (TF0c) are discrepancy columns kept at every level: Q is the recorded potential of the trace, Ξ of the antisymmetric part and E of the equation term. The restored tensor (TF0d) is a section that makes two of the discrepancies exact again after localization. The slope constant enters the trace through $H_q \to -12 s_G$, which reads the *same* number $s_G = s$ that was read in (DR5) from the coupling drift; the two readings are obtained from different readers (the kinetic reader α_a and the cubic current reader H_q) of the same determinant films, and their agreement is a derived consistency of the realized history and is not imposed. All objects are gauge-invariant cylinders or coefficient vectors, so the frame lift is the identity on values and the identities hold for $J \circ \pi$, Y pushed forward by the horizontal lift of Theorem 9.2, and so on.

The candidate itself is a literal configuration function at a fixed finite representative, and its defining relation (TF2) is an algebraic identity before any limit or expectation. This is the form

required by the method for an equation residual: the equation term is retained as a column of the realized record, and its vanishing in separated readings is proved (Proposition 18.2), not built into the definition. The possible degree-five vector component is removed by Proposition 16.5: the raw-to-free vector map has kernel exactly $\mathbb{R}E$, and the unique remaining multiple of E is subtracted before the coefficient limit. The quiet compensation is a motion inside the fibre of the moment readout of Proposition 14.10: it changes the presentation of the candidate by a family with vanishing zeroth and first moments and does not change the readings that define the trace.

23.2 The interacting-to-Gaussian comparison used for the trace

This comparison concerns the interacting Wilson family and its selected continuum phase. It is distinct from the calculation of a Gaussian constant coefficient in the preceding chapter. We derive the passage between them.

At a fixed retained comparison station take two physical lattice insertions A_0, A_1, supported in its strict chamber, with their coefficient representatives converging along the common comparison filter. Let m_0, m_1 be the corresponding one-point Gaussian readings and let z be the centered Gaussian two-point reading. The corrected singleton and pair polynomials are

$$D_j = A_j - m_j, \qquad D_{01} = (A_0 - m_0)(A_1 - m_1) - z. \tag{IG1}$$

We next derive the finite interacting bound for these functions. This also specifies which Gaussian quantities are subtracted. For n marked insertions let f be the cumulant polynomial of the Gaussian profile at the rounded positions. The corrected cylinder is the coefficient polynomial obtained by multiplying the insertion generating polynomial by $\exp(-f)$, in the finite algebra in which each marker has square zero. Its singleton coefficients are $A_j - m_j$, and its pair coefficient is exactly D_{01} in (IG1). Thus the subtraction includes the Gaussian means as well as its connected pair coefficient.

Let k be a comparison station, $M \geq n$ its fixed maximum number of marks per chamber, and H_k its monitor. Define

$$I_k = \frac{\log H_k}{16M}, \qquad R_k = e^{2I_k}, \qquad \varepsilon_k = e^{-I_k}. \tag{IG4}$$

Here $H_k \to \infty$, so $\varepsilon_k \to 0$. The initial marked density is the Wilson density times the product of the reflected copies of the corrected cylinder over all chambers of the tiling. Push this density along the same exact renormalization course as its unmarked Wilson density. In particular its vacuum integral is the original Wilson partition function. The comparison uses this density identity, not a replacement of the interacting law by a Gaussian law.

We describe the bounds on this push. At the middle stop its regular marked sources have values $W_\alpha(U)$, with specified fired marker sets and support cubes. They occur together with the forced sources and the unmarked background in the finite density expansion described in the marked-integration chapter. If C_α is the corresponding coefficient of the negative Gaussian cumulant, then on the healthy local chart

$$|W_\alpha(U) + C_\alpha| \leq b_{-31/32}(t), \qquad |C_\alpha| \leq b_{1/4}(t). \tag{IG5}$$

Here t is the station coupling, $b_a(t) = \exp(a \operatorname{boil}(t))$, and $\operatorname{boil}(t) \to \infty$. To obtain the first bound, expand the actual insertion product by its finite cumulant expansion. The local Gaussian comparison of its calibrated chart coefficients gives an error at most $b_{-3/2}(t)$ for each coefficient; the finite exponential map from cumulants to moments is continuous on this fixed finite coefficient space. Its chosen neighborhood makes the cumulant-polynomial discrepancy plus that error at

most $b_{-31/32}(t)$. The exact negative cumulant coefficient cancels the corresponding free coefficient. The second bound follows because the rounded Gaussian profile stays in a compact subset of the generic position domain, where its finitely many coefficients are bounded, while $b_{1/4}(t) \to \infty$.

Here is the polynomial estimate behind the local Gaussian comparison just used. Denote coefficient ℓ^1 size by coin, and let k_0 bound the degree of each insertion polynomial. After the actual linear substitution into the common key and residual coordinates, write these polynomials as P_i; write F_i for their calibrated free polynomials, included in the same coordinates. The calibration is the exact identity that their weighted truncations through the assigned degree d_i agree. If the substitution has weighted size at most b, the original polynomial has size at most a_i, and the weight radius is $\rho \geq 1$, every remaining monomial has weight at least ρ^{d_i+1}. Hence

$$\operatorname{coin}(P_i - F_i) \leq a_i b^{k_0} \rho^{-(d_i+1)} \leq \xi. \tag{IG8}$$

This is the high-degree tail estimate; it uses the actual solved calibration maps. If $\operatorname{coin}(F_i) \leq B_0$, $B_0 \geq 1$, telescoping a product over any subset J of the n insertions gives

$$\operatorname{coin}\left(\prod_{i\in J} P_i - \prod_{i\in J} F_i\right) \leq |J|\xi(B_0+\xi)^{|J|} \leq n\xi(B_0+\xi)^n. \tag{IG9}$$

Substitution of the Gaussian variables commutes exactly with their Wick functional: the substituted covariance is the pullback of the original covariance and its mean is the pullback of the original mean. These are equalities of finite Gaussian moments, including the residual coordinates, not limiting approximations.

Let all means and covariances have absolute value at most $B \geq 1$, and let E bound their discrepancies from the zero-mean continuum free data in the key coordinates. Put $N = nk_0$ and $C_{B,N} = [2B(N+1)]^N$. Wick's recursion bounds the functional on a degree-N polynomial P by $C_{B,N} \operatorname{coin}(P)$, and its change under these mean and covariance discrepancies by $EC_{B,N} \operatorname{coin}(P)$. One proves both assertions by induction on the monomial degree: remove one variable, pair it with each remaining variable or replace it by its mean, and, for the difference, telescope the resulting products. The at most $N+1$ choices at each step and the bound B fit the displayed constant.

Combining this bound with (IG9), and then comparing the free-key polynomials under the two Gaussian data, gives total error at most

$$C_{B,N}\left[n\xi(B_0+\xi)^n + B_0^n E\right]. \tag{IG10}$$

The common covariance estimates and small-mean estimates in the Gaussian comparison chapter give precisely these bounds on the pulled-back data, uniformly over the allowed small middle-stop means. The station choices make (IG10) at most $b_{-3/2}(t)$. Thus the estimate feeding (IG5) follows from the actual covariance comparison, exact calibration and a finite polynomial calculation. It is not an additional interacting-to-free hypothesis.

These coefficient estimates are weighted in the density, as follows. A regular source in a support cube of radius QB has support charge at most $\exp(e(2Q+1)^4)$, since that cube occupies at most $(2Q+1)^4$ budget blocks. Its marker weight at radius R_k is at most $R_k^M = b_{1/8}(t)$, by the identity between the monitor and the station coupling. Consequently multiplication of (IG5) by the support charge and marker weight fits the prescribed regular-source bound. The same calculation with the additional unit of support charge fits the Gaussian compensation source bound. The forced sources already satisfy their full-configuration bounds. The one-step marked integration estimate therefore propagates this compensated density from the middle stop down to the width stop, preserving the unmarked background and bounding the total source weight per marker and the total forced weight

per site by 1. This is finite induction over the stops: the initial insertion estimates give the terminal case, and at each earlier stop the chart transport estimate and its forced-source estimate give the next case. The delay conditions put every coupling in this finite induction in the same interval $[t/2, t]$, so the chosen budgets apply throughout.

For completeness, the final conversion of these density bounds to a rooted Wilson bound can be written explicitly. Let Z be the number of width-stop sites, which is also the number of reflected chambers. If $q \neq \varnothing$ is a fired subset of the n markers, the weighted coefficient estimate for the pushed density and the positive partition-function lower bound give

$$\|\langle \prod_{\text{chambers}} \text{copy } D_q \rangle_{\text{Wilson}}\| R_k^{Z|q|} \leq \exp\{ZP - 4 \dim \mathfrak{g} \log g\}\,. \tag{IG6}$$

The parameter $g > 0$ here is the width-stop small-field parameter, with $\beta g^2 = 1$; P is its per-site protection bound with source count at most $1 + n \leq 1 + M$. The partition estimate applies to the actual positive course density. Its comparison with the reference density uses the two-sided exponent error bounded by $Z\delta$: one side bounds the numerator and the other side bounds the positive denominator, so their errors are both included in P.

Taking the Z-th root in (IG6) is legitimate because $Z > 0$. The width-stop allocation was chosen so that

$$P - \frac{4 \dim \mathfrak{g} \log g}{Z} \leq I_k.$$

The rooted chamber seminorm therefore obeys

$$\|D_q\|_{\text{root}} \leq \frac{e^{I_k}}{R_k^{|q|}} \leq \frac{e^{I_k}}{R_k} = e^{-I_k} = \varepsilon_k. \tag{IG7}$$

The middle inequality uses $|q| \geq 1$ and $R_k \geq 1$. The allocation is eventually available because the finite source-count ticket at the width stop is bounded by I_k; this is the delayed station choice, made simultaneously with the finitely many chart, insertion and transport inequalities. Applied to the two singleton modes and the pair mode, (IG7) is the asserted common bound $c = \varepsilon_k$ for (IG1). These are bounds for the actual Wilson law, before any character is selected.

Here is why this gives the same bound for the character reading of each corresponding continuum polynomial. Let D be any finite linear combination of distinct-point words with the rooted bound c, and let B be its compressed whole-word operator. For every integer $n \geq 1$, place $2n$ copies of D, with alternating reflected conjugation, in chambers separated by a distance parameter t. At sufficiently large volume these are distinct chambers. The finite chessboard inequality bounds the absolute expectation of their product by c^{2n}. At fixed n, t, the ordered lattice limit preserves this inequality; the comparison filter lies below that same ordered-limit filter and the coefficient representatives converge there. Then let $t \to \infty$. The long-word compression limit, expanding the finite linear combinations without commuting their operator factors, gives

$$\langle \Omega_0, (B^* B)^n \Omega_0 \rangle \leq c^{2n}.$$

The character decomposition therefore gives

$$\int |\chi(B)|^{2n}\, d\mu(\chi) \leq c^{2n}. \tag{IG2}$$

It follows that $|\chi(B)| \leq c$ for every character. Indeed, if $|\chi_0(B)| > c$, continuity provides an open neighborhood of χ_0, a number $a > c$, and a positive measure h of that neighborhood, on which

$|\chi(B)| \geq a$. Full support gives $h > 0$. Then (IG2) implies $ha^{2n} \leq c^{2n}$ for every n, an impossibility as $n \to \infty$. This includes $c = 0$. The two limits here have their stated order: first the lattice limit at fixed copy count and separation, then separation to infinity, then the integer $n \to \infty$.

Let u_j and u_{01} denote the continuum one- and two-point readings in any one character. Applying this argument to (IG1), and using the unit reading 1, gives

$$|u_j - m_j| \leq c, \qquad |u_{01} - m_0u_1 - m_1u_0 + m_0m_1 - z| \leq c.$$

The exact decomposition

$$u_{01} - (m_0m_1 + z) = [u_{01} - m_0u_1 - m_1u_0 + m_0m_1 - z] + m_0(u_1 - m_1) + m_1(u_0 - m_0)$$

proves

$$|u_{01} - (m_0m_1 + z)| \leq [1 + |m_0| + |m_1|]c. \tag{IG3}$$

Thus the Gaussian comparison follows from three specified interacting rooted estimates, not from an assertion that the interacting measure converges to a Gaussian measure.

At comparison index i, the station construction supplies these rooted estimates for every pair of basis labels through degree i, with $c = \epsilon_i \to 0$, uniformly over its fixed position window. The Gaussian readings m_j, z are the singleton and pair coefficients of the Gaussian moment profile. In particular $m_0m_1 + z$ is its full two-point coefficient, by the defining cumulant identity. Write $Q_i(x)$ for the rounded position tuple and $r_iQ_i(x)$ for the actual physical tuple. If x is generic and has a neighborhood in the comparison window, then $Q_i(x) \to x$, so it is eventually generic and stays in that window. Smoothness of the free hierarchy implies convergence, and hence boundedness, of its two singleton readings at $Q_i(x)$. Equation (IG3) yields convergence of the interacting two-point reading to the free two-point reading at x.

The finite-cutoff representatives of a free label converge, along the retained ultrafilter, to a linear lift $Y_{d,i}$. Projecting that lift gives the continuum map $U_{d,i}$. The finite station bounds pass to these limits because the lattice moments with converging finite-dimensional coefficient labels have the calibrated ordered limits. The station construction also preserves the unit, conjugation and signed motions. The free detector proves eventual injectivity of $U_{d,i}$; the field-dimension bound gives surjectivity, so these maps are eventually isomorphisms at each fixed degree. Replacing finitely many exceptional initial maps by any fixed isomorphism leaves all limits unchanged.

The conclusion also holds for convergent moving free labels $v_{i,0}, v_{i,1}$. To see this without assuming a uniform norm bound on all interacting bilinear forms, fix a basis e_a of the degree-d free space and expand

$$B_i(v_{i,0}, v_{i,1}) = \sum_{a,b} v_{i,0}^a v_{i,1}^b B_i(e_a, e_b).$$

Each basis-pair value converges by (IG3), and every displayed coefficient converges. There are finitely many summands. Their limits are therefore exactly the free bilinear form evaluated at the limiting labels. This is the moving-label comparison used for the normalized centered trace.

We finally justify the universal radius selection used in (TLim1). Given an arbitrary positive sequence $s_n \to 0$, a fixed finite degree horizon, any lower bound on the sequence index and station depth, and any positive radius ceiling, the station construction supplies a later sequence member below that ceiling whose value is **exactly** the station's scale radius. Its purchased comparison bounds include the three modes in (IG1), the bounded coefficient lifts and the smeared comparison estimates through that horizon. All their error thresholds can be met together because only finitely many basis pairs and modes occur at this horizon. Recursively increase the horizon to i, require a larger sequence index and station depth, and set the next radius ceiling equal to half the radius

just chosen. This produces a strictly increasing subsequence map, strictly decreasing chosen radii, and station depths tending to infinity. At each fixed degree all its comparisons hold eventually.

The exact radius matching has an elementary interval argument. Write $M_k \geq 1$ for the calendar's magnification monitor and $\rho_k(b) = a_k M_k^b$ for the station gap. Eventually $M_k \leq M_{k+1}$, and the purchased block bound gives $L_k \leq M_{k+1}$. Thus, for $b \geq 0$,

$$\rho_k(b) = a_{k+1} L_k M_k^b \leq a_{k+1} M_{k+1}^{b+1} = \rho_{k+1}(b+1).$$

Also $\rho_k(b) \to 0$ for each fixed b. Given a lower station index J, choose $i \geq J$ beyond these eventual bounds. For $0 < s \leq \rho_i(b)$, take the first $k > i$ with $\rho_k(b) \leq s$. Minimality gives $s \leq \rho_{k-1}(b) \leq \rho_k(b+1)$, also when $k = i+1$. The continuous function $u \mapsto \rho_k(b+u)$, $0 \leq u \leq 1$, therefore attains s. The station family was purchased for this entire interval of exponents; its other charges and interpolation parameters stay fixed over the interval. Consequently this adjustment of u preserves its estimates. Given a sequence $s_n \to 0$, choose a later sequence index below the radius ceiling supplied by this argument, and repeat with a larger J. This proves the asserted simultaneous escape of sequence and station indices, with exact equality of radii.

The bounded lifts are then extracted along the already retained ultrafilter and projected as above. Their finite-dimensional limits assemble into one scaling family with exactly those chosen radii. The trace and local-coordinate estimates apply to every such compatible scaling family; they retain the previously fixed physical table and phase. Hence this construction works for the arbitrary sequence supplied to it, including the violating sequence in the proof of the full trace limit.

Source–readout formulation. The comparison never replaces the interacting law by a Gaussian law. Its structure is: (i) a finite identity of densities (the marked density is pushed along the same course, with the same partition function); (ii) rooted bounds (IG7) on three specified corrected modes of the *interacting* law; (iii) Theorem 23.1, which transfers a rooted bound to every character reading because the phase measure has full support. Step (iii) is a statement about the fibre of phases: a bound valid for the unselected hierarchy descends to *each* point of the character space, which is stronger than almost-everywhere and is what permits statements about the selected phase. The Gaussian quantities m_j, z are comparison data of the free sector read on the same labels; the residual of the comparison is ε_k and is retained.

Theorem 23.1 (From rooted bounds to every character)**.** *Let D be a finite combination of distinct-point words with rooted bound c and compressed whole-word operator B. Then*

$$\int |\chi(B)|^{2n}\, d\mu(\chi) \leq c^{2n} \quad (n \geq 1), \qquad \textit{hence } |\chi(B)| \leq c \textit{ for every character } \chi.$$

Consequently, for the continuum one- and two-point readings u_j, u_{01} in any character,

$$|u_j - m_j| \leq c, \qquad |u_{01} - (m_0 m_1 + z)| \leq [1 + |m_0| + |m_1|]\, c.$$

Proof. Place $2n$ copies of D with alternating reflected conjugation in chambers at separation t; the finite chessboard inequality bounds the expectation of their product by c^{2n}; the ordered lattice limit at fixed n, t preserves it; letting $t \to \infty$, the long-word compression limit gives $\langle \Omega_0, (B^*B)^n \Omega_0 \rangle \leq c^{2n}$ and the character decomposition gives the integral bound. If $|\chi_0(B)| > c$, continuity gives a neighbourhood and $a > c$ with $|\chi(B)| \geq a$ there, of measure $h > 0$ by full support, and $h a^{2n} \leq c^{2n}$ fails for large n. The order of limits is: lattice limit at fixed copy count and separation, then separation to infinity, then $n \to \infty$. For (IG3) apply this to (IG1) and use $u_{01} - (m_0 m_1 + z) = [u_{01} - m_0 u_1 - m_1 u_0 + m_0 m_1 - z] + m_0(u_1 - m_1) + m_1(u_0 - m_0)$. □

The exact radius matching at the end of the derivation is geometric: the requested physical radius s equals the purchased station radius $\rho_k(b+u)$ for some $u \in [0,1]$. It allows the station estimates to be applied at that radius on the same retained physical table and phase. It does not make interacting and Gaussian moments equal at a finite station. Their actual two-point readings satisfy

$$|u_{01} - (m_0 m_1 + z)| \le (1 + |m_0| + |m_1|)c. \tag{5}$$

At each fixed degree the purchased errors tend to zero; boundedness of the free singleton readings then gives equality in the specified limit. Prescribed determining-frame normalization elsewhere is an exact finite coordinate calibration, and is distinct from equality of actual hierarchy moments. The universal selection retains this distinction when it is applied to an arbitrary sequence of radii in the full trace-limit argument.

23.3 One calibration matrix for the three panels

The three comparison panels used in (WS1)–(WS17) differ in their point and smear geometry, but have the same finite Taylor order and the same calibration test system. We justify the common matrix, rather than choosing three unrelated solutions. At a fixed degree and sufficiently late stop, the detector matrix of that calibration system is invertible by (PC1)–(PC13). Select the solved chart at one anchor panel. Its defining equations prescribe its pairings with that test system. For any other panel, transport the same local representative and the same Taylor card to its allowed geometry. The support, threshold, and budget equalities make its equations identical to the anchor equations. Uniqueness of their solution then gives the same field matrix T at that panel. Signed covariance follows by transporting the solved equations under a signed permutation and again invoking uniqueness. This proves simultaneously the tested-chart and signed-intertwining properties needed in (WS1).

More concretely, the inverse chart is T^{-1} after the field dilation and its engineering-weight factor. Applying the defining calibration equations to a basis proves that it solves those equations; the unit equation gives the unit, and conjugating or turning the equations gives conjugation and signed covariance. Finite dimensionality bounds the finitely many inverse coefficients after the detector lower bound has been paid. These are precisely the coefficients used in the rooted comparison estimate, including on the smear panel. There is no separate choice of an inverse for each spectator word.

For a fixed finite horizon each panel has finitely many local demands, all eventually paid. Take their maximum together with the common detector and monitoring thresholds. At this stage the anchor solution transfers to all three panels at every finer cutoff and every admitted volume. Choose the stops by the cofinal advance already used in (WS16), satisfying these finite maxima and each earlier horizon. Passage to the common coefficient subsequence preserves the tested equations because their finite pairings converge. Therefore the point, displaced-point and smeared comparisons have the same limiting field identification and may be combined in the trace and Ward arguments.

23.4 Common scaling maps and convergence of every derivative

The comparison of a physical field with a free field must use one map for all its readings. Here is the construction, including the compactness and interpolation steps. Fix a finite maximum number of slots, a finite field horizon and a bounded separated configuration window. During this construction these bounds may be enlarged to include detectors and derivatives. The physical table, its null quotient and its limiting correlations remain those already constructed.

Choose radii $r_k \downarrow 0$ and lattice levels n_k with

$$\delta_k = \frac{a_{n_k}}{r_k} \longrightarrow 0. \tag{WS1}$$

For each free coefficient basis vector at each requested degree, the insertion construction gives a column of physical reference coefficients. Its bound is finite for each fixed k, d, uniformly in the late mesh and large period. These bounds are not asserted uniform in k. Extract the finitely many coordinates of each column on their compact balls, first in the mesh limit at a retained period and then in the period limit. Diagonalize over the countable pairs (k, d). The resulting maps $U_{k,d} : F_d^{\mathrm{free}} \to F_d$ initially need not be invertible or commute exactly with the inclusions. Reality, the signed permutation action and the unit pass to the limit since they are finite linear equalities.

The estimates that permit this extraction also supply numbers $A_k \geq 0$, for every fixed word length n, with

$$\delta_k A_k^n \longrightarrow 0. \tag{WS2}$$

The center and comparison estimates are imposed at levels $m \leq j$ satisfying

$$a_m < r_k/Q, \qquad 6na_j < a_m, \qquad a_j < r_k/(4Q).$$

The insertion supports are therefore separated before the comparison is taken. On the rescaled lattice, the aligned comparison tends to the free correlation. On a configuration window in which all the required coordinate projections have a fixed strict gap, the long finite-difference estimate gives

$$|S(U_k v; r_k x)| \leq A_k^n, \qquad |\partial_{x_{\ell,\alpha}} S(U_k v; r_k x)| \leq 4QA_k^n. \tag{WS3}$$

Rounding to the rescaled lattice changes a coordinate by $O(\delta_k)$; the resulting error vanishes by (WS2). Thus the correlations converge to the free ones on these coordinate-generic windows. All bounds here are for a fixed finite word and fixed window. No uniform bound on all fields or all degrees is being taken.

To fit the coefficient maps, use compactly supported smooth tests in the coordinate-generic part of the separated configuration space. Their readings separate the free quotient. Indeed, a field whose every such reading vanishes has zero correlations there by the fundamental lemma for test functions, and then on the entire separated space by continuity and density. Finite dimension now gives finitely many separating readings at each receiving degree. Spectators in these readings can have larger degree than the field being detected; enlarge the common horizon accordingly. If E_d denotes this finite injective reading map, the physical readings of $U_{k,d}$ converge, in matrix norm, to E_d. Injectivity is open for maps on a finite-dimensional domain: choose a left inverse L_d of E_d; when $\|L_d(E_{k,d} - E_d)\| < 1$, the matrix $L_d E_{k,d}$ is invertible. Thus $U_{k,d}$ is eventually injective. The previously constructed spanning family gives $\dim F_d \leq \dim F_d^{\mathrm{free}}$, so it is eventually an isomorphism.

We make explicit the finite-horizon estimates preceding these readings. A centered budget is the reflection-positive seminorm for every signed or conjugated copy of the one insertion in a reporting chamber. For a word of n insertions, take n chambers whose centers may be shifted within a finer common chamber. If distinct points have at least one coordinate difference larger than r_k/Q, the inequalities preceding (WS3) permit these shifts so the inspected insertion cells are disjoint. Apply Cauchy–Schwarz in the positive reflection form successively in the separating coordinate directions. Each insertion contributes its centered seminorm at most A_k; no factor depends on the volume or the microscopic cutoff. Thus the finite Wilson word has magnitude at most A_k^n.

Take its ordered mesh and period limits at a coordinate-generic tuple. The fitted coefficients converge in their finite-dimensional balls, and multilinearity permits them to vary along the limit.

The same bound passes to the limiting correlation. On the open set of tuples with some separating coordinate for each pair, the correlation is continuous and the generic tuples have full measure and are dense. Therefore the bound extends to every tuple in that open set. This proves the first part of (WS3) on the entire separated window, including coordinate ties.

For a coordinate-generic tuple with all coordinate gaps larger than $1/Q$, move one coordinate a positive rescaled distance $t < (4Q)^{-1}$. On the microscopic lattice perform the integer displacement $\lfloor r_k t/a_N \rfloor$ in that coordinate. Its physical length tends to $r_k t$. The long finite-difference estimate for the same centered budgets, with available gap $r_k/(4Q)$, bounds the difference of the two Wilson words by

$$A_k^n \frac{r_k t}{r_k/(4Q)} = 4Q A_k^n t. \tag{WS8}$$

Both endpoint readings converge with their original placements and coefficients. Pass through the ordered limits, divide by t, and let $t \downarrow 0$ through endpoints that remain generic. Smoothness identifies the limit with the coordinate derivative and proves the second bound in (WS3). This estimate is uniform on a fixed strict coordinate-gap window. It is not used with Q tending to infinity near a projection tie.

The small quantity in (WS2) has the following precise use. The finite suppliers provide it separately for every word in a fixed finite-degree basis and all its signed and conjugated variants. Taking a maximum over finitely many words preserves convergence to zero. Indeed, if $a_{k,w} \geq 0$ and $\delta_k a_{k,w}^n \to 0$ for each of finitely many w, then $\delta_k \max_w a_{k,w}^n = \max_w(\delta_k a_{k,w}^n) \to 0$. The polynomial prices of the solved columns are (PC12)–(PC13); the rooted insertion and aligned Gaussian estimates provide these centered budgets and the aligned word comparison on the same station. The finite estimates must supply these products at their actual station radius and grid; the grid cannot be altered independently after a comparison has been purchased. Thus varying labels in a bounded finite-dimensional set costs a fixed finite sum of these basis bounds, rather than requiring uniformity over an infinite set of labels.

Here is the scale and price bookkeeping in that last statement. For each requested station k, all degrees through the horizon use one earlier index $a(k) \leq k$, with

$$a(k) \longrightarrow \infty, \qquad s_k - s_{a(k)} \leq \log \widehat{\beta}_k, \qquad \frac{a_{a(k)}}{a_k} \leq \widehat{\beta}_k. \tag{WS11}$$

The last inequality follows by exponentiating the preceding one, since $a_i = e^{-s_i}$. At the comparison radius r_k the actual rescaled mesh is $a_{a(k)}/r_k = \mathrm{mag}_{a(k)}^{-1}$. The common finite matrices are taken to sublimits along the same comparison filter below the retained ordered filter. Each pair of degrees, each spectator and each placement therefore uses these same matrices and this same radius.

The reverse coefficient transport from a level m to $k \geq m$ has price

$$\mathrm{quota}_k(v) \leq (a_m/a_k)\, \mathrm{quota}_m(v). \tag{WS12}$$

Indeed one step costs the column allowance $R_{i+1} \leq L_i$; multiplying those inequalities gives $\prod_{i=m}^{k-1} L_i = a_m/a_k$. The solved readout at $a(k)$, followed by its exact field-number and engineering dilations, consequently supplies a preimage y for every actual coefficient v, with

$$\|y\| \leq 2a_k^{-d}\, \mathrm{quota}_k(v).$$

To check the factor two, its approximate matrix differs from the exact awake coefficient by at most half the awake coefficient size, as in (PC7); the triangle inequality bounds the approximate size by twice the awake size. The field shrink is at most one and the comparison magnification at least one, so undoing their positive dilations cannot increase the weighted coefficient size. Formula (WS12)

then gives the displayed bound. This argument applies to the full coefficient space, including equation directions, before restricting to the physical quotient.

The magnification and each fixed finite sum of its engineering and field-number powers have rate zero relative to $\log \widehat{\beta}_k$. For example the label price is bounded by

$$2\sum_{\alpha}(2\,\mathrm{monitor}_{a(k)})^{f_\alpha}\mathrm{mag}_{a(k)}^{r_\alpha}.$$

For every $\varepsilon > 0$, each fixed power is eventually at most $\widehat{\beta}^{\varepsilon}_{a(k)} \le \widehat{\beta}^{\varepsilon}_{k}$; the finite sum has the same rate-zero property. These are the actual finite-horizon prices used with the rooted insertion estimates. The centered estimates assert jointly, for a word of length n, the existence of reporting levels $m \le j$ such that

$$a_m < r_k/Q, \quad n(6a_j) < a_m, \quad a_j < r_k/(4Q), \quad A_k^n(a_{a(k)}/r_k) \to 0. \tag{WS13}$$

The first three conditions give the movable disjoint chambers used in (WS8); the last is exactly the price needed in (WS9). They are imposed together with the aligned moment comparison, not inferred merely from the two-field detector limit. In passing from a basis to bounded arbitrary labels, expand every slot in the fixed finite basis and sum the products of the coordinate norms. The finite sum preserves both the vanishing comparison error and the last limit in (WS13).

The smeared station version says the same thing without committing to a grid interpolation: for every smooth test t in the stated separated window, with $\|t\|_\infty \le M$, the ordered upper error is at most $\varepsilon M \prod_\ell \|v_\ell\|$. Its finite-cutoff lift is bounded, real, signed-equivariant, and sends the unit to the actual reference unit. Its scale and normalization are fixed for all those tests. Enlarging the arity, window and degree and decreasing the error preserves the previous requests. The common-order selection first retains these finite data, then uses (WS4) and finite detecting matrices to obtain all derivative estimates. Thus neither the conditional formulation of a station estimate nor the existence of a two-point comparison is being used as a substitute for the joint estimates (WS13).

The simultaneous centered supplier can be described explicitly; it is stronger than a two-point Gram comparison. Fix a degree cap D, an arity cap $n \ge 2$, and a compact separated window. The actual construction provides three calibrated panels at exponents a, b, e, on one family and with the same coefficient matrices. The bounds used for a panel are its solved-column and marked insertion bounds (PC1)–(PC13), together with the rooted comparison of the entire requested finite word. Denote the common monitor by M_k, and write $H_k(t) = M_k^t$. In the source this is $\mathrm{bill}(t, \mathrm{clim}\, k)$; thus $H_k(t)H_k(u) = H_k(t+u)$. A primary comparison radius is $r_k = a_k H_k(a)$, its mesh spacing is $\delta_k = H_k(-a)$, and a small motion allowance is $p_k = H_k(c)$. Choose the exponents so that

$$c < 0, \quad b + 6 < a, \quad e + 5 < a + c, \quad (a-b)Dn + c < 0, \quad (a-e)Dn - a - c < 0. \tag{WS14}$$

There is no obstruction to these finite numerical choices. Put $x = 7Dn + 1$, $y = x + 6$, $a = 17 + D + x + yDn + y$, $b = a - 7$, $e = a - y$, and $c = -x$. Then $7Dn - x = -1$, $e + 5 = a - x - 1 < a + c$, and $yDn - a + x = -17 - D - y < 0$; also all three panel exponents exceed $D + 6$. These are the source's finite choices. The calibrated panel supplier is chosen with these exponents before extraction.

Here is how that supplier yields centered bounds, rather than merely ordinary moment bounds. For a single basis label, the rooted comparison applied to the singleton at the origin controls the observable minus its free scalar value by the word error $\varepsilon_k \to 0$. The centered root seminorm is subadditive, is homogeneous in scalar coefficients, and assigns norm one to the constant observable. Thus its full observable has root seminorm at most $1 + |r_{\mathrm{free}}|$, eventually. Interior support in the centered chamber holds at every translated origin by locality of the observable; translation of the

chamber preserves this root seminorm. This gives a uniform centered bound at the panel's own width, including its constant part.

Changing the panel magnification from b to a multiplies a coefficient of engineering degree $q \le D$ by $H_k(a-b)^q$. Expand in the fixed finite label basis. The triangle inequality therefore gives the centered price

$$A_k = C\, H_k((a-b)D). \tag{WS15}$$

The same argument at panel e gives $A'_k = C' H_k((a-e)D)$. To include every signed and conjugate variant, first apply these finite transformations to the free label, expand each in the same basis, and take the sum of their finitely many coordinate prices. The coefficient lift intertwines these transformations and sends the unit to the actual reference unit, so the resulting bound is for the required physical transformed token. All slots of the word are included before taking the maximum constant. Thus one reporting width works simultaneously for the word; it is not selected after seeing its offsets.

The available plateau geometry gives, for j_b the reporting width at panel b,

$$n(6a_{j_b}) < a_{j_b-1}, \qquad \frac{a_{j_b-1}}{a_k} \le H_k(b+6), \qquad \frac{a_{j_e}}{a_k} \le H_k(e+5). \tag{WS16}$$

The first follows because one blocking factor tends to infinity. For the other two, the principal plateau length costs $H_k(b)$ or $H_k(e)$; the rounded length and two buffer lengths cost at most five further monitor powers, and passing to the preceding block costs one more. The block factor is bounded by the monitor and its tenth power dominates the fixed covering-size buffers. Multiplying these inequalities gives exactly the exponents $b+6$ and $e+5$, rather than changing the comparison radius.

Equations (WS14)–(WS16) now give the simultaneous vanishing products needed in the moving-chamber proof:

$$A_k^n p_k \longrightarrow 0, \qquad (A'_k)^n \delta_k / p_k \longrightarrow 0, \qquad 4n\delta_k \le p_k \quad \text{eventually.} \tag{WS17}$$

Indeed their first two exponents are the last two negative numbers in (WS14); the third uses $-a < c$, which follows from the last inequality there. Also $a_{j_b-1} < r_k/(2R)$ and $a_{j_e} < r_k p_k$ eventually, by the two strict panel gaps in (WS14). These two widths are the two centered budgets needed for the shift across a grid tie and for the comparison away from it. The rooted whole-word error is $C_R \varepsilon_k \to 0$. The moving-chamber decomposition (WS8)–(WS9), with its strip of width p_k, is therefore valid jointly in all slots. Integrating it against a test of supremum M gives the station error $\varepsilon M \prod_i \|v_i\|$, including projection ties. This is the actual three-panel route to the joint station statement. The simpler conditions (WS13) record the part of that geometry used by its one-width formulation; no two-field estimate is being used to supply a whole-word centered estimate.

For clarity about existence on the chosen trajectory, the dense calibrated supplies retain fixed three-panel exponents at each horizon, the foregoing inequalities, and the actual solved charts at a predecessor $g(k) \to \infty$ satisfying (WS11). Their availability is paid by the same delayed local insertion and marked comparison schedule used for (PC1)–(PC13). The finite horizon does not depend on the requested extraction order. Passing their finite matrices to limits on a fixed common subfilter preserves those charts and gives the smeared comparisons for their continuum fields. At each horizon, the finite-dimensional derivative and inclusion adjustments in (WS4)–(WS7) act on these fields. Finally choose an increasing integer horizon whose every finite requirement has already become valid at index k: if K_D is a threshold for horizon D, use the largest $D \le k$ with $\max_{j \le D} K_j \le k$. It tends to infinity. This is the common diagonal; it preserves the physical index k, the fixed family, and the inherited subfilter. It does not retune the grid for individual spectators.

These same readings control the inclusions and derivatives without requiring a bound on the inverse in the physical coefficient norm. In a rescaled position variable, integration by parts is exactly

$$\int t(x)S(r_k\partial_\alpha B,\dots;r_kx)\,dx=-\int(\partial_{\ell,\alpha}t)(x)S(B,\dots;r_kx)\,dx. \tag{WS4}$$

The derivative of a compactly supported test has the same support constraint. Apply the converging injective reading matrix at the receiving degree to the two sides. A left inverse close to the fixed free reading matrix then proves

$$\begin{aligned} U_{k,d+1}^{-1}I_{d,d+1}^{\mathrm{phys}}U_{k,d}&\longrightarrow I_{d,d+1}^{\mathrm{free}},\\ U_{k,d+1}^{-1}r_k\partial_\alpha^{\mathrm{phys}}U_{k,d}&\longrightarrow \partial_\alpha^{\mathrm{free}}. \end{aligned} \tag{WS5}$$

Every inverse used in this deduction acts on a finite detector matrix with a fixed injective limit. It does not presume that the physical charts are uniformly conditioned.

One can make the first line of (WS5) exact. Choose left inverses P_d of the free adjacent inclusions. Average them over the finite signed permutation group and then average with their complex conjugates. The averaged maps are still left inverses and now respect both symmetries. Define recursively

$$X_{k,0}=U_{k,0},\qquad X_{k,d+1}=I_{d,d+1}^{\mathrm{phys}}X_{k,d}P_d+U_{k,d+1}(1-I_{d,d+1}^{\mathrm{free}}P_d). \tag{WS6}$$

Substitution of $I_{d,d+1}^{\mathrm{free}}v$ proves exact compatibility with the adjacent inclusion, hence with every inclusion by composition. Induction in d, using (WS5), gives $U_{k,d}^{-1}X_{k,d}\to 1$. Consequently these maps are eventually isomorphisms, preserve the unit and symmetries, and still satisfy the second line of (WS5). On the finitely many exceptional indices choose any isomorphisms; these choices do not affect a limit. Conjugation by matrices tending to the identity preserves all the finite-dimensional limits just obtained.

It remains to remove coordinate-generic restrictions and to pass to all position derivatives. For a fixed word let

$$R_k(x)=S(X_kv;r_kx).\qquad \partial_{\ell,\alpha}R_k(x)=S(X_kv_1,\dots,X_kD_{k,\alpha}v_\ell,\dots;r_kx),\quad D_{k,\alpha}=X_{k,d+1}^{-1}r_k\partial_\alpha X_{k,d}. \tag{WS7}$$

The matrices $D_{k,\alpha}$ converge to the free derivative matrix. By multilinearity, readings with any converging finite list of coefficient vectors have the same limiting readings as their limits. Also (WS2) applies to the finitely many successor-degree basis words in (WS7). It bounds δ_k times every needed gradient by a quantity tending to zero on a slightly larger separated window. Interpolate from nearby sampled points to arbitrary separated points along short coordinate segments in that larger window. Their total length is $O(\delta_k)$. The fundamental theorem of calculus and (WS7) make the error tend to zero. The argument does not require a positive gap between distinct coordinate projections: the points themselves stay separated.

Repeat (WS7) for each differentiation. For m derivatives use the m successor degrees for the differentiated labels and one additional degree for the interpolation gradient. Thus a horizon D proves the assertion when $d+m<D$. The finite sum over basis components proves convergence of the full derivative tensor in its norm. This convergence is uniform on every compact separated window. Otherwise a sequence of points violating uniform convergence has a convergent subsequence in that compact set; the moving-point interpolation just proved gives a contradiction. Enlarging the initial finite request supplies every prescribed derivative order, word and compact window, on the same retained physical hierarchy.

The interpolation near projection ties uses a different gradient bound from (WS8). On a slightly larger separated window each gradient in (WS7) is a finite sum of readings of successor-degree basis words, with coefficients from $D_{k,\alpha}$. These coefficient matrices converge and hence are bounded. The first, all-separated part of (WS3) applied to those basis words and the finite-maximum argument above therefore give

$$\delta_k \sup_{\text{larger separated window}} \|\nabla R_k\| \longrightarrow 0. \tag{WS9}$$

Round every coordinate to the rescaled mesh, then perturb the rounded coordinates by distinct bounded integer offsets to make every coordinate projection different. The number of forbidden equalities is finite for a fixed word, so one fixed finite list of offsets suffices. The displacement remains $O(\delta_k)$. For large k, all short joining coordinate segments stay in the larger separated window. The integral of the gradient along them is $o(1)$ by (WS9). This proves interpolation also at coordinate ties without losing a power through a shrinking coordinate gap.

There is one final sign choice when these maps are required to identify the normalized physical stress with the free stress. Each enlarged finite request allows one of the two signs of the physical conserved line. If one sign works for all requests, retain it. If a particular request excludes it, enlarge any subsequent request to include that one; its available map must use the opposite sign and restricts to a map for the subsequent request. Hence one fixed sign works for every finite request. Subsequent normalization fixes its amplitude as explained below.

For completeness, a single sequence realizing all the finite requests is obtained as follows. At arity n, exhaust the separated space by

$$K_{n,j} = \{x : \max_{\ell} |x_\ell| \le j,\ \min_{\ell \ne m} |x_\ell - x_m| \ge j^{-1}\}, \tag{WS10}$$

with the minimum condition omitted for zero or one slot. Enumerate the countable requests consisting of an arity, a finite tuple of degree bounds, a derivative order and an integer j. At stage s, enlarge one finite request to contain the first s requests, their finitely many basis words, the detecting spectators, and one extra derivative degree. Use the finite construction on the same physical hierarchy and the same chosen stress sign. Along its available comparison sequence choose a station late enough that the radius is below $1/s$, the rescaled mesh is below $1/s$, all the first s derivative errors on their compact sets are below $1/s$, and every required instance of (WS9) is below $1/s$. There are finitely many conditions, each eventually true, so such a station exists. Pad maps above this stage's horizon by arbitrary fixed isomorphisms; every fixed degree eventually lies below the horizon.

The selected stage-s maps and radii form one common sequence. For a fixed enumerated request, all later stages include it and give errors at most $1/s$; hence it has the claimed limit. Every compact separated set lies in some $K_{n,j}$, so this proves all the asserted compact limits. If ordered mesh-period coefficient extractions are refined during these constructions, retain their countable columns and inverse charts together by the ordered compact extraction already used in (WS1): at each period take mesh limits first, then refine the period sequence and diagonalize the countable coordinate list. This preserves the same table and stress. The station diagonal just described is taken after those ordered limits and does not identify them with an arbitrary joint cutoff-volume diagonal. At any one station only its finite horizon is asserted; every fixed request is eventually included.

Source–readout formulation. The scaling maps are the interface between the two realized sectors over the common label algebra. They are *not* canonical: each $X_{k,d}$ is selected by compactness and diagonalization and corrected by (WS6), and the statement proved is about the limit of readings, not about the maps. What is recovered is the short-distance *germ* of the interacting hierarchy at

scale r_k, read in free coordinates, to all derivative orders. What is not asserted: uniformity in the degree, a bound on $\|X_{k,d}\|$ or its inverse as $k \to \infty$, or that the interacting theory is free. The lift transports these maps as linear maps between label spaces, on which the frame realization acts trivially.

Theorem 23.2 (Scaling comparison)**.** *There are radii $r_k \downarrow 0$ and isomorphisms $X_{k,d} : F_d^{\mathrm{free}} \to F_d$, real, signed-covariant, unital and exactly compatible with inclusions, such that for every word, every derivative order and every compact subset of the separated configuration space,*

$$\partial_x^\gamma \, S(X_k v; r_k x) \longrightarrow \partial_x^\gamma \, M_n(x; v)$$

uniformly, where M_n is the free hierarchy (FM3)*, and $X_{k,d+1}^{-1} r_k \partial_\alpha X_{k,d} \to \partial_\alpha^{\mathrm{free}}$.*

Proof. This assembles (WS1)–(WS17), Theorem 23.1 (for the two-point comparison and moving labels) and the detector argument of Theorem 16.7. □

One calibration matrix serves the three panels (trace, stress and rotation): the same finite reader list and the same matrix of readings are used for all three, so the three conclusions are statements about one realized family and can be combined. This is the common-witness requirement of the method in the form in which it is needed here.

23.5 Constructing the physical stress and its shell charges

We first detail the rank transfer on centered tensors. A scaling isomorphism need not take the free expectation functional to the physical one. Let $c_0(v) = v - \langle v \rangle_0 1$ and $c(v) = v - \langle v \rangle 1$ be the two centering maps. Apply them componentwise to tensors. For a late scaling index, $U_{k,4}$ intertwines reality and signed symmetries and fixes the unit. It therefore induces the map

$$\mathcal{R}_k : \mathcal{K}_0 \longrightarrow \mathcal{K}, \qquad \mathcal{R}_k B = c(U_{k,4} B), \tag{SR1}$$

between the real centered symmetric signed tensor spaces. Its inverse is $A \mapsto c_0(U_{k,4}^{-1} A)$. Indeed the first centering subtracts only componentwise unit multiples. The inverse isomorphism carries those units to units, which the second centering kills; the remaining term is the original centered tensor. Thus these are exact inverse maps even though the two expectations differ. The corresponding degree-five isomorphism $\mathcal{Q}_k$ on real signed polar vectors needs no centering correction.

Write D_0, D for the free and physical divergence maps on these centered tensor spaces. On the fixed free spaces consider

$$Z_k = \mathcal{Q}_k^{-1} \, r_k D \mathcal{R}_k. \tag{SR2}$$

Derivatives of the unit are zero, so the centering correction in (SR1) contributes nothing to (SR2). The derivative comparison (WS7) gives $Z_k B \to D_0 B$ for every B. The free calculation makes D_0 onto and its kernel one dimensional. Choose a right inverse G of D_0. Then $Z_k G \to I$ in matrix norm, by convergence on a finite basis. Eventually $\|Z_k G - I\| < 1$; the geometric series makes $Z_k G$ invertible, hence Z_k onto. Since $r_k > 0$ and both surrounding maps in (SR2) are isomorphisms, D is onto. The exact dimension equalities from (SR1) and $\mathcal{Q}_k$, followed by rank–nullity, now give

$$\dim_{\mathbb{R}} \ker D = \dim_{\mathbb{R}} \ker D_0 = 1. \tag{SR3}$$

This establishes the conserved centered line without identifying the expectation functionals through U_k.

The physical stress is selected on the common field quotient, before its final amplitude is fixed. At weight four the real signed symmetric tensor space includes the vacuum tensor as well as (FC3),

so it has dimension four. The rank comparisons transfer this dimension from the free table to the physical table and identify it with the image of the four real reference coefficients

$$J_{ab}(w) = \begin{cases} w_0\,1 + w_1 \sum_i Q_{ai,ai} + w_2 \sum_{i,k} Q_{ik,ik}, & a = b, \\ w_3 \sum_i Q_{ai,bi}, & a \neq b. \end{cases} \tag{SC1}$$

In (SC1) take the actual physical classes of these reference labels, not their free polynomial values. The physical weight-five polar vector space has dimension at most two, by the same rank comparison. Differentiation on the quotient therefore gives a linear divergence map from a four-dimensional real space into a space of dimension at most two:

$$(\nabla J)_b = \sum_a \partial_a J_{ab}. \tag{SC2}$$

Its range is polar because the signed derivative transformation has one sign for each derivative index; summing over that index cancels its squared sign.

Adjoin the real one-point reading of J_{00} to (SC2). The resulting map has target dimension at most three and thus has a nonzero kernel. Choose w in that kernel and normalize its Euclidean coefficient norm to one. The tensor map (SC1) is injective by the four-dimensional rank comparison, so $J(w) \neq 0$. It is conserved and its 00 mean vanishes. Reality makes every diagonal mean real; translation makes each mean constant in position; unsigned permutations make all diagonal means equal; reflection in one of two different indices negates an off-diagonal mean. Thus every mean vanishes. This constructs a centered nonzero real symmetric conserved tensor. It does not yet select its sign or claim its Ward charge is the unit translation generator.

Here is the normalization of this line along the comparison curve. For any nonzero tensor J in it and the fixed nonzero free tensor T^0, put

$$B_k = U_{k,4}^{-1}(r_k^4 J), \qquad \xi_k = \frac{\|B_k\|}{\|T^0\|} > 0, \qquad \widetilde{B}_k = \xi_k^{-1} B_k. \tag{SR4}$$

The inverse and norm here are on the full tensor coordinate space, including its scalar directions. The norm of $\widetilde{B}_k$ is exactly $\|T^0\|$, so finite-dimensional compactness gives a convergent subsequence with limit B of that norm. No positive lower bound on ξ_k is needed. Closedness of the real signed symmetric space puts B in that space. Its free mean is zero: the one-point comparison for bounded converging labels applies to $\widetilde{B}_k$, while its physical image is the positive scalar multiple $\xi_k^{-1} r_k^4 J$, whose expectation is exactly zero. Applying the converging derivative matrices (WS7) to $\widetilde{B}_k$ similarly passes the exact physical divergence zero to the free divergence zero. Thus $B \in \ker D_0$. By its one-dimensionality, $B = cT^0$ for real c; equality of norms gives $|c| = 1$. Choose the single sign of J that makes $c = 1$. Restricting the comparison parameter to this subsequence preserves all earlier limits and exact compatibility laws. This proves the normalized stress comparison used above; its eventual Ward amplitude is fixed separately below.

The finite lattice Ward identity associated with this construction retains its residual explicitly. At spacing a_N, write $T_{ab}^{N,p}(x)$ for the calibrated cylinder (SC1). Define its symmetric divergence, a clover-based directed Wilson variation, and its residual by

$$\begin{aligned} \operatorname{div}_a T_b(x) &= (2a_N)^{-1} \sum_a [T_{ab}(x + e_a) - T_{ab}(x - e_a)], \\ \Delta_b F(x) &= (2a_N^5)^{-1} \sum_i [\mathrm{contact}_{x,+i}(X_{bi}, F) - \mathrm{contact}_{x,-i}(X_{bi}, F)], \\ \mathcal{A}_b(x) &= \operatorname{div}_a T_b(x) - Z_\delta \mathcal{B}_b(x). \end{aligned} \tag{SC3}$$

Here X_{bi} is the signed clover: one quarter of the four oriented plaquette curvature readings, with signs $+,-,-,+$ for $(+b,+i),(+b,-i),(-b,+i),(-b,-i)$. All are transported to the common base. Its adjoint covariance and antisymmetry follow by reversing the corresponding plaquette words. In a color orthonormal basis e_c, the contact is $\sum_c X^c D_{x,\pm i,e_c} F$. The corresponding equation insertion is $\beta\langle X, \mathrm{force}_{x,\pm i}\rangle - \sum_c D_{x,\pm i,e_c} X^c$; $\mathcal{B}_b$ is its same signed sum divided by $2a_N^5$.

Right or left Haar invariance in that one directed link gives

$$\left\langle \left(\beta\langle X, \mathrm{force}\rangle - \sum_c D_c X^c \right) F \right\rangle = \left\langle \sum_c X^c D_c F \right\rangle . \tag{SC4}$$

Differentiate the change of variables in the finite Haar integral; the derivative of the Wilson density gives the force, and the product rule gives the derivative of X and of F. Compactness and continuity of the finite word derivatives justify moving that derivative through the integral. Linearity over the color basis proves (SC4) for the group-valued clover vector field.

For a lattice cutoff χ, a vector field V, an enclosed insertion F and an exterior word E, let

$$W_{ab}(x) = \frac{(\chi(x+e_a)-\chi(x))V_b(x+e_a) + (\chi(x)-\chi(x-e_a))V_b(x-e_a)}{2a_N}. \tag{SC5}$$

The finite flux is $-a_N^4 \sum_{x,a,b} W_{ab}(x)\langle T_{ab}(x) F E\rangle$. Its exact decomposition is

$$\mathrm{Flux} = a_N^4 \sum_{x,b} \chi(x) V_b(x) Z_\delta \langle \Delta_b F(x) E\rangle + a_N^4 \sum_{x,b} \chi(x) V_b(x) \langle \mathcal{A}_b(x) F E\rangle. \tag{SC6}$$

Use (SC4), the product rule, and finite symmetric summation by parts. The term with the discrete derivative of V vanishes for the constant and skew-linear Killing fields in question, by symmetry of T. The shifted V's in (SC5) are essential for this algebra. The derivative of E vanishes when each active Ward center avoids its full local insertion footprint. This is a support condition on the whole cylinder, stronger than just excluding its marked point. Taking absolute values in the second sum gives its exact error bound. No smallness of $\mathcal{A}$, including at the enclosed insertion, follows from this identity alone.

The continuum charges are instead supplied by the paid short-distance product estimates. Their input is the actual isolated product

$$r^{\,h} I_{J_{ab}}(\lceil y/a_N \rceil) I_B(0) - I_u(0), \qquad D + h = d + 4, \tag{SC7}$$

with B of weight d and u in the one fixed horizon D. At a fixed shell its coefficient bound may depend on that shell. It is common to every exterior test at that shell. The estimate is a rooted cylinder estimate on the reporting cell and on all its specified aligned seams and offsets, not only a two-point expectation. The paid product construction and weighted mesh interpolation give, for each finite family of exterior tests annihilating every D-field,

$$\sup_{y \in \{r_m/4 < |y| < 2r_m\}} \left| \sum_\nu c_\nu r_m^h \langle J_{ab}(y) B(0)\, \mathcal{O}_\nu \rangle \right| \longrightarrow 0. \tag{SC8}$$

The schedule uses $r_m > 0$ tending to zero and a finer mesh level $j(m)$ with $s_{j(m)}/s_m \to t > 1$. The product price chooses $t = 1 + 1/[64(d+12)]$, so $(d+12)(1-t) = -1/64 > -1/32$; its exponential gain dominates the growth of the two insertion columns. The rooted remainder after the exterior stencil estimate is at most $2\,\mathrm{stakes}(k)/\,\mathrm{pitts}(m)$, where $\mathrm{pitts}(m) \to \infty$. The common all-order mesh bounds transfer this from the fine reporting grid to the whole annulus, with the finer mesh

small compared with $r_m/128$. These are the annular product estimates used here; the finite Ward residual in (SC6) is not substituted for them.

Here are the rate choices that supply (SC7), rather than leaving its annular approximation as an input. Write $q_i = \log(a_i^{-1})$. The calendar has $q_{i+1}/q_i \to 1$. For $t > 1$, let $j(i)$ be the first index with $q_{j(i)} \geq tq_i$. Minimality gives $q_{j(i)-1} < tq_i$, so squeezing between these inequalities and $q_{j(i)}/q_{j(i)-1} \to 1$ gives $q_{j(i)}/q_i \to t$; also $j(i) > i+1$ eventually. A positive sequence of logarithmic rate b here means $\log f_i/q_i \to b$. Products add rates, quotients subtract them, and composition with j multiplies them by t. The estimates remain valid with arbitrarily small rate losses.

For $D + h = d + 4$, the two-field product and shell rescaling bounds give the available local approximation radius a logarithmic rate

$$-D + M, \qquad M = (d+12)(1-t). \tag{SC8a}$$

The d and 4 are the field weights, the h is the shell rescaling, and the additional eight is the pair of four-dimensional placement bounds. The adjacent insertion remainder starting at level k obeys the explicit induction bound

$$m_{k,i} \leq b \left(\frac{a_i}{a_k}\right)^{D+1} \frac{f_i}{f_k}, \tag{SC8b}$$

where $f_i > 0$ is its retained error allowance. The induction step is exactly the allowance inequality $\eta_i L_i^{D+1} f_i \leq f_{i+1}$, combined with $a_i = L_i a_{i+1}$. Choose $t = 1 + [64(d+12)]^{-1}$. Then $M = -1/64 > -1/32$. The test normalization $c_i = e^{q_i/32}$ has rate $1/32$, while the accumulated coefficient-normalization gain from level k to i is $e^{(q_i - q_k)/8}$, of rate $1/8$. Substitution of (SC8a)–(SC8b) into the finite product approximation leaves the strict inequalities

$$\frac{1}{32} - (D+1) < -D + M, \qquad \frac{1}{32} - M - \frac{1}{8} < 0. \tag{SC8c}$$

All column counts, fixed-frame norms and logarithmic factors have rate zero. The two strict margins in (SC8c) therefore pay them eventually, give a positive admissible approximation radius, and make its rooted error at most 2 $\text{stakes}_{p,k}/c_i$, the bound in (SC7). The support requirements hold as well: the shell radius tends to zero, and the finest mesh times every finite stencil count tends to zero relative to the fixed physical chamber size. Thus the finite two-field approximation applies at every point in the required annulus, with bounded coefficient representatives. This is the asserted source of (SC7).

Why do the approximants in (SC7) disappear from (SC8)? For a finite exterior family, let $L_N : E_D \to \mathbb{C}$ be its combined finite-cutoff reading. Its limiting reading is zero by hypothesis, and convergence on a basis of E_D gives $\|L_N\| \to 0$. At each fixed shell $\|u_N\| \leq A$, so $|L_N(u_N)| \leq A\|L_N\| \to 0$, although the approximant itself need not converge. The rooted estimate bounds the remaining product discrepancy against each exterior word by reflection positivity and its spectator cell bounds. The mesh interpolation then gives (SC8). This argument integrates annihilators, rather than selecting a new field at every integration point.

Let χ be smooth, compactly supported, one near the enclosed point and zero near the other insertions. For a Killing field V, define

$$Q_V(B;\chi) = -\sum_{a,b} \int (\partial_a \chi)(y) V_b(y - x_0) \langle J_{ab}(y) B(x_0)\, \mathcal{O} \rangle \, d^4y. \tag{SC9}$$

Its integrand is supported on a compact shell disjoint from all insertions, so it is an ordinary integrable smooth function. Conservation and symmetry imply independence of the cutoff: for their difference integrate the divergence of $(\chi_1 - \chi_2) V_b \langle J_{ab} B \mathcal{O} \rangle$. The divergence of J is zero, and

the symmetric tensor contracts to zero with the skew derivative of V. The difference vanishes near every insertion and has compact support, so this integration by parts has no contact or infinity boundary term.

Take V homogeneous of degree $i = 0$ or 1 and shrink χ to radius r_m. Changing variables $y = x_0 + r_m z$ in (SC9) gives the factor r_m^{3+i}: four powers from volume, minus one from differentiating the cutoff, and i from V. Choose $h = 3 + i$ in (SC8). Its bound times the finite integral of $|\partial_a \chi(z) V_b(z)|$ tends to zero. Therefore every finite linear relation among the ordinary D-field exterior readings is satisfied by the flux readings, with $D = d + 1$ for translations and $D = d$ for rotations.

Finite-dimensional linear algebra now represents all these flux readings by one field. Indeed, if functionals $A_\nu \in E_D^*$ and values b_ν satisfy every finite relation $\sum c_\nu A_\nu = 0 \Rightarrow \sum c_\nu b_\nu = 0$, define a linear functional on their span by $A_\nu \mapsto b_\nu$, extend it to E_D^*, and use $E_D^{**} = E_D$. Faithfulness of the quotient makes the representing class unique. Initially use generic grid configurations; translation and continuity of the shell integral and of the kernels extend the same identity to every separated configuration, including coordinate ties.

Linearity of (SC9) in B and faithfulness prove that the representing field depends complex linearly on B. Its rotation charge depends real linearly on the skew matrix, by the same argument. These are the actual unnormalized translation and angular charge maps on the physical quotient. Their normalization and full rotation laws are established by the subsequent angular-flux and comparison arguments; centering and the finite-dimensional existence argument alone do not fix them.

The shell functional also has an ordinary face-flux interpretation. Let a closed coordinate box contain x_0 strictly in its interior and every exterior insertion strictly outside the closed box. Integrate each a-component of the current on its upper face minus its lower face, with three-dimensional Lebesgue measure in the remaining coordinates. This sum equals (SC9). To prove it choose a smooth window supported strictly inside the box and equal to one near x_0. Multiply the current by $1 - \chi$. It is zero near the only enclosed insertion and smooth at all points of the box. Its divergence equals $-\sum_a \partial_a \chi \, F_a$, by conservation and the Killing identity. The ordinary coordinate divergence theorem applies to this smooth truncated current. On every face $\chi = 0$, so its face flux is the original one. The volume integral is (SC9), and cutoff independence gives the equality for any allowed window. This argument treats faces with their outward orientations and does not integrate with four-dimensional ambient measure on the surface.

The bounded coefficient in (SC7) is required at each fixed shell before its lattice limit. It need not be uniform as the shell shrinks. The order is: fix the finite exterior annihilator and a reporting level; buy a sufficiently fine seam level; fix a shell index; then take the ordered mesh and period limits. At that shell the finite-dimensional annihilator norm tends to zero and the approximant coefficient is bounded. Only after this limit is taken does the shell radius go to zero. The rooted error tends to zero by (SC8a)–(SC8c). The same aligned-cell estimate applies on every seam and offset, so it permits moving a shell node freely in the annulus. This is the joint two-field approximation required for the shell construction.

Equation terms may be restored before this estimate is used. For one chamber their difference has zero expectation against every function supported outside that chamber, by the link Haar identity and avoidance of all the active links. Replace one chamber factor at a time in the full replicated array. All other factors are supported in complementary chambers, so each replacement leaves the integral unchanged. Finite induction replaces every chamber, including the conjugated reflected copies. Taking the reciprocal-cardinality power then gives equality of the two rooted seminorms. An equality only for a single unreplicated expectation would not suffice. No contact error at overlapping stencils is suppressed in this argument: overlap is excluded by the full footprint

condition, and actual contacts are kept in (SC4)–(SC6).

Source–readout formulation. Theorem 23.3 is a *representability* statement: the flux reader, a priori only a family of numbers indexed by exterior words, is the reading of a field because it respects every linear relation among field readings. This is exactly the method's criterion for a function on the readout to descend from the source (here: to lift to a label class), and uniqueness is faithfulness of the quotient. The amplitude and sign of the stress are fixed in two separate steps ((SR4) for the sign relative to T^0, and the Ward normalization c of Theorem 19.3); the finite lattice identity (SC6) keeps its residual $\mathcal{A}_b$ explicitly and is never used as if that residual were small.

Theorem 23.3 (The charges are fields)**.** *For each B of weight d and each Killing field V there is a unique class in E_D/N_D whose exterior readings equal the flux readings $Q_V(B;\chi)$ for all exterior words; it depends complex linearly on B and real linearly on the skew matrix. The same number is the ordinary face flux through any closed coordinate box containing x_0 in its interior and all exterior insertions outside.*

Proof. If functionals $A_\nu \in E_D^*$ and values b_ν satisfy $\sum c_\nu A_\nu = 0 \Rightarrow \sum c_\nu b_\nu = 0$, then $A_\nu \mapsto b_\nu$ is a well-defined linear functional on the span, extends to E_D^* and is represented by an element of $E_D^{**} = E_D$; faithfulness of the quotient (Theorem 18.3) makes the class unique and gives linearity. Generic grid configurations are used first and translation and continuity extend to all separated configurations. For the face flux multiply the current by $1-\chi$ with χ supported strictly inside the box: it is smooth on the box with divergence $-\sum_a \partial_a \chi F_a$, the divergence theorem applies, and $\chi = 0$ on every face. □

24 The centered trace and its comparison exclusions

A selected observable and an all-distance conclusion. The centered trace belongs to the selected theory on its common operator domain. Its two-point function is represented by positive spectral measures patched from the same physical hierarchy and phase. Universal selection along arbitrary radii, combined with monotonicity, gives the full short-distance limit. This is stronger than observing the value on one favorable scaling sequence and requires the station-selection argument just proved.

The measure, Laplace and Schwartz passages below retain their polynomial bounds and test classes. They recover

$$\lim_{r\downarrow 0} r^8 \log^2(1/r) G_\Theta(r) = 3 \dim_{\mathbb{R}} \mathfrak{g}/\pi^4.$$

The positive constant describes an ultraviolet asymptotic. It is not a positive mass-gap bound. The ensuing finite mass-shell obstruction uses the precise finite polynomial comparison class, while the Maxwell arguments use their own coherent-sector and locality hypotheses. Neither exclusion is silently enlarged to every conceivable free comparison.

24.1 The centered physical trace and its full short-distance limit

In the selected character vacuum use the same conserved physical stress tensor and its degree-four Euclidean trace $T = \sum_{\mu=0}^{3} T_{\mu\mu}$. Put $\Theta = T - \langle T\rangle 1$. On the common field domain,

$$\langle \Omega, \Theta(f)\Omega\rangle = 0, \qquad \Theta(f) = T(f) - \langle T\rangle \left(\int f\right) I. \tag{TR1}$$

Indeed the Euclidean unit law removes a unit slot. Strict complex-time functions agree on imaginary times and then by holomorphic uniqueness. Fubini smearing gives the factor $\int f$, and tempered boundary passage gives the Wightman identity. Pairing with the dense word span makes unit insertion exactly $(\int f)I$ on the domain. Linearity gives TR1; the zero Euclidean one-point function passes through the same continuation and boundary. Reality and scalar covariance give a Hermitian Lorentz-scalar pinned field. For $s > 0$ define $E(s) = \langle \Theta(-se_0/2)\Theta(se_0/2)\rangle_E$. The source normalization gives

$$\lim_{s\downarrow 0} s^8 \log^2(1/s) E(s) = A_G > 0,$$
$$A_G = 12\frac{d_2(G)^2}{b_0(G)^2}\frac{\dim \mathfrak{g}}{\pi^4}, \qquad d_2(G) = -b_0(G)/2, \qquad A_G = \frac{3\dim \mathfrak{g}}{\pi^4}. \tag{TR2}$$

The nonabelian simple algebra has positive dimension and $b_0 > 0$, so the coefficient is positive. These are the physical-stress and Wilson-calibrated running-charge conventions. Translation and orthogonal invariance reduce every separated trace pair to E at its Euclidean distance. An orthogonal reflection takes their difference to a time-axis vector of the same norm; a common translation then puts them at the opposed points above. Degree-four scaling identifies the rescaled trace with $4d_2$ times the free scalar label after logarithmic running normalization. The scalar Wick contraction is $3\dim \mathfrak{g}/(\pi^4|x_0 - x_1|^8)$. Multiplying by running factor $b_0^{-2}/4$ and trace factor $(4d_2)^2$ gives exactly $A_G|x_0 - x_1|^{-8}$ on the extracted radii and nearby generic plots. The positive spectral representation

$$E(s) = \int_{\overline{V}_+} e^{-sp_0}\, d\mu_\Theta(p) \tag{TR3}$$

makes E real, nonnegative and nonincreasing. Choose generic distances $a < 1 < b$ arbitrarily near one; their scaling limits are $A_G a^{-8}$ and $A_G b^{-8}$ and their nearby actual distances bracket the exact radius. Monotonicity sandwiches its normalized value; let $a, b \to 1$. If the all-distance TR2 limit failed, choose violating distances tending to zero. The scaling construction admits a refining subsequence of these very distances with compatible moving plots and degree-four comparison. The same sandwich contradicts the violations. This proves the full positive-distance limit, beyond the earlier weak-sequence comparison.

Centered-field and all-distance recovery. Let V be the reconstructed Hilbert-space unitary to its horizontal realization, with $\Omega^Z = V\Omega$ and common field domain $\mathcal{D}^Z = V\mathcal{D}$. Retain the selected character under the ground-algebra isomorphism and set $T^Z = J_4 T$ and $1^Z = J_4 1$. Their scalar vacuum readings give

$$\langle T^Z\rangle = \langle T\rangle, \qquad \Theta^Z = J_4(T - \langle T\rangle 1), \qquad \Theta^Z(f)V\psi = V\Theta(f)\psi \quad (\psi \in \mathcal{D}).$$

Unit removal through the same strict-time continuation, smearing and tempered boundary gives, on this actual domain,

$$1^Z(f)V\psi = \left(\int f\right) V\psi, \qquad \Theta^Z(f) = T^Z(f) - \langle T\rangle \left(\int f\right) I^Z.$$

Here $T^Z(f)$ denotes the reconstructed field of the lifted label. These identities recover TR1, its zero vacuum expectation, Hermiticity and scalar covariance. They assert operator equality on the common domain; no bounded-field extension is inserted. To check the identity operator in this formula, first insert a unit label in any Euclidean correlation and use the hierarchy's unit law to remove it.

The separated Euclidean readout is unchanged, so $E^Z(s) = E(s)$. Its amplitude retains all normalization factors separately:

$$\frac{3\dim\mathfrak{g}}{\pi^4}\,\frac{1}{4b_0^2}\,(4d_2)^2 = \frac{12d_2^2}{b_0^2}\,\frac{\dim\mathfrak{g}}{\pi^4} = \frac{3\dim\mathfrak{g}}{\pi^4} = A_G > 0.$$

This uses exactly $d_2 = -b_0/2$ and the physical stress/running-charge conventions, which are retained rather than independently rescaled. The transported positive spectral measure gives the same Laplace reader; since $p_0 \geq 0$, for $s_2 \geq s_1 > 0$ it satisfies $0 \leq E^Z(s_2) \leq E^Z(s_1)$.

To expose the all-distance quantifiers, let $r_j \downarrow 0$ be any proposed violating array and take its source-permitted refining scaling subsequence. For fixed generic $a < 1 < b$ near one, retain the actual nearby distances $d_j^- < r_j < d_j^+$ supplied by the moving plots, with the corresponding normalized limits. Put $N(r) = r^8 \log^2(1/r) > 0$ for sufficiently small r. Exactly

$$N(r_j)E^Z(d_j^+) \leq N(r_j)E^Z(r_j) \leq N(r_j)E^Z(d_j^-).$$

The outside limits are $A_G b^{-8}$ and $A_G a^{-8}$ from the retained scaling comparison. Taking a, b to one forces the middle limit to A_G, contradicting a fixed positive violation tolerance. All distances, refinements and scalar readings are identical after recovery. Thus the lift retains the full positive-distance limit TR2 and its dependence on the positive spectral representation TR3, without replacing the earlier weak-sequence statement by itself.

The unique positive trace measure has the reconstruction normalization

$$\begin{aligned} \|\Theta(f)\Omega\|^2 &= \int |\widehat{f}_+(p)|^2 d\mu_\Theta(p), \\ \langle \Theta(f)\Omega, \Theta(g)\Omega\rangle &= \int \overline{\widehat{f}_+(p)}\widehat{g}_+(p) d\mu_\Theta(p). \end{aligned} \tag{TR4}$$

It has forward-cone support, Lorentz invariance and temperate growth; every positive time damping has finite mass. Its relative Fourier transform gives the full two-point distribution on joint Schwartz tests. The joint momentum measures on dense smeared trace vectors patch to this positive measure; polarization and compact-test separation give uniqueness. The regularized point-vector construction TR5 and the measure argument TR10–TR17 below give the patching and the ultraviolet limit explicitly. The lifted trace source retains the selected vacuum, stress label, unit insertion and actual common domain. Centering readout recovers TR1 exactly. The spectral reader retains its positive measure and monotonicity, so the generic scaling reader extends to all positive distances without selecting a new theory. The constants in TR2 retain both the running and trace factors, recovering the stated logarithmic coefficient; this asserts no positive mass gap.

24.2 The full short-distance limit: monotonicity, universal radius selection and the sandwich

It is real and nonnegative. Since $p_0 \geq 0$ almost everywhere for this measure, $0 < s \leq t$ implies $e^{-tp_0} \leq e^{-sp_0}$, and integration proves $E(t) \leq E(s)$. Finiteness at every positive s is part of the positive-time spectral representation. Thus this monotonicity is available before proving the short-distance limit; it is not inferred from that limit.

We give the precise selection statement needed here. Fix the physical table, its phase decomposition and its stress tensor. The trace construction selects a comparison ultrafilter below the table's mesh-then-volume filter, its retained local data, and one free hierarchy. For **every** sequence $s_n > 0$

tending to zero, it then provides a strictly increasing map $j : \mathbb{N} \to \mathbb{N}$, a scaling family on this same physical table, and its comparison data, whose radii satisfy the exact identity

$$r_i = s_{j(i)}. \tag{TLim1}$$

The scaling comparison applies to these radii. In particular the statement is not merely that one favorable sequence $r_i \to 0$ exists. Nor does it select a different physical hierarchy or phase when the sequence s_n changes. The construction and this universal sequence property were established in the preceding interacting-to-Gaussian comparison.

First prove convergence along the radii of any such retained comparison. For $1/2 < u < 2$, put

$$x_0^{(u)} = 0, \qquad x_1^{(u)} = (u/2, u/2, u/2, u/2).$$

The four coordinates of the two points are distinct, so this is a generic two-point configuration, and its Euclidean separation is u. Its maximum-coordinate separation is $u/2 > 1/4$, while its configuration norm is less than 2. Consequently it has a neighborhood in the fixed comparison window with norm bound 100 and maximum-coordinate separation at least $1/100$.

Let $\widehat{x}_i^{(u)}$ be the rounded configurations in that comparison, and set $d_i(u) = |\widehat{x}_{i,0}^{(u)} - \widehat{x}_{i,1}^{(u)}|$. Rounding convergence gives $d_i(u) \to u$. The actual physical configuration is exactly $r_i \widehat{x}_i^{(u)}$, not just an asymptotic replacement of it. The interacting comparison, followed by the running-clock identity and the free scalar Wick contraction, gives

$$c(r_i)E(r_i d_i(u)) \longrightarrow A_G u^{-8}, \qquad c(r) = r^8 \log^2(1/r). \tag{TLim2}$$

The centering and scalar homogeneity used in this identity are retained in the trace-label calculation below. The same normalization $c(r_i)$, independent of u, is used on both sides of the coming inequalities.

Fix $a < 1 < b$ in $(1/2, 2)$. Eventually $0 < d_i(a) < 1 < d_i(b)$. Monotonicity and $c(r_i) \geq 0$ give

$$c(r_i)E(r_i d_i(b)) \leq c(r_i)E(r_i) \leq c(r_i)E(r_i d_i(a)).$$

By (TLim2), their outer limits are $A_G b^{-8}$ and $A_G a^{-8}$. Given $\epsilon > 0$, first choose a, b close enough to 1 that both outer limits differ from A_G by less than $\epsilon/2$. Then choose i large enough for the two convergences and the bracketing inequalities. This proves $c(r_i)E(r_i) \to A_G$. The complex pairing has zero imaginary part by (TR3), so this also proves complex convergence.

Finally, suppose the full right-hand limit at zero failed. There would be $\epsilon > 0$ and, for every n, a radius $0 < s_n < 2^{-n}$ such that

$$|c(s_n)E(s_n) - A_G| \geq \epsilon.$$

Apply the universal selection statement to this particular sequence. Equation (TLim1) retains this inequality at every r_i, whereas the preceding sandwich proves that its left side tends to zero. This contradiction proves (TR2) for the full small-scale limit. No density or ratio condition on a single favorable sequence of radii is being assumed.

Source–readout formulation. The reader $r \mapsto c(r)E(r)$ of the selected phase is recovered at all small r, although each comparison witness (a scaling family) covers only a sequence of radii. The witnesses form a cover of every sequence, and the recovered quantity is independent of the witness: the scaling comparison supplies, for an arbitrary sequence tending to zero, a subsequence along which the limit is A_G, on the same table and phase, and the sandwich by monotonicity then forces the common value. No density or ratio condition on a favourable sequence of radii is assumed. Monotonicity of E is available before, and independently of, the short-distance limit, because it is read from the positive-time spectral measure.

Theorem 24.1 (Full limit from a universal subsequence property)**.** $\lim_{r\downarrow 0} c(r)E(r) = A_G$.

Proof. Fix $a < 1 < b$ in $(1/2, 2)$. Eventually $0 < d_i(a) < 1 < d_i(b)$, and by monotonicity and $c(r_i) \geq 0$,

$$c(r_i)E(r_i d_i(b)) \leq c(r_i)E(r_i) \leq c(r_i)E(r_i d_i(a)),$$

with outer limits $A_G b^{-8}$ and $A_G a^{-8}$. Choosing a, b close to one and then i large gives $c(r_i)E(r_i) \to A_G$ along every retained comparison. If the full limit failed there would be $\epsilon > 0$ and radii $0 < s_n < 2^{-n}$ with $|c(s_n)E(s_n) - A_G| \geq \epsilon$; applying the universal selection to this sequence, (TLim1) keeps the inequality at every r_i, contradicting the convergence just proved. □

With $A_G = 3\dim_{\mathbb{R}} \mathfrak{g}/\pi^4$ from the scalar Wick contraction this is the last displayed clause of the supplied public conclusion. The exactness class is exact recovery of a limit; the subsequences and scaling families are retained witnesses, not part of the recovered quantity.

24.3 Trace measure patching and the ultraviolet comparison measure

Let v_s, $s > 0$, be the trace shot at imaginary-time height s, centered at zero. Shot identities give $v_{s+t} = e^{-tH} v_s$ and the two-point reading at heights $(-s, s)$ equals $\langle v_s, U(x_1 - x_0)v_s\rangle$. Its diagonal momentum measure obeys $\mu_{s+t} = e^{-2tp_0}\mu_s$. Define

$$d\mu_\Theta = e^{2p_0} d\mu_1. \tag{TR5}$$

Comparing any two heights through their sum gives equivalently $d\mu_\Theta = e^{2sp_0} d\mu_s$ for every $s > 0$. Hence its damping $e^{-2sp_0}\mu_\Theta$ is finite and has the regularized two-point characteristic function. Taking height $s/2$ proves TR3. Smearing and the established boundary limit give TR4 and joint-test Fourier representation. Cone support passes from shot measures; scalar Lorentz covariance and uniqueness identify every Lorentz pushforward with the same measure. The common polynomial shot-height bound gives its polynomial growth, detailed below.

Shot-measure and undamping recovery. Let P be the joint momentum spectral measure. Under the same Hilbert unitary V, put $\mathsf{P}^Z(B) = V\mathsf{P}(B)V^{-1}$ and $v_s^Z = Vv_s$. For every Borel set B,

$$\mu_s^Z(B) = \langle v_s^Z, \mathsf{P}^Z(B)v_s^Z\rangle = \langle v_s, \mathsf{P}(B)v_s\rangle = \mu_s(B).$$

Functional calculus transports the heat semigroup. Comparing heights s, t through $s + t$ therefore gives

$$e^{-2tp_0}\mu_s = e^{-2sp_0}\mu_t, \qquad e^{2sp_0}\mu_s = e^{2tp_0}\mu_t.$$

Interpret the second equality first on compact momentum sets, where weights are bounded, then on all Borel sets by monotone exhaustion. This defines the possibly infinite undamped measure without applying an inverse heat operator to a Hilbert vector. Its horizontal measure is exactly $\mu_\Theta^Z = \mu_\Theta$, with

$$e^{-2sp_0}\mu_\Theta^Z = \mu_s^Z, \qquad \int e^{-2sp_0} d\mu_\Theta^Z = \|v_s^Z\|^2 < \infty.$$

Taking height $s/2$ recovers TR3's time damping. Smearing and the same boundary map recover TR4 since $\Theta^Z(f)\Omega^Z = V\Theta(f)\Omega$. Polarization and compact-test separation retain uniqueness; scalar Lorentz pushforwards therefore recover the same measure. Forward-cone support is unchanged. Temperate growth and joint Schwartz Fourier representation retain the polynomial shot-height and boundary dependencies developed below.

For the free contraction put $R_{ai}(x) = \delta_{ai} - 2x_a x_i/|x|^2$, an orthogonal reflection, since multiplying its two rows and summing the common index gives δ_{ab}. The curvature covariance is

$$\mathrm{Curl}_{abij}(x) = D(x)(R_{ai}R_{bj} - R_{aj}R_{bi}), \qquad D(x) = 4\,\mathrm{normal}\,|x|^{-4}.$$

Orthogonality gives $\sum_{a,b,i,j} \mathrm{Curl}^2_{abij} = 24D^2$: summing j, i first gives $D^2(2 - 2\delta^2_{ab})$, then summing a, b gives $24D^2$. Scalar curvature-square has factor $1/4$ in each field, two cross Wick contractions and $\dim \mathfrak{g}$ colors. Its two-point constant is therefore $(1/16)2\dim\mathfrak{g}\,24D^2 = 3\dim\mathfrak{g}/(\pi^4|x|^8)$ in the source normalization.

Free contraction reader recovery. The horizontal free-label map retains the finite component indices, color contraction and scalar Wick reader. Its curvature covariance is the same $D(x)$ times the wedge of $R(x)$. For fixed a, b, orthogonality gives

$$\sum_{i,j}(R_{ai}R_{bj} - R_{aj}R_{bi})^2 = 2\left(\sum_i R^2_{ai}\right)\left(\sum_j R^2_{bj}\right) - 2\left(\sum_i R_{ai}R_{bi}\right)^2 = 2 - 2\delta^2_{ab}.$$

Summing the sixteen ordered pairs (a, b) gives $32 - 8 = 24$. The two scalar labels retain their $1/4$ factors, the cross Wick contractions their multiplicity two and the color sum its $\dim\mathfrak{g}$. The recovered coefficient is therefore $(1/4)^2\,2\,24D(x)^2\dim\mathfrak{g}$, exactly $3\dim\mathfrak{g}/(\pi^4|x|^8)$ with the source normalization. This transports the finite calculation used in TR2 with no new stress or Gaussian normalization.

For $\rho > 1$ set $Z(\rho) = \rho^{-8}\log^2\rho$ and $\nu_\rho = Z(\rho)(p \mapsto p/\rho)_*\mu_\Theta$. These are positive Lorentz-invariant cone measures, and TR2 yields

$$\int e^{-sp_0}d\nu_\rho = Z(\rho)E(s/\rho) \longrightarrow A_G s^{-8} \quad (s > 0), \tag{TR6}$$

since $Z(\rho)/Z(\rho/s) \to s^{-8}$. Lorentz invariance gives the corresponding limit at every strictly future timelike damping vector. The candidate is

$$d\nu_\infty(p) = \frac{A_G}{1536\pi}(p_0^2 - |\mathbf{p}|^2)^2\mathbf{1}_{\{p_0 \geq |\mathbf{p}|\}}d^4p. \tag{TR7}$$

Its Laplace normalization follows explicitly from

$$4\pi\int_0^{p_0}(p_0^2 - r^2)^2 r^2 dr = \frac{32\pi}{105}p_0^7, \qquad \int_0^\infty e^{-sp_0}p_0^7 dp_0 = 7!s^{-8}.$$

The product is $1536\pi s^{-8}$. The cone-supported positive Laplace convergence argument below identifies this compact-test limit and supplies the stronger Schwartz-test passage. The lifted spectral source retains all shot measures and their exact semigroup relation. TR5 is an undamping reader independent of the chosen positive height, so it recovers the same trace measure and boundary pairings. The ultraviolet reader retains both the scaling pushforward and logarithmic weight, while TR7 records the normalized candidate with its spatial and temporal integral factors. No finite height measure is identified prematurely with the undamped one.

24.4 Polynomial trace bounds and Laplace-to-Schwartz continuity

On $p_0 \leq R$, $e^{-p_0/R} \geq e^{-1}$. TR2 at $(\rho R)^{-1}$ and $Z(\rho) \leq R^8 Z(\rho R)$ therefore give, uniformly for large ρ,

$$\nu_\rho\{p_0 \leq R\} \leq e\int e^{-p_0/R}d\nu_\rho \leq CR^8 \quad (R \geq 1). \tag{TR8}$$

The same holds for ν_∞. On the cone $\|p\|$ is comparable to p_0; dyadic shells and Schwartz decay give uniformly small tails. Compact-test convergence thus implies

$$Z(\rho)\int \varphi(p/\rho)d\mu_\Theta(p) \longrightarrow \int \varphi\, d\nu_\infty, \quad \varphi \in \mathcal{S}(\mathbb{R}^4). \tag{TR9}$$

For $f_\rho(x) = \rho^d f(\rho x)$, change of variables gives $\widehat{f}_{\rho,+}(p) = \widehat{f}_+(p/\rho)$; the joint relative wave transform has the same identity. Hence TR9 gives the stated relative and joint Wightman limits. The polynomial growth and Laplace continuity assertions used in TR6–TR9 are proved next, followed by the finite mass-shell obstruction; the subsequent stress correction and Ward calculation derive the trace coefficient and its Wilson normalization.

Ultraviolet pushforward and test recovery. The spectral source retains the actual trace measure, dilation and positive logarithmic weight. Since $\mu_\Theta^Z = \mu_\Theta$, its ultraviolet measure is identical on every Borel set:

$$\nu_\rho^Z(B) = Z(\rho)\mu_\Theta^Z(\rho B) = \nu_\rho(B), \quad \int \varphi\, d\nu_\rho^Z = Z(\rho)\int \varphi(p/\rho)\, d\mu_\Theta(p).$$

Thus the damping reader and all selected scaling parameters recover exactly. For fixed $s > 0$ and sufficiently large ρ,

$$\frac{Z(\rho)}{Z(\rho/s)} = s^{-8}\left(\frac{\log\rho}{\log\rho - \log s}\right)^2 \longrightarrow s^{-8}.$$

Combining with TR2 gives TR6 with no change of positive-distance quantifiers. The Lorentz action commutes with positive dilation and preserves the same cone, so timelike damping limits are transported by the identical measure action.

The comparison density retains its normalization independently of this limit. Substituting $r = p_0 u$ gives the spatial factor

$$4\pi p_0^7 \int_0^1 (u^2 - 2u^4 + u^6)\, du = 4\pi p_0^7\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) = \frac{32\pi}{105}p_0^7.$$

Multiplication by 7! and $A_G/(1536\pi)$ recovers $A_G s^{-8}$. The horizontal scalar measure reader therefore keeps both the candidate density and its exact Laplace normalization. Its compact-test identification retains the positive Laplace continuity theorem proved next.

For the uniform TR8 bound, the exact scale inequality is

$$\frac{Z(\rho)}{R^8 Z(\rho R)} = \left(\frac{\log\rho}{\log\rho + \log R}\right)^2 \le 1 \quad (R \ge 1,\ \rho > 1).$$

Consequently the same large-ρ threshold from TR2 controls all $R \ge 1$. To make the Schwartz tail recovery explicit, choose $L > 8$ and set $M_L = \sup_p (1+\|p\|)^L |\varphi(p)|$. The uniform energy-slab bound controls each Euclidean dyadic shell and gives, for $R \ge 1$,

$$\int_{\|p\|\ge R} |\varphi(p)|\, d\nu_\rho(p) \le \frac{2^8 C M_L}{1 - 2^{8-L}} R^{8-L}.$$

The same holds for ν_∞. A fixed compact cutoff followed by $R \to \infty$ therefore transports compact-test convergence to TR9 with uniform tails. Finally the normalized position dilation has its exact Fourier readout $\widehat{f}_{\rho,+}(p) = \widehat{f}_+(p/\rho)$; the retained joint relative transform obeys the same change of

variables. This recovers the relative and joint distribution limits with the original wave convention and positive-measure dependencies intact.

Explicitly, if a positive cone measure has

$$\mu\{p : p^0 \le R\} \le AR^k \quad (R \ge 1), \tag{TR10}$$

then its damped-vector bound supplies this with $A = e^2 A_0$ when the squared height-R^{-1} norm is at most $A_0 R^k$, since $e^{-2p^0/R} \ge e^{-2}$ on the slab. Euclidean radius-R balls are in that slab. On $2^j \le 1 + \|p\| \le 2^{j+1}$ the mass is at most $A2^{k(j+1)}$, so

$$\int (1 + \|p\|)^{-k-1} d\mu \le 2^{k+1} A. \tag{TR11}$$

Boundary overlap only enlarges the bound; this proves temperateness. Let positive μ_i, ν on the closed cone C have finite Laplace transforms on its open dual C° and satisfy

$$\int e^{-u\cdot p} d\mu_i \longrightarrow \int e^{-u\cdot p} d\nu \quad (u \in C^\circ). \tag{TR12}$$

Here damping uses the real Euclidean pairing. On the one-point compactification $C \cup \{\infty\}$, $s_u = e^{-u\cdot p}$ extends continuously by zero because $u \cdot p \ge \frac{1}{2}(u^0 - \|\mathbf{u}\|)\|p\|$. These real functions multiply by $s_u s_v = s_{u+v}$. With constants they separate points: positivity separates infinity; perturbations of the time unit by small real vectors separate any two finite points. Stone–Weierstrass gives uniform density of their linear span. This is the real algebra approximation theorem of Stone [10], applied on the stated compactification with the point-separation and constant hypotheses just verified. Fix $u \in C^\circ$ and damp all measures by s_u. Their masses converge by TR12, and their integrals of constants and s_v converge by the transforms at u and $u + v$. Uniform approximation and bounded masses give convergence on the compactification's continuous functions. For ambient compact continuous f, the function $e^{u\cdot p} f$ on C, zero at infinity, is such a test and its damped integral is $\int f \, d\mu_i$. This proves compact-test convergence, extending to complex tests by real and imaginary parts. For common TR10 bounds on all measures and the limit, put $q = 2^{k+1}A$, and choose B with $|f(p)| \le B(1 + \|p\|)^{-k-2}$. Smooth χ_R equal to one on radius R and zero beyond $R+1$, $0 \le \chi_R \le 1$, gives

$$\int |f - \chi_R f| d\mu_i \le Bq/R, \qquad \int |f - \chi_R f| d\nu \le Bq/R.$$

First choose R for the tails, then apply compact convergence to $\chi_R f$. This proves TR9 with uniform tails. The same density argument identifies a positive cone measure uniquely from all its future Laplace transforms; no signed-measure convergence is asserted. The lifted measure source retains positivity, cone support, the common polynomial budget and damping family. Its compact reader is recovered by bounded-mass approximation after damping, and its Schwartz reader by the explicit Bq/R tail. Keeping the normalized dilation and joint relative transform transports the same ultraviolet measure to the full tested two-point distributions without exchanging uncontrolled limits.

Positive Laplace continuity and Schwartz recovery. All measure operations here are readouts of the retained positive cone measure, so the height bound and its spectral integral agree under the horizontal Hilbert realization. Explicitly

$$\mu\{p^0 \le R\} \le e^2 \int e^{-2p^0/R} d\mu = e^2 \|v_{1/R}\|^2 \le e^2 A_0 R^k.$$

The same identity holds for its lifted shot. On the recorded dyadic sets the weighted contributions are bounded by

$$2^{-(k+1)j}A2^{k(j+1)} = A2^k2^{-j}, \qquad \sum_{j\ge0} A2^k2^{-j} = 2^{k+1}A.$$

This recovers TR11 and its temperate bound with the original constant.

For the Laplace continuity step fix $u \in C^\circ$ and write $\lambda_i = s_u\mu_i$, $\lambda = s_u\nu$ on the one-point compactification. Their total masses are eventually bounded by some M, and the limit mass is bounded by M as well, increasing M if necessary. The functions s_v and constants retain their exact multiplication and separating properties after readout. If h is continuous there and a finite linear combination a of this algebra satisfies $\|h-a\|_\infty \le \delta$, then

$$\left|\int h\,d\lambda_i - \int h\,d\lambda\right| \le 2M\delta + \left|\int a\,d\lambda_i - \int a\,d\lambda\right|.$$

The last term tends to zero by the finite collection of Laplace readings at u and $u+v$. First choosing a and then the measure index proves convergence for h. For a compact continuous ambient test take $h = e^{u\cdot p}f|_C$, extended by zero at infinity; its damped readout is exactly the original f integral. Thus the compact-test recovery is quantitative and uses bounded positive masses, rather than an uncontrolled undamping of test functions at infinity. The same argument for two measures with identical Laplace readings gives their identical compact-test readings and uniqueness.

Finally set $q = 2^{k+1}A$. Outside radius $R \ge 1$ the Schwartz bound gives

$$|f - \chi_R f| \le \frac{B}{R}(1+\|p\|)^{-k-1},$$

so TR11 supplies each tail bound Bq/R. In particular

$$\left|\int f\,d\mu_i - \int f\,d\nu\right| \le \frac{2Bq}{R} + \left|\int \chi_R f\,d\mu_i - \int \chi_R f\,d\nu\right|.$$

Choose R before the index. All weights, cutoffs and constants are unchanged scalar readouts, so this also recovers the lifted Schwartz limit TR9. The argument retains positive measures, the common polynomial budget and the full future damping family throughout.

24.5 The finite mass-shell polynomial comparison class

There is no bound on colors or on the degrees of the two finite comparison polynomials. Their elementary contractions, however, are finite sums of fixed mass-shell kernels with finite polynomial momentum numerators. More generally allow a real continuous homogeneous spatial numerator P of integer degree $j \ge 0$, energy power $k \ge 0$ and mass $m \ge 0$, with

$$\omega_m(p) = \sqrt{m^2+|p|^2}, \qquad a(p) = \frac{\omega_m(p)^k P(p)}{2\omega_m(p)}, \qquad p_+ = (\omega_m(p), p). \tag{TR13}$$

The isolated massless zero value does not affect integration. A track of n factors is

$$L(\varphi) = \int_{(\mathbb{R}^3)^n} \prod_{\alpha=1}^{n} a_\alpha(p_\alpha)\varphi\left(\sum_\alpha (p_\alpha)_+\right)\prod_\alpha dp_\alpha. \tag{TR14}$$

The empty track evaluates at zero. Finite complex linear combinations allow units, contraction coefficients and statistics signs without positivity. Expanding the two polynomials into monomials,

cross-vertex occurrence bijections, each arc's finite shell sum and each numerator's energy/spatial monomials gives exactly finite tracks. Each resulting term is (TR14); all arc coefficients, monomial amplitudes and pairing coefficients multiply into its constant coefficient. Only finitely many generator labels occur, even if infinitely many are available. At positive imaginary time damping makes a track absolutely integrable:

$$L(e^{-sp^0}) = \prod_{\alpha=1}^{n} \int_{\mathbb{R}^3} a_\alpha(p) e^{-s\omega_{m_\alpha}(p)} dp. \tag{TR15}$$

Polar coordinates give each factor as

$$\tfrac{1}{2} \left(\int_{\mathbb{S}^2} P(\theta) d\theta \right) \int_0^\infty r^{j+2} (m^2 + r^2)^{(k-1)/2} e^{-s\sqrt{m^2+r^2}} dr. \tag{TR16}$$

Compact total-momentum tests are absolutely integrable too, being bounded by a constant times this damping on their support.

Mass-shell track source and damping recovery. For a fixed track retain its finite ordered arc list, all masses, energy powers, homogeneous numerators and constant contraction coefficients. Its source space is $(\mathbb{R}^3)^n$, with the realization and weighted readout

$$Q(p_1, \dots, p_n) = \sum_{\alpha=1}^{n} (\omega_{m_\alpha}(p_\alpha), p_\alpha), \qquad L(\varphi) = \int (\varphi \circ Q) \prod_\alpha a_\alpha(p_\alpha)\, dp_\alpha.$$

The source also retains the absolute weight $\prod_\alpha |a_\alpha|$ for integrability estimates. Horizontal copying of each momentum coordinate with its transported Lebesgue measure leaves this realization and readout identical. The empty list has the one-point source, $Q = 0$ and weight one, recovering the unit track.

Each total energy is the sum of nonnegative shell energies, so

$$e^{-sQ^0} = \prod_\alpha e^{-s\omega_{m_\alpha}(p_\alpha)}.$$

For one factor, continuity of P on the sphere bounds it by a constant times r^j. At $m = 0$ the absolute radial integrand is at most a constant times $r^{j+k+1} e^{-sr}$, integrable at zero and infinity. At $m > 0$ it is locally bounded by a constant times r^{j+2} near zero and has polynomial times exponential decay at infinity. Thus the absolute damped track integral is finite. Product-law Fubini recovers TR15 with its actual signed or complex factors. If a compact momentum test is supported where $q^0 \le M$, then on the source

$$|\varphi(Q)| \le \|\varphi\|_\infty e^{sM} e^{-sQ^0}.$$

This proves the required compact-test absolute integrability and the locally finite total-momentum readout, without requiring its weight to be positive.

For a single arc, $P(r\theta) = r^j P(\theta)$ and $dp = r^2 dr\, d\theta$, while the shell denominator contributes $1/(2\omega_m)$. These three exact factors recover TR16, including its $1/2$ and radial power r^{j+2}. Its angular integral may vanish or have either sign; it is retained as part of the track coefficient.

Finally the finite monomial expansion, occurrence pairing bijections, arc-shell choices and numerator expansion are recorded as a finite index set I. For the two chosen polynomial fields their comparison reader is exactly $\sum_{\iota \in I} c_\iota L_\iota$, with every statistics sign and contraction coefficient in c_ι. The lift recovers each summand and its coefficient before finite addition. There is no uniform bound on field degree or color count across the comparison class; only the finite data of each chosen

comparison are used. The subsequent radial-expansion obstruction applies to this same retained finite class.

The radial expansion proved below has, for every integer N, a finite sum of $s^a(\log s)^b$, $a \in \mathbb{Z}$, $b \in \mathbb{N}$, with remainder $O(s^N)$. Finite sums and products preserve it: bound one factor by an integer power and expand the other deeply enough to pay it. In particular each real finite damped track combination has such an expansion with bounded remainder. The radial expansion itself will be derived below from the massive integral; it is not an assumption on the comparison fields. No such real function has a nonzero finite limit after multiplication by $s^8 \log^2(1/s)$. The bounded remainder vanishes. Combine equal terms and take $s = 2^{-n}$; each remaining term is a nonzero constant times

$$(2^{-(8+a)})^n n^{b+2}. \tag{TR17}$$

Order the finite terms by exponential base then polynomial degree. If the largest base is below one all vanish. Otherwise the leading term diverges and every lower term divided by it tends to zero; dividing any supposed finite limit forces its nonzero coefficient to vanish, a contradiction. To transfer the theorem's product-test equality to these damped values, represent any compact smooth momentum test v as the relative wave transform of two Schwartz tests. Choose one Fourier test equal to one on a ball containing its support, and the other by Fourier surjectivity to give v; equivalently multiply by that first test and its conjugate, whose squared modulus is one there. Equality on tensor tests therefore gives equality on all such v. Insert $v_n = \chi_n e^{-sp^0}$ with compact smooth $0 \le \chi_n \le 1$ tending pointwise to one. Finite trace damped mass and each track's absolute damping justify dominated convergence, giving $E(s) = \sum_{\ell=1}^{M} c_\ell L_\ell(e^{-sp^0})$. Real parts and TR17 contradict the positive TR2 limit. This excludes precisely these finite polynomial comparisons, without extending the claim to arbitrary free or generalized-free classes. The lifted comparison source retains occurrence pairings, finite shell choices, coefficients and damping. Its track reader is TR14 and its imaginary-time reader is TR15. Product-test recovery and dominated cutoff removal identify the same reader as the trace Laplace transform under any proposed equality. The asymptotic obstruction therefore preserves the exact comparison class and its stated limitation.

Finite asymptotic obstruction and tested recovery. The track source retains each radial function and its finite expansion at the requested order, including its remainder bound. To make the product rule explicit, suppose $f = O(s^{-A})$, $g = O(s^{-B})$ for nonnegative integers A, B. Choose finite approximants F, G with

$$f - F = O(s^{N+B}), \qquad g - G = O(s^{N+A}).$$

The orders may be increased so that $F = O(s^{-A})$ as well. Then $fg - FG = (f - F)g + F(g - G) = O(s^N)$. Finite products and sums therefore retain a finite power/log expansion to any requested order. Every identity here concerns the same scalar radial readout after lifting; the existence of its expansions retains the convergent-integral argument below as a dependency.

At bounded remainder order, write the real finite combination as

$$h(s) = \sum_{(a,b)\in I} c_{a,b} s^a (\log s)^b + O(1), \qquad a \in \mathbb{Z}, \quad b \ge 0,$$

with equal pairs combined and zero coefficients removed. At $s = 2^{-n}$ the normalized term is exactly

$$s^8 \log^2(1/s) c_{a,b} s^a (\log s)^b = c_{a,b}(-1)^b (\log 2)^{b+2} (2^{-(8+a)})^n n^{b+2}.$$

This retains TR17's nonzero coefficient and sign. If the finite index set is empty the normalized remainder tends to zero. Otherwise the largest exponential base and then largest polynomial

degree determine the leading term. A base below one gives zero; a base at least one gives divergent magnitude, since $b+2 \geq 2$. All lower terms divided by it vanish. The same exact scalar sequence is recovered by the lift, so none has TR2's positive finite amplitude.

For product-test recovery take $\eta \in C_c^\infty$ equal to one on $\operatorname{supp} v$, and use Fourier surjectivity to choose Schwartz f, g with $\widehat{f}_+ = \eta$ and $\widehat{g}_+ = v$ in the retained convention. Then $\overline{\widehat{f}_+}\widehat{g}_+ = v$ on all momentum space. Thus the actual tensor-test reader recovers every compact smooth momentum test without a density limit in the pair of fields. For fixed $s > 0$, insert $v_n = \chi_n e^{-sp^0}$. On the trace side its limiting error is controlled by the finite positive damped mass; on each finite track it is controlled by the absolute damped product weight. Dominated convergence therefore recovers

$$E(s) = \sum_{\ell=1}^{M} c_\ell L_\ell(e^{-sp^0})$$

from any proposed equality of the stated tested two-point functions. Taking real parts gives exactly the preceding finite obstruction. The source–readout translation thus preserves the exclusion of the specified finite polynomial mass-shell comparison class and every coefficient or statistics sign, without changing that class's scope.

24.6 Convergent-integral derivation of the radial expansion

For $s > 0$ define

$$M_{j,k}(s) = \int_0^\infty r^{j+2}(m^2+r^2)^{(k-1)/2} e^{-s\sqrt{m^2+r^2}}\, dr.$$

For $k = 0, 1$ the bounds are respectively $r^{j+1}e^{-sr}$ and $r^{j+2}e^{-sr}$; all higher cases follow from

$$M_{j,k+2} = m^2 M_{j,k} + M_{j+2,k}. \tag{TR18}$$

The gamma integral gives $M_{j,0} \leq \Gamma(j+2)s^{-j-2}$. Local differentiation is justified by damping $s/2$ and gives $M'_{j,k} = -M_{j,k+1}$. Integrate the radial derivative of $r^{j+3}e^{-s\sqrt{m^2+r^2}}$, whose boundary terms vanish at both ends and whose derivative is integrable, to obtain

$$sM_{j+2,0} = (j+3)M_{j,1}, \qquad (s^{j+3}M_{j,1})' = -m^2 s^{j+3} M_{j,0}. \tag{TR19}$$

Say expansion to integer order a means a finite sum of integer powers and nonnegative log powers with remainder $O(s^a)$. If $f' = g$, g has order $a \neq -1$, then f has order $a+1$. Indeed each finite term has an antiderivative of the same type: $s^{-1}(\log s)^b$ integrates directly; other powers use integration by parts. Subtract this primitive. For the remaining derivative bounded by Cs^a, integrate to a fixed positive point if $a < -1$, giving $O(s^{a+1})$; if $a > -1$, integrability at zero gives a finite constant plus that remainder. The integration constant is an allowed term. Start $M_{j,0}$ at order $b = -j-2$. Given order $a \geq b$, multiply by s^{j+3} and apply the integration rule to TR19, then divide: $j+3+a \geq 1$ avoids the exceptional exponent and gives $M_{j,1}$ to order $a+1$. Integrating $-M_{j,1} = M'_{j,0}$ improves $M_{j,0}$ to at least order $a+1$. If $a+1 = -1$, weaken its derivative remainder to order -2 before integration to obtain order -1; otherwise the rule gives $a+2$, which can be weakened to $a+1$. Induction gives all integer orders for $M_{j,0}$ and then $M_{j,1}$; TR18 gives every k. This proves the finite expansions used after TR16, without summing an infinite formal series, and closes that comparison exclusion. The lifted radial source retains its actual damped integrals, derivative identities and finite-order remainder. Integration is the reader transport, with constants retained as allowed terms. The induction recovers each requested finite asymptotic horizon and its explicit error, exactly the data required by the finite-track obstruction.

Finite radial-horizon recovery. Retain (m, j, k, s) and the actual radial Lebesgue integral in the source record. Its horizontal scalar readout is the identical $M_{j,k}(s)$ for every $s > 0$. Multiplying the integrand by $m^2 + r^2$ recovers TR18 before integration. Differentiating on a compact positive s interval uses the same damping bound and gives the identical derivative reader $-M_{j,k+1}$. In particular

$$\begin{aligned}(s^{j+3}M_{j,1})' &= (j+3)s^{j+2}M_{j,1} - s^{j+3}M_{j,2}\\ &= s^{j+3}(M_{j+2,0} - M_{j,2}) = -m^2 s^{j+3}M_{j,0}.\end{aligned}$$

This explicitly recovers the second TR19 identity from the first and the finite recurrence, retaining all boundary and domination conditions.

The finite primitive reader for a term is explicit. For $b \geq 0$,

$$\int s^{-1}(\log s)^b ds = \frac{(\log s)^{b+1}}{b+1},$$

and for an integer $\ell \neq -1$ it is

$$\int s^{\ell}(\log s)^b ds = s^{\ell+1}\sum_{r=0}^{b}\frac{(-1)^r b!}{(b-r)!(\ell+1)^{r+1}}(\log s)^{b-r}.$$

Both formulas retain an arbitrary additive constant. For a derivative remainder $|R'(s)| \leq Cs^a$, $a > -1$, integration from zero gives $|R(s) - R(0)| \leq Cs^{a+1}/(a+1)$. For $a < -1$, integration to a fixed positive point gives $R(s) = O(s^{a+1})$, with the finite constant absorbed because s^{a+1} diverges. These bounds recover the source's precise primitive rule without integrating an infinite series or discarding an integration constant.

Starting with the zero approximant and remainder order $-j-2$, record each finite primitive and its error. If $M_{j,0}$ has order $a \geq -j-2$, the derivative of $s^{j+3}M_{j,1}$ has remainder order $a+j+3 \geq 1$. Its primitive therefore has order $a+j+4$; division by s^{j+3} gives order $a+1$ for $M_{j,1}$. A further primitive gives order $a+2$ for $M_{j,0}$ unless the remainder exponent is -1. In that exceptional case weaken the remainder to order -2 and recover order -1, exactly as in the source. Thus each cycle raises the guaranteed order by at least one. TR18 then supplies every fixed k by finitely many linear operations. The horizontal reader retains every finite approximant, constant and error bound. This completes the required translation of the radial-expansion dependency in the finite polynomial comparison obstruction.

24.7 Maxwell comparison and common sector amplitudes

For m copies use $d\lambda(k) = d^3k/(2|k|)$ and

$$\mathcal{F}_{\mathrm{amb}} = \bigoplus_{n\geq 0}\bigoplus_{s:\{1,\dots,n\}\to\{1,\dots,m\}\times\{1,2,3\}} L^2((\mathbb{R}^3)^n, \lambda^{\otimes n}). \tag{TR20}$$

The isolated zero-momentum convention is irrelevant. Physical waves are symmetric under simultaneous label/momentum permutations and transverse under $\Pi_{ab}(k) = \delta_{ab} - k_a k_b/|k|^2$ in each slot. Translations multiply by total-momentum plane waves with $P(k) = \sum_i(|k_i|, k_i)$; the vacuum is the unit zero-sector wave. The comparison assumes a complex-linear isometry $U : \mathcal{H} \to \mathcal{F}_{\mathrm{amb}}$ with physical image, preserving the selected vacuum and intertwining translations. Neither onto-ness nor a full field identification is required. For real electric tests h set $z_h(c, a; k) = \sum_b \Pi_{ab}(k)|k|\widehat{h}_{c,b,+}(|k|, k)$ and

$$\mathcal{E}(h)_{n,s}(k) = e^{-\|z_h\|^2/2}\frac{i^n}{\sqrt{n!}}\prod_{\alpha=1}^{n} z_h(s_\alpha; k_\alpha). \tag{TR21}$$

Schwartz decay and $\lambda\{|k| \le R\} = \pi R^2$ give square integrability; the exponential sector norm series defines a physical vector. With $\sigma(f) = U\Theta(f)\Omega$, the excluded condition is

$$\langle \mathcal{E}(-h), \sigma(f)\rangle = \langle \sigma(\bar{f}), \mathcal{E}(h)\rangle. \tag{TR22}$$

It is required for compact smooth f and real compact smooth h with mutually spacelike supports. The proof needs f only in one fixed ball of any positive radius, giving the small-region variant. The pushforward $P_*(|Uv|^2 d\Lambda)$ is the translation measure of v by equality of characteristic functions. For trace vectors it is $|\widehat{f}_+|^2 d\mu$. Polarization with bounded momentum multipliers gives

$$\widehat{g}_+(P)\sigma(f) = \widehat{f}_+(P)\sigma(g) \quad \text{almost everywhere.} \tag{TR23}$$

The difference's norm equals its $L^2(\mu)$ counterpart, identically zero, so no exceptional-point Hilbert evaluation is used. For $R_j = 2^j$, $B_{n,j} = \{k : |k_i| \le R_j \ \forall i\}$ and $\nu_q = |q|^2 \lambda^{\otimes n}$, the resulting common amplitudes obey

$$\nu_{q_{n,s}}(B_{n,j}) \le \mu\{p^0 \le (n+1)R_j\}. \tag{TR24}$$

TR2 makes this upper bound $o(R_j^8)$. The local sector argument below kills every $n \ge 4$ and bounds the remaining finite sectors by

$$\nu_{q_{n,s}}(B_{n,j}) \le C_{n,s} R_j^6. \tag{TR25}$$

An energy slab lies in its sector barrel, so summing gives CR_j^6, contradicting the previously obtained $R^8/\log^2 R$ high-energy law. For the common amplitude choose countably many Schwartz v_j with $|\widehat{v}_{j,+}| = 1$ on radius-j balls. On the measurable partition where j is the first nonzero Fourier weight at P, put

$$q = \sigma(v_j)/\widehat{v}_{j,+}(P). \tag{TR26}$$

For each fixed f, intersect the countably many TR23 full-measure sets to get $\sigma(f) = \widehat{f}_+(P)q$, with a separate almost-everywhere interpretation for each test. Weighted measure equality and exhaustion by those balls remove the weights, proving

$$\mu = P_*(|q|^2 d\Lambda), \qquad \mu(B) = \sum_{n,s} \int_{\{P(k)\in B\}} |q_{n,s}(k)|^2 d\lambda^{\otimes n}. \tag{TR27}$$

On a barrel $P^0 \le nR_j$, proving TR24; $n{+}1$ also covers vacuum. The lifted comparison source retains the isometry, physical sector constraints, coherent probes and measurable amplitude partition. TR23 is its compatible smeared reader; TR26–TR27 recover a single amplitude measure from all test-weighted spectral readings without claiming simultaneous pointwise identities for uncountably many tests. The following local sector reader supplies the remaining TR25 bound; its scope remains precisely the TR22 comparison hypothesis.

Comparison-isometry and common-amplitude recovery. Keep the Maxwell ambient space, photon measures, labels and momentum map unchanged. If $V : \mathcal{H} \to \mathcal{H}^Z$ is the reconstruction unitary, the transported comparison isometry is

$$U^Z = UV^{-1}, \quad (U^Z)^* U^Z = I_{\mathcal{H}^Z}, \qquad U^Z \Theta^Z(f)\Omega^Z = U\Theta(f)\Omega = \sigma(f).$$

Its physical image, vacuum and translation intertwining are therefore identical. No surjectivity is added. The transverse projector and simultaneous permutations are the same operators on the ambient wave space. The coherent probes retain their normalization explicitly:

$$\sum_{n,s} \|\mathcal{E}(h)_{n,s}\|^2 = e^{-\|z_h\|^2} \sum_{n\ge 0} \frac{\|z_h\|^{2n}}{n!} = 1.$$

Thus TR22 has exactly the same two scalar matrix elements after transport, for the same spacelike-separated tests and the same permitted fixed small ball.

Let M_a denote bounded multiplication by $a(P)$ on the ambient space. Polarized spectral equality for trace vectors gives

$$\langle M_a\sigma(f), M_b\sigma(g)\rangle = \int \overline{a(p)\widehat{f}_+(p)}b(p)\widehat{g}_+(p)\,d\mu(p).$$

Applying this to the two products in TR23 gives the exact squared norm of their difference as $\int |\widehat{g}_+\widehat{f}_+ - \widehat{f}_+\widehat{g}_+|^2 d\mu = 0$. Both multiplication operators are bounded since Fourier transforms of Schwartz tests are bounded. This recovers the wave-space identity without evaluating vectors at exceptional points.

Retain the countable test sequence v_j and the measurable pieces

$$A_j = \{\widehat{v}_{j,+}(P) \neq 0\} \cap \bigcap_{\ell<j} \{\widehat{v}_{\ell,+}(P) = 0\}.$$

They exhaust the wave space, since each finite total momentum lies in some ball where a chosen weight has modulus one. The ratio reader $q|_{A_j} = \sigma(v_j)/\widehat{v}_{j,+}(P)$ is identical under transport. For each fixed f, the same countable intersection of full-measure sets recovers $\sigma(f) = \widehat{f}_+(P)q$. On any momentum ball use a v_j with unit-modulus weight there; weighted measure equality then recovers TR27 on that ball, and countable exhaustion recovers it on all Borel sets.

Every labeled sector retains this same amplitude and squared-wave measure. Since $P^0 \leq nR_j$ on $B_{n,j}$, its barrel mass obeys TR24 with the same harmless $n+1$ allowance for the zero sector. Conversely $P^0 \leq R_j$ forces every $|k_i| \leq R_j$. Once the subsequent local sector result supplies vanishing for $n \geq 4$ and TR25 for the remaining finite labeled sectors, summing those barrel bounds gives the same energy-slab estimate. Thus the final comparison obstruction retains precisely its original local-sector dependency and high-energy trace-law dependency in the lifted representation.

24.8 Sector dilation algebra and projected comparison variants

Amplitudes are measurable functions modulo almost-everywhere equality. Define the commuting slot dilations by

$$D_i q(k) = q(k_1, \ldots, 2k_i, \ldots, k_n).$$

Photon measure scaling $(k \mapsto rk)_*\lambda = r^{-2}\lambda$ gives

$$\nu_{D_i q}(B(d)) = \tfrac{1}{4}\nu_q(B(d_1, \ldots, 2d_i, \ldots, d_n)). \tag{TR28}$$

Amplitudes finite on every dyadic barrel with $R_j^{-8}\nu_q(B_{n,j}) \to 0$ form a real vector space, by $|q+r|^2 \leq 2(|q|^2 + |r|^2)$, stable under D_i by TR28 and containment in the next barrel with fixed loss 2^6. Thus every finite polynomial in the D_i preserves this restriction. The local distribution argument below supplies a common finite K with

$$\prod_{a=0}^{K-1} (D_i - 2^{a-1})q = 0 \quad (1 \leq i \leq n). \tag{TR29}$$

The shift $a-1$ includes the electric factor $|k_i|$. Distinct real roots split q into finite eigencomponents within the invariant space. Explicitly, if $v = \prod_{a<t}(D_i - 2^{a-1})q$, it has eigenvalue 2^{t-1}; subtract $v/\prod_{a<t}(2^{t-1} - 2^{a-1})$ to remove that factor, with positive denominators. Commuting slot operators

permit successive splitting inside already chosen eigenspaces and the other polynomial kernels. Each simultaneous eigenwave retains the growth bound. For $D_i w = 2^{a_i-1} w$, TR28 gives

$$\nu_w(B(d_1,\dots,2d_i,\dots,d_n)) = 2^{2a_i}\nu_w(B(d)),$$
$$\nu_w(B_{n,j}) = R_j^{2\sum_i a_i}\nu_w(B_{n,0}). \tag{TR30}$$

If $a_i = 0$, shrinking that radius by halves leaves masses equal; continuity from above for finite parcel mass gives the mass at $k_i = 0$, which is zero. Exhaust barrels to get $w = 0$. If $\sum_i a_i \geq 4$, the sub-R^8 restriction forces zero base barrel mass and again zero wave. Nonzero waves therefore satisfy $n \leq \sum_i a_i \leq 3$, and have mass at most CR_j^6. Finite squared triangle bounds yield TR25, including sector zero. This completes its algebraic conclusion once TR29 is established. For projected variants, a bounded self-adjoint map acting identically on $U\mathcal{H}$ moves across TR22's inner products and is removed from coherent vectors. Compact spacelike-separated supports have open neighborhoods with spacelike-separated closures; the trace neighborhood can have closure inside the chosen ball. Thus the region identity implies the same ball-local identity. For probability-indexed maps P_z preserving inner products between physical waves and fixing $U\mathcal{H}$ pointwise, physical coherent vectors satisfy $\langle P_z\mathcal{E}(h), Uv\rangle = \langle \mathcal{E}(h), Uv\rangle$ and its reversed counterpart. Scalar integrands are constant in z, so averaging removes the maps and gives the excluded identity. This uses precisely these conditions, without adding boundedness, linearity or a projection condition for individual P_z. The weaker projected and averaged versions require no new sector estimate. The lifted sector source retains the invariant growth space, all commuting dilation operators and finite polynomial kernels. Eigencomponent readout stays within that same growth restriction; TR30 recovers the parcel masses with exact photon-measure scaling. The zero-slot and large-degree branches are removed by finite-measure continuity and the retained ultraviolet bound, not by pointwise wave evaluation. Projection and averaging readers preserve exactly the matrix hypothesis specified above.

Dilation eigencomponents and comparison-variant recovery. The horizontal Maxwell comparison has the identical wave amplitudes on the same photon measure space. Thus every slot dilation D_i and almost-everywhere equivalence class is unchanged by readout. Since $d\lambda(2k) = 4d\lambda(k)$, change of variables in one slot recovers TR28 with factor $1/4$. In particular

$$R_j^{-8}\nu_{D_i q}(B_{n,j}) \leq 2^6 R_{j+1}^{-8}\nu_q(B_{n,j+1}) \longrightarrow 0.$$

This proves that all finite polynomial readers in the commuting D_i remain within the same growth space.

A fully explicit simultaneous splitting of TR29 uses its distinct roots $\lambda_a = 2^{a-1}$, $0 \leq a < K$. On the cyclic polynomial space generated by q, set

$$E_{i,a} = \prod_{\substack{0\leq b<K\\ b\neq a}} \frac{D_i - \lambda_b}{\lambda_a - \lambda_b}, \qquad w_{a_1,\dots,a_n} = \prod_i E_{i,a_i} q.$$

The denominators are nonzero real constants. Polynomial interpolation modulo the annihilating polynomials gives $q = \sum_{a_1,\dots,a_n} w_{a_1,\dots,a_n}$ and $D_i w_{a_1,\dots,a_n} = \lambda_{a_i} w_{a_1,\dots,a_n}$. These are finite algebraic operations, with no bounded-operator spectral theorem assumed for D_i on this growth space. Readout preserves every component and its growth restriction.

For an eigenwave TR28 gives the exact parcel ratio $4|\lambda_{a_i}|^2 = 2^{2a_i}$. If $a_i = 0$, decreasing only its radius to zero leaves the parcel mass constant; finite-measure continuity identifies it with the zero mass of $k_i = 0$. If $\sum_i a_i \geq 4$, TR30 and the vanishing normalized barrel mass force the base

mass to be zero. The surviving finite components have $n \le \sum_i a_i \le 3$. For M such components, the explicit bound $|\sum_{r=1}^{M} w_r|^2 \le M \sum_r |w_r|^2$ recovers TR25. Sector zero retains its one-point wave and constant mass. The local distribution proof of TR29 remains the required dependency for this finite algebraic conclusion.

For a bounded self-adjoint comparison map P fixing $U\mathcal{H}$, the exact scalar equality

$$\langle P\mathcal{E}(h), Uv\rangle = \langle \mathcal{E}(h), PUv\rangle = \langle \mathcal{E}(h), Uv\rangle$$

is unchanged under $U^Z = UV^{-1}$. For the probability-indexed variant, physical inner-product preservation instead gives

$$\langle P_z\mathcal{E}(h), Uv\rangle = \langle P_z\mathcal{E}(h), P_zUv\rangle = \langle \mathcal{E}(h), Uv\rangle.$$

The reversed pairing obeys the same identity. Its scalar integrand is constant, so its probability integral has precisely that value. These recover the projected, averaged and open-region reductions with their original hypotheses and support geometry intact.

24.9 Point-supported data and locality-to-dilation recovery

Polynomial slab growth and TR27 give, for some integer d,

$$\int |q(k)|^2 \prod_i (1+|k_i|)^{-2d} d\lambda^{\otimes n} < \infty. \tag{TR31}$$

A sector ball lies in energy slab $(n+1)R$; apply TR11 and a sufficiently large product weight. Cauchy–Schwarz and $\int (1+|k|)^{-4} d\lambda < \infty$ show that $|q| \prod_i (1+|k_i|)^{-d-2}$ is integrable, controlling all ensuing integrals by fixed Schwartz seminorms. Slot dilation preserves the bound by $1+2|k| \le 2(1+|k|)$. With $\widehat{g}_-(p) = \widehat{g}_+(-p)$, sector extraction from TR22 gives

$$\int q \prod_j |k_j| \widehat{g}_{j,-}((k_j)_+) d\lambda^n = \int \bar{q} \prod_j |k_j| \widehat{g}_{j,+}((k_j)_+) d\lambda^n. \tag{TR32}$$

Each compact g_j is spacelike to zero; no mutual separation is required. Smear the trace with a normalized real bump $a_\ell(x) = \ell^4 a(\ell x)$, eventually inside the fixed ball and remote from all supports. Its Fourier transform tends to one with modulus at most one. TR31 permits its removal by dominated convergence from the coherent-sector identity, extracted below. Keep the other slots tested and define spatial distributions

$$\begin{aligned} A_b(f) &= \int q |k_i|^{b+1} \widehat{f}_-(k_i) \prod_{j\neq i} |k_j| \widehat{g}_{j,-}((k_j)_+) d\lambda^n, \\ B_b(f) &= \int \bar{q} |k_i|^{b+1} \widehat{f}_+(k_i) \prod_{j\neq i} |k_j| \widehat{g}_{j,+}((k_j)_+) d\lambda^n, \\ T_b(f) &= A_b(f) - (-1)^b B_b(f). \end{aligned} \tag{TR33}$$

Their common controlling seminorm is

$$2^J (\|\widehat{f}_-\|_{0,0} + \|\widehat{f}_-\|_{J,0} + \|\widehat{f}_+\|_{0,0} + \|\widehat{f}_+\|_{J,0}), \quad J = d+2+b+1. \tag{TR34}$$

Its order is independent of the other tests, though its multiplier may depend on them. For compact f away from zero, use spacetime $f(\mathbf{x})\partial_t^b a_\ell(t)$ in TR32. Eventually it is remote; its time Fourier

factors are $(i|k_i|)^b$ and $(-i|k_i|)^b$. Dominated convergence gives $T_b(f) = 0$, so T_b has point support. A distribution with this fixed seminorm depends on a uniformly finite jet: dominate it by finitely many Schwartz seminorms of maximal order L. If the L-jet of f vanishes, point support permits replacement by $\chi(sx)f$. Taylor bounds derivatives of order $L - j$ on $|x| \le C/s$ by $C_f s^{-j-1}$; cutoff derivatives cost s^j. Leibniz and bounded weights give every needed seminorm $O(s^{-1})$. Thus the distribution vanishes on such f. This gives L_b depending only on TR34, uniformly in remote tests and slots.

Weighted locality-data recovery. For a fixed nonzero sector retain its amplitude and photon coordinates. Its horizontal wave readout is the same q, so every weighted norm in TR31 is unchanged. Writing $k = (k_1, \dots, k_n)$, one has $\prod_i (1 + |k_i|) \ge 1 + \max_i |k_i|$; choosing $2d$ large enough therefore bounds that product weight by a constant times the inverse radial weight furnished by polynomial ball growth. In particular Cauchy–Schwarz gives the explicit integrability budget

$$\begin{aligned}&\int |q| \prod_i (1 + |k_i|)^{-d-2} d\lambda^n \\ &\quad \le \left(\int |q|^2 \prod_i (1 + |k_i|)^{-2d} d\lambda^n \right)^{1/2} \left(\int (1 + |k|)^{-4} d\lambda(k) \right)^{n/2} < \infty.\end{aligned}$$

This same majorant controls the test products after the prescribed finite Schwartz seminorms are taken. The normalized shrinking trace bump and its Fourier transform are retained exactly, with limit one and modulus at most one. Thus the ball-local reader recovers TR32 by dominated convergence, keeping each remote support and imposing no separation between different remote tests. The coherent-sector extraction below remains its explicit dependency.

The two Fourier signs, electric factors and $(-1)^b$ in TR33 are unchanged scalar readouts. Consequently $A_b^Z = A_b$, $B_b^Z = B_b$ and $T_b^Z = T_b$ as continuous spatial distributions, with precisely the TR34 seminorm and its uniform derivative order. The time-bump derivatives retain their two factors $(i|k_i|)^b$ and $(-i|k_i|)^b$. Their common dominated limit therefore transports point support without altering either frequency branch.

For explicit finite-jet recovery choose a spatial cutoff $\chi = 1$ near zero and put $\eta_\alpha(x) = \chi(x) x^\alpha / \alpha!$ for $|\alpha| \le L_b$. These tests have $\partial^\beta \eta_\alpha(0) = \delta_{\alpha\beta}$ through that order. The shrinking-cutoff estimate in the source gives $|T_b(\chi(s\,\cdot) f)| \le C_f / s$ for a vanishing L_b-jet. It follows exactly that

$$T_b(f) = \sum_{|\alpha| \le L_b} T_b(\eta_\alpha)\, \partial^\alpha f(0).$$

The coefficients and jet are identical under horizontal readout. The order L_b depends only on the fixed seminorm; its coefficients may depend on the remote tests. This preserves the uniformity needed for the subsequent single annihilating dilation polynomial.

For $C_2 f(x) = 2^3 f(2x)$, photon scaling gives

$$T_b[D_i q](f) = 2^{-3-b} T_b[q](C_2 f). \tag{TR35}$$

Order-j jets of $C_2 f$ scale by 2^{3+j}, yielding roots 2^j for $b = 0$ and 2^{j-1} for $b = 1$. With $K = L_0 + L_1 + 2$, TR29's polynomial kills all these jets. Its amplitude r therefore satisfies $T_0[r] = T_1[r] = 0$ for every spatial test and every other remote test. For compact smooth momentum w away from zero, use inverse negative Fourier tests of $|k|w$ for $b = 0$ and of w for $b = 1$. The negative parts agree and positive parts have opposite signs; adding isolates

$$\int r |k_i|^2 w(k_i) \prod_{j \ne i} |k_j| \widehat{g}_{j,-}((k_j)_+) d\lambda^n = 0. \tag{TR36}$$

Small translations preserve remoteness. Multiplying w by its slot's shell plane wave then makes the Fourier transform of this integrable integrand vanish near zero in all translation variables. For an integrable function on a product of forward massless shells, such vanishing implies zero almost everywhere. Restrict its transform to strictly future lines: positive energies give a bounded continuous upper-half-plane function, holomorphic inside, with differentiation justified by $ue^{-yu} \le 1/y$. A zero boundary interval extends by Schwarz reflection and analytic uniqueness to the whole line. Continue along future lines, then write every displacement as the difference of two sufficiently large future displacements. The transform is zero everywhere. Spatial Fourier uniqueness of the finite complex measure proves the claim. Choose countably many shrinking normalized remote bumps around a fixed spacelike point, whose shell transforms are eventually nonzero at every fixed momentum, and spatial cutoffs avoiding zero eventually equal to one at nonzero momenta. Intersect their full-measure zero-integrand sets in TR36. Shell momenta are nonzero almost everywhere and all factors are eventually nonzero, so $r = 0$ almost everywhere. In each slot this proves TR29. The lifted local-sector source retains weighted integrability, both equal-time data and their uniform jet orders. TR35 transports the point-supported reader under slot dilation with the electric factor included. Retaining both $b = 0, 1$ isolates TR36; Fourier uniqueness and countable probe exhaustion recover the polynomial amplitude itself, rather than only its separately tested values. The coherent coefficient extraction required for TR32 follows next.

Dilation polynomial and zero-amplitude recovery. The slot readout retains q, its photon measure and both data functionals. Changing variables $u_i = 2k_i$ gives a factor 2^{-2} from the measure and $2^{-(b+1)}$ from the electric/energy power; $\widehat{C_2 f_\pm}(u_i) = \widehat{f_\pm}(u_i/2)$ supplies the remaining test transformation. This recovers exactly 2^{-3-b} in TR35. For the real polynomial $P_K(z) = \prod_{a=0}^{K-1}(z - 2^{a-1})$, iteration gives

$$T_b[P_K(D_i)q](f) = T_b[q](P_K(2^{-3-b}C_2)f).$$

The scalar jet eigenvalue on order j is 2^{j-b}. For $b = 0$ its root index is $j + 1$, and for $b = 1$ it is j. With $K = L_0 + L_1 + 2$ all required indices lie in $0, \dots, K - 1$. Thus the finite jet reader vanishes for both data of $r = P_K(D_i)q$. The horizontal copy retains these exact finite polynomials and identities, including the electric-factor shift.

For the two tests in the source's isolation step, denote the common negative-frequency integral by N and positive-frequency integral by P. The $b = 0$ equation is $N - P = 0$ and the $b = 1$ equation is $N + P = 0$. Adding gives $2N = 0$, exactly TR36 with its $|k_i|^2$ factor. No frequency branch or conjugation is removed before this step. All integrands obey the retained weighted absolute bound. Their small-translation readers are the Fourier transform of that same integrable function, with unchanged shell momenta and phase.

For a strictly future direction in the product of shell variables, its nonnegative energy pairing u gives the analytic factor e^{izu}, bounded by one for $\operatorname{Im} z \ge 0$. On $\operatorname{Im} z = y > 0$ its derivative is controlled by $ue^{-yu} \le 1/y$ times the integrable amplitude. Thus the source's half-plane continuation, zero-interval uniqueness and spatial Fourier uniqueness all apply to the identical lifted scalar transform. This recovers its almost-everywhere zero integrand, not a pointwise assertion at exceptional shell values.

More explicitly, a normalized bump shrinking about a fixed spacelike point x_* has shell transform equal to its nonzero plane-wave factor times $\widehat{a}(\varepsilon p)$, with $\widehat{a}(0) = 1$. For every fixed finite shell momentum this is eventually nonzero as $\varepsilon \downarrow 0$. Use a countable shrinking sequence and countable compact spatial cutoffs exhausting nonzero momenta. On the common full-measure set for these probes, all their factors are eventually nonzero, and $|k_i|^2$ and the other electric factors are nonzero

almost everywhere. Hence $r = 0$ as an amplitude class. The same countable readout recovers TR29 in every slot, completing its locality-to-dilation dependency while retaining the coherent-extraction dependency established next.

24.10 Coherent coefficient extraction and physical sector selection

For physical x define

$$L_x(g_1, \dots, g_n) = \sum_s \int \overline{x_{n,s}(k)} \prod_i z_{g_i}(s_i; k_i) d\lambda^n.$$

Cauchy–Schwarz gives integrability. Orthogonal sector decomposition gives the convergent expansion

$$\langle x, \mathcal{E}(h) \rangle = e^{-\|z_h\|^2/2} \sum_{n \geq 0} \frac{i^n}{\sqrt{n!}} L_x(h, \dots, h). \tag{TR37}$$

The reversed $\mathcal{E}(-h)$ inner product has coefficients $i^n \overline{L_x(h, \dots, h)}/\sqrt{n!}$. Apply TR22 to th, $t > 0$; cancel the identical positive Gaussian factors. Every power coefficient vanishes. To justify uniqueness directly, after lower coefficients vanish divide by the next power of t; for $0 < t \leq 1$ the tail is dominated by the absolutely summable series at one and tends to zero, isolating the next coefficient. Each L_x is multilinear and symmetric by simultaneous permutation of momenta and labels, physical symmetry and product-measure invariance. Diagonal identities polarize to arbitrary real lists: expansion at $h + tu$ and symmetry isolate the mixed slot with its multiplicity n, then induction gives all terms. The spacelike-supported test space is complex linear; real/imaginary decomposition preserves support. Complex multilinearity gives the general identity with conjugated lists on its conjugate side. Expand $z_g = \Pi z_g^{\mathrm{raw}}$ in one slot and move the real symmetric matrix through its finite spatial-label sum onto the transverse wave. It leaves that component unchanged. Repeat in every slot. Exchanging selected label and summed matrix index is a finite bijection and leaves other factors fixed. Only finitely many almost-everywhere transverse identities are used at each particle number; no regular wave representative is assumed. Now give slot i only the component s_i with scalar test g_i. All other sectors vanish, leaving

$$\int x_{n,s} \prod_i |k_i| \widehat{g}_{i,-}((k_i)_+) d\lambda^n = \int \overline{y_{n,s}} \prod_i |k_i| \widehat{g}_{i,+}((k_i)_+) d\lambda^n,$$

where $x = \sigma(f)$, $y = \sigma(\bar{f})$. The identity $\widehat{\bar{g}_+(p)} = \widehat{g}_-(p)$ fixes Fourier signs. The real shrinking trace bump then gives TR32 by its previously paid dominated convergence. With TR26–TR36 and the high-energy contradiction, this completes the specified Maxwell exclusions. The lifted coherent source retains the entire convergent sector series, its physical symmetry and transverse constraints. The scalar parameter reader extracts each coefficient with controlled tail; finite polarization and projector transfer recover the individual sector reading. This proves compatibility of the local matrix reader with TR32 without selecting pointwise representatives or adding separation between different probe tests.

Convergent coherent coefficient and component recovery. The horizontal comparison has the same physical vector x in the same ambient wave space. Its L_x, coherent probes and scalar series are therefore identical. Write x_n for its full labeled n-sector and $r = \|z_h\|$. Cauchy–Schwarz gives, for finite $u \geq 0$,

$$\sum_{n \geq 0} \frac{u^n}{\sqrt{n!}} |L_x(h, \dots, h)| \leq \left(\sum_n \|x_n\|^2 \right)^{1/2} \left(\sum_n \frac{u^{2n} r^{2n}}{n!} \right)^{1/2} = \|x\| e^{u^2 r^2/2}.$$

Thus coefficient recovery uses an actual absolutely convergent series with the original factorials. If its lower coefficients through $N-1$ vanish, division by t^N leaves the Nth coefficient and a tail whose terms are bounded for $0 < t \le 1$ by the summable coefficient series at one. Dominated convergence of that tail as $t \downarrow 0$ isolates the coefficient. The canceled Gaussian factor is positive and identical on both sides of TR22; the reversed inner product retains its stated i^n and conjugation.

For an explicit finite real polarization reader, if B is the symmetric n-linear difference of the recovered coefficients and $b(h) = B(h, \dots, h)$, then

$$n!B(g_1, \dots, g_n) = \sum_{S \subset \{1,\dots,n\}} (-1)^{n-|S|} b\left(\sum_{j \in S} g_j\right).$$

Each real sum has the same remote support condition; the empty sum is zero for $n > 0$. Hence the diagonal equality recovers every mixed real list. Real/imaginary decomposition then transports the complex identity with conjugated tests on its conjugate side, as specified in the source.

For fixed momentum and all other labels, projector removal is the finite sum identity

$$\sum_a \overline{x_a} \sum_b \Pi_{ab}(k) v_b = \sum_b \overline{(\Pi(k)x)_b} v_b = \sum_b \overline{x_b} v_b.$$

It uses real symmetry of Π and the almost-everywhere transverse constraint. Repetition over the finite slots and finite labels preserves all other factors. A single chosen component per test therefore selects the indicated label sector exactly. Together with $\widehat{\overline{g}_+} = \widehat{g}_-$, this recovers the source's isolated two-frequency formula. The already paid shrinking real trace bump yields TR32 with the same domination. Thus the coherent-extraction dependency in the locality-to-dilation argument is recovered without imposing regular representatives of the waves.

24.11 Running trace labels and calibrated clock cancellation

In the fixed finite free space choose basis e_a and scaling identifications U_i. Along the same refinement and selected phases,

$$\langle U_i e_a(x_i) U_i e_b(y_i) \rangle_\chi \longrightarrow \langle e_a(x) e_b(y) \rangle_{\text{free}}. \tag{TR38}$$

Rounding error is at most $4n/\mathfrak{m}_i \to 0$, generic separated plots remain admissible, and the comparison error $o_i(1 + M(x_i, y_i)) \to 0$ since M is continuous at the separated target. For convergent labels v_i, w_i, bilinearity gives the finite sum $\sum_{a,b} (v_i)_a (w_i)_b \langle U_i e_a(x_i) U_i e_b(y_i) \rangle$. Coordinate convergence and TR38 give the free moment of their limits. This is the passage used for the running centered trace, rather than an interchange of an uncontrolled varying field with a distributional limit. No infinite-dimensional uniformity is used. Centering $T_{\mu\nu}$ by subtracting its expectation times the unit preserves symmetry, reality, conservation and charges: unit derivatives vanish and a constant symmetric unit tensor has zero Ward charge. Its trace differs from the phase-centered trace by a constant unit. Inverse scaling gives $B_i r_i^4 c1$, vanishing by the running band below, with exact unit identifications. In the source conventions,

$$b_0 = \frac{11 C_{\text{adj}}}{48\pi^2}, \qquad \beta_{\text{slope}} = 2b_0, \qquad d_2 = -b_0/2. \tag{TR39}$$

The last identity is the stress coefficient forceR with this slope, derived in the stress comparison below. The running trace limit is $4d_2 \mathcal{O}_{\text{scale}}$; finite-basis passage and free Wick contraction give the coefficient

$$(2b_0)^{-2} (4d_2)^2 \frac{3 \dim \mathfrak{g}}{\pi^4} = \frac{12 d_2^2}{b_0^2} \frac{\dim \mathfrak{g}}{\pi^4} = \frac{3 \dim \mathfrak{g}}{\pi^4}.$$

Each running label carries $B_i r_i^4$ and the remaining prefactor is $(\log(1/r_i)/B_i)^2$, so the powers and logarithms are accounted for before the limit. Explicitly, let $\widehat{\beta}_i = \beta_* + g t_i$, $r_i = e^{-t_i} a_i$, $g = 2b_0 > 0$. The magnifications obey eventually $a_i \geq 1$ and $0 \leq \log a_i \leq \mathrm{Gate}(\widehat{\beta}_i)$, with $\mathrm{Gate}(x)/x \to 0$. Calibration and band comparison give $\beta_i/\widehat{\beta}_i \to 1$, $\beta_i/B_i \to 1$. Hence

$$\frac{\log(1/r_i)}{B_i} = \frac{t_i - \log a_i}{\widehat{\beta}_i} \frac{\widehat{\beta}_i}{\beta_i} \frac{\beta_i}{B_i} \longrightarrow 1/g. \tag{TR40}$$

Here $t_i/\widehat{\beta}_i = (1 - \beta_*/\widehat{\beta}_i)/g$ and the magnification quotient vanishes. Since $r_i \downarrow 0$, $\ell = \log(1/r_i) \to \infty$ and $\ell e^{-4\ell} \to 0$, giving

$$B_i r_i^4 = \frac{\log(1/r_i) r_i^4}{\log(1/r_i)/B_i} \longrightarrow 0.$$

This proves both the logarithmic normalization and vanishing unit shift. The lifted running source retains the finite label coordinates, calibrated clock, magnification gate and band ratios. Its moment reader commutes with finite coordinate limits; TR40 transports the running normalization with every factor visible. Retaining the exact unit map recovers the selected-vacuum centering from the convenient stress centering without adding a residual constant to the limit.

Running-label and calibrated-clock recovery. Keep the free label coordinates fixed and transport the interacting scaling map as $U_i^Z = J_i U_i$. The running-clock cancellation used here follows directly from the calibrated clock, including the extra magnification in the scaling sequence. For the selected character readout, the finite matrix of moments is identical:

$$F_{i,ab}^Z = \langle U_i^Z e_a(x_i) U_i^Z e_b(y_i) \rangle_{\chi^Z} = \langle U_i e_a(x_i) U_i e_b(y_i) \rangle_\chi = F_{i,ab}.$$

All rounding addresses, magnifications and the continuous profile norm are retained. Their comparison error therefore still tends to zero on the separated moving plots. For $v_i \to v$ and $w_i \to w$ in this finite space, the recovered moment error has the explicit bound

$$\begin{aligned} &\left| \sum_{a,b} (v_i)_a (w_i)_b F_{i,ab} - \sum_{a,b} v_a w_b F_{0,ab} \right| \\ &\quad \leq \sum_{a,b} |(v_i)_a (w_i)_b| \, |F_{i,ab} - F_{0,ab}| \\ &\qquad + \sum_{a,b} |(v_i)_a (w_i)_b - v_a w_b| \, |F_{0,ab}| \longrightarrow 0. \end{aligned}$$

Thus the same finite coordinate reader recovers TR38 for the running trace label. Its actual coefficient limit remains a dependency of the stress correction below; this passage does not replace that coefficient calculation.

The scalar clock source retains $t_i, a_i, \widehat{\beta}_i, \beta_i, B_i$ and the gate inequality. In particular

$$0 \leq \frac{\log a_i}{\widehat{\beta}_i} \leq \frac{\mathrm{Gate}(\widehat{\beta}_i)}{\widehat{\beta}_i} \longrightarrow 0.$$

Together with the two retained band ratios this gives precisely TR40's $1/g$ limit. All these are identical scalar functions of the lifted record. Let $\ell_i = \log(1/r_i)$. Then the unit contribution obeys

$$\|B_i r_i^4 c 1\| = |c| \|1\| \frac{\ell_i e^{-4\ell_i}}{\ell_i/B_i} \longrightarrow 0.$$

The exact unit identification and unchanged zero Ward charge therefore recover the selected-vacuum trace from the convenient centered tensor. No extra unit term survives the scaling readout.

Finally, bilinearity gives the exact normalization cancellation

$$\left(\frac{\ell_i}{B_i}\right)^2 \langle (B_i r_i^4 \Theta)(r_i x)(B_i r_i^4 \Theta)(r_i y)\rangle = \ell_i^2 r_i^8 \langle \Theta(r_i x)\Theta(r_i y)\rangle.$$

Applying the finite-label recovery and the clock limit retains $(2b_0)^{-2}(4d_2)^2$ multiplying the free scalar contraction. With the source's TR39 conventions it yields exactly the displayed A_G. Thus the lift preserves the running charge, centering and scale factors on the same refinement and selected phase.

25 Exact stress correction and local quantitative estimates

Exact correction and quantitative near-kernel recovery. The three finite candidates share a filter and a common extraction. The correction inverse is applied to the actual detector and divergence coordinates, so the repaired stress obeys exact conservation and the specified scalar amplitude. A small detector reading is useful only after the quantitative near-kernel argument controls the other paid coordinate; it is not an exact kernel statement by itself.

The six local contributions below retain their separate estimates until their sum earns the required trace bound. Moving positive masses are chosen before the volume estimates, and the resolvent selectors keep both their finite windows and their errors. Determinant differentiation, vector/ghost poles and the angular factor eleven are evaluated with those same conventions. This construction recovers a Ward-normalized scalar coefficient from actual finite objects. It does not replace the finite calculation by the formal value of a continuum determinant.

25.1 The three finite candidates and their common extraction

Here are the actual finite functions behind the three columns used below. Write $C_{ab}(x) = \mathrm{curv}(\mathrm{Hol}_{\square(a,b)}(x))$, with the curvature coordinate and normalization of (SQ1)–(SQ12), and put $P_{ija}(x) = \langle C_{ia}(x), C_{ja}(x)\rangle$. At a fixed issued ticket, the three tensors are

$$\begin{aligned} J^{\varnothing}_{ij}(x) &= \mathrm{Meld}(J)_{ij}(x), \\ J^{\mathrm{sh}}_{ij}(x) &= \begin{cases} 0 & i = j, \\ \sum_a P_{ija}(x) & i \neq j, \end{cases} \\ J^{\mathrm{sc}}_{ij}(x) &= \frac{\delta_{ij}}{4} \sum_{a,b} P_{aab}(x). \end{aligned} \tag{TF11}$$

The first is exactly the restored, localized principal tensor (TF0d), including its six antisymmetric differences and its scalar forward difference. The other two use elementary plaquette holonomies; there is no new stationary current in their definition. Their free coefficient vectors in the three-coordinate tensor basis are respectively the principal vector, $(0, 0, 1)$, and $(0, 1/4, 0)$. In the source's three-coordinate convention the principal free vector is $-\,\mathrm{weyl}$. Its physical sign is fixed together with the Ward charge in (TR41)–(TR44); it cannot be changed independently of that normalization.

All sixteen entries of each tensor, all sixteen individual stencil derivatives, and all four divergences are extracted on the same filter. Explicitly these are the origin functions $J^b_{ij}(0)$, $\mathrm{elm}_i\, J^b_{ij}(0)$, and $\sum_i \mathrm{elm}_i\, J^b_{ij}(0)$, for $b = \varnothing, \mathrm{sh}, \mathrm{sc}$. The tensor entries have degree-four coefficient records and the derivatives have degree-five records. These records include both the finite-cutoff label and its

retained limit; extraction of the joint correlation book gives the corresponding fields L_i, Ω_i, Δ_i. Thus extraction of a divergence is extraction of the sum of its actual finite derivatives, not an independently selected vector limit.

The local bounds needed here have a direct source. A plaquette has four links. If each matrix link has derivatives through order k bounded by the chart price d, its holonomy has price $\mathrm{travelCost}(G, k, 4, d)$. Applying the curvature map gives price $z(d) = \mathrm{curvatureCost}(G)\,\mathrm{travelCost}(G, k, 4, d)$. The bilinear product has price $2^k \|\langle\cdot,\cdot\rangle\| z(d)^2$. Sum the finitely many axes and multiply the scalar branch by $1/4$. This gives a bound independent of the torus and the ticket, supported in radius four. One stencil derivative enlarges the radius to eight. The scheduled small-field extension makes these bounds into the coefficient and correlation bounds used in extraction. The issued principal tensor has these same bounds by (TF0d) and its localization estimates. The condition $20(\mathrm{high}(x) + \mathrm{rim}(x) + 10) < n$ makes all integer offsets used in the coefficient tests nonwrapping.

Taylor projection commutes with the stencil divergence on these records: expand the finite translates and use the stencil moments from (TF0b). The degree-four stress coordinate therefore maps to its degree-five divergence coordinate. For the principal branch the antisymmetric difference corrections have zero relevant tensor coordinate; the scalar difference has zero trace coordinate. Consequently its free divergence scalar coordinate vanishes. This is the specific low-degree statement used later; it does not say that its entire finite divergence vanishes. The finite action, localized current, surgery polynomial and coefficient equations are issued by one ticket, so the six terms of (LCE6) concern this same first branch. The two elementary branches supply the correction columns in (TR41).

With $\mathcal{Y}_i T = r_i^4 U_i^{-1} T$ and $\mathcal{P}_i = B_i \mathcal{Y}_i$, their normalized limits and correction are precisely (TR41)–(TR44): solve the two divergence coordinates for $c_{i,0}, c_{i,1}$, then set $\widetilde{T}_i = B_i L_i + c_{i,0}\Omega_i + c_{i,1}\Delta_i$. This specifies the finite origin, normalization, and limiting meaning of all three candidates and both correction objects.

Source–readout formulation. The three candidates are three realized representatives read on *one* filter with one family of coefficient records. The correction is a motion along the two elementary columns, which span, together with the principal one, the three-dimensional tensor space of (FC3); by (FC14) the divergence reader separates exactly these directions modulo the equation direction E, so the corrected tensor is determined by its two divergence coordinates. Extraction of a divergence is extraction of the sum of the actual finite derivatives, on the same filter, and not an independently selected vector limit; this is what makes the corrected tensor a single realized object and not a combination of limits taken along different subsequences. The sign of the principal vector is not a free presentation choice: it is tied to the Ward normalization of the charge and cannot be changed independently. For the principal branch only the low-degree free divergence scalar coordinate is shown to vanish; the whole finite divergence is not asserted to vanish.

25.2 Exact stress correction and its scalar amplitude

The coefficient TR39 is obtained by finite-dimensional correction. The centered real symmetric covariant candidates L_i, Ω_i, Δ_i have free limits Maxwell stress, a traceless correction and a scalar tensor of trace $4\mathcal{O}_{\mathrm{scale}}$. For their two-component divergence detector K_i, put $S_i(c_0, c_1) = c_0\Omega_i + c_1\Delta_i$, $X_i = B_i K_i L_i$. The matrices $K_i S_i$ converge to an invertible free map. In fixed bases factor a nearby matrix as $A(I + A^{-1}E_i)$; for $\|A^{-1}E_i\| < 1/2$ the geometric inverse gives uniform bounds. Solve exactly

$$(K_i S_i) c_i = -X_i. \tag{TR41}$$

The detector estimate $\|X_i\| \le CR_i$ gives $\|c_i\| \le C'R_i$, $R_i/B_i \to 0$. Since K_i has exactly the true divergence kernel, $\widetilde{T}_i = B_iL_i + S_ic_i$ is exactly conserved eventually. For $t_i \ge R_i \ge 0$, the scalar detector has

$$\ell_i(X_i) \to -g, \quad t_i\ell_i(K_i\Omega_i) \to 0, \quad t_i(\ell_i(K_i\Delta_i) + 4) \to 0. \tag{TR42}$$

Coefficient bounds kill both error products. Applying ℓ_i to TR41 and rearranging gives

$$4c_{i,1} = \ell_i(X_i) + c_{i,0}\ell_i(K_i\Omega_i) + c_{i,1}(\ell_i(K_i\Delta_i) + 4).$$

Hence $c_{i,1} \to -g/4 = \text{forceR}$, with sign and factor four fixed by the detector equation. The primitive detector estimates are derived below. Write $\mathcal{Y}_iT = r_i^4U_i^{-1}T$ componentwise and $\mathcal{P}_i = B_i\mathcal{Y}_i$. The candidate comparisons give $\mathcal{P}_iL_i \to T_{\text{Maxwell}}$, $\mathcal{P}_i\Omega_i \to T_\Omega$ and $\mathcal{P}_i\Delta_i \to T_\Delta$. Thus

$$\mathcal{Y}_i\widetilde{T}_i = \mathcal{P}_iL_i + \frac{c_{i,0}}{B_i}\mathcal{P}_i\Omega_i + \frac{c_{i,1}}{B_i}\mathcal{P}_i\Delta_i \longrightarrow T_{\text{Maxwell}}. \tag{TR43}$$

The stronger traceless estimates are $B_i \operatorname{tr}(\mathcal{P}_iL_i) \to 0$ and $c_{i,0} \operatorname{tr}(\mathcal{P}_i\Omega_i) \to 0$. The remaining scalar contribution yields

$$B_i \operatorname{tr}(\mathcal{Y}_i\widetilde{T}_i) \longrightarrow 4\,\text{forceR}\,\mathcal{O}_{\text{scale}}. \tag{TR44}$$

Only the first coefficient needs a growth estimate; the second has the finite limit just established. Onto-ness of K_iS_i makes K_i onto. Scaling and exact unit identifications equate the centered physical and free stress dimensions; the detector target has the free divergence-target dimension. Kernel equality and rank–nullity therefore give a one-dimensional physical conserved centered stress space, as in the free calculation. The actual centered Ward-normalized stress R is nonzero. The free derivative map is nonzero because it attains a nonzero target coordinate; otherwise separating observers would make that target zero. Choose an observer with derivative value $y \ne 0$. Its free translation-flux test has value y; the physical test of R, scaling positions, window and fields together with the prescribed single derivative factor r_i, tends to y. Vanishing R would contradict this. Consequently $\widetilde{T}_i = v_iR$ with real v_i. The shell test of both R and $\widetilde{T}_i$ tends to the same nonzero y, for the latter by TR43 and tested derivative comparison. Linearity and division give $v_i \to 1$, hence eventual nonzero v_i. Multiplying TR43–TR44 by v_i^{-1} transfers both limits to R. TR40 removes its unit shift and gives the selected trace limit used in TR2. The lifted correction source retains the three candidates, exact kernel detector, inverse matrix and Ward shell test. TR41 is its finite correction reader, and TR42 recovers its scalar coordinate with the numerical sign intact. Rank–nullity and the nonzero charge reader identify the same physical stress line; normalization by v_i then recovers the actual Ward-normalized tensor rather than an arbitrarily scaled conserved representative.

Exact detector correction and Ward-line recovery. Let $J_{\text{st}}, Q_{\text{st}}$ and $J_{\text{det}}, Q_{\text{det}}$ be the horizontal maps on the finite stress and detector spaces. Transport $K_i^Z = J_{\text{det}}K_iQ_{\text{st}}$, $S_i^Z = J_{\text{st}}S_i$, and $X_i^Z = J_{\text{det}}X_i$. Then the exact inverse on the horizontal detector image gives

$$c_i^Z = -(K_i^ZS_i^Z)^{-1}X_i^Z = -(K_iS_i)^{-1}X_i = c_i, \qquad \widetilde{T}_i^Z = J_{\text{st}}\widetilde{T}_i.$$

In the first expression the output coefficient space is unchanged. For $A_i = A + E_i = K_iS_i$ and $\|A^{-1}E_i\| < 1/2$, $\|A_i^{-1}\| \le 2\|A^{-1}\|$. Thus the same detector bill bounds both correction coordinates by $C'R_i$. Moreover $K_i^Z\widetilde{T}_i^Z = J_{\text{det}}(X_i + A_ic_i) = 0$. Transport of the exact kernel equality makes this genuine divergence zero, not merely zero at a selected detector.

Use $\ell_i^Z = \ell_iQ_{\text{det}}$. With $\varepsilon_i = \ell_i(K_i\Omega_i)$ and $\delta_i = \ell_i(K_i\Delta_i) + 4$, the exact scalar identity yields

$$|4c_{i,1} + g| \le |\ell_i(X_i) + g| + C'R_i(|\varepsilon_i| + |\delta_i|) \longrightarrow 0.$$

Indeed $R_i \le t_i$ and TR42 pays each error product, including when $t_i = 0$. This recovers the actual finite scalar limit forceR $= -g/4 = -b_0/2$ with the original sign and factor four.

The transported scaling readers satisfy $\mathcal{Y}_i^Z J_{\mathrm{st}} = \mathcal{Y}_i$ and $\mathcal{P}_i^Z J_{\mathrm{st}} = \mathcal{P}_i$, so TR43 is identical term by term. Before the trace limit the exact decomposition is

$$B_i \operatorname{tr}(\mathcal{Y}_i \widetilde{T}_i) = B_i \operatorname{tr}(\mathcal{P}_i L_i) + c_{i,0} \operatorname{tr}(\mathcal{P}_i \Omega_i) + c_{i,1} \operatorname{tr}(\mathcal{P}_i \Delta_i).$$

The first two terms have their retained stronger vanishing estimates; the third uses the actual scalar coefficient limit and trace $4\mathcal{O}_{\mathrm{scale}}$. This recovers TR44 without replacing the stronger trace estimates by ordinary candidate convergence.

Stress-space dimensions, detector rank and kernel equality are preserved by the horizontal isomorphisms. The nonzero Ward shell reading also remains identical. If F_i is that common linear shell test, the exact ratio is

$$v_i = \frac{F_i(\widetilde{T}_i)}{F_i(R)} \longrightarrow \frac{y}{y} = 1, \qquad y \neq 0.$$

The denominator is eventually nonzero; the same ratio is recovered from the lifted tensors. Thus the physical stress line and its Ward normalization, including the subsequent v_i^{-1} transfer and TR40 unit removal, are retained. The primitive detector and stronger trace estimates remain explicit dependencies developed next.

25.3 Scalar coordinates and quantitative detector transfer

The real invariant degree-four scalar basis $1, 4\mathcal{O}_{\mathrm{scale}}$ has an injective coordinate map and a continuous finite-dimensional left inverse. Apply it to the same running trace limit to obtain simultaneously

$$U_i^{-1}(r_i^4 \Theta) = e_i 1 + 4 d_i \mathcal{O}_{\mathrm{scale}}, \qquad B_i d_i \to d_2, \quad B_i e_i \to 0.$$

Exact scalar membership ensures these coefficients without unrelated subsequences. The free surjection $V : \mathbb{R}^3 \to \mathcal{D}$ has kernel $\mathbb{R}(1, -1/4, 1)$. Columns $\Omega = (0, 0, 1)$ and $\Delta = (0, 1/4, 0)$ are independent modulo that line: coordinate zero forces its coefficient to vanish. They span the quotient because

$$(w_2 - w_0)\Omega + (4w_1 + w_0)\Delta = w - w_0(1, -1/4, 1). \tag{TR45}$$

This proves the free invertibility used in TR41. Let s_i map bare degree-five coordinates to the free divergence target, converging in operator norm. Candidate detector $p_i = K_i A_i$ and bare coordinates u_i satisfy

$$H_i^N(p_i - s_i u_i) \to 0 \quad \text{for every fixed } N \ge 0. \tag{TR46}$$

The clock dominates t_i. For the correction columns $u_i \to u_*$, with $s_\infty u_*$ respectively $V\Omega, V\Delta$; $N = 0$ gives their limits. Uniformly bounded scalar detectors obey

$$\sup_{\|u\| \le 1} |t_i(\ell_i(s_i u) - \mathcal{L}(u))| \to 0. \tag{TR47}$$

The bare correction comparisons also give $t_i(\mathcal{L}(u_i) - \mathcal{L}(u_*)) \to 0$. Split the full error as

$$\begin{aligned} t_i(\ell_i(p_i) - \mathcal{L}(u_*)) = {} & t_i \ell_i(p_i - s_i u_i) \\ & + t_i(\ell_i(s_i u_i) - \mathcal{L}(u_i)) \\ & + t_i(\mathcal{L}(u_i) - \mathcal{L}(u_*)). \end{aligned}$$

TR46 at $N = 1$ and $t_i \le H_i$ kill the first, TR47 and bounded u_i the second, bare comparison the third. Exact readings $\mathcal{L}(u_\Omega) = 0$, $\mathcal{L}(u_\Delta) = -4$ give TR42. For the principal renormalized bare divergence,

$$X_i - s_i v_i \to 0, \quad \|v_i\| \le CR_i, \quad R_i \le t_i, \qquad \mathcal{L}(v_i) \to -g. \tag{TR48}$$

Even if v_i grows, TR47 gives error at most $\varepsilon_i\|v_i\|/t_i \le C\varepsilon_i \to 0$. Bounded ℓ_i handles $X_i - s_i v_i$, proving $\ell_i(X_i) \to -g$. Bounded s_i and eventual $R_i \ge 1$ also yield $\|X_i\| \le C'R_i$, the same norm bound used by the correction solver. The elementary absorption principle used throughout this step is: if $g_i \ge 0$, $|a_i| \le Cg_i$, and $g_i x_i \to 0$ in a normed space, then $a_i x_i \to 0$. The lifted detector source retains the bare coordinates, moving map, clock rates and scalar functionals. TR46–TR48 commute the divergence reader with the scaling identification with paid growth, while TR45 recovers the exact two-column quotient inverse. The continuous scalar left inverse then recovers both trace coordinates on the same refinement.

Scalar coordinates and paid detector-map recovery. On the invariant scalar space let $C(e,d) = e1 + 4d\mathcal{O}_{\mathrm{scale}}$ and choose its continuous left inverse L_C. Horizontal transport gives $C^Z = JC$ and $L_C^Z = L_C Q$, so $L_C^Z C^Z = 1$. The inverse scaling reader satisfies $(U_i^Z)^{-1} J_i = U_i^{-1}$, hence the same simultaneous pair (e_i, d_i) is recovered at every finite i. Applying the fixed linear map to the running trace limit recovers $B_i e_i \to 0$ and $B_i d_i \to d_2$ on the same refinement.

In the free correction quotient, the coefficient reader is explicitly

$$[w] \longmapsto (w_2 - w_0,\, 4w_1 + w_0), \qquad (x,y) \longmapsto [x\Omega + y\Delta].$$

It is well defined because adding any multiple of $(1, -1/4, 1)$ changes neither coordinate. TR45 proves that these maps are inverse. Transporting the detector target therefore recovers the same free two-column inverse, not just its rank.

Equip the horizontal finite spaces with the transported norms. The bare coordinates can be retained unchanged, with $s_i^Z = J_{\mathrm{det}} s_i$, $p_i^Z = J_{\mathrm{det}} p_i$ and $\ell_i^Z = \ell_i Q_{\mathrm{det}}$. Their composition obeys $\ell_i^Z s_i^Z = \ell_i s_i$, and all TR46 remainder norms agree. Write $M = \sup_i \|\ell_i\|$ and $\varepsilon_i = \|t_i(\ell_i s_i - \mathcal{L})\| \to 0$. The three-term correction error then has the explicit bound

$$\begin{aligned} |t_i(\ell_i(p_i) - \mathcal{L}(u_*))| &\le MH_i\|p_i - s_i u_i\| + \varepsilon_i\|u_i\| \\ &\quad + |t_i(\mathcal{L}(u_i) - \mathcal{L}(u_*))| \longrightarrow 0. \end{aligned}$$

This is an identical bound after horizontal readout. It retains the exact free scalar values zero and -4 for the two correction columns. For the growing principal vector, $t_i > 0$ and $\|v_i\| \le CR_i$ with $R_i \le t_i$ give

$$|\ell_i(X_i) - \mathcal{L}(v_i)| \le M\|X_i - s_i v_i\| + \varepsilon_i\|v_i\|/t_i \le M\|X_i - s_i v_i\| + C\varepsilon_i \longrightarrow 0.$$

Thus the same controls recover both the scalar limit and the solver's norm bill. Finally if $B_i \le 2H_i$ and $H_i^N e_i \to 0$ for every fixed N, then $\|B_i e_i\| \le 2H_i\|e_i\| \to 0$. Applying this to both preexisting detector errors recovers the paid running-factor transfer used in TR42–TR44. No unbounded coordinate error is suppressed by merely taking an unweighted limit.

25.4 Near-kernel detector repair and paid coordinate estimates

The absorption rule is $g_i \ge 0$, $|a_i| \le Cg_i$ and $g_i x_i \to 0 \Rightarrow a_i x_i \to 0$, by norms. Both preexisting bare/physical detector and renormalized-divergence errors decay faster than every inverse power of H_i. Since $B_i \le 2H_i$, absorption permits their multiplication by B_i in TR42–TR48. For a real

signed-symmetry invariant scalar u, the tensor $J_{ab} = \delta_{ab}u$ belongs to the classified free stress space. Its coordinate trace is $4w_0 1 + 4(w_1 + 4w_2)\mathcal{O}_{\mathrm{scale}} = 4u$. Thus scalar membership follows. Expectation kills $\mathcal{O}_{\mathrm{scale}}$ and fixes the unit; its nonzero free two-point norm then proves the scalar coordinate map injective. Inverse scaling preserves those real symmetries and membership. The physical detector is exactly

$$(K_i T)_a = B_i r_i^5 U_{i,5}^{-1}((\operatorname{div} T)_a). \tag{TR49}$$

Positive prefactor and invertibility give equality of detector and divergence kernels. The map s_i is the componentwise renormalized bare lift followed by $U_{i,5}^{-1}$; symmetry places its image in the free target. Componentwise comparison gives TR46.

For the finite repair let $f : E \to H$ be onto with $\ker f = \mathbb{R}x$, $x \neq 0$, and right inverse U. Suppose s_i has exact right inverses $u_i \to U$, and $w_i \to x$, $s_i w_i \to 0$. A linear A gives $y = Ufy + A(y)x$ by the kernel coordinate. Set $V_i = u_i f + A \otimes w_i \to I_E$. Eventually its inverse exists and tends to identity. Since $s_i V_i y = fy + A(y)s_i w_i \to fy$, multiplication by V_i^{-1} proves $s_i \to f$ in operator norm; finite-basis reconstruction makes pointwise convergence sufficient. For a bare functional $\mathcal{L}$ and precision e_i, assume

$$e_i s_i w_i \to 0, \qquad e_i \mathcal{L}(w_i) \to 0. \tag{TR50}$$

Define $\ell_i = \mathcal{L}u_i$, uniformly bounded, and $Q_i = e_i(\ell_i s_i - \mathcal{L})$. Exact right-inverse cancellation gives

$$Q_i V_i y = A(y)[\mathcal{L}u_i(e_i s_i w_i) - e_i \mathcal{L}(w_i)] \to 0.$$

Thus $Q_i \to 0$ in norm, proving TR47 for $e_i = t_i$ and the needed s_i convergence without assuming prior uniformity near its kernel. Here x is the nonzero free equation-of-motion vector. Lifting target vectors at the same retained scale and undoing coordinates gives $s_i u_i = I$ exactly and $u_i \to U$. Issued equation coordinates give w_i and the two paid TR50 estimates. The bare functional annihilates x. The resulting precision is $t_i = \mathrm{Age}_i^{1/5}$. Correction coordinates differ prelimit from ideal orbit coordinates by at most $\mathrm{Age}_i^{-1/4}$; ideal scalar readings are exact. Bounded linearity therefore bounds the scalar discrepancy by a constant times loss_i. Bounded linearity and multiplication by t_i give $\mathrm{Age}_i^{-1/20} \to 0$, proving the third error in TR47. The same joint degree-four/five orbit comparison and unweighted loss give qualitative limits on the original extraction filter.

True-divergence and near-kernel repair recovery. The scalar membership argument is transported by the same real signed-symmetry maps: the diagonal tensor of u and its classified trace recover $u = w_0 1 + (w_1 + 4w_2)\mathcal{O}_{\mathrm{scale}}$. Expectation and the nonzero free scalar two-point norm retain the injectivity used by the scalar coordinate reader. For the detector let J_4, J_5 denote the horizontal tensor maps. Its exact formula gives

$$K_i^Z J_4 T = B_i r_i^5 U_{i,5}^{-1} Q_5 J_5 \operatorname{div} T = K_i T.$$

Here the detector output is in the unchanged free coordinates. Since $B_i r_i^5 > 0$ and $U_{i,5}$ is invertible, zero detector value recovers zero true divergence in both realizations. Equivariance keeps the real target restriction and componentwise TR46 comparison.

To expose the repair maps, the source retains f, U, A, s_i, u_i, w_i with $s_i u_i = I$, and the actual decomposition $y = Ufy + A(y)x$. Their finite-dimensional horizontal transports satisfy the identical composition laws. In the original coordinates put $V_i = u_i f + A \otimes w_i$ and $b_i = s_i w_i$. Then exactly

$$s_i V_i = f + A \otimes b_i, \qquad s_i = (f + A \otimes b_i)V_i^{-1}.$$

As $V_i \to I$, eventually $\|V_i^{-1}\| \le 2$, and hence

$$\|s_i - f\| \le \|f\|\|V_i^{-1} - I\| + \|A\|\|b_i\|\|V_i^{-1}\| \longrightarrow 0.$$

This recovers operator convergence from the actual right inverses and near-kernel vector, including its previously uncontrolled direction.

For the precision scale e_i put $Q_i = e_i(\mathcal{L}u_i s_i - \mathcal{L})$. Right-inverse cancellation gives the exact rank-one map

$$Q_i V_i = A \otimes [\mathcal{L}u_i(e_i s_i w_i) - e_i \mathcal{L}(w_i)].$$

Its norm is at most

$$\|Q_i\| \le \|A\|\|V_i^{-1}\|(\|\mathcal{L}u_i\|\|e_i s_i w_i\| + |e_i \mathcal{L}(w_i)|) \longrightarrow 0.$$

Thus both weighted near-kernel payments in TR50 are retained. Transported norms and maps give the same estimate, so $\ell_i = \mathcal{L}u_i$ recovers TR47 without assuming prior uniform convergence on the near-kernel direction. In this application its exact right inverse uses the same retained scale and projection-lift identity. The issued/ideal scalar error is paid explicitly by $\mathrm{Age}_i^{1/5}\,\mathrm{Age}_i^{-1/4} = \mathrm{Age}_i^{-1/20} \to 0$, on the original extraction filter and joint degree-four/five coordinates. The subsequent TS1–TS5 estimates remain the dependencies supplying TR50 itself.

For principal coordinates the prelimit bounds are

$$\begin{aligned} \|v_i\| &\le 2\|\mathcal{O}\|\,\mathrm{Age}_i^{1/6}, \\ |\mathcal{L}(v_i) + g| &\le \|\mathcal{M}\|\,\mathrm{Age}_i^{-1/4} + 4\|\mathcal{M}\|\,\mathrm{Age}_i^{-1} + \tfrac{1}{12}H_i^{-30} + \tfrac{1}{12}H_i^{-100}. \end{aligned} \tag{TR51}$$

Here $\mathcal{O}$ is the fixed degree-five orbit map and $\mathcal{M} = \mathcal{L}\mathcal{O}$. Compare the issued representative to its retained-core projection at error $\mathrm{Age}^{-1/4}$; the latter has norm at most $\mathrm{Age}^{1/6}$. Apply the bounded orbit map and scalar functional. The finite scalar calculation below supplies the other terms and $-g$. Continuous linear maps and closed norm inequalities pass through extraction; every error tends to zero. TS1–TS5 below supply the weighted near-kernel hypotheses. The lifted repair source retains exact right inverses, near-kernel vectors, precision scales and closed prelimit inequalities. Its coordinate reader V_i reconstructs the missing kernel direction; the scalar reader $\mathcal{L}u_i$ then recovers the required weighted accuracy. Keeping the issued and projected coordinates together pays both qualitative and weighted limits on the same extraction filter.

25.5 Local estimates for the coefficient calculation and the six contributions

Here are the local estimates underlying the coefficient calculation. Write x for the large coupling parameter and put

$$\begin{aligned} \Omega_a(x) &= \exp\{a(\log\log x)^2\}, \quad \theta = x^{-1/2}\Omega_1(x), \\ c_x = \theta^2\Omega_{-2}(x), \quad t_x &= \theta^2\Omega_{1/8}(x), \quad \sigma_x = \theta^2 x^{-1/100}. \end{aligned} \tag{LCE1}$$

All assertions in this paragraph concern sufficiently large $x > 1$. Products of the Ω_a add their subscripts; for every fixed a and $\varepsilon > 0$, $\Omega_a(x) \le x^{\varepsilon}$ eventually. This follows by dividing $a(\log\log x)^2$ by $\log x$, whose ratio tends to zero.

The regular local fields entering the exact split have the following derivative envelopes on the common healthy chart: the nonlinear principal discrepancy and the finite Taylor surgery error have envelope c_x; the principal comparison field and its loop field have envelope t_x; the finite-loop tail has envelope c_x. These are envelopes through the prescribed chart order, on the stated local

support, including measurability, gauge covariance and chart regularity. The large-x threshold is common to the stated local estimates. At each fixed admitted x, exhaust the finite loop and tail shelves after specifying the required positive error allowance. Absolute convergence makes their errors arbitrarily small. These shelves may depend on x; a single finite truncation uniform for all large couplings is not needed.

For clarity, the nonlinear estimate has a quantitative Taylor proof. Let $s = \theta(1 + \mathrm{high}(x))^P$, where P is the fixed stage power. In exponential link coordinates the discrepancy K between the nonlinear curvature expression and its linear comparison has derivatives through order four at zero bounded by x^{-30}. Its derivatives through the required higher order on $\|y\| \le Ds$ are bounded by $2\Omega_{1/200}(x)$; $D \ge 1$ is a fixed chart radius. The change from color coordinates to link directions has norm at most one, so these low-jet bounds persist under that change. Taylor's formula with integral remainder, after rescaling $y = sp$, gives, uniformly for $\|p\| \le D$,

$$\|\operatorname{minor}_s(\beta K)\|_{C^k} \le 4x\, 2\Omega_{1/200}(x)\, C_{5,D} s^5 \le \sigma_x, \qquad k \ge 5, \quad |\beta| \le 4x. \tag{LCE2}$$

The low-order terms fit this same estimate because $x^{-30} \le 2\Omega_{1/200}(x) s^{5-j}$ for $0 \le j \le 4$. The final inequality follows from $s^5 = x^{-5/2}$ times a fixed subpower factor, whereas $\sigma_x = x^{-1-1/100}\Omega_2(x)$. Thus the fifth-order Taylor remainder has a strictly decaying power to spare. The finite-difference surgery uses $h = \theta x^{-1/24}$. Since $h/s \le x^{-1/24}$, choosing its order $m \ge 48$ makes every fixed subpower multiple of $s(h/s)^{m+1}$ at most $\sigma_x h$ eventually. This controls the surgery error after division by its difference step. The support margins and nonwrapping conditions used here are the finite chart conditions already imposed when choosing the station; none is a new continuum hypothesis.

We spell out how these local bounds survive projection and summation. Fix the degree-five frame and its local section, and denote their fixed reading prices by $r \ge 1$ and $v \ge 1$. If the section has growth exponent $p \ge 0$, put $K_0 = (1+p)r + v$. A field with derivative envelope c_x has weighted coefficient size at most rc_x. This is the coefficient projection estimate: each coordinate is a bounded derivative reading, its fixed reading weight is included in r, and summing the coordinate absolute values bounds the coefficient norm.

For the secondary surgery terms indexed by a finite set I, let Q_i be their supports and g_i their derivative envelopes before multiplying by the difference coefficient. The actual summable tail bound is

$$\sum_{i \in I} e^{s_0 \operatorname{cover}(Q_i)} \left| \frac{\mathrm{teas}_i}{h} \right| g_i \le \sigma_x. \tag{LCE3}$$

Here s_0 is the fixed support weight of the local budget, distinct from the chart radius s. Multiplying each summand by its projection price at most $r+v$ and using the triangle inequality gives coefficient size at most $(r+v)\sigma_x$ for their sum. The same summation constructs its local representative and bounds its approximation remainder; this is why a bound for each individual term is not used in place of the required bound for their sum.

The principal local support has bounded weighted cover, so its support weight is at most a fixed $A = \exp(P_0\ \mathrm{TileNo})$. Its derivative envelope is $2c_x + 2t_x \le 4t_x$. Combining it with (LCE3) bounds both the total projected coefficient and its local representative error by

$$K_0(4At_x + \sigma_x) \le (4A+1)K_0\theta^2\Omega_{1/8}(x) \le \theta^2\Omega_{1/6}(x). \tag{LCE4}$$

The last inequality follows from $\Omega_{1/8}/\Omega_{1/6} \to 0$. Similarly,

$$K_0\sigma_x \le c_x, \qquad rc_x \le \theta^2\Omega_{-1}(x). \tag{LCE5}$$

Indeed their respective ratios are $K_0 x^{-1/100}\Omega_2(x)$ and $r\Omega_{-1}(x)$, both tending to zero. These inequalities prove the total envelope and all four small coefficient envelopes used next. They also show explicitly why the local errors still vanish after division by θ^2.

We identify those six contributions in terms of the finite functions, to avoid ambiguity between this split and the six antisymmetric tensor differences in (TF0d). At the current scale let Y_b be the localized current, $\widetilde{J}$ the restored tensor, $\mathcal{P}_A$ the finite principal action, and L_Q the finite ordinary loop sum. Write $\mathcal{Q}_Q$ for its augmented quiet sum, with the zero and first path moments removed as in (TF6a)–(TF6c). Define $D_Y f$ by differentiating f along the left logarithmic link current, and write $\operatorname{div}_\nu Y$ for its Haar divergence. The six objects are

$$\begin{aligned} W_b &= \operatorname{altoD}\{\beta(\textstyle\sum_a \mathscr{D}_a \widetilde{J}_{ab} - D_{Y_b}\mathcal{P}_A)\}, & B_b &= \operatorname{div}_\nu Y_b - (\operatorname{div}_\nu Y_b)(1), \\ K_b &= D_{Y_b}\mathcal{Q}_Q, & D_b &= D_{Y_b}(\mathcal{Q}_Q - L_Q), \\ E_b &= \operatorname{altoD}\{h^{-1}\mathcal{T}_{m,h}(Y_b, -\beta\mathcal{P}_A - L_Q)\}, & F_b &= \sum_{\gamma\in\mathcal{F}} \operatorname{weighted}_{P_\gamma} H_\gamma. \end{aligned} \tag{LCE6}$$

In this display W, E, F already denote coefficient vectors; B, K, D are scalar configuration functions until altoD is applied. The sign identity is $D_Y L_Q = K - D$, so the loop contribution is $-K + D$. The operator altoD means the degree-five Taylor reader with its actual threshold weights, on the common local support; the secondary readers retain their own supports P_γ. The current here is the localized version of (TF0a), transported through the stationary descent, not an independently chosen continuum vector.

The term $\mathcal{T}_{m,h}$ is a concrete finite-difference remainder. If $t_i > 0$ are the distinct nodes and ℓ_i their Lagrange polynomials on m nodes, use

$$w_i = \ell_i(0)/t_i, \qquad \mathcal{D}_{m,h} f = \sum_{i<m} w_i\{f(t_i h) - f(0)\}.$$

Their exact moment identities are $\sum_{i<m} w_i t_i^{j+1} = 1_{\{j=0\}}$ for $0 \le j < m$, by interpolation of t^j at zero. Therefore $\mathcal{D}_{m,h} f - h f'(0)$ starts in order h^{m+1}, with its integral Taylor remainder. Apply this to the changed current pairing times the exponential action ratio on each active link, and average the color directions over the Haar adjoint orbit of a unit color basis vector. Sum over the finitely many active links. This is precisely $\mathcal{T}_{m,h}$, including its current derivative and density derivative. Its bounds after division by h are (LCE2).

The low-jet estimate for the first term in (LCE6) uses the quantitative stationary error as well as the identity (TF0c). Here is that step in detail. Write $F = \operatorname{boom} J_b(0)$, $R' = D_Y \mathcal{P}_{\mathrm{main}}$, and E for the stationary error. Equality of the analytic jets of $F - E$ and R' through order four gives

$$D^j(F - R')(0) = D^j E(0), \qquad 0 \le j \le 4. \tag{TF12}$$

The issued stationary chart bounds the analytic coefficient norm of E by $\operatorname{shock}(x) = x^{-100}$ on radius $\sigma = \operatorname{east}(x)$. Converting the coefficient bound to ordinary derivatives through order four costs the previously defined analytic derivative price $g_G(\sigma/5, 4)$, called gable in the source. Localizing the tensor to $\operatorname{Meld} J$ costs at most $x^{-70} g_G(v/5, 4)$, where $v = \sigma/100$.

There are two further replacements: the original current Y by its localized current Y_c, and the full principal action by its finite retained action $\mathcal{P}_A$. Use the exact identity

$$D_Y \mathcal{P}_{\mathrm{main}} - D_{Y_c}\mathcal{P}_A = D_{Y - Y_c}\mathcal{P}_{\mathrm{main}} + D_{Y_c}(\mathcal{P}_{\mathrm{main}} - \mathcal{P}_A). \tag{TF13}$$

The current difference has weighted price $p = x^{-70}$; the localized current has price at most $q = \operatorname{trophy}(x) + p$. The full action derivative has bound

$$\begin{aligned} U(x) &= \operatorname{bankCost}(B(x)) K \operatorname{momentPrice}_P(D/2)[\operatorname{GeoSize} + 2 \text{ paths}], \\ T(x) &= \operatorname{bankCost}(B(x)) K x^{-100} + e^{-D \operatorname{rim}(x)/4} U(x). \end{aligned} \tag{TF14}$$

Here $D > 0$ is the issued path width; K, P depend on G and the fixed derivative order four, and the geometric and path prices are fixed by the stage. Cofinal finite path cuts give the tail derivative bound T. The weighted product rule for contracting a current with an action derivative bounds (TF13) by

$$V(x) = \mathrm{vocation}_G(v, p, U) + \mathrm{vocation}_G(v, q, T). \tag{TF15}$$

The prices in (TF15) have the following explicit meanings. Set

$$\begin{gathered} c_G(a) = \#\,\mathrm{Loc}(G,1)\,\frac{e^{2a}6!\,2^4}{a^6}, \qquad g_G(a,k) = (1 + c_G(a))^k, \\ \mathrm{vocation}_G(a, C, D) = 16\, c_G(a/5) C\, g_G(a/5, 3) D. \end{gathered} \tag{TF20}$$

Here $\mathrm{Loc}(G,1)$ is the fixed finite set of first-order link direction types (axis and color); its cardinality has no volume dependence. The lattice shell count bounds the sum of $e^{-a\,\mathrm{distance}}$ over these directions by $c_G(a)$. Expanding a multilinear derivative in its coordinate directions and summing these decays gives $g_G(a,k)$. For a current-action contraction, write $D_Y F = \sum_\ell Y_\ell\, \partial_\ell F$. The localized current has its first three derivatives bounded by $Ce^{-(a/5)\,\mathrm{distance}(\ell,0)} g_G(a/5, 3)$; the action derivative has four derivatives bounded by D and vanishes at the identity. In its fourth product derivative the term requiring a fourth derivative of the current is multiplied by that zero. The remaining binomial coefficients sum to $2^4 = 16$. Summing in ℓ gives exactly (TF20). The vanishing action derivative follows because its one-color linear coefficient is invariant, and simplicity excludes an invariant color vector; it holds for both the full action and its invariant finite tail. This proves the contraction bound in (TF15) using only three current derivatives and four action derivatives. The scheduled radii and cuts give $T = O(x^{-98})$, $V = O(x^{-65})$, and

$$x^{-100} g_G(\sigma/5, 4) + x^{-70} g_G(v/5, 4) + V(x) = O(x^{-64}) \le x^{-30} \tag{TF16}$$

eventually. These bounds are uniform in the admitted finite torus and hold for every sufficiently large finite cut at that fixed stage. Equations (TF12)–(TF16) supply the second, third, and fourth derivatives as well as the zeroth and first ones. They are the quantitative hypothesis used in the rescaled Taylor remainder (LCE2). In that application the higher derivative bound and $x^{-30} \le 2\ \mathrm{thinM}(x) s^{5-j}$, $0 \le j \le 4$, give the order-five rescaled remainder, with $s = \theta(1 + \mathrm{high})^{\mathrm{stage\ power}}$. Summing the Taylor terms gives the stated $\theta^2 x^{-1/100}$ envelope.

The quiet derivative K has zero scalar contraction by the two path-moment cancellations (TF6a)–(TF6c), whose full-family justification is given immediately below. The derivative of the appended quiet correction is D. The term B keeps the Haar Jacobian, and its centered constant is explicitly removed. The last term is obtained by expanding each finite surgery exponential into support factors, with coefficient w_i/h; its entire weighted sum is bounded in (LCE3). Thus the four small terms are exactly $W, \mathrm{altoD}\, D, E, F$, and the two non-small terms have their actual scalar calculations.

For completeness, the retained quiet moments are small because the *full* issued loop family has zero moments. For each ordered pair a of plaquette types and pair b of color indices, define

$$c_\gamma^{a,b} = \mathbf{1}_{\{\text{types of } \gamma = a\}}\, \mathrm{coeff}_\gamma(b), \qquad z_\gamma = \mathrm{offset}_{\gamma,1} - \mathrm{offset}_{\gamma,0}\,. \tag{TF17}$$

The path bounds include absolute summability of these coefficients and of each $c_\gamma^{a,b}(z_\gamma)_l$, together with their normalization

$$\sum_\gamma c_\gamma^{a,b} = 0, \qquad \sum_\gamma c_\gamma^{a,b}(z_\gamma)_l = 0 \quad (l = 1, \dots, 4). \tag{TF18}$$

These are curvature loop value and first-jet normalizations, distinct from the potential jet normalization. They are carried in the issued path data as the summable loop terms and their zero sums; joining path families preserves them by absolute summation. The path-data construction retains these identities along with the quadratic metric normalization, so (TF18) applies to the family actually used in (LCE6).

Let L be the retained finite set. Subtracting its sum from (TF18) gives

$$\sum_{\gamma\in L} c_\gamma^{a,b} = -\sum_{\gamma\notin L} c_\gamma^{a,b}, \qquad \sum_{\gamma\in L} c_\gamma^{a,b}(z_\gamma)_l = -\sum_{\gamma\notin L} c_\gamma^{a,b}(z_\gamma)_l. \tag{TF19}$$

Since $|z_{\gamma,l}| \le \operatorname{length}\gamma$ and the coefficient is bounded by its atom budget, an omitted bound $\sum_{\gamma\notin L}\operatorname{budget}_a(\gamma)\,(1+\operatorname{length}\gamma) \le A$ bounds every retained moment in (TF19) by A. Thus the coefficients appended at displacement zero and the four unit displacements in (TF6a) have total coefficient magnitude at most $5\,\#\{\text{color pairs}\}\#\{\text{type pairs}\}(A+4A)$. Indeed there are five displacement choices for each type pair, and each coefficient has magnitude at most $A+4A$; the unit-displacement coefficients individually have the sharper bound A. Their paths have length at most ten, so a weighted atom budget at width a costs at most an additional e^{10a}. Fixed-order local function and derivative bounds follow by the finite plaquette product estimate used for (TF11). The exponential path estimate gives $A = q + e^{-\sigma r/4}\operatorname{momentPrice}_1(\sigma/2)B$ after a sufficiently large cut and radius r, for width $\sigma > 0$ and full loop price B. This proves the small appended-tail bound used for D. The augmented finite family, meanwhile, has *exactly* zero constant and first moments by its explicit coefficients. Its Hessian against a supported direction and a direction with affine curl therefore vanishes by (TF6b), proving the scalar cancellation for K. Smallness of the correction and exact cancellation of the augmented family are separate consequences of (TF18) and (TF6a), respectively.

The issuance is constructive: for each of these regular scalar functions subtract the local section of its weighted Taylor card and put the masked difference in one regular activity slot, as in (TF5). Add the four principal formatted insertions and the finitely many secondary ones. Their coefficient is exactly the sum of their Taylor cards by linearity, and their support-weighted prices add. Equations (LCE3)–(LCE5) then prove both the bound on the issued coefficient and the paid remainder of its representative. This proves existence of the candidate with its coefficient identity and its budgets simultaneously.

Source–readout formulation. The six-term split (LCE6) is an exact decomposition of one configuration function into retained source columns, each with its own reader, support and bill. Four are small after division by θ^2 (W, altoD D, E, F); two are not small and are computed (B, the Haar Jacobian with its centred constant removed, and K, which has zero scalar contraction exactly). The lift keeps all six, and keeps the distinction between the exact statement ((TF6b) for the augmented family) and the estimate ((TF19) for the appended part). The quiet moments used for the appended part are those of Proposition 14.10, obtained from the full family and not from a truncation of it.

25.6 Invariant scalar reading and the six-term local payment

For the bare divergence coordinates the scalar functional is

$$\rho(\text{bareVector}\,w) = -4(w_0 + 4w_1). \tag{TR52}$$

The coordinate map is an isomorphism for simple $\mathfrak{g}$. The equation vector $(1,-1/4,1)$ is annihilated; more generally ideal membership gives $2(w_0-w_2)=0$, $2w_1+w_2/2=0$, forcing the same. Composing

ρ with the degree-five isomorphism gives $\mathcal{L}$, including the correction-column readings zero and -4. For real regular local vector V, its degree-five card reader obeys

$$\mathcal{M}(\theta^{-2}(\mathrm{altoD}\, V_a)_a) = \tfrac{1}{12}\,\mathrm{Re}\,\mathrm{Holt}(V). \tag{TR53}$$

The actual positive threshold cancels through $\mathrm{altoD}\, V = \theta^2\,\mathrm{DCard}\, V$ exactly. The card contraction equals the local polynomial contraction by reality and nonwrapping, without periodic aliases. This is an exact coefficient identity, not an error estimate. The principal vector splits

$$V = W + \mathrm{altoD}\, B - \mathrm{altoD}\, K + \mathrm{altoD}\, D + E + F. \tag{TR54}$$

The regular split $y + D = K$ and additivity of cards prove exactness. The central K has zero Holt contraction by local cancellation. The B contraction differs from midPay /12 by at most calorie /12. The four other terms $W, \mathrm{altoD}\, D, E, F$, after θ^{-2}, each have coefficient norm at most $\mathrm{Wave}(-1)$. Thus

$$\left|\mathcal{M}(\theta^{-2}V) - \frac{\mathrm{midPay}}{12}\right| \le 4\|\mathcal{M}\|\,\mathrm{Wave}(-1) + \frac{\mathrm{calorie}}{12}. \tag{TR55}$$

The total coefficient-size envelope is $\theta^2\,\mathrm{Wave}(1/6)$, hence $\|\theta^{-2}V\| \le \mathrm{Wave}(1/6)$, since coefficient size dominates norm. Finally $|\,\mathrm{midPay} - \mathrm{landPay}\,| \le \mathrm{shock}$ and $\mathrm{landPay}\,/12 = -g$ supply TR51's remaining errors and limit. This reduces the continuum scalar reading to the local cancellation, four paid envelopes and payment comparison on the same threshold. The lifted scalar source retains the invariant coordinate isomorphism and exact regular split. Its card reader commutes with the threshold cancellation by TR53; TR54–TR55 recover the principal payment and its residual with all six contributions retained.

Principal scalar functional and local payment recovery. Retain the fixed orbit map $\mathcal{O}$, bare functional $\mathcal{L}$ and their composition $\mathcal{M} = \mathcal{L}\mathcal{O}$. Transport of the finite coordinate spaces preserves this composition and its operator norm in transported norms. The issued-to-core error and retained norm therefore recover separately the first TR51 error and the principal norm bill. Closed norm inequalities and continuous linear maps give the same bounds along the original extraction. The remaining scalar payments stay attached to their finite-level calculation, rather than being replaced by an assumed continuum value.

For invariant coordinates the retained row is explicitly $\rho \circ \mathrm{bareVector} = (-4, -16, 0)$. On an equation ideal vector, $w_0 = w_2$ and $w_1 = -w_2/4$ give

$$-4(w_0 + 4w_1) = -4(w_2 - w_2) = 0.$$

In particular its values on $\Omega = (0, 0, 1)$ and $\Delta = (0, 1/4, 0)$ are exactly 0 and -4. The coordinate isomorphism and horizontal readout preserve the full row, its kernel annihilation and these two column values.

The local card source retains the actual positive θ, regular support and nonwrapping margins. Before any estimate,

$$\theta^{-2}\,\mathrm{altoD}\, V = \mathrm{DCard}\, V, \qquad \mathcal{M}(\mathrm{DCard}\, V) = \frac{1}{12}\,\mathrm{Re}\,\mathrm{Holt}(V).$$

Horizontal transport of the cards leaves this literal coefficient identity unchanged. For TR54 write the four residual contributions as $R_1 = W$, $R_2 = \mathrm{altoD}\, D$, $R_3 = E$, $R_4 = F$. Additivity and the central cancellation give exactly

$$\mathcal{M}(\theta^{-2}V) - \frac{\mathrm{midPay}}{12} = \sum_{a=1}^{4} \mathcal{M}(\theta^{-2}R_a) + \frac{\mathrm{Re}\,\mathrm{Holt}(B) - \mathrm{midPay}}{12}.$$

The omitted central scalar is exactly zero by TR53 applied to K; its minus sign remains in the full vector decomposition. Each of the four retained residual norms is bounded by Wave(-1). Thus this equality recovers TR55 with precisely four operator-norm payments and the calorie $/12$ term. Adding the actual comparison to landPay yields explicitly

$$|\mathcal{M}(\theta^{-2}V) + g| \le 4\|\mathcal{M}\|\,\mathrm{Wave}(-1) + \frac{\mathrm{calorie} + \mathrm{shock}}{12}.$$

The same retained threshold cancels in the total norm envelope. Together with issued-to-core error and the source's finite payments, these identities recover all TR51 error terms and their limiting $-g$ value, keeping the subsequent massive calculation and TS near-kernel estimates as the specified dependencies.

25.7 Massive payment comparison and exact quiet-loop cancellation

Keep mass strictly positive through the volume estimates. The scalar resolvent primitive and differentiated shell satisfy

$$\frac{d}{dx}\operatorname{Re}\mathcal{R}_i(e^x) = -\operatorname{Re}\mathcal{S}_i(e^x), \qquad |\operatorname{Re}\mathcal{R}_i(e^x) + Lx| \le C\ (x \le 0), \quad L = -12g. \tag{TR56}$$

Each fixed x has an eventual estimate with the same C; the massive matrix is strictly positive and its finite-volume resolvent derivative converges. For fixed $a < b \le 0$, the mean value theorem and the two endpoint estimates give $\mu_i \in [e^a, e^b]$ with

$$|\operatorname{Re}\mathcal{S}_i(\mu_i) - L| \le 2C/(b-a). \tag{TR57}$$

The mass can depend on volume; no uniform mass convergence is assumed. The uncorrected local payment obeys, for every moving mass in a fixed $[\delta, 1]$, $\delta > 0$,

$$\mathcal{X}_i - \mathcal{S}_i(\mu_i) - \mu_i^2\mathcal{K}_i(\mu_i) \to 0, \qquad |\mu_i^2\mathcal{K}_i(\mu_i)| \le C_1\mu_i. \tag{TR58}$$

Given tolerance first choose b negative with $C_1e^b < \varepsilon/4$, then $b-a$ large with $2C/(b-a) \le \varepsilon/4$, so that the right side of (TR57) is at most $\varepsilon/4$. The resulting positive e^a is fixed before the volume limit, so TR58 applies and $\operatorname{Re}\mathcal{X}_i \to L$. Finite local resolvent-jet bounds $|\mathcal{K}_{d,i,b}(\mu)| \le C_{d,i,b}\mu^{-1}$ give the remainder after summing finite indices and multiplying by μ^2. The same bound makes the local second-derivative correction vanish: choose small fixed μ first for its $C\mu$ bound, then large volume for the comparison error. Uniform analytic jets identify the local difference quotient with the second derivative. Therefore

$$\mathrm{midPay}_i = \operatorname{Re}\left(\mathrm{crest}_i - \tfrac{1}{2}\sum_{d,a}\mathrm{floft}_{d,a,i}\right) \longrightarrow -12g. \tag{TR59}$$

The crest has the $\mathcal{X}_i$ limit and every correction vanishes.

For a finite gauge-invariant loop roster its periodization is

$$\mathcal{P}(U) = c_{\mathrm{reset}}\sum_{\alpha}\sum_{\text{seats } s} p_\alpha(s\cdot U). \tag{TR60}$$

It is analytic and gauge invariant. For analytic gauge-covariant Y_a in a nonwrapping neighborhood, $r_a = Y_a\mathcal{P}$ is local, analytic, gauge invariant and zero at identity because Y_a is zero there. Covariance follows by differentiating the invariant identity with conjugated direction; locality by differentiating

supported links. The quiet roster Hessian pairs to zero between a supported direction and the prescribed distant polynomial directions through degree two. With loop length bound M, $2(t+1+1+M)+6 < n$ prevents wrapping. The derivative of Y_a remains supported there, so differentiating r_a twice gives two zero Hessian pairings. In the third difference entering Holt, constant-direction terms vanish by gauge invariance and the local Ward identity; all others are these Hessian pairings. The finite difference is zero, and the coefficient identity summing those differences gives $\mathrm{Holt}((Y_a\mathcal{P})_a) = 0$. Add the finite boundary compensation U to the original truncated sum S, giving quiet $P = S+U$. Then $Y_aS = Y_aP - Y_aU$, precisely $y = K - D$ in TR54. The compensation has the small calorie envelope; the original derivative retains its larger local envelope. Gauge invariance and loop-length bounds persist, accounting for the exact zero term and its separately paid boundary correction. The lifted massive source retains positive mass, endpoint estimates, mean-value choice and local jet errors. Its payment reader recovers TR59 in the required tolerance–mass–volume order. The quiet-loop source retains the compensation roster and nonwrapping supports; its exact directional reader recovers the zero central contraction while preserving the separate boundary residual.

Positive-mass choice and quiet-roster recovery. Retain the finite-volume primitive, shell quantity, local correction and their actual positive mass in the source record. Their scalar readouts are identical after horizontal transport. Put $h_i(x) = \operatorname{Re}\mathcal{R}_i(e^x) + Lx$. At fixed $a < b \le 0$, the two eventual bounds $|h_i(a)|, |h_i(b)| \le C$ give a mean-value point $c_i \in (a,b)$ with

$$h_i'(c_i) = L - \operatorname{Re}\mathcal{S}_i(e^{c_i}), \qquad |h_i'(c_i)| \le \frac{2C}{b-a}.$$

This recovers TR57 from only two endpoint bounds. The same points and masses are retained in the lifted scalar reader. Writing the TR58 comparison error as $e_i = \mathcal{X}_i - \mathcal{S}_i(\mu_i) - \mu_i^2\mathcal{K}_i(\mu_i)$, one has explicitly

$$|\operatorname{Re}\mathcal{X}_i - L| \le |e_i| + C_1e^b + \frac{2C}{b-a}.$$

Choose b, then a, then the volume threshold; the fixed lower mass $e^a > 0$ makes the moving-mass comparison applicable. This retains the exact tolerance–mass–volume quantifiers rather than assuming a mass-uniform primitive limit.

Every local jet correction retains its finite index list and μ^{-1} bound. Multiplication by μ^2 gives $C\mu$; first fixing a sufficiently small positive μ and then the volume recovers its vanishing. The recorded analytic-jet estimate identifies the local difference quotient with the actual derivative in that comparison. Hence the finite sum of corrections in TR59 vanishes, and its crest reader recovers the same $L = -12g$ payment.

For the quiet roster keep the actual finite loop functions, seats, reset coefficient and compensation. Configuration realization gives $\mathcal{P}^Z = \mathcal{P} \circ \pi$. A horizontal lift of the supported covariant direction satisfies exactly

$$Y_a^Z\mathcal{P}^Z = (Y_a\mathcal{P}) \circ \pi.$$

The finite local jet readers intertwine by the same directional chain rule. The roster length, support and separation inequality are retained, so the source's zero supported/distant Hessian pairings and local Ward cancellation are identical scalar readings. Their finite third-difference sum therefore recovers $\mathrm{Holt}((Y_a\mathcal{P})_a) = 0$ exactly. Finally the retained equality $P = S+U$ gives $Y_aS = Y_aP - Y_aU$ before estimating either term. Thus the central zero and the separate boundary compensation of size calorie recover the same split $y = K - D$ in TR54, with no loss of its sign or nonwrapping condition. The determinant and massive estimates below remain the dependencies for TR56–TR59.

25.8 Determinant differentiation and finite roster moments

The local primitive contracts

$$\tfrac{1}{2}\log\det(A(U)+\mu^2 I)-\log\det(B(U)+\mu^2 I). \tag{TR61}$$

The first matrix includes physical quadratic and gauge-fixing terms; the second is the gauge comparator. At fixed $\mu>0$ both are invertible near identity and

$$\frac{d}{d\mu}\log\det(F+\mu^2 I)=2\mu\,\mathrm{tr}(F+\mu^2 I)^{-1}. \tag{TR62}$$

Joint mass/chart analyticity commutes this derivative with the finite coefficient derivatives. Apply the normalized contraction $\mathrm{yard}_{d,a}=|\mathrm{sites}|^{-1}\,\mathrm{cob}_{d,a}$ and sum ordered $d\neq a$. Its derivative is $-\mu$ times the resolvent contraction; $\mu=e^x$ supplies the other mass factor, proving TR56. The period-limit comparison has eventual error at most one from the massive integral satisfying $|\alpha(\mu)+g\log(1/\mu)|\le C_0$, uniformly in $0<\mu\le 1$ and allowed regulator segment. Twelve ordered-axis contributions give

$$|\operatorname{Re}\mathcal{R}_i(\mu)+12g\log(1/\mu)|\le 12(C_0+1)$$

eventually for each fixed mass, the bound in TR56. The massive-symbol calculation below supplies this uniform integral estimate. For each axis/color pair the quiet charges and relative offsets obey

$$\sum_\alpha c_\alpha=0,\qquad \sum_\alpha c_\alpha(\delta_\alpha)_a=0\quad\text{for all } a. \tag{TR63}$$

Hence $\sum_\alpha c_\alpha F(\delta_\alpha)=0$ for every additive real offset function, by its four coordinate values. Each two-leg loop Hessian is the sum of two cross-leg curvature products. Shift seat summation so the first leg has common base. Its nonzero supported curvature lies in the one-link enlargement; the other curvature is affine on the loop-enlarged neighborhood: $\mathrm{curl}\,T(x+\delta_\alpha)=C+F(x)+F(\delta_\alpha)$. Charge summation gives zero by TR63 for both leg orders. Signed rotations preserve the argument, with at most one-link support growth. Summing seats and rotations proves the exact Hessian zero used in TR55, retaining the original nonwrapping margins. The derivative in (TR56) can also be read directly from its determinant definition. The lifted determinant source retains positive mass, matrix factors and normalized ordered contractions. Its derivative reader recovers TR56 with the correct mass factors and twelvefold normalization. The roster reader retains offsets and both finite moments, making its zero Hessian contraction an exact source identity rather than a small-error replacement.

Massive determinant and finite-moment recovery. Retain the finite physical and gauge-comparison matrices with their actual positive mass. On their horizontal copies, use the transported orthonormal bases, so determinant and trace are the original scalar readouts. In particular

$$\begin{aligned}\partial_\mu\left[\tfrac{1}{2}\log\det(A+\mu^2 I)-\log\det(B+\mu^2 I)\right]&=\mu\,\mathrm{tr}(A+\mu^2 I)^{-1}\\&\quad-2\mu\,\mathrm{tr}(B+\mu^2 I)^{-1}.\end{aligned}$$

This is an identity on the same positive-mass chart, before local coefficient contraction. Joint analyticity allows the retained finite local derivatives to commute with ∂_μ. The site-normalized reader and ordered axis sum are unchanged by horizontal transport. The chain rule $\partial_x=\mu\partial_\mu$ at $\mu=e^x$ therefore retains both mass factors and the sign in TR56.

The four-dimensional ordered off-diagonal axis list has twelve entries. Its fixed-mass comparison error at most one and the source's uniform massive coefficient estimate give exactly

$$|\operatorname{Re}\mathcal{R}_i(\mu) + 12g\log(1/\mu)| \le 12(C_0+1).$$

With $L=-12g$ and $\log(1/e^x)=-x$, this is the stated $|\operatorname{Re}\mathcal{R}_i(e^x)+Lx|$ bound. Its index threshold may depend on the fixed mass; the same constant is retained for every permitted regulator segment. The massive integral below remains its quantitative dependency.

For the roster, retain the finite signed charges and relative offset vectors. If $F(\delta)=\sum_a \delta_a F(e_a)$ is additive, then

$$\sum_\alpha c_\alpha F(\delta_\alpha) = \sum_a F(e_a)\sum_\alpha c_\alpha(\delta_\alpha)_a = 0.$$

After the source's exact seat reindexing, the supported first-leg curvature is fixed. Multiplying it by the second affine curvature and summing charges gives that fixed factor times

$$(C+F(x))\sum_\alpha c_\alpha + \sum_\alpha c_\alpha F(\delta_\alpha) = 0.$$

The other leg order, finite signed rotations and seats obey the same identity. Horizontal configuration readout retains their actual links, offsets and nonwrapping margins, so it also retains this zero before any limit or estimate. Consequently TR63 recovers the quiet Hessian cancellation used in TR55 and its full local third-difference reader, with the original compensation kept separate.

25.9 The local resolvent bound

We derive the local resolvent bound in (TR58). Use a periodic lattice of side n, angular momentum spacing $h=2\pi/n$, and normalized momentum weight n^{-4}. Write $k_x = h\,\mathrm{lift}(x)$ for its centered momenta. If $R\ge h$, any subset of momenta with $|k_x|\le R$ has at most

$$(3R/h)^4 \tag{LR1}$$

members. Indeed their centered integer coordinates have absolute value at most $\lfloor R/h\rfloor$. The lift is injective, and the integer coordinate box has at most $(2\lfloor R/h\rfloor+1)^4 \le (3R/h)^4$ points. This upper bound remains true when the coordinate box wraps on the torus.

For $h\le\mu\le 1$, group momenta according to

$$2^j\mu \le |k_x|+\mu \le 2^{j+1}\mu, \qquad j\ge 0.$$

Assign boundary points to one of their adjacent groups. A group has at most $(6\cdot 2^j\mu/h)^4$ points by (LR1); its summand $(|k_x|+\mu)^{-5}$ is at most $(2^j\mu)^{-5}$. Therefore

$$n^{-4}\sum_x (|k_x|+\mu)^{-5} \le \frac{6^4}{(2\pi)^4\mu}\sum_{j\ge 0}2^{-j} \le \frac{2\cdot 6^4}{(2\pi)^4\mu}. \tag{LR2}$$

Only finitely many groups are nonempty at a given n; extending their sum by zeros justifies the infinite geometric majorant.

Here is the integrand estimate to which (LR2) applies. At $z=|k|+\mu>0$, scale an external momentum direction p by z. The zero-field matrix, one-field vertex and two-field vertex have sizes $z^2, z, 1$ with bounded scaled derivatives through the finite orders in use. These follow from

their vanishing origin jets and the analytic symbol bounds proved in the massive-symbol section. Ellipticity gives the inverse size z^{-2}. Differentiating $A^{-1}A = I$ gives

$$D(A^{-1}) = -A^{-1}(DA)A^{-1},$$

$$D^2(A^{-1})[u,v] = A^{-1}(DA[v])A^{-1}(DA[u])A^{-1} + A^{-1}(DA[u])A^{-1}(DA[v])A^{-1} - A^{-1}(D^2A[u,v])A^{-1}.$$

These formulas, and one further differentiation when needed, retain the same inverse size under scaled derivatives. Hence the contracted resolvent-insertion integrands have scaled analytic-jet bounds of size z^{-2}, with constants depending on the fixed directions and finite matrix dimensions but not on k, μ, n.

For such an analytic integrand V, let $[V]_j(p) = D^jV(0)[p,\dots,p]/j!$ be its homogeneous Taylor coefficient. The scaled bound gives

$$z^3|[V]_3(p)| \le Cz^{-2}/3!, \quad \text{hence} \quad |[V]_3(p)| \le C'z^{-5}.$$

For the second coefficient it gives $|[V]_2(p)| \le C'z^{-4}$. The angular cube and $\mu \le 1$ give a fixed upper bound $z \le R_*$, so this too is bounded by $C'R_*z^{-5}$. Summing the finite matrix entries and applying (LR2) bounds both contracted coefficients by a constant times μ^{-1}.

The local readings are explicit finite coefficient selectors. For a scalar function on integer arguments set

$$\operatorname{Pan} f = \tfrac{1}{2}(f(1) - 2f(0) + f(-1)), \qquad \operatorname{Flop} f = \tfrac{1}{12}[8(f(1) - f(-1)) - (f(2) - f(-2))].$$

On powers of degree at most two, Pan selects the quadratic coefficient; on powers of degree at most three, Flop selects the linear coefficient. The mixed selector is $i \operatorname{Pan}_a \operatorname{Flop}_b$, evaluated on the specified two-direction momentum contour, with the corresponding powers of h^{-1}. Its fixed finite coefficients bound it by a finite sum of the directional third coefficients just estimated. The local second selector is Pan on the opposed-momentum contour, with h^{-2}. Analytic Taylor remainder bounds show that each selector differs from its indicated derivative coefficient by a term tending to zero as $h \to 0$, with mass restricted to a fixed $[\delta, 1]$. For $\delta > 0$, the massive matrices have a common positive lower bound, so their analytic inverse neighborhoods and the higher derivatives needed for these remainders are bounded throughout this interval. This also allows a mass μ_n depending on n. Absorb a final selector error at most 1 into μ_n^{-1}, since $\mu_n \le 1$.

The physical local lens is the prescribed half-weighted physical resolvent contraction minus its gauge comparison contraction. Both have the preceding bounds; their matrix tickets identify the actual finite matrices with the Fourier inverse matrices. Linearity and the triangle inequality therefore prove, with constants fixed before the positive lower mass cutoff is requested,

$$\exists C_{d,a,b} \ge 0 \;\; \forall \delta > 0 \; \forall (\mu_n)_n, \quad [\delta \le \mu_n \le 1 \; \forall n] \implies |\mathcal{K}_{d,a,b,n}(\mu_n)| \le C_{d,a,b}\mu_n^{-1} \quad \text{eventually.} \tag{LR3}$$

The eventual threshold may depend on δ and the moving mass family. Once δ is fixed, $h \le \delta \le \mu_n$ eventually, as required in (LR2). Sum (LR3) over the finitely many tensor indices. Multiplication by μ_n^2 yields $C_1\mu_n$, the remainder bound in (TR58).

For the local second-derivative correction, its exact massive comparison has error tending to zero at each fixed $0 < \mu \le 1$, and its massive term is at most $C\mu^2\mu^{-1} = C\mu$ by the second-selector version of the same argument. Given $\epsilon > 0$, choose $0 < \mu \le 1$ with $C\mu < \epsilon/2$, and only then take the volume large enough that the comparison and selector errors total less than $\epsilon/2$. This proves vanishing of that correction, including the actual second Taylor coefficient rather than merely its finite difference. Consequently

Source–readout formulation. In the lift the masses, the selectors Pan and Flop and the local lens are numerical readers of the realized determinant films of Section 14. The mass is an auxiliary spectator parameter with no target counterpart; it is removed by the two-step limit and is never set to zero inside an integral. The quantifier order in (LR3), with the constant chosen before the lower mass cutoff and the threshold after it, is part of the recovered statement: for the local second-derivative correction the massive term is at most a constant times μ, and one chooses μ first and the volume afterwards. The selectors are exact on polynomials of the stated degrees and differ from the corresponding derivative coefficients by terms vanishing with the momentum spacing, uniformly for masses in a fixed interval $[\delta, 1]$; that discrepancy is retained until the volume limit is taken.

25.10 The logarithmic momentum coefficient and its uniform remainder

Use $k \in \mathcal{B} = (-\pi, \pi]^4$, $r = |k|$ and $c_4 = (2\pi)^{-4}$. For the actual massive symbol's half second external derivative F_μ,

$$\alpha(\mu) = c_4 \int_{\mathcal{B}} \operatorname{Re} F_\mu(k) d^4k. \tag{TR64}$$

The change $k = 2\pi x$ from the unit-period torus has Jacobian $(2\pi)^4$. At fixed positive mass the symbol is analytic and its first two external derivatives are bounded on a compact external interval times the closed internal cube. Dominated differentiation moves them under the integral; the half coefficient and definition's factor two give precisely TR64. Uniform symbol estimates are

$$|F_\mu(k)| \le A(r+\mu)^{-4}, \qquad |F_\mu(k) - P_G(k)| \le Dr^{-3} + D\mu^2 r^{-6} \quad (\mu \le r), \tag{TR65}$$

with common $A, D \ge 0$ for $0 < \mu \le 1$ and

$$P_G(k) = 4C_{\mathrm{adj}} \left(k_1^2/r^6 - 4k_1^2 k_0^2/r^8 - r^{-4} \right). \tag{TR66}$$

The distinct axes are the external direction and polarization. Packaging the admissible symbol/mass families at scales $r + \mu$ and r supplies the same-mass estimates without shifting to a preferred radius. On $\mathcal{A}_\mu = \{\mu < r \le 1\}$, polar integration gives

$$\begin{aligned} J_\mu = \int r^{-4} d^4k &= 2\pi^2 \log(1/\mu), \\ \int r^{-3} d^4k &= 2\pi^2 (1-\mu), \\ \int r^{-6} d^4k &= \pi^2(\mu^{-2} - 1). \end{aligned} \tag{TR67}$$

Rotational invariance and $\sum_a k_a^2 = r^2$ give $\int k_a^2/r^6 = J_\mu/4$. Rotate to $(e_a \pm e_b)/\sqrt{2}$ and add fourth-power identities; odd terms cancel, giving diagonal fourth moment three times the mixed one. Summing over b gives $\int k_a^2 k_b^2/r^8 = J_\mu/24$ for $a \ne b$. Consequently

$$\begin{aligned} c_4 \int_{\mathcal{A}_\mu} P_G d^4k &= c_4\, 4C_{\mathrm{adj}} (\tfrac{1}{4} - \tfrac{4}{24} - 1) 2\pi^2 \log(1/\mu) \\ &= -\frac{11 C_{\mathrm{adj}}}{24\pi^2} \log(1/\mu) = -g \log(1/\mu). \end{aligned} \tag{TR68}$$

Split the cube into $r \le \mu$, this annulus and $r > 1$. The inner integral is at most $A\mu^{-4}$ times the radius-μ ball volume, giving $A\pi^2/2$. The outer integrand is at most A, giving $A \operatorname{vol}(\mathcal{B})$. Annular

difference error integrates to at most $2D\pi^2(1-\mu)+D\pi^2(1-\mu^2)\le 3D\pi^2$. Thus absolute bounds before region summation yield

$$|\alpha(\mu)+g\log(1/\mu)|\le c_4[A\pi^2/2+A\,\mathrm{vol}(\mathcal{B})+3D\pi^2]. \tag{TR69}$$

This is TR56's mass-independent constant, with the factor eleven and $g=2b_0$ normalization explicit. The lifted momentum source retains the actual symbol, angular measure, same-mass comparison and three regions. Its pole reader recovers TR68 by finite angular moments; its residual reader retains all absolute integral errors in TR69, so logarithmic recovery preserves the uniform mass bound and constants.

Angular pole and uniform remainder recovery. Retain the actual massive symbol, external derivative, angular cube and normalized measure c_4d^4k. Rotational invariance determines the required angular moments without a sphere normalization convention. The horizontal matrix realization uses transported bases, so its scalar half second derivative is the same $F_\mu(k)$ at every fixed positive mass. Thus

$$\alpha^Z(\mu)=c_4\int_{\mathcal{B}}\mathrm{Re}\,F_\mu^Z(k)d^4k=\alpha(\mu),\qquad P_G^Z(k)=P_G(k).$$

The coordinate change $k=2\pi x$ retains its Jacobian, and the analytic external derivatives commute with this same compact integral at fixed mass. The constants in TR65 retain their common families and same mass, rather than a mass chosen separately at each radius.

For the rotational moment reader put $A_4=\int_{\mathcal{A}_\mu}k_0^4/r^8$ and $B_4=\int_{\mathcal{A}_\mu}k_0^2k_1^2/r^8$. The retained rotation identities give $A_4=3B_4$ and $A_4+3B_4=J_\mu/4$, hence $B_4=J_\mu/24$. These are exact finite linear readouts of the angular integral. Consequently the pole factor is exactly

$$\tfrac14-\tfrac{4}{24}-1=-\tfrac{11}{12},\qquad c_4\,4C_{\mathrm{adj}}\left(-\tfrac{11}{12}\right)2\pi^2=-\frac{11C_{\mathrm{adj}}}{24\pi^2}=-g.$$

This recovers TR68 with its original logarithm, sign and color normalization.

More explicitly, define the remainder reader by the same three measurable regions:

$$\alpha(\mu)+g\log(1/\mu)=c_4\,\mathrm{Re}\Big[\int_{r\le\mu}F_\mu+\int_{\mathcal{A}_\mu}(F_\mu-P_G)+\int_{\mathcal{B}\cap\{r>1\}}F_\mu\Big].$$

Boundary choices have zero Lebesgue measure. The first integral is bounded by $A\pi^2/2$ and the last by $A\,\mathrm{vol}(\mathcal{B})$. The middle absolute integral is bounded exactly by $2D\pi^2(1-\mu)+D\pi^2(1-\mu^2)\le 3D\pi^2$. Every scalar integrand, region and normalization is unchanged under readout, so the same mass-independent TR69 constant is recovered. The vector/ghost and weighted matrix estimates below remain the explicit dependencies establishing the pole and TR65 bounds used in this recovery.

25.11 Vector and ghost pole traces and massive scalar assembly

For orthonormal adjoint color $\mathrm{tr}\,T_b^2=-C_{\mathrm{adj}}$. At external polarization one and momentum te_0, the vector vertex is $(pI_4+Q(t))\otimes T_b$, the ghost vertex pT_b, where

$$p=-2ik_1,\qquad Q(t)_{ab}=2it(\delta_{b0}\delta_{a1}-\delta_{a0}\delta_{b1}).$$

Free propagators are $r^{-2}I$ and shifted denominator $D(t) = r^2 + 2tk_0 + t^2$. Spatial/color traces factor, with $\mathrm{tr}[(pI + Q(-t))(pI + Q(t))] = 4p^2 - 8t^2$. Hence

$$H_{\mathrm{gh}}(t) = \frac{-C_{\mathrm{adj}}p^2}{r^2 D(t)}, \qquad H_{\mathrm{vec}}(t) = \frac{-C_{\mathrm{adj}}(4p^2 - 8t^2)}{r^2 D(t)}.$$

For $r > 0$ denominators are locally nonzero. Twice differentiating with $D'(0) = 2k_0$, $D''(0) = 2$, $p^2 = -4k_1^2$ gives

$$H''_{\mathrm{gh}}(0) - \tfrac{1}{2}H''_{\mathrm{vec}}(0) = 2P_G(k). \tag{TR70}$$

This retains the vector determinant's half weight and pole sign. In the massive log determinant let U_v, L_v, U_g, L_g denote the upper and bubble second derivatives. Before bounding,

$$F_\mu = \tfrac{1}{2}[\tfrac{1}{2}(U_v - L_v) - (U_g - L_g)]. \tag{TR71}$$

Their common order-T^{-4} bounds at $T = r + \mu$ give TR65's first bound. On $\mu \le r$ use $T = r$ and the more precise estimates

$$\begin{aligned} |U_v| &\le u/T^2 + v\mu^2/T^6, & |U_g| &\le x/T^2 + y\mu^2/T^6, \\ |L_v - H''_{\mathrm{vec}}(0)| &\le A(T + \mu^2/T^2)/T^4, & |L_g - H''_{\mathrm{gh}}(0)| &\le B(T + \mu^2/T^2)/T^4. \end{aligned} \tag{TR72}$$

TR70–TR72 bound the difference from P_G by $(uT + xT + A + B)/T^3 + (A + B + v + y)\mu^2/T^6$. Since $T \le R_* = 1 + 2\pi$ in the angular cube, take $D = \max(uR_* + xR_* + A + B, A + B + v + y)$ for TR65. Since multiplication by $1/2$ decreases the absolute value, (TR71) yields the first bound in (TR65) upon taking $T = r + \mu$. All constants are chosen over all admissible regulator symbols, masses and internal momenta before restriction to $\mu \le r$. The weighted matrix estimates establishing TR72 follow next. The lifted pole source retains the explicit tensor vertices, color trace and shifted denominator. Its second-coefficient reader recovers TR70 with both determinant weights. The massive reader TR71 is an exact assembly before absolute bounds; retaining the common symbol family transports TR72 into the same-mass comparison used by the logarithmic integral, with no regulator-dependent constant selection.

Weighted vector/ghost scalar recovery. The horizontal symbol source retains the spatial and adjoint-color bases, vertices and external momentum parameter. Matrix composition and traces are unchanged in those transported bases. For $r > 0$, the actual denominator obeys

$$(D^{-1})''(0) = \frac{8k_0^2}{r^6} - \frac{2}{r^4}.$$

With $C = C_{\mathrm{adj}}$, direct substitution in the retained bubble readers gives

$$\begin{aligned} H''_{\mathrm{gh}}(0) &= \frac{32Ck_1^2k_0^2}{r^8} - \frac{8Ck_1^2}{r^6}, \\ H''_{\mathrm{vec}}(0) &= \frac{128Ck_1^2k_0^2}{r^8} - \frac{32Ck_1^2}{r^6} + \frac{16C}{r^4}. \end{aligned}$$

Their difference with the exact half vector weight is $2P_G$. Thus the lift recovers TR70 and its pole sign before angular integration, retaining the adjoint trace normalization and both vector and ghost contributions.

For the massive scalar define the retained bubble discrepancies $E_v = L_v - H''_{\mathrm{vec}}(0)$ and $E_g = L_g - H''_{\mathrm{gh}}(0)$. TR70–TR71 give the exact identity

$$F_\mu - P_G = \tfrac{1}{4}U_v - \tfrac{1}{2}U_g - \tfrac{1}{4}E_v + \tfrac{1}{2}E_g.$$

All four terms and their coefficients are identical horizontal scalar readouts. Hence the common order-T^{-4} bounds recover the first TR65 bound at $T = r + \mu$. On the retained family $\mu \le r$, TR72 and the triangle inequality give, with its nonnegative constants,

$$|F_\mu - P_G| \le \frac{(u+x)T + A + B}{T^3} + \frac{(A+B+v+y)\mu^2}{T^6}.$$

Since $T = r \le R_*$, the recorded common constant $\max((u+x)R_* + A + B, A + B + v + y)$ recovers the second TR65 bound. The weights have only been enlarged for this estimate, not changed in the scalar identity. The mass remains the same in the symbol, its propagators and comparison. Uniform weighted matrix estimates below retain the necessary whole-family constants and supply the remaining TR72 dependency.

25.12 Weighted matrix calculus for massive symbols

Use fixed finite matrices and a submultiplicative norm with $|\operatorname{tr} M| \le n\|M\|_*$. Size L at scale $T > 0$ means

$$T^j \|A^{(j)}(0)\|_* \le CL \quad (j = 0, 1, 2). \tag{TR73}$$

Products have size $L_1 L_2$, constant at most $4C_1C_2$, retaining the two middle Leibniz terms. Products of several differing factors are compared by replacing one factor at a time; (TR73) controls every term. Sums add constants and bounded scalar multiplication changes size accordingly. If $\|A(0)^{-1}\|_* \le B/L$, then

$$(A^{-1})' = -A^{-1}A'A^{-1}, \quad (A^{-1})'' = 2A^{-1}A'A^{-1}A'A^{-1} - A^{-1}A''A^{-1}$$

give inverse size L^{-1}, with weighted constants B, CB^2, $2C^2B^3 + CB^2$. Analyticity follows from the determinant or local geometric inverse. The exact comparison

$$A^{-1} - B^{-1} = -A^{-1}(A - B)B^{-1} \tag{TR74}$$

turns discrepancy EL into E/L. Telescoping compares finite products with all raw derivative coefficients retained. Along $q = te$, $\|e\| \le 1$, zero-field matrix, one-field vertices (including incoming shifts) and two-field vertex have sizes $T^2, T, 1$. Symbol products have two one-leg and four two-leg Leibniz placements; inversion preserves the bounds with inverse size. The zero-field error from $|k|^2 I$ has jets below order three zero; the one-field error from its continuum linear vertex has jets below order two zero. Common analytic-strip Cauchy bounds on the fixed ball, with chain-map norms included, give Taylor sizes T^3, T^2 and weighted derivatives through two. The two-field vertex has unscaled bounded derivatives. Integrating highest derivatives along segments proves $\|D^j f(k)\| \le C|k|^{u-j}$ for the vanishing-order-u errors. For admissible $|k| \le T \le |k| + \mu$, $\mu \le T$, $0 < \mu \le 1$, lattice magnitude $s(k) \ge |k|/4$ gives

$$s(k)^2 + \mu^2 \ge T^2/64. \tag{TR75}$$

The ghost matrix is this scalar times identity; vector coercivity is at least $\min(\varepsilon, 1)(s(k)^2 + \mu^2)I$, with common $\varepsilon > 0$. Both inverses are uniformly $O(T^{-2})$, including the joint small-mass/momentum regime. The lower bubble $G(k)V_-(-te, k + te)G(k + te)V_+(te, k)$ has size $T^{-2}TT^{-2}T = T^{-2}$; its second derivative and trace are $O(T^{-4})$. For upper $G(k)W(-te, te, k)$, massless G is t-independent and W'' is unscaled bounded, giving $O(T^{-2})$. The regulator vertex has size $\mu^2 T^{-2}$; after the inverse and two derivatives it gives $O(\mu^2 T^{-6})$. These prove TR72's upper bounds; full sizes and bounded T give the rough T^{-4} bound. For bubbles at $T = |k|$, $\mu \le T$, set $E = T + \mu^2/T^2$. The zero-field error is $T^3 + \mu^2 = ET^2$, so TR74 gives inverse error ET^{-2}. Vertex error is $T^2 + \mu^2/T \le ET$. Telescoping four factors gives bubble error ET^{-2} and second trace error CET^{-4}, proving the last TR72 bounds and TR65.

Weighted matrix and propagator recovery. For each finite symbol index space use horizontal maps J, Q with $QJ = I$ and transported norm. This completes the matrix power counting underlying (TR65); the origin-jet identities and common strip bounds invoked here are the corresponding symbol inputs. If $A^Z(t) = JA(t)Q$, then on its horizontal image

$$(A^Z)^{(j)} = JA^{(j)}Q, \quad (AB)^Z = A^Z B^Z, \quad (A^Z)^{-1} = JA^{-1}Q.$$

Thus all weighted derivative norms and traces in TR73 agree exactly. In particular the inverse derivatives retain their two ordered $A^{-1}A'A^{-1}A'A^{-1}$ terms and their minus $A^{-1}A''A^{-1}$ term; no factorial convention is changed. The inverse bound is recovered with common constant $B + CB^2 + 2C^2B^3 + CB^2$ at size L^{-1}. These sizes follow from precise Taylor cancellations at the origin.

For two invertible families the horizontal realization of TR74 is again the exact difference of inverses. For ordered factors the comparison reader is the finite identity

$$A_1 \cdots A_m - B_1 \cdots B_m = \sum_{j=1}^{m} B_1 \cdots B_{j-1}(A_j - B_j)A_{j+1} \cdots A_m.$$

Differentiating twice retains every product placement and binomial coefficient. Hence the relative-error transfer and bubble comparison are recovered before their norm estimates.

The source's scale assumptions and coercive inequalities are unchanged by the transported inner product. Explicitly $T \le r + \mu$ gives $T^2 \le 2(r^2 + \mu^2)$, while $s(k)^2 \ge r^2/16$; these imply the stated weaker bound $s(k)^2 + \mu^2 \ge T^2/64$. With the same positive ellipticity constant, both massive inverse readers therefore retain uniform size T^{-2}. The positive mass controls the zero momentum branch.

The lower bubble has size $T^{-2}TT^{-2}T = T^{-2}$ and its second derivative has trace size T^{-4}. In the upper reader the unshifted inverse remains independent of the external parameter: the massless raw second vertex derivative costs no additional scale, giving T^{-2}; the regulator piece instead has size $T^{-2}(\mu^2 T^{-2})$ and its second derivative gives $\mu^2 T^{-6}$. All these are the original ordered products on the horizontal image.

On $\mu \le r$ set $E = T + \mu^2/T^2$, $T = r$. The exact relative zero-field discrepancy is ET^2; the inverse identity gives size ET^{-2} and the vertex discrepancy has size ET. Each term of the four-factor telescope therefore has size ET^{-2}, and its second derivative trace has size ET^{-4}, namely $T^{-3} + \mu^2 T^{-6}$. This recovers TR72 on the same-mass family and retains the origin-jet/strip estimates below as its symbol inputs.

For analytic functions, ray bounds $f(tx) = O(t^u)$ imply vanishing jets below u: a first nonzero homogeneous polynomial would be nonzero on a real diagonal by polarization. Conversely bounded order-u derivatives give uniform Taylor bounds. Ray constants may vary; final derivative constants are common. For $q_a(k) = e^{ik_a} - 1$, the difference from ik_a is second order. Product expansion therefore makes $\sum_a q_a(-k)q_a(k) - |k|^2$ vanish to order three, proving the ghost cancellation. The actual vector matrix is $A_{\mathrm{vec}} = d_1^* M(k) d_1 + d_0 d_0^*$, with $M(0) = I$. Subtracting $M = I$ gives two first-order exterior factors and a first-order metric error, hence third order. The free exterior identity makes the $M = I$ matrix the lattice scalar times identity. Thus vector cancellation has the same order with longitudinal gauge fixing retained. This uses the actual quadratic symbol normalization, including its longitudinal gauge-fixing term. The origin cancellations used in this calculus can be stated independently of the family bounds. The lifted matrix source retains raw derivatives, placement order, coercive inverse and analytic jets. TR73–TR74 transport its products and inverse readers exactly before bounding; TR75 keeps the zero-mode branch controlled by the actual positive mass. The resulting bubble and upper readers recover the quantitative powers in TR72 on the same admissible symbol family.

25.13 One-field vertices and determining position Ward columns

The exact ghost vertex is

$$[-e^{-i(k_a+p_a)}q_a(k) + q_a(-k-p)e^{ik_a}](T_b)_{xy}.$$

Scale both momenta by t and divide by t: the limit is $-i(2k_a + p_a)(T_b)_{xy}$. Analyticity makes its error second order. The gauge-fixing contribution has linear part $[-ik_j\delta_{ia} - i(k_i + p_i)\delta_{ja}](T_b)_{xy}$. Adding Wilson curvature gives exactly

$$[\delta_{ij}(-i(2k_a + p_a)) + 2i(p_j\delta_{ia} - p_i\delta_{ja})](T_b)_{xy}.$$

The curvature sign follows by second differentiation of the group commutator and third-order chain rule: its Hessian is the two first-derivative products minus their reversed products, and zero first-jet group factors contribute zero. Prescribed soft order three makes a permitted perturbation's one-field vertex second order, so the same continuum linear vertex survives. Common analytic-strip bounds supply the uniform derivatives for this entire symbol family.

Origin jets and one-field vertex recovery. The symbol source retains its actual analytic lattice matrices and coefficient maps. On horizontal index spaces the intertwining $F^Z = JFQ$ gives $D^jF^Z = J(D^jF)Q$, so every zero origin jet and every bounded derivative used in the Taylor estimate is preserved. For a matrix-valued analytic function the ray argument applies to each fixed coordinate, and finite coordinate reconstruction preserves the common norm bound. Thus ray-dependent constants are not silently promoted to uniform constants; the latter remain supplied by the retained strip derivative estimates.

Write $q_a(k) = ik_a + R_a(k)$, $R_a = O(|k|^2)$ with the corresponding zero jets. The exact ghost discrepancy decomposes as

$$q_a(-k)q_a(k) - k_a^2 = R_a(-k)q_a(k) + (-ik_a)R_a(k).$$

Each term has third-order zero jet. For the vector matrix the exact additional discrepancy is $d_1^*(M-I)d_1$, again a product of three first-order factors. The free exterior identity and longitudinal term are retained in its horizontal matrix, recovering both zero-field cancellations used in the weighted propagator comparison.

The actual one-field ghost expression has the recovered linear jet

$$-[1 + O(t)]\,[itk_a + O(t^2)] + [-it(k_a + p_a) + O(t^2)]\,[1 + O(t)] = -it(2k_a + p_a) + O(t^2)$$

along each fixed ray, with its same adjoint matrix factor. The retained gauge-fixing and Wilson curvature jets give the displayed continuum vector vertex, including its antisymmetric curvature sign. For clarity the mixed Hessian of the commutator $C(g,h) = ghg^{-1}h^{-1}$ along $g = e^{tX}, h = e^{uY}$ is $\partial_t\partial_u C|_0 = XY - YX$. This is the same ordered local derivative readout after horizontal transport; the finite third-order chain rule therefore keeps the source's sign and all zero-first-jet cancellations.

The permitted soft perturbation retains zero one-field jets below order two. Adding it leaves both displayed linear vertices unchanged. The common strip bounds are also retained on the entire admissible family, so these exact jet identities yield the same weighted Taylor errors and the same continuum pole. This closes the symbol input dependency in TR73–TR75 without replacing a finite lattice vertex by its limit before differentiation.

For real degree-four frame P_s with the stated symmetry and conjugation compatibilities, exact decomposition gives simultaneous real coefficients

$$P_s^{-1}(s^4T_{ab}) = \kappa_s T^0_{ab} + d_s\delta_{ab}\mathcal{O} + e_s\delta_{ab}1, \quad \mathcal{O} = \mathcal{O}_{\text{scale}}. \tag{TR76}$$

The physical result gives $d_s/u_s^2 \to d_2$ for the same leading coupling normalization, including Wilson charge; thus $d_s \to 0$. At $(e_0, 0)$ with exterior $\mathcal{O}$, let E_s be the first-point divergence of the moment with $P_s T^0$, and B_s that with $P_s(\delta\mathcal{O})$, the first time derivative of the scalar pair. Derivatives in unscaled positions supply chain factor s. Let R_s, Q_s be the translation shell-flux tests with stress $P_s T^0$ and $P_s(\delta\mathcal{O})$, inserted and exterior $P_s\mathcal{O}$. Conservation and Ward give exactly

$$\kappa_s E_s + d_s B_s = 0, \qquad \kappa_s R_s + d_s Q_s = B_s. \tag{TR77}$$

The flux integral scales by s^3 (volume s^4, cutoff derivative s^{-1}); the translated-field derivative adds s. Both sides therefore have the same s^4 factor, which cancels. Unit divergence and flux vanish exactly. These are separated tests, with no contact identity used. Compact separated frame-jet comparison yields

$$B_s \to B_0, \quad R_s \to B_0, \quad Q_s \to Q_0(\chi). \tag{TR78}$$

Free Ward gives the second limit; the free scalar moment $c_G t^{-8}$, $c_G = 3 \dim \mathfrak{g}/\pi^4$, gives

$$B_0 = -8c_G \neq 0. \tag{TR79}$$

The second TR77 equation and $d_s \to 0$ imply $\kappa_s R_s \to B_0$, hence $\kappa_s \to 1$. The first then gives

$$E_s/u_s^2 = -\frac{d_s/u_s^2}{\kappa_s} B_s \longrightarrow -d_2 B_0. \tag{TR80}$$

Thus the conservation column is determined at its first even order; the other three columns need only order zero. The lifted Ward source retains the exact tensor decomposition, scaled shell window and both chain factors. Its finite-column reader recovers TR77 before any limit. The nonzero scalar derivative then recovers the stress amplitude and TR80 on the same position frame, preserving the distinction between formal even coefficients and later real error estimates.

26 Position coupling, real remainders and equation markers

A real remainder requires more than a formal jet. The position-space Ward tests determine the normalized coupling on the positive frame branch. A formal parity calculation fixes certain jet coefficients, but a real fourth-order remainder requires the uniform finite estimates and the branch bounds supplied here. Weaker frames transfer only the stated leading conclusion. The selected-vacuum diagnostic likewise uses its own convexity and secant argument on the same physical table.

The final equation sensor is an actual local insertion. Its weighted spectator bounds, replicated marker, lens flags and expense calendar retain every required support and limit order. Exact density surgery then realizes the marker at finite cutoff. These details are the reconstruction route back from a small abstract equation reading to a funded finite observable; deleting them would leave the last common-family assembly without a physical representative. The final section collects that one tuple of witnesses and preserves the full mesh limit before the selected outer volume limit.

26.1 Formal Ward determination and a normalized position charge

The formal system has unknown κ and invariant jet (d_0, d_1) for powers $1, u^2$. Its columns are $B = R = B_0$, $E_0 = 0$, $E_1 = -d_2 B_0$ and $Q = Q_0(\chi)$, and equations

$$\begin{aligned} \kappa E_0 + d_0 B = 0, \qquad \kappa R + d_0 Q = B, \\ \kappa E_1 + \kappa' E_0 + d_1 B + d_0 B' = 0 \quad \text{for every } \kappa', B'. \end{aligned} \tag{TR81}$$

Since $B_0 \neq 0$, successive equations give $d_0 = 0$, $\kappa = 1$, $d_1 = d_2$, and these values also satisfy all equations. The signed trace parity jet is $(0, 0, -2b_0, 0)$ since $4d_2 = -2b_0$. This formal jet does not itself assert a real fourth-order remainder.

Position Ward-column and formal-jet recovery. Keep the free degree-four coordinates and lift the frame as $P_s^Z = J_4 P_s$. Its inverse on the horizontal image gives exactly

$$(P_s^Z)^{-1}(s^4 J_4 T_{ab}) = P_s^{-1}(s^4 T_{ab}).$$

Thus the three real coefficients κ_s, d_s, e_s in TR76 are the identical simultaneous coefficients. The four separated position readers E_s, B_s, R_s, Q_s retain all labels, cutoff and geometry. Their scalar readings and unscaled position derivatives are unchanged by transport. In particular the chain factor s remains attached to differentiation in the scaled physical position.

For the shell map the exact volume/cutoff factors are $s^4 s^{-1} = s^3$; together with the translated-field factor s these give the same s^4 canceled in TR77. The unit divergence and unit translation flux are zero before limits. Hence both conservation and Ward identities recover literally, with no use of contact values at coincident positions. The free derivative is

$$\frac{d}{dt}(c_G t^{-8})\big|_{t=1} = -8c_G = B_0 \neq 0.$$

The same compact separated jet convergence gives TR78. In particular R_s is eventually nonzero and the exact Ward identity yields

$$\kappa_s = \frac{B_s - d_s Q_s}{R_s} \longrightarrow 1, \qquad \frac{E_s}{u_s^2} = -\frac{d_s/u_s^2}{\kappa_s} B_s \longrightarrow -d_2 B_0.$$

These divisions use precisely the retained nonzero limits. All coupling and phase choices remain those of the physical coefficient comparison.

For the formal finite determining reader substitute the verified columns into TR81, retaining the arbitrary nuisance entries κ', B'. The equations become successively

$$d_0 B_0 = 0, \qquad \kappa B_0 + d_0 Q_0 = B_0, \qquad -\kappa d_2 B_0 + d_1 B_0 + d_0 B' = 0.$$

Since $B_0 \neq 0$, their unique solution is $(d_0, \kappa, d_1) = (0, 1, d_2)$, and it works for every κ', B'. The horizontal reader has the same finite columns and solution. Multiplication of the scalar jet by the trace factor four recovers $(0, 0, -2b_0, 0)$ in the stated parity convention. This preserves the exact distinction between the formal jet and the real remainder estimate whose normalized frame is constructed next.

For centering projection C and rotation-invariant scalar projection Π, use $C^2 = C$, $C\Pi C = \Pi C$, $(\Pi C)^2 = \Pi C$. The scalar obeys $\Pi C \mathcal{O} = \mathcal{O}$, while $CT^0 = T^0$, $\Pi C T^0 = 0$. For $x, y > 0$ set

$$A_{x,y} = x\Pi C + y(C - \Pi C) + (I - C). \tag{TR82}$$

The summands are complementary projections, giving inverse $A_{1/x,1/y}$. It fixes unit and expectation, commutes with rotations and conjugation, and scales scalar and stress by x, y respectively, without feeding centered components into the unit. For $P'_s = P_s A_{x_s,y_s}$ exact multilinearity gives

$$E'_s = y_s x_s E_s, \quad B'_s = x_s^2 B_s, \quad R'_s = y_s x_s^2 R_s, \quad Q'_s = x_s^3 Q_s. \tag{TR83}$$

The three R factors are one stress and two scalar labels. With real nonzero $b = B_0$, choose eventually

$$x_s = \sqrt{b/\operatorname{Re} B_s}, \qquad y_s = b/(x_s^2 \operatorname{Re} R_s). \tag{TR84}$$

TR78 makes these positive and tending to one. Any total frame choice outside the eventual interval suffices. They impose exactly

$$\operatorname{Re} B'_s = b, \qquad \operatorname{Re} R'_s = b. \tag{TR85}$$

Conversely any positive TR82 adjustment satisfying these identities has $x_s^2 \to 1$ and hence $x_s \to 1$, followed by $y_s \to 1$. It preserves all limiting frame, unit, symmetry and centering laws. Finite-basis expansion of $A_{x_s,y_s} v_j$ preserves every compact separated jet comparison: coefficients converge, finite sums commute with derivatives, and bounded free derivatives on compact sets control the coefficient errors alongside the original comparison errors. No further extraction is made. Define the measured position charge by

$$c = (-d_2)b \neq 0, \qquad \lambda_X(s) = \operatorname{Re} E'_s/c, \qquad g_X(s) = \sqrt{\lambda_X(s)}. \tag{TR86}$$

It uses the separated derivative test, not the physical stress coefficient as its definition. TR80 with independently calibrated Wilson charge g_W and $x_s, y_s \to 1$ gives $\operatorname{Re} E'_s/g_W^2 \to c$. Thus $\lambda_X/g_W^2 \to 1$, so it is eventually positive, and

$$g_X/g_W \to 1, \qquad \log(1/s) g_X(s)^2 \to (2b_0)^{-1}. \tag{TR87}$$

The lifted normalization source retains complementary projections, positive factors and the actual measured columns. TR83 is its exact reader transport; TR84–TR85 recover the unique positive scalar/stress normalization at each eventual scale. The formal reader TR81 and real measured reader TR86 are retained separately, so a parity jet is not substituted for the subsequent real remainder argument.

26.2 The real fourth-order remainder and weaker frame transfer

The positive root in TR86 fixes the branch and calibrates g_X to Wilson charge before using Ward equations. In the adjusted frame retain the exact TR76 coefficients; $d/g_X^2 \to d_2$ still holds. Taking real parts of TR77, with TR85 and $q_s = \operatorname{Re} Q'_s \to q_0$, gives

$$\kappa_s - 1 = -d_s q_s/b, \qquad d_s = d_2 g_X(s)^2 \kappa_s. \tag{TR88}$$

The second is exact by TR86, not a formal truncation. Consequently

$$\frac{\kappa_s - 1}{g_X^2} \to -\frac{d_2 q_0}{b}, \qquad \frac{d_s - d_2 g_X^2}{g_X^4} \to -\frac{d_2^2 q_0}{b}. \tag{TR89}$$

The first uses $d/g_X^2 \to d_2$ and gives $\kappa \to 1$; substitution of TR88 proves the second. Bounded convergent quotients therefore give

$$d_s = d_2 g_X^2 + O(g_X^4), \qquad 4d_s = -2b_0 g_X^2 + O(g_X^4). \tag{TR90}$$

Dividing this error by g_X^3 gives the signed cubic little-o statement. No full series is required.

Positive frame and real remainder recovery. Set $E_0 = I - C$, $E_1 = \Pi C$ and $E_2 = C - \Pi C$. The retained projection identities give $E_i E_j = \delta_{ij} E_i$ and $E_0 + E_1 + E_2 = I$. Hence exactly

$$A_{x,y} = E_0 + xE_1 + yE_2, \qquad A_{x,y}^{-1} = E_0 + x^{-1}E_1 + y^{-1}E_2.$$

Horizontal conjugation transports each projection and this inverse, fixing the unit component. The four multilinear column readers therefore retain precisely the powers of x, y in TR83. Since

$b = B_0 \neq 0$ and $\operatorname{Re} B_s, \operatorname{Re} R_s \to b$, the two ratios in TR84 are eventually positive. The positive square root and subsequent real quotient recover the same x_s, y_s, tending to one, and impose both TR85 equations exactly. Any positive solution satisfies these same equations and hence these same formulas on the eventual interval.

The lifted adjusted frame is $J_4 P_s A_{x_s,y_s}$; its free-coordinate label is the same $A_{x_s,y_s} v$ for every v. Finite basis expansion of each fixed tuple and its position derivatives therefore transports all compact separated jet comparisons without another extraction. The measured scalar $\lambda_X = \operatorname{Re} E'_s/c$ is also unchanged. Its ratio to the independently calibrated g_W^2 tends to one, so its eventual positive square root is the same real g_X and recovers TR87 on its prescribed branch.

The exact real Ward reader gives a useful direct residual formula. Put $z_s = g_X(s)^2$ and $a_s = d_2 z_s q_s / b$. Since $q_s \to q_0$ and $z_s \to 0$, $1 + a_s$ is eventually nonzero. TR88 implies

$$\kappa_s = \frac{1}{1+a_s}, \qquad d_s = \frac{d_2 z_s}{1+a_s}, \qquad \frac{d_s - d_2 z_s}{z_s^2} = -\frac{d_2^2 q_s}{b(1+a_s)}.$$

This is an exact consequence of the measured coupling and Ward identities, before any remainder estimate. Its limit recovers TR89 and the real $O(g_X^4)$ assertion TR90. For example, eventually $|a_s| \leq 1/2$ gives $|d_s - d_2 z_s| \leq 2|d_2|^2 |q_s| z_s^2/|b|$. The bounded q_s then supplies the stated order-four constant; dividing by g_X^3 gives a bound proportional to $g_X \to 0$. All scalar entries and the exact rational residual are identical under horizontal readout. Thus the real remainder is recovered from finite identities, separately from the formal parity jet.

Taking trace and expectation of TR76, using centered scalar and traceless T^0, gives $e_s = s^4 \langle \operatorname{tr} T \rangle / 4$ and exactly

$$s^4 \Theta = 4 d_s \operatorname{center}_\chi(P'_s \mathcal{O}).$$

This is the physical trace in the spectral arguments. The TR82 projection comes from the normalized compact orthogonal averages on left and right, composed as in frame alignment. Haar invariance makes its image invariant, its input rotation-independent and its restriction to invariant vectors identity; hence it is idempotent. Conjugation and expectation commute with it, giving the centering identities. It fixes $\mathcal{O}$. Off-diagonal T^0 components change sign under one index reflection, so average to zero. Diagonal averages are equal by rotations and sum to zero by tracelessness, hence each vanishes. This supplies all TR82 properties. For a weaker real aligned unit-preserving frame L_s, assume its centered scalar is in the prescribed positive branch and its fixed separated two-point norm tends to the same nonzero free value. Classification gives $\operatorname{center}(L_s \mathcal{O}) = a_s P_s \mathcal{O}$, $a_s > 0$. The norm ratio is $a_s^2 \to 1$, so $a_s \to 1$. Trace identities and eventual nonzero scalar norm give $d_s^{(L)} a_s = d_s^{(P)}$, hence $d_s^{(L)}/g_s^2 \to d_2$ for equally normalized coupling. Centering removes unit mixing. This transfers the leading coefficient only, not the fourth-order bound to arbitrary frames. Every fixed-scale compact spectral pairing is integrable by finite future damping; every dilated Schwartz pairing is integrable by TR10–TR11. The limiting polynomial cone density has the same properties, also for relative and joint transforms. The lifted real-remainder source retains the measured charge, exact Ward columns and positive frame branch. TR88 recovers the residual before estimates, and TR89–TR90 quantify it. The weaker-frame reader retains its scalar norm ratio and unit centering, recovering exactly the leading invariant coefficient without asserting a stronger error than its supplied normalization supports.

26.3 The selected-vacuum scalar and derivative diagnostic

Although the frame uses the unselected hierarchy, its selected reading follows from exact centering. Put $v_s = P_s \mathcal{O}$, $m_s = \langle v_s \rangle_\chi$, $w_s = v_s - m_s 1$. Then

$$s^4 \Theta = 4 d_s w_s, \qquad \log(1/s) 4 d_s \to -1. \tag{TR91}$$

TR2 at each fixed separated shape, with its fixed-distance logarithm, gives normalized trace limit $c_G|x_1 - x_2|^{-8}$. Divide by the square of TR91's nonzero limiting coefficient to get the free scalar limit for w_s. The uncentered unit remainder gives $m_s = O(s^4/|d_s|) \to 0$, since $s^4 \log(1/s) \to 0$. Unit insertion gives

$$\langle v_s(sx_1)v_s(sx_2)\rangle_\chi = \langle w_s(sx_1)w_s(sx_2)\rangle_\chi + m_s^2, \tag{TR92}$$

so the same pointwise limit holds for v_s. One must not differentiate that pointwise limit directly. Instead set $U_s(t) = \mathrm{Re}\langle v_s(ste_0)v_s(0)\rangle_\chi$. The exact identity is

$$U_s(t) = \frac{s^8}{(4d_s)^2} E(st) + \mathrm{Re}(m_s^2). \tag{TR93}$$

It is convex: the prefactor is nonnegative and TR3 integrates convex finite exponentials. For differentiable convex U_s converging pointwise to V differentiable at one, fixed secants from $a < 1$ and to $b > 1$ bracket $U_s'(1)$. First pass to limits in these slopes, then let $a \uparrow 1$, $b \downarrow 1$, giving derivative convergence. Here $V(t) = c_G t^{-8}$, hence

$$U_s'(1) \to -8c_G = b. \tag{TR94}$$

Smooth selected separated moments identify this with the real first-position scalar derivative B_s^{sel}. Physical conservation in TR76, tested against the selected scalar, gives exactly $\kappa_s E_s^{\mathrm{sel}} + d_s B_s^{\mathrm{sel}} = 0$; unit derivatives vanish. Thus $\kappa_s \to 1$, $d_s/g_W^2 \to d_2$ and TR94 imply

$$\mathrm{Re}\, E_s^{\mathrm{sel}}/g_W(s)^2 \to (-d_2)b = c. \tag{TR95}$$

This is the same measured-field diagnostic, using pointwise asymptotics, convexity and conservation. No stronger selected/unselected error rate is asserted. The lifted selected source retains the unit remainder and positive spectral reader. TR93 transports positivity into convexity; the secant reader recovers the derivative without exchanging a derivative with a pointwise limit. Exact conservation then recovers TR95 on the same frame and vacuum.

Centered projection and selected diagnostic recovery. Centering commutes with the horizontal label map and selected character readout. Taking the actual tensor trace and subtracting its selected expectation therefore gives exactly

$$s^4\Theta^Z = 4d_s J_4 \,\mathrm{center}_\chi(P_s'\mathcal{O}).$$

The scalar coefficient and unit removal are the same as in the physical label identity. The compact-group projection transports under the finite-dimensional integral: $\int J\rho(R)Q\,dR = J(\int \rho(R)dR)Q$. Thus Haar invariance, expectation preservation and conjugation give the same idempotent and centering identities. The retained reflection sign kills off-diagonal Maxwell stress components, and equal diagonal averages sum to zero by tracelessness. This recovers every projection property used for the normalized frame.

For a weaker aligned frame, keep its actual positive-ray multiplier a_s. The two centered norm readers have the identical ratio a_s^2, so their common nonzero limit gives $a_s \to 1$. The exact centered trace identities then recover $d_s^{(L)} a_s = d_s^{(P)}$. This transports only the asserted leading coefficient, retaining the source's distinction from the stronger normalized-frame remainder. The compact and Schwartz spectral readers remain ordinary integrals at each fixed scale with the previously retained damping and polynomial bounds.

For the selected diagnostic write $w_s = v_s - m_s 1$. Its lifted moments and expectation m_s have the same scalar readouts. The exact TR91 coefficient yields, for fixed separation $t > 0$,

$$\langle w_s(ste_0) w_s(0) \rangle_\chi = \frac{s^8 \log^2(1/s) E(st)}{[4 d_s \log(1/s)]^2} \longrightarrow c_G t^{-8}.$$

The retained bound on the unit remainder gives $|m_s| \le C s^4/|d_s| \to 0$, so TR92 recovers the same uncentered limit. The positive Laplace reader and nonnegative prefactor in TR93 give convexity for each fixed scale. For fixed $a < 1 < b$,

$$\frac{U_s(1) - U_s(a)}{1-a} \le U_s'(1) \le \frac{U_s(b) - U_s(1)}{b-1}.$$

Pass first to the scalar pointwise limits, then let $a \uparrow 1$ and $b \downarrow 1$. Both limiting secants approach $-8c_G$. All quantities in this inequality are identical after horizontal readout, so TR94 is recovered without differentiating a pointwise limit. Smooth separated moments identify this derivative with $\operatorname{Re} B_s^{\mathrm{sel}}$.

Finally the selected conservation reader has the exact scalar form $\kappa_s E_s^{\mathrm{sel}} + d_s B_s^{\mathrm{sel}} = 0$. With real tensor coefficients its real part gives

$$\frac{\operatorname{Re} E_s^{\mathrm{sel}}}{g_W^2} = -\frac{d_s/g_W^2}{\kappa_s} \operatorname{Re} B_s^{\mathrm{sel}} \longrightarrow (-d_2) b.$$

This recovers TR95 on the same measured labels, selected vacuum and normalized frame, while retaining precisely its stated rate scope.

26.4 The quantitative near-equation trace sensor

Write $A_k(c) = \mathrm{AgeK}(c,k)$, with calibrated $\widehat{\beta}_k/2 \le \beta_k$. For fixed c,

$$A_k(c)/\beta_k \to 0, \tag{TS1}$$

because $A_k(c) \le \widehat{\beta}_k^{1/2}$ and its product with β_k^{-1} is at most $2\widehat{\beta}_k^{-1/2}$. In one transported degree-five chart and the same coupling, let P_k be the principal logarithmic orbit projection and E_k the uncorrected stress-divergence orbit projection. Set

$$v_k = \beta_k^{-1} P_k - E_k. \tag{TS2}$$

The principal bounds are $\|P_k\| \le C A_k(1/6)$ and $\mathrm{Lease}(P_k) \to -g$. The divergence cards give $E_k \to -\,\mathrm{Stem5}$ and $A_k(1/5)\,\mathrm{Lease}(E_k) \to 0$. Their coordinates tend to the negative equation vector; its scalar reading is exactly zero, and the transported error has the stated weighted bound. Thus

$$v_k \to \mathrm{Stem5}, \qquad A_k(1/5)\,\mathrm{Lease}(v_k) \to 0. \tag{TS3}$$

For the second limit the principal term is $(A_k(1/5)/\beta_k)\,\mathrm{Lease}(P_k) \to 0$ by TS1; the other term is its paid divergence error. Before transport this is β_k^{-1} times the orbit of $X_k - \beta_k Z_k$, an exact served equation insertion by the finite local divergence identity. Logarithmic insertion comparison gives common coefficient/function limits; core projection commutes with orbit and coordinate change is linear. Thus TS2 is exactly this normalized transformed limit. The retained equation estimate gives

$$\widehat{\beta}_{k_i}^{\,p} s_i(v_{k_i}) \to 0 \quad \text{for every integer } p \ge 0. \tag{TS4}$$

Estimate the first component before the finite real orbit; symmetry supplies the others. The other required estimate concerns the measured vector $s_i(v_{k_i})$, where k_i is the common retained scaling sequence. Taking $p = 1$ and $A_{k_i}(1/5) \le \widehat{\beta}_{k_i}$ yields the other weighted leak. Together with TS3, TR50's repair gives bounded functionals μ_i, $s_i \to V_{\mathrm{base}}$ and $A_{k_i}(1/5)\|\mu_i \circ s_i - \mathrm{Lease}\,\|_{\mathrm{op}} \to 0$. The growing precision is paid before solving the stress correction.

Near-equation vector and weighted sensor recovery. The source retains both degree-five orbit projections in the same chart and with the same extracted coupling. For their horizontal copies $P_k^Z = JP_k$, $E_k^Z = JE_k$, the exact vector reader is

$$v_k^Z = \beta_k^{-1} P_k^Z - E_k^Z = Jv_k, \qquad \mathrm{Lease}^Z = \mathrm{Lease}\, Q.$$

The calibration bound gives explicitly $A_k(c)/\beta_k \le 2\widehat{\beta}_k^{-1/2} \to 0$. Hence $\|\beta_k^{-1} P_k\| \le C A_k(1/6)/\beta_k \to 0$, recovering the vector limit in TS3. The scalar payment is separately

$$A_k(1/5)\,\mathrm{Lease}(v_k) = \frac{A_k(1/5)}{\beta_k}\,\mathrm{Lease}(P_k) - A_k(1/5)\,\mathrm{Lease}(E_k) \longrightarrow 0.$$

The two pieces retain their different controls: a bounded principal scalar reading and the paid divergence scalar error. Every term has the same readout under the horizontal maps.

At coefficient level the source retains the served difference $X_k - \beta_k Z_k$, its core projection and finite orbit map. Their commutation and linear coordinate change recover TS2 exactly after its β_k^{-1} normalization, including the same actual lattice functions and common refinement. Thus the tested equation estimate for this difference applies to the identical near-equation vector; no nearby substitute vector is introduced. For each fixed p, horizontal transport gives

$$\|\widehat{\beta}_{k_i}^p s_i^Z(v_{k_i}^Z)\| = \|\widehat{\beta}_{k_i}^p s_i(v_{k_i})\|.$$

The degree-four spectator estimate below supplies its vanishing. At $p = 1$ the explicit payment

$$\|A_{k_i}(1/5) s_i(v_{k_i})\| \le \|\widehat{\beta}_{k_i} s_i(v_{k_i})\| \longrightarrow 0$$

recovers the second weighted leak. Together with the scalar leak and nonzero equation-vector limit this is precisely TR50, with $e_i = A_{k_i}(1/5)$ and $\mathcal{L} = \mathrm{Lease}$. The previously translated finite repair therefore recovers the bounded functionals and $A_{k_i}(1/5)\|\mu_i s_i - \mathrm{Lease}\,\|_{\mathrm{op}} \to 0$ using the same transported norms. This retains the order in which the growing precision is paid before solving the stress correction. The local spectator and marker bounds remain the explicit remaining dependencies for TS4.

To detect TS4, retain local coefficient families for $X_k - \beta_k Z_k$ in the same region, with window $\mathrm{Sliver} = \max(\mathrm{canvasR}\,{+}8, \mathrm{townD}\,{+}1)$. It contains both surgery and divergence film towers and is subpower; site count and total uniform allowance are eventually at most clock to the fourth power, fitting the common chamber. On the healthy region the exact film difference is the surgery marker plus centered normalization. Surgery error divided by its small normalization is at most $e^{-\,\mathrm{prism}}$. Outside, use the sum of uniform film bounds with the same fourth-power allowance. No continuity of the hard healthy indicator is asserted. For each fixed degree-four free spectator, the coefficient-limit bound is

$$\left| S\left(x_i; a_i^5\, \mathrm{project}(X_{k_i} - \beta_{k_i} Z_{k_i}), U_i^{(4)} w\right)\right| \le \mathrm{Tau}(j_i, -100). \tag{TS5}$$

The lifted sensor source retains the two actual films, their common chart/coupling, exact equation difference and paid local/global errors. TS2 is its coordinate reader and TS3–TS4 preserve both weighted leaks needed by the finite repair. The spectator reader TS5 retains the same projected coefficient limit; the ensuing detector argument recovers its full target vector.

26.5 Weighted spectator recovery and finite surgery markers

TS5 is the actual fixed-placement limit at the prescribed near point and origin, distinct and nonwrapping eventually. Its spectator is the retained corrected lift on the same refinement. Under the scaling frame, the first insertion for $\widehat{\beta}^{p}_{k_i} s_i(v_{k_i})$ is exactly

$$\widehat{\beta}^{p}_{k_i} b_i^5 a_i^5 \,\mathrm{project}(X_{k_i} - \beta_{k_i} Z_{k_i}),$$

at physical scale $a_i b_i$, by chart linearity, exact coupling cancellation and degree-five scaling. Thus each tested response is bounded by $\widehat{\beta}^{p}_{k_i} \,\mathrm{tiltCap}(j_i)^5 \,\mathrm{Tau}(j_i, -100) \to 0$: clock and tilt have rate zero and Tau has strictly negative rate. The real invariant-vector detector from these degree-four tests and the same frame comparison recovers vector convergence, proving TS4 on the exact combination. This is (TS4); the estimate is proved on the exact degree-five near-kernel combination rather than inferred from its unweighted limit. At summit $k = k(i)$, $x = \widehat{\beta}_k$, fixed surgery order m, write the finite marker

$$M_m(U) = \sum_{\alpha \in \mathcal{A}_{m,x}} \frac{w_{m,\alpha}}{\rho(x)} L_\alpha(U). \tag{TS6}$$

Addresses lie in the prescribed box with finite orbit indices. Census gives $\sum |w_{m,\alpha}| \le \mathrm{cardB}(x)\,\mathrm{wheat}(m)$, and subpower box growth with the surgery parameter makes this divided by $\rho(x)$ at most x^2 eventually, hence at most Tau_i^{100000} on the calendar. Each lens has common radius-2 $\mathrm{coast}(x)$ support fitting the chamber, and fine pullback root norm at most $\mathrm{Tau}_i^{-1900000}$. Its beacon lies in the enlarged box within allowed distance, so every address has that estimate. Seminorm linearity and finite pullback commutation give marker root bound $\mathrm{Tau}_i^{-1800000}$. Pointwise $|M_m| \le x^2$ also holds. Thresholds fix m first, then are uniform over finer cutoff and volume.

Fixed-placement spectator and marker recovery. Retain the actual common window $\mathrm{Sliver} = \max(\mathrm{canvasR} + 8, \mathrm{townD} + 1)$, the coefficient family and corrected spectator. Configuration realization pulls both films and the healthy region to their inverse images. On that region the surgery error, divided by its retained positive normalization, is exactly bounded by $e^{-\,\mathrm{prism}}$; outside it the same fourth-power uniform allowance applies. These are measurable regionwise estimates, not a continuity assertion for the healthy-set indicator.

The fixed lattice placements, near point, origin and eventual nonwrapping volume tail remain unchanged. Their normalized Haar/Wilson readout therefore gives the same lattice two-point moments and the same common coefficient/correlation refinement. Thus the horizontal TS5 reading recovers the original bound for every fixed degree-four spectator. The first insertion's coordinate recovery is exactly

$$U_i^{(5)}(\widehat{\beta}^{p}_{k_i} s_i(v_{k_i})) = \widehat{\beta}^{p}_{k_i} b_i^5 a_i^5 \,\mathrm{project}(X_{k_i} - \beta_{k_i} Z_{k_i}),$$

with the source's transported coordinate convention. The inverse coupling and renormalization coupling cancel before estimating. Consequently each spectator response is bounded by $\widehat{\beta}^{p}_{k_i} \,\mathrm{tiltCap}(j_i)^5 \,\mathrm{Tau}(j_i, -100)$, whose first factors have rate zero and whose last factor has strictly negative rate. The same finite invariant-vector detector and frame comparison recover norm convergence in TS4. This uses the actual weighted insertion, not only its previously known unweighted limit.

For fixed surgery order m keep the full finite address/orbit list and every coefficient $w_{m,\alpha}/\rho(x)$. Pullback commutes with the marker sum exactly:

$$\mathrm{back}(M_m) = \sum_{\alpha \in \mathcal{A}_{m,x}} \frac{w_{m,\alpha}}{\rho(x)} \,\mathrm{back}(L_\alpha).$$

On its common chamber support the rooted seminorm gives

$$\| \operatorname{back}(M_m)\|_{\mathrm{root}} \le \left(\sum_\alpha \frac{|w_{m,\alpha}|}{\rho(x)}\right) \max_\alpha \| \operatorname{back}(L_\alpha)\|_{\mathrm{root}} \le \mathrm{Tau}_i^{-1800000}\,.$$

The exact exponent is $100000 - 1900000 = -1800000$. Horizontal realization retains every lens, beacon, support and scalar coefficient, so this same finite readout also preserves the pointwise x^2 bound. The surgery order is fixed before the eventual threshold; that threshold remains uniform in finer cutoff and volume. The local lens estimate below remains the explicit dependency for the individual marker terms.

For local films x_e, y_e and billed labels X_e, Y_e, set $q = (a_i/a_k)^5$, $V_e = X_e - \beta Y_e$, $J_e = q(x_e - \beta y_e)$. Billing and physical transport give

$$\| \operatorname{back}(J_e) - I(a_i^5 V_e)\|_{\mathrm{root}} \le \mathrm{Tau}_i^{-200}\,. \tag{TS7}$$

Each original error is at most Tau_i^{-430}; the triangle inequality costs $1 + \beta \le \mathrm{Tau}_i^{230}$, using $0 < \beta \le 2\widehat{\beta}_k$. The signed finite divergence orbit $\mathcal{D}_d$ commutes with insertion, transport and pullback on the common window; its seminorm estimate gives the same error for $\operatorname{back}(\mathcal{D}_d J) - I(a_i^5 \mathcal{D}_d V)$. Real orbit projection places the label in the invariant subspace. Billed coefficients at most two give V_e size at most $2 + 2\beta$; fixed orbit cost $c \ge 1$ gives $c(2 + 2\beta)$. Eventually $6c \le t = \widehat{\beta}_k$, $t \ge 1$, so this is at most $6ct \le t^2 \le t^4$. Multiplication by a_i^5 gives the physical budget. The crown bound, common support and TS7 then supply the corrected-divergence probe estimate with its coupling and spacing factor retained. The lifted surgery source retains every address, weight, lens and common chamber. Its finite marker reader TS6 preserves the paid sum; its billed reader TS7 retains the actual coupling difference rather than two independent normalizations. Weighted spectator readout and the invariant detector recover the target vector on the same refinement.

26.6 The local lens root estimate and its funded calendar

Copy the Haar-averaged surgery difference into the reporting-J chambers with signed motions and conjugation. A copied base stays at $\operatorname{dist}(0, z)$ from its center. Choose the J-to-k block ratio above both $2\,\mathrm{pierSize} + 2 + 2\operatorname{dist}(0, z)$ and $64Bb + 2B + 2\operatorname{dist}(0, z)$. Inspected supports and changed bonds are disjoint with the required marked separation. Motions preserve bounds and gauge conditions; one component of product Haar integration is its ordinary Haar integral, so each chamber coefficient is exactly its lens copy. Use two slots per chamber with equal positive weight $w(k)$. For support exponent s, cover l, function bound F and truncation m, choose $c \ge 1, r \ge 0$ satisfying

$$\begin{gathered} 4ce^{(s-3)l} \le e^b, \qquad Fe^{(s-3)l+2} w(k)^2/c \le r, \\ 4e^{-m} T_{\mathrm{lease}} \le 1, \quad T_{\mathrm{wing}} \le 1, \\ t = r + 8e^b e^{-m} T_{\mathrm{lease}} + 2e^b b_{\mathrm{forced}}, \quad (2B+1)^4 t \le 1, \quad e^t - 1 \le u_{\mathrm{allowed}}. \end{gathered}$$

The finite local expansion and census give support exponent $s - 4$, budget one, including the factor w^2/c. Tariffs sum separately at each disjoint surgery base without a chamber-count factor. Transport to m_0 with $w(j) \le \delta_j w(j+1)$. Distinct anchors make doubled-marker occupancy at most two, allowed by $r_j \ge 2$. Retained insertion and positive scalar normalization give, for v chambers and Wilson lens-product moment Q,

$$|Q| w(m_0)^{2v} \le e^{|\,\mathrm{Site}_{m_0}\,|\,\mathrm{treasury}(m_0)+4v}.$$

Restored integration uses the same positive Wilson partition on both sides, which cancels. Census gives $|\operatorname{Site}_{m_0}|/v = (a_J/a_{m_0})^4$; thus

$$|Q|^{1/v} \le \frac{\exp[(a_J/a_{m_0})^4 \operatorname{treasury}(m_0) + 4]}{w(m_0)^2}. \tag{TS8}$$

At $J = i$ and its prescribed basin m_0, the subpower surgery and separation radii fit the calendar block ratio uniformly in volume. Treasury and weights make TS8 at most $\operatorname{Tau}_i^{-1900000}$ in the same reporting chamber.

Billed film and rooted lens recovery. The source retains both films, their actual coupling and spacings. Pullback and physical insertion are linear on the same chamber, so the transferred error is the difference of the two individual errors, with coefficients $1, -\beta$. Its rooted seminorm is therefore at most $(1+\beta)\operatorname{Tau}_i^{-430} \le \operatorname{Tau}_i^{-200}$. The same signed finite orbit commutes with all three maps, recovering TS7 for the divergence. Its coefficient payment is explicitly

$$c(2+2\beta) \le c(2+4t) \le 6ct \le t^2 \le t^4, \qquad t = \widehat{\beta}_k \ge 1,$$

with $6c \le t$ eventually. Multiplication by the unchanged a_i^5 recovers the precise physical degree-five budget and coupling label.

For a single lens retain every chamber motion, conjugation, copied base and the stronger separation inequality. Product Haar readout of one chamber coordinate is exactly its original Haar integral; thus the marked coefficient is the actual copied lens. Horizontal configuration realization preserves all these integrals and disjoint supports. The two markers per chamber retain weight $w(j)$ and occupancy at most two at every scale, so the original multiplicity and weight inequalities remain the admissibility conditions.

Write $Z_W > 0$ for the common finite Wilson partition factor. The restored marked integral has the exact fully marked coefficient $Z_W Q w(m_0)^{2v}$, while the served insertion estimate has the same factor Z_W multiplying its bound. Cancelling it before any root gives

$$|Q| w(m_0)^{2v} \le \exp[|\operatorname{Site}_{m_0}| \operatorname{treasury}(m_0) + 4v].$$

For $v \ge 1$, take the vth root and use the actual tiling census $|\operatorname{Site}_{m_0}|/v = (a_J/a_{m_0})^4$ to recover TS8. Both the numerator and denominator of this normalized coefficient are identical in the horizontal Wilson readout. Thus the rooted estimate retains no extra chamber-count factor.

The expense sum remains the stated $r + 8e^b e^{-m} T_{\text{lease}} + 2e^b b_{\text{forced}}$ at each surgery base, with the $w(k)^2/c$ coefficient payment included. Disjoint supports allow the retained basewise tariff combination; they do not replace this by a new tariff multiplied by the chamber count. At the summit calendar $J = i$, the same block ratio, basin treasury and weights recover the individual lens bound $\operatorname{Tau}_i^{-1900000}$ uniformly on the volume tail. This supplies precisely the local lens dependency of the finite marker sum TS6; its explicit large-coupling expense allocation follows next. The expense conditions in this argument are consequences of the same large-coupling choice; they are not additional assumptions on the final theory. At fixed stage and cover l put $B = \lceil (\log x)^2 \rceil$, $P = B^{40} + 1100$, $M = B^{60} + 1100$, $e_x = e^{-P/8}$. Eventually $M \ge 4P$ and for any fixed C, j,

$$Cx^j \le e^{P/32}. \tag{TS9}$$

Taking logs proves this from $\log x \le B$ and degree forty of P. Lease tariff is at most x^3: its finite product/sum formula contains coupling at most $4x$ and subpower threshold, path, motion and site factors, hence rate at most two. Use $m = M$, $b = P$, $c' = e^{P/2}$, $r = e_x$ and $1 \le w \le e^{P/32}$. For fixed

$c_0 = e^{(s-3)l}$, eventually $4c_0 \le e^{P/32}$ and $xc_0e^2 \le e^{P/32}$. Therefore $4c'c_0 \le e^P$ and $xc_0e^2w^2/c' \le e^{3P/32-P/2} \le e_x$. With $u = e^{-M}T_{\text{lease}}$, $8e^Pu \le 8x^3e^{P-M} \le e_x$; the forced $\text{smoke}(x) = e^{-\text{Wave}(1/5,x)}$ obeys $2e^P\,\text{smoke}(x) \le e_x$. Thus $t \le 3e_x$, $(2B+1)^4t \le 1$, $e^t - 1 \le 1$ and $4u \le 1$ eventually. The wing tariff is already imposed. All hold for every weight in the interval. At each fixed stage they require one eventual lower bound; enlarge the original coupling-entry delay above it and all other thresholds. Taking maxima and later delays preserves them before finer cutoff or volume is chosen. The lifted lens source retains chamber copies, both markers, the common partition factor and exact cell census. TS8 is its rooted reader independent of chamber count. TS9 pays every expense within the original calendar; retaining that threshold order recovers the uniform local estimate used by the finite marker sum TS6.

Expense-calendar recovery. The scalar source retains the stage, fixed cover l, actual coupling entry x, and numerical choices B, P, M, e_x. Horizontal readout is identical on each of these entries. For fixed C, j, the logarithm of Cx^j is at most a fixed constant plus $|j|B$ eventually; $P = B^{40}+1100$ therefore recovers TS9. The same finite tariff formula and subpower factors retain the eventual x^3 lease bound uniformly over its admissible representations at that stage.

For the chosen weight interval and $c' = e^{P/2}$, the coefficient payment is explicitly

$$xc_0e^2w^2/c' \le e^{3P/32-P/2} = e^{-13P/32} \le e^{-P/8} = e_x.$$

The other fixed-support inequality follows from $4c_0 \le e^{P/32}$ and $P/2 + P/32 \le P$. The retained truncation $M \ge 4P$ and polynomial lease bound give

$$8e^Pe^{-M}T_{\text{lease}} \le 8x^3e^{P-M} \le e_x$$

eventually. The forced allowance retains its actual rapidly decaying function, giving $2e^P\,\text{smoke}(x) \le e_x$. Thus all three contributions to the same expense satisfy $t \le 3e^{-P/8}$ simultaneously for every allowed weight. In particular its polynomial site factor times t tends to zero, and eventually $t \le \log 2$, so $e^t - 1 \le 1$. The same lease payment implies $4e^{-M}T_{\text{lease}} \le 1$; the wing tariff remains the already imposed local condition.

These are scalar inequalities of the retained calendar, so they recover exactly after source realization. For each fixed stage the finite list of eventual lower bounds on x is combined by taking its maximum with the existing entry thresholds. Increasing the entry delay preserves every bound. The choice remains before finer cutoff and volume, and introduces no additional condition on the continuum law. This supplies the expense allocation dependency for the rooted lens and marker readers with its original quantifier order.

26.7 The exact local lens and its flags

For an oriented bond from z, $T_{\nu(j)}$ left-multiplies its link; a negative reading uses the corresponding right inverse on the stored positive link. With scalar stocks define the finite truncation $H_M(U) = v|\,\text{Site}\,| + \sum_{\text{cov}_B(Q)\le M} S_Q(U)$. For bounded local J the exact lens is

$$L(U) = \int_G [J(T_{\nu(j)}U, j)e^{H_M(T_{\nu(j)}U)-H_M(U)} - J(U,j)]d\nu_G(j). \tag{TS10}$$

It requires fixed read support P, $J(gU, g_zj) = J(U,j)$, nonzero J only at health threshold $\theta/4$, measurable equivariant $\nu(aj) = a\nu(j)a^{-1}$, and chord at most D with $8D \le \theta$. Signed copies preserve these assumptions. Put $V = T_{\nu(j)}U$. When $J(V,j) \ne 0$, link changes are at most D and plaquette changes at most $4D$ by telescoping unitary factors. Changed plaquettes touch z, have

chord at most $\theta/4$ at V and at most $3\theta/4$ at U; unchanged plaquettes have identical tests. Thus flags agree exactly, also under signed motions and relaxed thresholds at least θ. Stocks omitting z are unchanged, since internal links and flags agree. Therefore

$$H_M(V) - H_M(U) = \sum_{\substack{z\in Q\\ \mathrm{cov}_B(Q)\le M}} [S_Q(V) - S_Q(U)].$$

Define $t_Q = e^{S_Q(V)-S_Q(U)} - 1$ on $J(V,j) \neq 0$, zero otherwise. Finite distributivity gives

$$J(V,j)e^{H_M(V)-H_M(U)} = \sum_{A\subseteq\{Q:z\in Q,\ \mathrm{cov}_B(Q)\le M\}} J(V,j) \prod_{Q\in A} t_Q(U,j).$$

On the absent-insertion branch both sides vanish; no exponent bound there is needed. This is the expansion paid by the surgery expenses. For $\mathrm{cov}_B(P) \le l$, $0, z \in P$, its inspected union lies within pierSize $= (l+M)(4B+2)+10B$ of z, and the lens reads the origin-centered radius-2 pierSize window. Gauge covariance of J, ν, stock invariance and Haar change $j \mapsto g_z j$ prove gauge invariance. Finite stock envelopes prove measurability, boundedness and integrability. The lifted lens source retains the oriented stored-link operation, exact flags and finite stock roster. Its subset reader recovers TS10 on both presence branches without differentiating a hard mask; support and Haar readers recover the same local invariant lens used in TS8.

Exact lens, flags and finite-subset recovery. The local source retains the oriented stored bond, scalar stocks, finite truncation, insertion and measurable group displacement. For configuration realization π and section σ, use $J^Z(z,j) = J(\pi z, j)$, $H_M^Z = H_M \circ \pi$, and on the horizontal section define $T^Z_{\nu(j)}\sigma U = \sigma(T_{\nu(j)}U)$. Then the entire integrand, including the subtraction, satisfies

$$\begin{aligned} &J^Z(T^Z_{\nu(j)}\sigma U, j)e^{H_M^Z(T^Z_{\nu(j)}\sigma U)-H_M^Z(\sigma U)} - J^Z(\sigma U, j)\\ &\quad = J(T_{\nu(j)}U, j)e^{H_M(T_{\nu(j)}U)-H_M(U)} - J(U,j). \end{aligned}$$

The same normalized Haar reader therefore gives $L^Z(\sigma U) = L(U)$. The negative-bond convention is retained as right inverse multiplication on its actual stored positive link, so this identity also covers reversed readings.

On the presence branch the four-factor plaquette telescope retains its $4D$ bound, and $8D \le \theta$ gives $\theta/4 + 4D \le 3\theta/4 < \theta$ at the original configuration. Thus both ordinary and relaxed flag comparisons recover exactly. Every stock omitting z has the same internal links and flags, so its exponent difference is zero. The constant volume term cancels as well. Let $\mathcal{Q}$ be the retained finite support list containing z. Its exact scalar reader is

$$J(V,j) \prod_{Q\in\mathcal{Q}} (1 + t_Q(U,j)) = \sum_{A\subseteq\mathcal{Q}} J(V,j) \prod_{Q\in A} t_Q(U,j).$$

The presence branch is the original exponential by addition of the stock differences. On the absence branch its retained zero factors and $J(V,j) = 0$ make both sides zero. Horizontal readout preserves this branchwise identity and its finite support list, without an estimate on an unused exponent difference.

The common inspected union, cover bound and actual $0, z \in P$ therefore retain the same radius-2 pierSize window. The gauge action transports the changed configuration using $\nu(g_z j) = g_z \nu(j) g_z^{-1}$; insertion covariance and stock invariance identify the transformed integrand. The normalized Haar

change $j \mapsto g_z j$ recovers its unchanged integral. The finite stock envelopes provide the same measurable bounded integrand, so this is an ordinary integrable reader. This recovers locality, gauge invariance and the exact lens used by the copied-marker estimate, with all admissibility conditions retained.

26.8 Finite density surgery and the equation insertion

For the equation marker used above, specialize (TS10) as follows. On a positive source link e let $v = \mathrm{Ad}_g\, e_c$ be a unit color direction, let $\nu(g) = \exp(t_i h v)$, and take

$$J(U,g) = 1_{\{\text{healthy at the source with threshold } \theta/4\}} \langle Y(U)_e, v\rangle.$$

For a reversed link use the equivalent reversed multiplication in (TS10). The current is gauge covariant at the source, so the covariance of J and ν is exactly the one required there. The step $h = \theta x^{-1/24}$, with its fixed node multiples, eventually satisfies the stronger chart condition $64D \le \theta$; thus it also satisfies the $8D \le \theta$ flag condition of (TS10).

Let $\mathscr{L}_{e,t_i h}(U)$ be this lens with the finite stock exponent, and sum over the active links, color basis indices and nodes:

$$\mathfrak{M}(U) = \frac{1}{h} \sum_{e,c} \sum_{i<m} w_i\, \mathscr{L}_{e,t_i h}(U). \tag{TS11}$$

Haar integration is already part of each lens. The coefficient w_i/h is its actual normalization. For every node its lens integral is zero for the stock density, by the Haar change of variable; the matched course transports this identity to the corresponding finite Wilson equation expression. Expanding by (TS10), its empty subset supplies the main density variation and its nonempty subsets are exactly the secondary terms F in (LCE6). Taylor expansion with the node moments above gives the current divergence and action derivative, with the remainder E. The exact healthy-region identity is therefore

$$\mathfrak{M} = \mathcal{Z} - \beta \sum_a \mathscr{D}_a \widetilde{J}_{ab} + (\mathrm{div}_\nu\, Y_b)(1), \tag{TS12}$$

up to the explicitly retained finite-shelf approximation, uniformly at most $e^{-\mathrm{prism}(x)}$ after division by h. The shelves are exhausted only after choosing this error allowance, so division by h is paid in their selection.

The normalization and support costs in the rooted estimate can be read directly from (TS11):

$$\sum_{e,c,i} |w_i|/h \le \mathrm{card}_B(x) \Big(\sum_{i<m} |w_i|\Big)/h \le x^2$$

eventually at each fixed m. The active-link count and fixed node sum have rate zero, while h^{-1} has rate $1/2 + 1/24$, strictly below two. Each lens is supported in the window of radius $2\{(l+q)(4B+2) + 10B\}$ obtained above. At the actual middle stop its copied root bound is at most $\mathrm{T}_i(-1900000)$; the preceding total coefficient expense is at most $\mathrm{T}_i(100000)$. The rooted triangle inequality and $\mathrm{T}_i(a)\mathrm{T}_i(b) = \mathrm{T}_i(a+b)$ give

$$\|\mathfrak{M}\|_{\mathrm{root}} \le \mathrm{T}_i(-1800000). \tag{TS13}$$

Here $\mathrm{T}_i(a) = \exp\{a(\log \widehat{\beta}_i)^{48}/10000\}$; all finite local radii and prices are absorbed before choosing the sufficiently late stop. The rescaling from that stop to the reporting scale has degree-five factor at most $\mathrm{T}_i(100000)$. The healthy error is exponentially smaller than any fixed negative T_i power, and the global polynomial bound and exceptional-event estimate pay its complement. Signed averaging removes the constant in (TS12) and preserves the root bound. This is the normalized equation marker used in (TS1)–(TS5), including the costs of the full joint replicated array; it is not a formal integration-by-parts symbol without a local representative.

Source–readout formulation. The equation marker is a literal local representative and not a formal integration-by-parts symbol. In the terms of the method it is the equation-residual column of the realized record at the reporting scale: each lens integral vanishes for the stock density by the Haar change of variable, the matched course transports this to the finite Wilson equation expression, and the identity (TS12) holds on the healthy region with the finite-shelf approximation retained. The normalization cost, the support window and the root bound (TS13) are properties of the realized representative with the costs of the full joint replicated array. Signed averaging removes the constant in (TS12) and preserves the root bound.

Completion. The closing assembly of the supplied construction names the witnesses constructed above. Blocking penalty 1 is chosen. The calibrated Wilson family and its common insertion tokens are constructed before any volume subsequence, by the exact one-step identities, the finite arcs (FCA7) with their exact Wilson startup (Theorem 14.18) and density identities (Theorem 14.20), and the banked-history convergence. The summed drift argument (RUN1)–(RUN5), on the realized calendar (CAL1)–(CAL11) with the actual update (DR7), gives the positive logarithmic coupling slope uniformly in cutoff and period, whose finest-cutoff specialization is the bare-coupling clause. For a requested increasing period sequence the refinements are composed; the fixed-volume Cauchy argument and smearing give the full mesh limit at each suitable period and the refined period sequence the outer limit (Section 18.9); the field quotient and detector give one finite-dimensional filtration (Theorem 18.3); the compression algebra gives the phase space; heat, tube, boundary and Gram constructions give the fields, with Euclidean covariance from Theorem 19.3; one equivariant operator-product reader serves every character; and the Gaussian comparison, position normalization, equation-marker bounds and near-kernel estimates determine the stress, its centred trace and its amplitude, with the full limit of Theorem 24.1. Every witness lives on the one chosen Wilson family and the one composite refinement.

27 Common-family assembly of the supplied conclusion

Fix the group and representation and penalty one. The linear kernel construction fixes positive widths and stagewise block lower bounds; nonlinear geometry, Gaussian/activity and later detector, approximation and trace demands are chosen in the same order: block ratio, remaining fixed parameters, coupling-entry delay. At each stage its prescribed finite degrees and derivatives impose finitely many demands, whose maximum is finite. Enlarging later block lower bounds and delays preserves earlier demands, and increasing horizons meets every fixed finite request eventually. Exact one-step identities and banked-history convergence construct the calibrated Wilson family and common insertion tokens before selecting volume subsequences. The finest represented action is exactly Wilson; positive partition and local insertions give probability laws, continuity, gauge invariance and finite-stencil locality. The scalar running slope is positive logarithmic and uniform in period. The summed drift argument (RUN1)–(RUN5), using the realized calendar (CAL1)–(CAL11) and the actual update (DR7), gives the positive logarithmic coupling slope uniformly in cutoff and period; its finest-cutoff specialization is the bare-coupling statement of the theorem. For any requested increasing period sequence retain coupling limits, extract the common hierarchy and coefficients and compose refinement maps. Fixed-volume Cauchy convergence and smearing give the full inner mesh limit; the refined periods give its outer limit. Auxiliary compact selections therefore leave the full inner limit intact. The quotient and physical detector identify one finite-dimensional filtration on this same table. Its compression algebra gives compact metric phases and full-support probability with smooth character hierarchies and barycenter. The same heat/tube/boundary/Gram

construction gives common-domain Hilbert fields, adjoints, locality, continuous spacetime action and cone spectrum. Heat-ground and separation arguments give each character's unique vacuum, faithful labels and nonzero unit. Strict imaginary-time and boundary identities recover the corresponding Euclidean moments. Fix one equivariant OPE reader. Separated, nested and integrated expansions use its same coefficients for every phase and mean law; collision extensions retain its basis and finite projections across simultaneous truncations and collision patterns. Gaussian comparison and position normalization fix the same physical stress. Equation markers and weighted near-kernel estimates identify its centered trace and amplitude, giving in the selected phase

$$\lim_{r\downarrow 0} r^8 \log^2(1/r) S_2^{\chi}(\Theta(-r/2,\mathbf{0}),\Theta(r/2,\mathbf{0})) = 3\dim_{\mathbb{R}} \mathfrak{g}/\pi^4 > 0.$$

The same scalar diagnostic is retained under the established phase identifications. The mass-shell argument excludes the specified finite class; coherent countertests give the stated Maxwell exclusions. All witnesses remain on one Wilson family and one composite refinement. Public fields, operators, moments and channels are their identifications on that common table. This assembles the supplied theorem within its stated scope, with no positive mass gap asserted. In the source lift the retained assembly tuple includes every finite stage choice, refinement map and reader already constructed. Its public readout acts componentwise on this one tuple. Exact operation compatibilities and the preserved order of limits recover the source's common-witness conclusion.

Common-witness assembly recovery. Retain the single stage-choice record, Wilson family, insertion tokens, refinement maps, field quotient and all reader data as one source object $\mathfrak{W}^Z$. Its realization $\mathcal{R}$ is the corresponding physical tuple $\mathfrak{W}$. For every finite construction operation F already lifted, the recovery identity is

$$\mathcal{R}_{\mathrm{out}} F^Z(\mathfrak{W}^Z) = F(\mathcal{R}\mathfrak{W}^Z).$$

Finite composition preserves this identity; the same composite refinement therefore carries every required component, rather than choosing a separate subsequence for each conclusion. At stage n the finite list of demands for its fixed finite degree/derivative horizons is combined by a maximum in the specified choice order. The same numerical choices and later monotone enlargements are retained under realization, so every fixed finite request has the same eventual stage in both tuples.

For any requested period sequence, write $C_{N,L}$ for a fixed correlation or coefficient reader and $C^Z_{N,L}$ for its lift. At each finite mesh and suitable period they have the same readout. The retained fixed-period Cauchy bounds and smearing maps give

$$\mathcal{R}\Big(\lim_{N\to\infty} C^Z_{N,L}\Big) = \lim_{N\to\infty} C_{N,L}, \qquad \mathcal{R}\Big(\lim_{j\to\infty}\lim_{N\to\infty} C^Z_{N,L_j}\Big) = \lim_{j\to\infty}\lim_{N\to\infty} C_{N,L_j}.$$

These statements use the previously transported convergence domains and norms; the inner limit remains the full mesh limit. The compact phase readout and its probability pushforward recover the same character hierarchies and barycenter. The Hilbert unitaries preserve actual common domains, adjoints, locality and momentum spectral measures, while their label maps preserve the filtration and unit. The unique ground-vacuum conclusion retains precisely its source scope.

The fixed OPE reader, homogeneous basis and finite projections are entries of $\mathfrak{W}^Z$, so their separated, nested, integrated and collision readers recover the same public coefficients for simultaneous finite requests. The stress correction, spectral trace measure and selected character are also entries of this one tuple. Thus the displayed all-distance trace amplitude and both specified comparison exclusions are recovered with the same witnesses. This completes the assembly of the supplied conclusion as a source–readout statement. The claim-status register, the correspondence with the input and the limitations of the reconstruction are given in Appendices H–J.

28 Conclusion

The Wilson Yang–Mills construction admits an explicit source–readout realization across its full chain of finite, effective and continuum objects. An admitted fan supports the compatible histories. The finite frame reader has a complete fibre description, a reverse section and an exactly normalized measure pushforward. Activities and their budgets are recovered in the transported norm. A stationary section eliminates vertical variables while its retained residual controls the off-shell discrepancy. Calibrated histories and their running belong to one recipe and calendar, and the continuum hierarchy retains the full inner mesh limit before the outer volume refinement.

At the field interfaces, radial reconstruction, coefficient readers and the spectator-null quotient make explicit which local representatives are identified. Inclusions, derivatives, Hilbert completion, character evaluation and product readers act compatibly on those retained objects. The stress and trace calculation uses the same physical table, phase and comparison labels through the universal-radius and spectral arguments. Its result is the precise short-distance asymptotic and the stated comparison exclusions of the supplied construction.

The mathematical purpose of Rodgers's framework is realized in these interfaces: it supplies a connected account of the operations that survive projection, the reverse coverage available, and the information, residuals and choices that a reduction must retain. The result is a complete conditional recovery of the specified Yang–Mills derivation. Together with the standalone companion in this release, it provides both the construction and its reflective source–readout analysis in fully editable mathematical form.

29 Attribution of the source and mathematical background

Jeremy Rodgers originated Shadow Theory, the source–readout architecture and the Source Lift protocol, and directed this research. The framework's role here is the explicit realization and recovery analysis developed throughout the paper. OpenAI is credited for the underlying Lean Yang–Mills construction, represented in this edition by the expanded mathematical translation [2]. AI assisted in carrying out the research and preparing the manuscript. The earlier Einstein reconstruction [4] is a methodological and expository example; the present Yang–Mills derivations are supplied locally.

Wilson's lattice gauge formulation is credited to Wilson [5], compact-group invariant measure to Haar [6], and the pairing formula to Wick [7]. The Euclidean reconstruction framework and reflection positivity are credited to Osterwalder and Schrader [11, 12], with the lattice-gauge setting of [13] and the chessboard estimates of [14]. The positive-functional construction is credited to Gelfand, Naimark and Segal [8, 9], and the character-algebra approximation theorem to Stone [10]. Renormalization and cluster-expansion background is provided by [17, 18, 19, 15], the axiomatic field framework by [16], and the one-loop coefficient by [20, 21]. The precise hypotheses, parameter orders, domains and readout definitions used in this application are stated in the corresponding derivations.

Funding and availability. No external funding. This flagship paper and the standalone companion *A Wilson Yang–Mills Construction: A Natural-Language Mathematical Exposition* form one Zenodo release under DOI 10.5281/zenodo.24173681. The release contains both editable source trees, the expanded mathematical input and the coverage, amendment and verification records.

A Tangent coverage, forms and horizontal operations

Let $d_0\phi$ be the edge cochain $(d_0\phi)_e = \phi_y - \phi_x$. At the identity readout, differentiating $r_x(t) = \exp(t\phi_x)$ and $A_e(t) = \exp(ta_e)$ gives

$$p(\phi, a) = a + d_0\phi, \qquad j(u) = (0, u), \qquad pj = I.$$

The kernel is exactly $\{(\phi, -d_0\phi)\}$: this follows directly by solving $p(\phi, a) = 0$. Oriented boundary cancellation gives $d_1 d_0 = 0$, hence $d_1 p(\phi, a) = d_1 a$. This kernel is infinitesimal frame redundancy; it is distinct from any subsequent physical gauge or harmonic restriction in the supplied construction.

Theorem A.1 (Operator and quadratic-form recovery)**.** *For a target operator $T : H_1 \to H_2$ on the specified finite cochain spaces, put $T_Z = j_2 T p_1$. Then*

$$p_2 T_Z = T p_1, \qquad T_Z j_1 = j_2 T.$$

Compositions are preserved: $S_Z T_Z = j_3 S T p_1$. A symmetric target form with matrix Q instead pulls back as $Q_Z = p^* Q p$. Its nullspace is $p^{-1}(\ker Q)$, and its restriction to the section is the original form.

Proof. Insert $p_i j_i = I$ in the two intertwining identities and the composition. For the form, its value on v, w is $\langle pv, Qpw\rangle$ by the definition of the adjoint. Thus its radical consists precisely of v for which $Qpv = 0$, since p is onto. In particular every vertical vector is null. Taking $v = ju, w = jv'$ recovers the original form. The transported section inner product makes j an isometry. It does not assert that j is an isometry for an unrelated ambient inner product or that jTp equals p^*Tp. □

For example the source stationary identity

$$Q_c = T^* Q_f T + b(I - PT)^*(I - PT)$$

pulls back to

$$p_c^* Q_c p_c = T_Z^* p_f^* Q_f p_f T_Z + b(p_c - P p_f T_Z)^*(p_c - P p_f T_Z).$$

This follows by substitution and $p_f T_Z = T p_c$. Replacing the last readout difference by $I - P_Z T_Z$ on the whole redundant space changes the form on vertical directions. Any positive lower bound is retained only on the original restricted space or its isometric section, with the original norm and restrictions.

For a differentiable finite-lattice curve let $k_x = \dot{r}_x r_x^{-1}$. Exact matrix differentiation gives

$$\dot{U}_e = r_x^{-1}(\dot{A}_e - k_x A_e + A_e k_y) r_y.$$

Every target tangent $h_e \in T_{U_e} G$ lifts, for any differentiable frame curve, by

$$\dot{A}_e = r_x h_e r_y^{-1} + k_x A_e - A_e k_y.$$

Indeed differentiate $A_e = r_x U_e r_y^{-1}$; this also proves group-tangent membership. At fixed target input the two frame terms are still required. The flat target configuration at arbitrary frames is $A_e = r_x r_y^{-1}$.

Let P be any actual tagged source predictor from fine links to coarse links on its stated domain. With coarse frames s, define

$$B_b = s_y P(\pi(r, A))_b s_z^{-1}, \qquad b : y \to z.$$

Then $\pi_c(s, B) = P(\pi_f(r, A))$ identically. On a smooth branch, for coarse rates $l = \dot{s}s^{-1}$,

$$\dot{B}_b = s_y DP_U[h]_b s_z^{-1} + l_y B_b - B_b l_z.$$

Differentiating the coarse readout cancels the last two terms and recovers $DP_U[h]$. The proof uses the actual predictor, with its routes, minimizers, fallback tags and finite stencil. A multivalued target branch is lifted as a tagged relation; no new selection is introduced. Norm estimates transfer with the transported color-block norms on fixed-frame or horizontal derivatives. They do not bound arbitrary unbounded frame rates.

B Integral, residual and limit transport

For each tagged finite lattice λ, let π_λ and σ_λ be the maps above. The tag includes every source parameter used by the operation. An insertion F_λ has lifted representative $F_\lambda^Z = F_\lambda \circ \pi_\lambda$. Coefficients, support labels, multiplicities and histories remain attached to this representative; equality of functions alone is not permission to discard those records.

Theorem B.1 (Exact transport of tested limits)**.** *Suppose the supplied construction forms a scalar tested quantity*

$$I_\lambda = \int F_\lambda \, d\mu_\lambda$$

on its stated integrability domain. Then $I_\lambda^Z = I_\lambda$ at every tag. Every convergence assertion, Cauchy estimate, uniform bound or iterated limit for these quantities therefore holds for their lifted representatives with precisely the same index set, quantifier order and constants. The same statement applies componentwise to finite-dimensional coefficient vectors and to distributional limits tested on the original test class.

Proof. The measure recovery theorem gives equality at each tag, including for complex integrable insertions. Consequently every difference $I_\lambda^Z - I_\kappa^Z$ equals its target difference and every tested residual has the same value. For example, if for every $\varepsilon > 0$ there is an original threshold beyond which $|I_\lambda - I| < \varepsilon$, the identical threshold proves the lifted assertion. Supremum bounds over any specified parameter subset are identical. Applying this argument first to the original inner index and then to the original outer index proves the iterated assertion without exchanging limits. A distributional assertion is this same argument for each allowed test function; uniformity over a test set transfers only when it was present in the original assertion. Finite coordinate sums commute with these equalities. No new majorant, compactness assertion or test class is inferred. □

A source operation with input space Q_1, output space Q_2 and map T has the canonical section representative $\sigma_2 T \pi_1$. More generally, relations are lifted by retaining all target branches and their original tags. A target residual R is represented by $R\pi$, so its zero and nonzero values are both preserved. For a composable finite chain, insertion of $\pi_i \sigma_i = I$ proves the whole-chain readout identity. These identities provide recovery for operations already defined by the supplied construction; they do not replace its definitions or derivations, which must also be preserved in the complete correspondence.

In particular the original full mesh limit at a suitable fixed period is retained before its outer refined-period limit. Auxiliary compact extractions retain their actual refinement maps and common insertion tokens. This theorem does not turn a subsequential statement into a full limit, nor does it identify different phase choices. A common positive denominator is carried through numerator and denominator together; a normalized equality is not obtained by cancelling unrelated

partition functions. Hard masks and health indicators remain measurable insertion data on their original domains. No derivative of such an indicator is licensed by integral transport.

For parameter derivatives the equality is differentiated only on the source's actual differentiability domain, with its given domination or finite-dimensional smoothness hypotheses. Moving-frame derivatives use the horizontal formula already proved. For infinite sums the original absolute majorant or other stated convergence mechanism is retained before passing the operation through the sum. Thus the map preserves quantitative estimates while leaving every analytic warrant at its original location in the dependency chain.

C Reflection quotients and field recovery

Fix the common continuum hierarchy produced by the supplied construction, with its actual phase and test labels. Let $\mathcal{V}$ be its algebraic future-word space and q its positive semidefinite reflection form. Take $\mathcal{V}_Z$ to be the linear span of the corresponding realized words, with their complete labels and common-witness records, and let $p : \mathcal{V}_Z \to \mathcal{V}$ read those words. The identity-frame word section j satisfies $pj = I$. Define

$$q_Z(v, w) = q(pv, pw), \quad N = \{u : q(u, u) = 0\}, \quad N_Z = \{v : q_Z(v, v) = 0\}.$$

These word maps are distinct from the finite cochain maps of the tangent chapter.

Theorem C.1 (Unitary recovery of the reflection space)**.** *The map $[v] \mapsto [pv]$ is a well-defined isometric bijection $\mathcal{V}_Z/N_Z \to \mathcal{V}/N$. It extends uniquely to a unitary $\mathscr{U} : \mathcal{H}_Z \to \mathcal{H}$. The empty word is preserved. Bounded operators intertwined on these dense quotients extend by $A_Z = \mathscr{U}^{-1}A\mathscr{U}$, preserving norms and adjoints.*

Proof. For any positive semidefinite form the Cauchy–Schwarz inequality follows by requiring $q(u + tw, u + tw) \geq 0$ for every complex t. Thus a zero-norm vector is orthogonal to every vector. The defining equality gives $N_Z = p^{-1}N$ and equality of quotient inner products, proving well-definedness and injectivity. Every $[u]$ has preimage $[ju]$, proving onto recovery before completion. The isometry therefore extends to the completions, with dense and closed range, hence onto. The empty word maps to itself. Conjugation by this unitary proves the bounded-operator assertions, including the adjoint identity by the inner-product definition. □

Unbounded fields require the supplied common cyclic domain $\mathcal{D}$ and its actual null-preservation proof. Once a target field $\Phi(f)$ is defined there, set

$$\mathcal{D}_Z = \mathscr{U}^{-1}\mathcal{D}, \qquad \Phi_Z(f)v = \mathscr{U}^{-1}\Phi(f)\mathscr{U}v \quad (v \in \mathcal{D}_Z).$$

This specifies its domain as well as its values. Products on the invariant cyclic domain, conjugate-label adjoint pairings and locality identities are recovered by inserting $\mathscr{U}\mathscr{U}^{-1}$ between successive fields. No bounded extension of a field to the entire Hilbert space is inferred. If the target field is closable, taking convergent graph sequences proves that its lifted graph closure is the unitary image of the original graph closure.

The same conjugation transports the constructed strongly continuous spacetime action and heat semigroup: continuity follows from $\|\mathscr{U}^{-1}(V(a) - V(a_0))u\| = \|(V(a) - V(a_0))u\|$. If E is the source's momentum projection-valued measure, then $E_Z(B) = \mathscr{U}^{-1}E(B)\mathscr{U}$ is countably additive in the strong topology, has the same support restrictions and recovers its scalar spectral pairings. In particular the ground projection and its rank are preserved. A rank-one vacuum projection does not imply a positive lower bound on the nonzero energy spectrum.

Boundary and reflection spaces use their respective unitary recovery maps. A constructed attachment J is lifted as $J_Z = \mathscr{U}_{\mathrm{ref}}^{-1} J \mathscr{U}_{\mathrm{bdry}}$. This preserves its isometry property and, when the supplied density argument proves onto range, its surjectivity. Isometry alone supplies no surjectivity argument. Tested boundary distributions remain those of the same hierarchy, with their original gap bounds and derivative requirements, by Theorem B.1.

D Compression algebras and phase recovery

Retain the supplied heat-fixed subspace, its bounded compressed word operators and the norm-closed unital commutative algebra $\mathcal{A}$ they generate. These constructions are made only after their source commutativity and boundedness arguments. The unitary $\mathscr{U}$ restricts to a unitary $\mathscr{U}_0$ on the heat-fixed spaces, since it intertwines the heat semigroups. Define the realized algebra as the norm closure of the corresponding conjugated generators.

Theorem D.1 (Character and phase-measure recovery)**.** *The map $\alpha(B) = \mathscr{U}_0 B \mathscr{U}_0^{-1}$ is an isometric unital star-isomorphism $\mathcal{A}_Z \to \mathcal{A}$. It induces a homeomorphism*

$$h : \mathrm{Spec}(\mathcal{A}) \longrightarrow \mathrm{Spec}(\mathcal{A}_Z), \qquad h(\chi) = \chi \circ \alpha.$$

For the source phase measure ρ, the measure $\rho_Z = h_* \rho$ has the same barycenter evaluations. Full support of ρ implies full support of ρ_Z.

Proof. Unitary conjugation preserves sums, products, the unit, adjoints and norms. It maps the specified generating set onto its target set, so continuity and inverse conjugation give onto recovery of the norm closures. Composition with α maps characters to characters; composition with α^{-1} is its inverse. The character topology is pointwise evaluation on algebra elements. For each $B \in \mathcal{A}_Z$, the function $h(\chi)(B) = \chi(\alpha B)$ is continuous, and the same argument applies to the inverse, proving the homeomorphism. By the definition of pushforward,

$$\int h(\chi)(B)\, d\rho(\chi) = \int \eta(B)\, d\rho_Z(\eta).$$

Thus the original vacuum-state barycenter identity transports exactly. If O is a nonempty open set in the realized character space, $h^{-1}(O)$ is nonempty and open. Full support gives $\rho_Z(O) = \rho(h^{-1}(O)) > 0$. □

This theorem transports full support after it has been established by the supplied cyclicity and separating-vector argument; probability normalization and compactness alone do not prove it. The character-dependent hierarchy is read from the same compressed whole-word element before character evaluation. Reading selected factors separately would replace $\chi(B)$ by a different expression unless the required product identity has first been proved in this algebra.

Countable dense families, exceptional null sets and the original continuity upgrade remain part of the realized phase construction. An almost-everywhere property transfers to an almost-everywhere property by h; extending it to every character requires the supplied full-support and continuity argument. In particular vacuum uniqueness and rank-one ground-space recovery are retained in their original scope. No new phase is chosen by the frame section, and no change in phase measure is hidden in a normalization. The selected trace field remains centered with respect to the same selected vacuum.

E Operator-product readers and collision extensions

Let $\mathcal{F}$ be the supplied filtered field space, with its fixed finite-dimensional complements, bases and projections. Use an isomorphic realized field-label space $\mathcal{F}_Z$ with readout q and section $i = q^{-1}$. This is the field-label isomorphism after the source's physical detector quotient; it is not an inverse on all unreduced frame histories. Set

$$\mathcal{F}_{Z,\le m} = q^{-1}\mathcal{F}_{\le m}, \qquad \Pi_m^Z = i\Pi_m q.$$

Every channel label and cluster geometry is retained. For the actual source reader K_m define, on its stated separated-configuration domain,

$$K_m^Z(\xi; v_1, \dots, v_n) = iK_m(\xi; qv_1, \dots, qv_n).$$

The same fixed complements are used at every truncation degree, rather than refitted to each test.

Proposition E.1 (Reader and remainder recovery)**.** *All algebraic reader identities and tested expansion estimates of the supplied construction are recovered by q, with unchanged coefficients and truncation degrees. The full expansion residual is preserved when it is retained as a separate component. Compatibility between finite packets transfers with the same projections.*

Proof. Applying q to the displayed reader cancels $qi = I$ and recovers K_m exactly. Similarly $q\Pi_m^Z = \Pi_m q$, and every finite identity obtained by composing these maps follows by inserting these inverse identities at each slot. A tested residual is the difference of the original tested cluster and its finite reader expansion. Each term has the same scalar reading, so the residual and its absolute bound are identical. Infinite summations use the original convergence warrant, as in Theorem B.1. No inversion of Π_m is used: its discarded tail is retained in the residual. The argument applies to each original nested cluster and spectator domain without altering its scale ratios or error exponents. □

The collision-extension device is also retained as data. To make its recovery explicit, suppose one stage of the source subtraction is a continuous linear map W on its specified test space, and the punctured kernel acts integrably on Wf. Write that stage, including its chosen contact functional, as

$$T(f) = t(Wf) + C(f).$$

Here this expression is used only on the domain and with the combined-integrability bounds supplied by that stage; separate nonsummable pieces are not assigned integrals. For corresponding realized tests let L be the test-space isomorphism with its transported topology. Define

$$W_Z = L^{-1}WL, \quad t_Z(g) = t(Lg), \quad C_Z(g) = C(Lg).$$

Then $T_Z(g) = t_Z(W_Z g) + C_Z(g) = T(Lg)$. This is a direct substitution and proves continuity in the transported topology. It preserves the contact choice, not just the punctured restriction.

For the full finite forest of nested subtractions use this identity at each node in the source's prescribed order. Induction on the number of nodes proves equality of the combined subtracted test and every retained contact contribution. All normal-test spaces, cutoff functions, density Jacobians, overlap records and finite projections remain fixed. The source's combined L^1 majorant justifies its integrations and gluing; those steps are not replaced by termwise integration of divergent summands. If two extensions differ by a diagonal-supported distribution, their punctured restrictions agree, but their full readouts agree only when that contact distribution agrees on every allowed test. Thus punctured values alone are not a recovery map for the chosen extension.

F Gaussian and ordered inverse operations

The supplied Gaussian constructions retain their actual finite coordinate spaces, precision matrices, means and support windows. In particular a nonsymmetric auxiliary matrix is not identified with its symmetric covariance. If the source uses

$$C = (P + P^*)/2, \qquad \mu = -P^* b,$$

then both C and P remain in the realized history. For any comparison vector v and $E = AP - I$ the exact identity is

$$\mu - v = E^* v - P^*(Av + b),$$

when $A = A^*$. Indeed $E^* = P^* A - I$, so the right side is $-v - P^* b$. Replacing μ by $-Cb$ generally changes this identity. A coordinate isomorphism transports the mean as a vector and covariance as a form with the stated metric; the original Gaussian integral and all normalized insertions then recover by change of variables, with its Jacobian retained. No probability law is assigned to a nonsymmetric covariance.

For the source's approximate inverse C of A, put $S = AC - I$. On its original convergence domain, for example $\|S\| < 1$ in the stated submultiplicative norm,

$$C^{-1} - A = \sum_{k \geq 1} (-S)^k A.$$

To prove this, $AC = I + S$ gives $C^{-1} = (I + S)^{-1} A$. The finite geometric identity has residual $(-S)^{n+1}(I + S)^{-1} A$, whose norm tends to zero. Subtracting the zeroth term gives the displayed ordered series. The terminal factor A cannot be removed or commuted through S without an additional identity. Under an isomorphic coordinate readout each finite ordered product conjugates exactly, and the norm-convergent series follows in the transported norm.

Local determinant diagnostics retain the original supported contribution together with every complementary arm required by the complete determinant identity. A diagnostic contribution alone is not the full log determinant. Likewise a quiet-coordinate condition is evaluated in the actual source coordinates: under $z = Oz'$ it becomes the condition on Oz', not the same componentwise bound on z'. Orthogonal invariance of a Euclidean norm does not imply invariance of a coordinate-wise box. All whole-color-block restrictions and source window conditions are therefore retained before inverse or determinant transport.

G Assumptions, dependency chain and source–readout contracts

Three registers collect what the reconstruction must preserve and how the conclusion depends on it. Table 3 lists the assumptions and scope controls of the supplied construction, with the lines of the supplied text at which each is stated; rows A19–A27 are those of the constructions of Sections 9, 14 and 16 and of the rotation and trace arguments. Table 4 is the chain of stages through which the public conclusion of Section 4 is reached, each with its input contract, the operations that are lifted, and the recovered output. Table 5 lists the source–readout contracts of the general transport statements, with what each readout fails to invert and what is therefore retained. Together with Table 2 they are the summary form of the hypotheses under which every recovery statement of this paper holds; none of them adds a hypothesis to the supplied construction.

Table 3: Assumptions and scope controls of the supplied construction, as preserved in this paper.

	Assumption	Input lines	Preserved condition	Scope control
A01	Group and representation	9	Closed faithful matrix subgroup of U(m); specified exponential real Lie algebra is nonabelian simple.	No connectedness hypothesis is added.
A02	Finite prior and Wilson order	11–26	Independent normalized Haar link prior and literal ordered Wilson plaquette; insertion support offsets depend only on token.	Gauge invariance/continuity and uniform-period slope refer to one family.
A03	Volume and mesh quantifiers	26–28	Every increasing period sequence admits common refinement; suitable fixed periods have full inner mesh limit before outer period limit.	Auxiliary compact selections do not weaken full inner limit.
A04	Field domain and spectrum	30	Common dense invariant cyclic word domain, Schwartz continuity, adjoints/locality, proper orthochronous covariance and forward-cone PVM.	Unique invariant vacuum is not a positive-gap claim.
A05	OPE parameters	32–45	Fixed finite words/degrees, $M \geq 0$, $\eta > 0$, $\varepsilon > 0$; separated cluster and spectators; same phase/truncation-independent channels.	Radius threshold may depend on fixed words/ parameters; uniformity is in phases and stated configurations.
A06	Public comparison class	47–60	Centered degree-four trace and finite complex combinations/ products of stated fixed mass-shell integrals.	Does not enlarge exclusion to every generalized-free comparison; penalty is one.
A07	Analytic conventions	64–108	Four dimensions; physical metric/chord/color factors; odd block ratios ≥ 3; connected-union cover; absolute weighted bounds.	Rate-zero is subpower, not boundedness; thresholds independent of period.
A08	Contact continuation	12526–12543	Tested rate inequality on every allowed logarithmic path; palette continuity and constancy in assigned normals; common detector collar.	Remaining branch retains support $t \leq s$, full finite remainder; projected gain is actual degree.
A09	Weak-sequence Gaussian comparison	12755–12767	Same weak scaling sequence/ frames and compact distinct configurations with each fixed derivative order.	No interpolation to every positive real scale; Ward sign remains fixed.
A10	Private Gaussian marks	12945–12973	Finite regulated potential law, positive semidefinite infinite-lattice subtraction matrix; independent private marks by insertion.	Covariance is bilinear; same-slot value need not vanish at finite regulator; no subtraction inverse required.

	Assumption	Input lines	Preserved condition	Scope control
A11	Laplace continuity	14113–14136	Positive future-cone measures with finite transforms on full open dual cone; common polynomial budget for Schwartz passage.	No arbitrary signed-measure convergence assertion.
A12	Maxwell isometry countertest	14220–14252	Complex-linear isometry with physical image, vacuum/translation intertwining, and precise local coherent matrix identity.	Neither surjectivity nor identification of every field required; fixed-ball variant retained.
A13	Projected and averaged variants	14326–14329	Bounded self-adjoint fixed-image map for projected variant; physical inner-product preserving pointwise fixed-image maps for probability average.	Individual averaged maps need no extra linearity, boundedness or projection property.
A14	Near-kernel repair	14648–14674	Finite real onto map with nonzero one-dimensional kernel, exact convergent right inverses and both weighted near-kernel errors.	No prior uniform convergence on near-kernel direction assumed; precision paid before correction.
A15	Mass and volume order	14890–14982	Strictly positive mass through volume estimates; moving masses confined to a fixed positive interval after tolerance choice.	Separate eventual endpoint estimates suffice; no uniform mass convergence is inserted.
A16	Position normalization branch	15317–15408	Complementary scalar/stress/unit projections and positive adjustment factors; independent Wilson calibration of measured charge.	Real fourth-order remainder belongs to normalized frame; weaker positive-ray norm alignment transfers leading coefficient only.
A17	Marker and threshold order	15556–15647	Finite surgery order fixed before eventual thresholds; common windows, two marker slots and positive weights, exact common Wilson partition.	Uniformity in finer cutoff/volume follows retained threshold order; no chamber-count tariff factor.
A18	Local lens admissibility	15686–15766	Fixed read support, measurable equivariant displacement, health threshold $\theta/4$, chord D with $8\mathrm{D}\leq\theta$, finite stocks and nonwrapping.	Flag equality is branchwise exact; absent insertion requires no exponent estimate or hard-mask differentiation.
A19	Activity domains and budgets	110–226	Threshold $0 < \vartheta \leq 1$ with equality healthy; ordered left derivatives on healthy domains only; budgets at a fixed finite torus; early reserve $s + 3$.	Masks are never differentiated; no volume-independent bound on the total scalar action is asserted.
A20	Stationary corridor	5515–5709	Fixed derivative order $n \geq 2$; counting gap and exponential weights; contraction on $\|x\| < R_X$, $\|v\| \leq R_V$.	Uniqueness of the stationary point only inside the corridor; off-shell values and the residual are retained.

	Assumption	Input lines	Preserved condition	Scope control
A21	Calendar and order of choices	6226–6430	Stage data and length floors, then lengths, then entry delays; one calendar chosen before every history, cutoff, period and observable.	Later enlargements are monotone and preserve earlier payments; estimates on different calendars are not combined.
A22	Calibration of finite histories	6617–6819, 7504–7575	Bare parameter in a closed interval; bottom coupling $\bar{\beta}$ at level zero; deterministic forward recipe with continuous admissible levels.	Existence by the intermediate value theorem only; uniqueness within the retained recipe; no inverse shooting map is assumed.
A23	Reference tori and beacons	6843–7259	Read radius r with $2r \le w$; five folded reference views; compact coupling band; orders and stage parameters, then block length, then delay.	Constants are tied to fixed reference tori and are not uniform in the physical period.
A24	Labels and calibrated readers	8399–8753	Simple color Lie algebra, with no connectedness of the group; finite slot family; degree and level fixed before the eventual threshold.	The label reader is inverted at each fixed level; its inverse bound is not uniform in the level.
A25	Free comparison hierarchy	8754–9247	Explicit Gaussian kernel on the fixed real symbol space; one exterior degree bound for all weights in detection.	Statements are exact in the free comparison theory; they pass to the interacting hierarchy only by rank transfer along convergent readings.
A26	Euclidean rotations	10373–10674	Bands, window and tolerance fixed before the limit removing the outer flux; three response signs $f \ge 0$, $g \le 0$, $h \ge 0$; contact subtraction.	No positivity is claimed for the six individual band readings; the unsubtracted stress pair is not asserted conserved at coincident points.
A27	Scaling comparison and trace limit	13237–13994	Selected radii $r_k \downarrow 0$ and scaling maps; for every sequence $s_n \downarrow 0$ a subsequence with a scaling family on the same table and phase; monotonicity from the positive-time spectral measure.	No uniformity in the degree and no bound on the scaling maps or their inverses; the interacting theory is not asserted to be free.

Table 4: Dependency chain of the supplied conclusion: each stage with its input contract, the operations lifted, and its recovered output.

	Stage (parents)	Input contract	Operation	Recovered output
D01	Finite Wilson realization	fixed group/ representation, finite lattice, normalized Haar	frame realization and section; Wilson density pushforward	physical finite-lattice law
D02	Multiscale calibrated family (D01)	positive widths, prescribed stage ordering, banked histories	exact one-step identities and common insertion-token transport	one Wilson family before volume subsequence

	Stage (parents)	Input contract	Operation	Recovered output
D03	Common continuum table (D02)	requested increasing period sequence; fixed-period Cauchy bounds	full mesh limit then outer refined-period limit	common hierarchy, coefficients and finite field filtration
D04	Phases and Hilbert reconstruction (D03)	compression algebra, character hierarchy, positive Gram/heat data	phase readout and same heat/tube/boundary construction	common-domain fields, cone momentum measure and unique vacuum
D05	Fixed OPE and contact extension (D03, D04)	same finite homogeneous reader, basis and projections	nested replacement, tested integration and combined contact resolution	same public channels across finite truncations/ collisions
D06	Gaussian comparison (D03)	same weak sequence, free frames and moving covariance limits	private marks, corrected covariance, Wick recursion and profile sums	mean Gaussian comparison with mixed stress jets
D07	Local equation sensor (D02, D03)	same served coefficient difference, chart/coupling and retained scale	finite marker estimates and weighted degree-four spectator detection	both paid near-kernel leaks
D08	Physical stress amplitude (D06, D07)	exact divergence detector; same nonzero Ward shell test	finite correction inverse and near-kernel functional repair	Ward-normalized stress and scalar coefficient $-b_0/2$
D09	Selected trace ultraviolet measure (D04, D08)	selected stress, exact unit centering, positive spectral measure	all-distance sandwich; dilation/Laplace/ Schwartz recovery	trace amplitude and normalized cone measure
D10	Specified finite polynomial exclusion (D09)	finite mass-shell tracks with fixed finite data	damped radial expansions and product-test recovery	source-specified comparison exclusion
D11	Specified Maxwell countertests (D09)	physical-image translation isometry and precise local coherent identity	common amplitudes; local dilation polynomial; finite sector growth	source-specified Maxwell/projected/ averaged exclusions
D12	Normalized position diagnostic (D08, D09)	positive frame branch; separated Ward columns	measured Wilson-calibrated charge and exact rational residual	normalized-frame real fourth-order remainder; weaker leading transfer

Table 5: Source–readout contracts of the general transport statements.

	Contract	Readout	Non-invertibility and retained data	Where
M01	Native admission and literal preparation	The realized branch supplies physical configuration/history readouts only after group, lattice, law and analytic data are specified.	Does not recover or classify all possible native antecedents; admission does not select Wilson law or identify abstract vertices with physical sites.	Prop. 6.1

	Contract	Readout	Non-invertibility and retained data	Where
M02	Finite frame fibre and section	$\pi(r,A)_e = r_x^{-1} A_e r_y$; reverse links are inverses.	Readout forgets frame representatives; it does not quotient physical gauge equivalence. Physical gauge acts by $r \mapsto rg^{-1}$, A fixed.	Thm. 7.1
M03	Paths and physical gauge action	Interior frames cancel; open paths transform at endpoints and loop traces read unchanged.	A trace alone discards conjugacy and other matrix information; full link history remains retained.	§7
M04	Normalized Wilson measure and insertions	$\pi_* \widetilde{\mu} = \mu_W$; insertion F reads $F \circ \pi$.	No unnormalized gauge volume; normalization and integrability remain part of the contract.	Thm. 7.2
M05	Tangent operators and quadratic forms	$p_2 T_Z = T p_1$ and $T_Z j_1 = j_2 T$; compositions insert $p_i j_i = I$.	Vertical frame directions remain null. Operator transport differs from adjoint form pullback. No ambient vertical coercivity follows.	Thm. A.1
M06	Actual nonlinear predictors and horizontal derivatives	Coarse $\pi(s,B) = P(\pi(r,A))$; differentiated frame terms cancel in the coarse readout.	Multivalued branches remain tagged relations; no invented selection or bound on arbitrary unbounded frame rates.	§A
M07	Tested limits and quantitative estimates	Exact scalar identity $I_\lambda^Z = I_\lambda$ at every tag.	No exchange of limits, stronger test class, derivative of hard masks, new compactness or mixing of unrelated partition functions.	Thm. B.1
M08	Operations, relations and residual histories	$\pi\sigma = \mathrm{id}$ at every composable intermediate stage.	No discard of branch tags or auxiliary histories merely because functions coincide.	§B
M09	Reflection quotient, fields and spectrum	Unitary $U : [v] \mapsto [pv]$ between completed reflection spaces.	Null histories are quotiented; an unbounded field is not promoted to a bounded whole-space operator. Rank-one vacuum is not a positive spectral gap.	Thm. C.1
M10	Compression algebra and phase measure	Whole-word character evaluation; $\rho_Z = h_* \rho$.	Factors cannot be read separately without a proved product identity. Almost-everywhere statements require the original continuity/ full-support upgrade.	Thm. D.1

	Contract	Readout	Non-invertibility and retained data	Where
M11	Fixed OPE reader and retained remainder	Applying q recovers the same channel reader and finite projections.	Projection Π_m is not inverted; discarded tails are retained as residuals. Label inverse is not an inverse on unreduced histories.	Prop. E.1
M12	Contact extensions with subtraction data	$T_Z(g) = T(Lg)$ on the stated combined integrability domain.	Punctured restriction loses diagonal-supported contact information; separately nonsummable pieces are not assigned integrals.	§E
M13	Gaussian means, covariances and residuals	Normalized Gaussian integrals and $\mu - v = E^*v - P^*(Av + b)$, $E = AP - I$, $A = A^*$.	Replacing μ by $-Cb$ loses the specified mean identity. No probability covariance is assigned to nonsymmetric P.	§F
M14	Ordered inverse, determinant and quiet-coordinate data	$C^{-1} - A = \sum_{k\geq 1}(-S)^k A$ on the original norm-convergence domain.	Terminal A cannot be removed/commuted; determinant diagnostic alone is not full log determinant. Orthogonal norm invariance does not preserve coordinate boxes.	§F

H Claim-status register

The register separates the origin of a statement from the recovery established here. **A** denotes a native admission result; **B**, a result conditional on the realized branch; **C**, a result of the supplied Yang–Mills construction; **D**, a recovery statement proved in this reconstruction; and **E**, an open selection question. A row can combine classes when a supplied construction also receives a new recovery proof. The hypotheses and exactness classes are part of the stated result.

Item	Class	Reconstruction status and recovery class
Admitted antecedent and literal preparation (Prop. 6.1)	A	Admission of the walking marked dependency; selects no physics.
Admitted fan supporting the family; its automorphisms and fold map (Props. 8.1, 8.2)	A	From the admission closure alone: no relation between leaves, no preferred labelling, finite statements over finite admitted supports. Selects no physics.
Lift groupoid of a leaf; fibre law (Thm. 8.3, Cor. 8.4)	D	The realization is an equivalence of groupoids; lifts unique up to unique isomorphism; stabilizers preserved. Checked on a finite model.
Interfaces and their discrepancies (Table 2)	B	Classification of the supplied construction's interfaces by exactness class, each row proved where indicated.
Frame realization, fibre, section, measure (Thms. 7.1, 7.2)	D	Exact: quotient equivalence $\mathcal{Z}/G^V \simeq \mathcal{U}$ and normalized pushforward.

Item	Class	Reconstruction status and recovery class
Reflection positivity and rooted bounds	C	Lifted with identical Gram matrices on the retained probe algebra.
Activity domains, budgets, norm (AB1–AB12)	C, D	Definitions supplied; frame-blind flags, ordered-derivative recovery and isometry of the activity chart are new (Prop. 9.1, Thms. 9.2, 9.4). Exact embedding with section.
Quadratic history contract	C	Five-row identities lifted through $d_1 p$ (Prop. 11.1); amendment recorded as a retained choice.
Primitive integrand (PRI1–PRI9)	C	Regrouping is a bijection; marginal extension loses only unread coordinates (Props. 13.3, 13.5).
Stationary graph (AS, US)	C, D	Graph and response bounds supplied; recovery of the vertical discrepancy from the residual with period-free constants is new (Thm. 14.5). On-shell recovery with explicit off-shell defect.
Exterior division, two-leg primitive (ED, TEL)	C	Division is a section of contraction; primitive not unique (Props. 14.8, Thm. 14.9).
Loop moments (LM)	C	Quiet families are the zero fibre of an additive moment readout (Prop. 14.10).
One-loop drift and $s = 11C_A/(24\pi^2)$ (DR, DP, MA)	C	Coefficient derived inside the supplied construction from an exact pole integral (arithmetic recomputed here); actual couplings tied to the reference clock by inequality (Thm. 14.12). Not a physical-coupling identification.
Common calendar (CAL)	C	One common witness for all histories, cutoffs, periods, observables (Thm. 14.16); non-unique, monotone under admissible enlargement.
Startup representative (SQ, FST)	C	Exact density identity with nonzero activities (Thm. 14.18); evaluation reader many-to-one.
Forward recipe and shooting (FCA, RSP, FCC, RET)	C, B	Operation lift with zero defect (Thm. 14.20); calibrated histories exist (Thm. 14.24); calibration is imposed at the bottom and propagated.
Beacons, reference chart (RB, PL, GL, RBG)	C	Controlled approximation with stated rates; constants tied to fixed reference tori.
Activity norm, step maps (LCN, ASP)	C	Descent of step maps to the common chart (Thm. 14.28).
Stability and running (FCB, RUN)	C	Bottom reader injective and Lipschitz-invertible within the retained recipe (Thm. 14.29); slope clause of the public theorem recovered (RUN5).
Protected variables (PV)	C, D	Reading as a horizontal lift with equivariance under the frame action is new.
Path books (SH)	C	Full-sequence limits for quadratic rows, subsequential for nonlinear books (Prop. 15.3).
Radial jets (RJ)	C, D	Curl/radial reconstruction supplied; explicit gauge parameter and fibre description are new (Thm. 16.1).

Item	Class	Reconstruction status and recovery class
Labels, calibration, free hierarchy (PF, PC, FM, GP, FC, FS, HC)	C	Finite-order recovery of labels; invertible readout at fixed level (Thm. 16.3); free sector is an imported comparison realization; faithful readout with bounded probes (Thm. 16.7). Numerical tables recomputed.
Physical quotient and derivatives (CVP, CVR, QD)	C, D	Supplied construction; its formulation as kernel of all spectator readers with operations descending is made explicit (Thm. 18.3). Quotient equivalence.
Slab insertions (BS)	C	Bounded on the reflection quotient (Prop. 19.1).
Euclidean rotations (RT, RSW, DB)	C	Covariance of the readout derived, not a symmetry of the lattice source (Thm. 19.3); elimination constant sharpened to $7K$ (Lemma 19.2).
Trace candidate, Gaussian comparison, scaling maps, stress, charges (TF, IG, WS, SR, SC, LCE, LR, TS)	C	Recovered with retained residuals; charges are representable fields (Thm. 23.3); scaling comparison (Thm. 23.2).
Full short-distance limit (TLim)	C	Recovered for the selected phase (Thm. 24.1); value $3\dim_{\mathbb{R}}\mathfrak{g}/\pi^4$ supplied by the scalar Wick calculation of the sequential part.
Wightman reconstruction, OPE, contact extensions, mass-shell exclusion (sequential part)	C	Lifted through Thms. C.1, D.1 and Prop. E.1.
Coverage and amendment record	–	Every expanded-source heading unit and its dependencies are mapped in both standalone papers. Four local recovery findings and two formula amendments are resolved with their calculations in Section I.1.
Selection of the gauge group, dimension, action or coupling history from source-side principles	E	Not obtained and not claimed; inequivalent realizations over the same support exist (Section 8).
Positive mass gap; identification of the regulator coupling with a physical coupling beyond the trace/Ward normalization	–	Not asserted by the source; not asserted here.

I Correspondence with the supplied text

Table 7 locates the expanded source's derivations by input range, equation tags and position in this paper. The sequential sections and mathematical appendices retain the complete construction and its dependencies; the additional source–readout statements give the maps, operations and recovery proofs at those same stages.

Table 7: Supplied derivations typeset in full, with their source–readout formulations.

Input lines	Subsection of the supplied text	Tags	Where
110–226	Local activity domains, derivatives and absolute budgets	AB1–AB12	§9
229–234	The finite Wilson law is a probability measure		§9
5057–5163	The primitive integrand and its local extension	PRI1–PRI9	§13.24
5515–5709	A common analytic stationary graph and its response estimates	AS1–AS8, US1–US8	§14
5710–5719	The Fourier normalization and the kinetic readout		§14
5720–5753	Analytic exterior division with the origin jets retained	ED1–ED2	§14
5754–5806	The two-leg correction with both momentum blocks retained	TEL1–TEL4	§14
5807–5827	Production of the fresh loop moments	LM0	§14
5828–5861	Production and transport of the inherited loop moments	LM1–LM4	§14
5862–6109	The one-loop drift used by the finite histories	DR1–DR7, DP1–DP12	§14
6110–6225	Massive auxiliary symbols and the uniform alias trace bound	MA1–MA12	§14
6226–6325	Constructing the common asymptotic calendar	CAL1–CAL11	§14
6326–6381	Countable local demands on the same calendar	CAL12–CAL18	§14
6382–6402	Paying the actual compulsory-emission forecast	EF1–EF3	§14
6403–6430	Uniform reserve at every step, including startup	CAL19–CAL21	§14
6431–6531	Inverse curvature coordinates and the startup fourth coefficient	SQ1–SQ12	§14
6532–6616	The actual finest-scale Wilson representative	FST1–FST8	§14
6617–6819	Finite coupling arcs and calibrated finite histories	FCA1–FCA11, RSP1–RSP6, FCC1–FCC2	§14
6820–6842	Exact density identity after retirement	RET1–RET2	§14
6843–6988	Finite reference beacons and weighted retirement continuity	RB1–RB8	§14
6989–7259	Derivation and transfer of the reference chart estimate	PL1–PL5, GL0–GL5, RBG1–RBG9	§14
7260–7503	The local activity norm and analytic comparison estimates	LCN1–LCN11, ASP1–ASP5	§14

Input lines	Subsection of the supplied text	Tags	Where
7504–7575	Stability of finite calibration and its comparison weights	FCB1–FCB5	§14
7576–7627	Uniform running of the actual coupling	RUN1–RUN6	§14
7738–7797	The protected change of variables and its determinant	PV1–PV7	§15.5
7947–8040	Extracting stationary path coefficients and their higher jets	SH1–SH10	§15.12
8399–8442	Radial Taylor variables and the free equation space	RJ1–RJ5	§16
8443–8561	Polynomial labels, coherent frames and local sections	PF1–PF13	§16
8562–8713	Calibrated coefficient changes and exact solutions	PC1–PC13	§16
8714–8753	The full free Maxwell polynomial hierarchy	FM1–FM3	§16
8754–8831	Positivity and strictness of the reflected free pairing	GP1–GP4, FM4–FM7	§16
8832–8997	The free low-degree stress and vector spaces	FC1–FC15	§16
8998–9001	Free shell charges and separation of the Ward kernel		§16
9002–9047	The compact stress shell identity	FS1–FS3	§16
9048–9098	Low-degree vector charges	FS4–FS10	§16
9099–9152	Fixed-displacement evaluation and the triangular Wick calculation	WE1–WE4	§16
9153–9230	Detection with a fixed exterior degree bound	HC0–HC4, FS11–FS13	§16
9231–9247	Elimination of higher-degree vector and angular defects	FS14	§16
9767–9788	Uniformly bounded symmetric projections, the calibrated equation image and the differential quotient	CVP1–CVP3	§18.9
9801–9801	Uniformly bounded symmetric projections, the calibrated equation image and the differential quotient		§18.9
9805–9862	The calibrated equation image and its silent representatives	CVR1–CVR7	§18.9
9863–9921	Inclusions and derivatives on the same physical quotient	QD1–QD6	§18.9
9924–9974	Bounded slab insertions and their norm derivatives	BS1–BS4	§19.1
10373–10578	Removing the outer angular flux and obtaining all Euclidean rotations	RT0a–RT16	§19.16
10579–10594	The order of choices in the spectral sign walk	RSW1–RSW2	§19.16
10595–10619	Normalized two-band diagonal estimate	DB1–DB2	§19.16

Input lines	Subsection of the supplied text	Tags	Where
10620–10674	Ward kernel transfer and the rotation action	RT10–RT19	§19.16
13237–13427	The finite trace candidate, its equation identity and its compensation	TF0a–TF10	§23.1
13428–13558	The interacting-to-Gaussian comparison used for the trace	IG1–IG10	§23.1
13559–13566	One calibration matrix for the three panels		§23.3
13567–13754	Common scaling maps and convergence of every derivative	WS1–WS17	§23.3
13755–13919	Constructing the physical stress and its shell charges	SR1–SR4, SC1–SC9	§23.3
13960–13994	The full short-distance limit: monotonicity, universal radius selection and the sandwich	TLim1–TLim2	§24.2
14485–14503	The three finite candidates and their common extraction	TF11	§25.1
14705–14869	Local estimates for the coefficient calculation and the six contributions	LCE1–LCE6, TF12–TF20	§25.5
14915–14972	The local resolvent bound	LR1–LR3	§25.9
15729–15765	Finite density surgery and the equation insertion	TS11–TS13	§26.8

Coverage and amendments. The accompanying source-to-both-papers coverage register has an entry for every heading unit of the expanded source, lines 1–15788, with its substantial arguments and dependencies. It connects the two standalone accounts without making either one supply missing mathematics for the other. The preserved input, equation tags and source ranges provide a reproducible correspondence to the complete derivation.

The following register records the formula amendments, typography clarifications and local recovery corrections, including their algebra and their domains of use.

I.1 Editorial amendments and focused recovery checks

The expanded mathematical source is reproduced unchanged in the shared release. This register distinguishes damaged-character repairs, typography clarifications, supported formula amendments and local recovery corrections, with their calculations and domains of use.

A1: principal predictor parenthesization. The extraction at line 1017 reads $m(q_0)^{-1}T_a/g_a - H/U$. The formula used in the construction is

$$G_{0,a} = m(q_0)^{-1}(T_a/g_a - H/U).$$

The multiplier convention matters here. Write $m_+ = m(q_0)$, $m_- = m(-q_0)$, and let the opposite-momentum product replace the real-slice absolute square. To exhibit the scalar cancellation without discarding the directional rail, set

$$a = \frac{|\operatorname{pullLine}_a(q_0)|^2 f_a}{m_+ m_-}, \qquad F = \sum_c f_c, \qquad T = T_a, \qquad g = g_a = m_+ m_- a + ST.$$

This is a local algebraic abbreviation on the regular principal collar, where $m_+m_- \neq 0$; it does not rename the original coefficient f_a. With $U = m_+m_-F + SH$, the corresponding covariance weight and directional rebate have the reduced form $E_0 = m_-F/U$, $r_0 = m_-a/g$. Direct subtraction gives

$$E_0 - r_0 = \frac{m_-S(FT - aH)}{Ug},$$
$$\frac{S}{m_+}\left(\frac{T}{g} - \frac{H}{U}\right) = \frac{S(TU - Hg)}{m_+Ug} = \frac{m_-S(FT - aH)}{Ug}.$$

Thus $E_0 - r_0 = SG_{0,a}$ with the amended parenthesization. In the real, even principal-filter convention this is the familiar calculation with $m_+ = m_- = m$ and the directional rail absorbed in a. The formula with the unparenthesized $-H/U$ does not give that cancellation. The remaining nonprincipal branch is unchanged. In particular the factor S cancels the longitudinal S^{-1} before Fourier extraction; no value at zero is assigned to that isolated singular factor. The filled principal value, its analytic collar, alias estimates and subsequent dual-predictor construction use the amended expression.

A2: finite Gaussian normalization. The extraction at line 1317 has exponent $-rh + d_v$; the correct exponent in the finite integral identity is $-(rh + d_v)$. Here r is the color dimension, h the active free-link count, f the total free tree-gauge link count, rh the active real dimension, and d_v the dormant real dimension. Thus $f - h$ free links are frozen and the ambient fluctuation dimension is $rh + d_v$. Set $K = \sqrt{2\pi}$ and $m_g = c(Kg)^r$, where c is the Haar chart scale. Cancellation of frozen factors gives the exact substitution

$$\begin{aligned} K^{-(rh+d_v)}m_g^f e^{-(f-h)\log m_g} &= K^{-(rh+d_v)}m_g^h \\ &= K^{-d_v}g^{rh}c^h. \end{aligned}$$

The extracted exponent instead gives $K^{d_v}g^{rh}c^h$, which differs by K^{2d_v}. The dormant density remains $K^{-d_v}e^{-\|y\|^2/2}$ with integral one, including $d_v = 0$. This amendment does not change the separate covariance determinant. The finite active/dormant factorization, frozen-link cancellation and later normalized Gaussian readers consistently use this convention.

SR1–SR4: local recovery statements. Equation (2) uses the ordered group product $\exp(\ell)BU^{-1}$, neutral on the protected chart and retained off it; the reversed factors would produce a commutator. Equation (3) retains distinct base and varied covariances, including both determinant and inverse derivatives, and specializes to a fixed covariance only when they coincide. The startup allowances are simultaneous on $x/2 \leq \beta \leq 2x$: the forced estimate needs the lower endpoint and the regular estimate also needs the upper endpoint. Finally exact station radius matching is an equality of geometric radii. Equation (5) retains the error between actual interacting and Gaussian moments, which vanishes only in the specified limiting comparison on the same table and phase. The finite startup identity, shooting argument, radius interval construction and full trace-limit sandwich retain their original roles.

Interpreted typography and a completed implication. Three malformed extracted expressions are interpreted using their local counting or norm arguments: line 1575 becomes $C_K = (n_a + n_b)AJ$; line 3218 becomes $4\,\mathrm{pcap}(4)(\sqrt{\mathrm{direct}_{\mathrm{top}}} + \mathrm{paths}_{\mathrm{top}})t =: C_1t$; and line 3428 becomes $HC(QR_{\mathrm{price}})^n$, supported by the repeated geometric estimate at line 4020. These are disclosed interpretations, not literal transcriptions. One further transpose-scope ambiguity in source line

815 is made explicit in both presentations. The extraction writes $V = d_2 B_1^{\mathsf{T}} F(d_2 B_1)$, whereas its preceding identities give

$$V = B_1 M B_1 = B_1 d_2^{\mathsf{T}} F d_2 B_1 = (d_2 B_1)^{\mathsf{T}} F(d_2 B_1),$$

because B_1 is symmetric. The reviewed flagship already used this whole-product adjoint. This is a type-consistent parenthesization of that local factorization, not a new positivity assumption. The packet normalization at source line 3121 prints a damaged exponent $\beta^{-1}/2$. It is read as $\beta^{-1/2}$ in $m_g(\beta) = c(\sqrt{2\pi}\,\beta^{-1/2})^r$, consistently with the already defined Gaussian coupling $g = \beta^{-1/2}$ and the finite normalization calculation above. The reviewed flagship already used this reading; it is explicitly retained in the companion. Equation (4) completes the truncated projection proof: $\|u\| \le \|Q - P\|\|u\|$ forces $u = 0$ when $\|Q - P\| < 1$; reversing the projections gives equality of their finite kernel dimensions. Source-range headings use the expanded extraction's 15788-line count.

The accompanying change ledger records the original and amended expressions, exact locations in both papers, and affected uses checked. The focused checks exercise noncommuting group elements, genuinely different Gaussian covariances, both shooting-band endpoints, station geometry and the two scalar substitutions. They provide reproducible checks of the local interventions documented here; the general arguments and their scope are stated in the corresponding proofs.

J Verification, limitations and reproducibility

Mathematical and computational evidence. The general source–readout results are proved in the text. The verification package also includes exact-arithmetic and finite-model checks of the formulas used by the reconstruction. The retained arithmetic suite records 28 passing checks: angular moments and the pole coefficient; the stress stencil moments; the free Hessian and third derivative, detector tables and conserved combination; the differentiated covariance; the Laplacian and regularized fundamental solution; the rotation idempotents; the exponent systems and activity rates; the beacon fourth difference; calendar plateaus; and shooting coefficients. Its script and receipt are `verification/check_lift_numbers.py` and the corresponding result record.

The retained structure suite records nine passing checks. It tests frame equivariance and fibre transitivity on the symmetric group on three letters over a triangle, the automorphism/stabilizer and holonomy-centralizer identifications, the fibre law, fan automorphisms and the single-axis exterior-division identity. Its script is `verification/check_lift_structure.py`. The focused amendment suite separately checks the ordered noncommutative residual, distinct base and varied Gaussian covariances, the two-sided shooting band, geometric radius matching, principal predictor cancellation and the Gaussian normalization. Records distinguish retained results from executions made for this release. Finite computations support these specific calculations; the general proofs are the displayed arguments.

Scope. The reconstruction takes the expanded translation of OpenAI's underlying Lean construction as its mathematical input; it does not re-execute or independently certify that formal development. The recovered conclusions retain the supplied hypotheses and comparison classes, and do not assert a positive mass gap. The group, dimension, Wilson action and physical preparation remain realization inputs. Native admission establishes the supporting structure, while the specific operators, estimates and choices give the Yang–Mills realization described in the body.

Several recovery statements have essential local qualifications. On gauge-invariant quantities the frame reader is an identity encoding; its larger role is to organize compatible non-invariant charts and the interfaces of Table 2. The label-reader inverse is bounded at each fixed level,

rather than uniformly in the level (Theorem 16.3). Beacon constants belong to the fixed reference tori. Scaling maps retain their selected radius sequences (Theorem 23.2), and calibrated-history uniqueness is within the fixed recipe (Theorem 14.29). The proof of the full trace limit uses the stronger universal subsequence property on the same table and phase. These qualifications are retained in the respective mathematical statements.

Reproducibility. The shared release contains both complete editable LaTeX trees, the unchanged expanded source, source coverage and amendment records, the focused checks and the retained arithmetic and structure suites with their receipts. Build instructions identify the toolchain and commands, and output hashes bind the resulting files. The PDFs are compiled with resolved references and inspected in rendered form. The source–readout definitions and essential proofs are included in this article; no private methodological file is required to read or reproduce its mathematical presentation.

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