Second-Order Detection by Boundary Balls in a Source–Readout Architecture

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Abstract

Can the same boundary balls that miss a metric perturbation at first order detect it at second order? We construct an explicit transverse solution of the linearized vacuum Einstein equations in Poincaré AdS_4 whose linear area and energy variations both vanish for every rest-frame ball on a fixed boundary time slice. Its wedge canonical energy is strictly positive for every radius. We compute its closed form by eliminating the second-order metric using the linearized constraint and check it independently by covariant phase space at two radii. We also obtain an explicit positive lower bound at every radius without solving the surface displacement equation. A bounded oscillatory mode supplies a second example, with numerically evaluated lower bounds at four frequency–radius products. The result distinguishes first-order invisibility from second-order detection on the same measurement regions. The equality with a boundary relative-entropy Hessian uses the holographic dictionary and requires an admissible state family.

1 Introduction

The first law of entanglement for boundary balls links two first variations to the bulk linearized Einstein equations [2]. This relationship raises a separate inverse question. A family of readouts can constrain an equation without identifying a metric perturbation. In particular, all centres and radii on one boundary time slice test the Hamiltonian constraint on the associated bulk slice, while individual area and energy readouts can both vanish on a nonzero on-shell mode [3]. A measurement family with such a kernel need not be enlarged to gain sensitivity. Its higher variations can supply additional information.

Lashkari and Van Raamsdonk identified the relative-entropy Hessian of a ball with the canonical energy of its AdS-Rindler wedge [1]. We use this identity to compare two orders of the *same* readout on the *same* regions. The comparison is carried out in the classical vacuum Einstein branch. The main example is a wedge-local, normalizable-falloff transverse perturbation. Every ball in the selected family gives zero first variations, while its canonical energy has a positive polynomial expression in the radius. A second, oscillatory example tests the role of the polynomial preparation.

In Shadow Theory's source–readout terminology, the source data in this realization are a reference solution, a metric tangent and, for a nonlinear response, an admissible completion. A ball selects the wedge, boundary port and extremal surface. The charge and area variations are the readouts. This construction uses one metric tangent across all balls and all readout orders. The relation between the linear kernel and the quadratic response is thus a property of the maps on

a common domain. The gravitational dynamics and the holographic identification enter as stated physical premises.

2 Geometry, source data and readouts

Use signature $(-+\cdots+)$, $\Lambda = -d(d-1)/(2\ell^2)$ and

$$\bar{g} = \frac{\ell^2}{z^2}(-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}\mathbf{x}^2), \qquad \mathcal{F}_{ab}(g) = G_{ab}(g) + \Lambda g_{ab}. \tag{1}$$

For a ball B of radius R centred at $\mathbf{x}_0$ at $t = 0$, write $r = |\mathbf{x} - \mathbf{x}_0|$. Its bifurcation surface $\widetilde{B}$ is the hemisphere $z^2 + r^2 = R^2$, and its bulk slice Σ_B is the enclosed half-ball. The reference Killing field is

$$\xi_B = \frac{\pi}{R}\left[(R^2 - t^2 - z^2 - r^2)\partial_t - 2t\,(z\partial_z + (\mathbf{x} - \mathbf{x}_0)\cdot\nabla_{\mathbf{x}})\right]. \tag{2}$$

Its surface gravity is 2π. We use the one-sided modular charge in this normalization. In a regulated CFT the reference modular Hamiltonian is

$$K_B = 2\pi \int_{r<R} \frac{R^2 - r^2}{2R} T_{tt}\, \mathrm{d}^{d-1}\mathbf{x} + \text{constant}. \tag{3}$$

The conformal vacuum ball construction underlying this expression is described in [4]. The full modular generator also acts on the complementary region.

Write the metric family as $g_\lambda = \bar{g} + \lambda h + \frac{1}{2}\lambda^2 k + o(\lambda^2)$. We hold the conformal boundary metric fixed and impose source-free Fefferman–Graham falloff on the variations. We assume sufficient differentiability and regularity up to the wedge boundaries in the respective gauges, and convergence of the regulated boundary expressions. The hemisphere's perturbed counterpart must be anchored at the same entangling sphere. For the gauge-shift calculation, boundary terms from the gauge vector at the conformal boundary must vanish. The examples have first-order surface displacement $O(z^3)$ in AdS_4, and $O(z^2)$ in the AdS_3 benchmark. We use these explicit falloffs, rather than a dimension-independent displacement criterion.

Set $F_B = E_B^{\mathrm{grav}} - A_B/(4G)$, with vacuum constants subtracted and with A_B evaluated on the extremal surface. For a classical vacuum family through second order the necessary equations are

$$D\mathcal{F}[h] = 0, \qquad D\mathcal{F}[k] + D^2\mathcal{F}[h,h] = 0. \tag{4}$$

We evaluate the quadratic form on linearized solutions. Interpreting it as an actual family derivative additionally requires a completion satisfying (4) on the relevant domain. Background divergence-freeness of the quadratic source is a compatibility check, and global completion is a separate existence condition.

3 The second-order identity

3.1 Covariant phase space

For $L = (\mathcal{R} - 2\Lambda)\epsilon/(16\pi G)$, define

$$\Theta(g; v) = \frac{1}{16\pi G}(\nabla_b v^{ab} - \nabla^a v^b{}_b)\epsilon_a, \tag{5}$$

$$Q_X = -\frac{1}{16\pi G}\nabla^a X^b \epsilon_{ab}, \qquad \chi(v; X) = \delta_v Q_X - \iota_X \Theta(g; v), \tag{6}$$

$$\omega(g; v, w) = \delta_v \Theta(g; w) - \delta_w \Theta(g; v). \tag{7}$$

The conventions follow covariant phase space as used in [6, 1]. The symplectic pairing W_{Σ_B} uses the charge-compatible orientation. Equivalently, its coordinate bulk term on the future slice is the integral of the densitized time component of the symplectic current. Boundary orientations below are induced by this pairing.

Theorem 3.1. *For a family satisfying the conditions above and* (4)*, the gravitational Hessian is*

$$F_B''(0) = \mathcal{E}_B(h) = W_{\Sigma_B}(\gamma, \mathcal{L}_{\xi_B}\gamma), \tag{8}$$

where $\gamma = h + \mathcal{L}_V \bar{g}$ fixes the extremal surface and obeys the linearized Hollands–Wald conditions there. In a boundary-preserving representative h, the right side equals the bulk symplectic pairing plus the oriented integral on $\widetilde{B}$ of

$$\rho(h, V) = \chi(h + \mathcal{L}_V \bar{g}; [\xi_B, V]) - \chi(\mathcal{L}_{\xi_B} h; V). \tag{9}$$

Proof. Vary the Noether current at fixed ξ_B. On shell its charge variation gives the boundary integral of χ, and its bulk variation is $\omega(g_\lambda; \partial_\lambda g_\lambda, \mathcal{L}_{\xi_B} g_\lambda)$. The asymptotic boundary term is the variation of E_B^{grav}. In surface-fixed gauge the inner boundary term is the area variation divided by $4G$, since ξ_B vanishes on the surface and its normal boost has surface gravity 2π. Off shell the identity also contains the Einstein and constraint residuals. Equation (4) removes those terms through the required order. Differentiation at the reference metric leaves only the pairing of the two first variations because $\mathcal{L}_{\xi_B}\bar{g} = 0$.

The gauge correction uses the commutator of Lie derivatives. Since $\mathcal{L}_{\xi_B}\bar{g} = 0$, we have $\mathcal{L}_{\xi_B}\mathcal{L}_V\bar{g} = \mathcal{L}_{[\xi_B, V]}\bar{g}$. For an on-shell tangent v the identity $\omega(\bar{g}; v, \mathcal{L}_X\bar{g}) = \mathrm{d}\chi(v; X)$ holds for general X. Applying it to the gauge terms in the expanded pairing and using antisymmetry gives $\mathrm{d}\rho$. The contribution at $z = 0$ vanishes by the boundary condition on V. The term at $\widetilde{B}$ yields (9). In surface-fixed gauge the mixed normal–tangential metric variations vanish and the normal block is boost invariant. The normal displacement is determined by the first variation of extremality. These conditions determine the surface data needed for (9). □

Under the stress-tensor and HRT identifications [5, 1], and for a state family whose relative entropy has two derivatives, F_B is identified to this order with $D(\rho_B(\lambda)\|\rho_B(0))$. Our convention for Fisher information is its second derivative. For a faithful regulated density matrix σ, with tangent u of trace zero, the logarithm derivative gives

$$\mathcal{I}_\sigma(u) = \int_0^\infty \mathrm{Tr}\big[u(\sigma + s)^{-1} u (\sigma + s)^{-1}\big]\, \mathrm{d}s. \tag{10}$$

The quadratic Taylor coefficient is $\mathcal{I}_\sigma(u)/2$. The first derivative vanishes at σ, so the second-order state correction cancels from the Hessian. In the continuum we require a well-defined relative-entropy Hessian and the corresponding dictionary. The unregulated formulation uses the relative entropy of states on the local algebra.

3.2 Eliminating the second-order metric

The linear charge–area identity in the convention above is

$$DF_B[v] = \frac{1}{8\pi G} \int_{\Sigma_B} \xi_B^t D\mathcal{F}_{tt}[v] (\ell/z)^{d-1}\, \mathrm{d}z\, \mathrm{d}^{d-1}\mathbf{x}. \tag{11}$$

It is an off-shell identity. The area at this order uses the reference hemisphere because its first shape derivative vanishes with fixed anchors. The boundary charge is linear in the normalizable

coefficient for the fixed flat boundary. Applying (11) to k and eliminating $D\mathcal{F}[k]$ by (4) gives the alternative evaluation

$$\mathcal{E}_B(h) = -\frac{1}{4G}A''_{\rm ext}[h] - \frac{1}{8\pi G}\int_{\Sigma_B} \xi^t_B D^2\mathcal{F}_{tt}[h,h](\ell/z)^{d-1}\,\mathrm{d}z\,\mathrm{d}^{d-1}\mathbf{x}. \tag{12}$$

Here $A''_{\rm ext}[h]$ is the area Hessian along the affine metric path $\bar{g}+\lambda h$ including the first-order motion of its extremal surface. This evaluation uses the first-order identity (11) but requires no second-order symplectic-current calculation. The identity also follows directly on the affine path, without constructing a second-order vacuum family. The differentiated Noether identity on that path gives the charge–area Hessian as canonical energy plus the integral in (12) with positive prefactor. This sign is fixed by the linear residual term in (11). Terms proportional to the background or first-order residual vanish. Thus the right side of (12) evaluates the wedge quadratic form on a linearized solution. An actual vacuum family derivative still requires a completion.

On a reflection-symmetric slice, let w denote the spatial surface displacement, A_{gg} the fixed-surface metric Hessian, $M[w]$ the mixed metric–shape derivative and $Q[w]$ the shape Hessian. Then

$$A''_{\rm ext}[h] = A_{gg} + \min_w\{2M[w]+Q[w]\} = A_{gg} - Q[w_*], \tag{13}$$

where w_* solves the linearized extremality equation. The hemisphere is totally geodesic in the hyperbolic slice. Its normal shape Hessian for proper normal displacement u is $\int_{\widetilde{B}}(|\nabla u|^2 + (d-1)u^2/\ell^2)\mathrm{d}A$, using the fixed-anchor domain. This proves nonnegativity of Q and yields a useful lower bound on (12) from any admissible trial displacement. The minimization is on the time-reflection slice. The Lorentzian extremal-area problem in general has a timelike variation as well.

4 An all-radii same-ball detection theorem

Theorem 4.1. *In AdS_4 with $G=\ell=1$, let $h_{xy}=h_{yx}=z(t^2+z^2/5+1/3)$ and set all other components to zero. For all rest-frame balls at $t=0$, both first-order readouts vanish. The wedge quadratic response is*

$$\mathcal{E}_B(h) = \frac{\pi R^6}{138600}(29R^4+110R^2+165) > 0 \qquad (R>0). \tag{14}$$

The calculation is wedge-local. For any admissible vacuum family with this tangent on the wedge, (14) *gives $F''_B(0)$.*

Proof. For the ansatz $h_{xy}=zH(t,z)$ the linearized vacuum equation reduces to $H_{tt}-H_{zz}-4H_z/z=0$. The polynomial solves it since $H_{tt}=2$ and $H_{zz}+4H_z/z=2$. The boundary energy variation vanishes because the normalizable coefficient has no tt component. In the area variation the contraction of h_{xy} with the hemisphere projector is proportional to $(x-x_0)(y-y_0)$ times an axially symmetric factor. Reflection about a diameter makes its integral zero. Translation invariance of h reduces the second-order calculation to the centred ball.

Use spherical coordinates (ϱ,θ,ϕ) on the slice with $z=\varrho\cos\theta$. Write the surface as $\varrho = R+\lambda w+O(\lambda^2)$. With $s=\sin\theta$ and $c=\cos\theta$, its minimizing first-order displacement is

$$w_* = f(\theta)\sin(2\phi), \qquad f = -\frac{R^4}{120}s^2c^3(2R^2c^2+R^2+5). \tag{15}$$

Indeed, after azimuthal integration the shape equation in the forced sector reads

$$R^4(21R^2c^2+25)s^4c^4 - 30(s^3+s)f' + 120cf - 30s^2cf'' = 0. \tag{16}$$

Substitution gives zero. The behaviour $O(s^2)$ at the pole and $O(c^3)$ at the anchor gives a regular fixed-anchor variation. The positive shape Hessian ensures uniqueness in the admissible space.

For clarity, the bulk curvature term in (12) uses

$$D^2\mathcal{F}_{tt}[h,h]\big|_{t=0} = \frac{z^4}{30}(69z^4 + 130z^2 + 45). \tag{17}$$

The transverse disk integration of ξ_B^t gives $\pi^2(R^2-z^2)^2/(2R)$. The surface terms follow by differentiating the induced area density $\sqrt{q_{\theta\theta}q_{\phi\phi} - q_{\theta\phi}^2}$ on the radial graph. With I denoting the integral including its prefactor $1/(8\pi)$ in (12), the resulting pieces are

$$A_{gg} = -\frac{4\pi R^6}{2027025}(3717R^4 + 15210R^2 + 16445), \tag{18}$$

$$Q[w_*] = \frac{\pi R^6}{4054050}(567R^4 + 2730R^2 + 3575), \tag{19}$$

$$I = \frac{\pi R^6}{41580}(69R^4 + 286R^2 + 297). \tag{20}$$

Thus $-A_{gg}/4 + Q[w_*]/4 - I$ equals (14). □

A simpler positivity proof sets the trial displacement to zero in (13). Its lower bound is

$$\mathcal{E}_B(h) \geq -A_{gg}/4 - I = \frac{\pi R^6}{8108100}(1413R^4 + 5070R^2 + 7865) > 0. \tag{21}$$

This holds for every $R > 0$ and separates the source of the sign from the precise value of the extremal-surface response. Surface relaxation adds $Q[w_*]/4$.

Independent evaluations. The direct symplectic calculation uses the displayed displacement as the normal gauge data at $\widetilde{B}$. The mixed and normal-block conditions give the required first normal derivatives of the other components. Integrating the current and the correction (9) yields $38\pi/17325$ at $R=1$ and $8552\pi/17325$ at $R=2$. At $R=1$ the split is $\pi/675 + 37\pi/51975$. Varying the free gauge data in this check leaves the sum unchanged. The AdS_3 black-brane benchmark of [1] has split $-2R^4/15 + 7R^4/45 = R^4/45$ in units $G = \ell = 1$. The latter is also the second derivative of the exact family's leading relative-entropy expression $\lambda^2 R^4/90$. Omitting the surface correction reverses the sign of this benchmark. The vacuum and a boundary-trivial pure-gauge tangent give zero response. These comparisons test the convention and the cancellation of gauge-dependent pieces.

5 Bounded oscillatory comparison

Consider the time-even solution

$$h_{xy} = z\cos(\omega t)j(\omega z), \qquad j(u) = 3\frac{\sin u - u\cos u}{u^3}, \qquad j(0) = 1. \tag{22}$$

The regular solution obeys $j'' + 4j'/u + j = 0$. It has bounded boundary shear, is smooth at every finite $z > 0$, and obeys the same first-order invisibility argument. Spatial homogeneity and regularity on individual wedges do not supply a global finite-norm CFT state family.

To evaluate (12), use trial shapes in the span of

$$\sin(2\phi)\sin^2\theta\cos^3\theta\,(\cos^2\theta)^m, \qquad 0 \leq m < N.$$

Minimizing $2M + Q$ in this nested space bounds the true minimum from above and hence bounds the response from below. For example, with $Q_{ij} = Q(b_i, b_j)$ and $M_i = M[b_i]$, the trial contribution to the response is $M^{\mathsf{T}} Q^{-1} M/(4G)$. The exact polynomial displacement lies in the first two basis functions and reproduces (14) at $R = 1$. For the oscillatory solution the evaluations below use $R = 1$ and high-precision quadrature. They are numerical evaluations of a lower-bound formula, without certified quadrature intervals.

ωR	$N = 1$	$N = 3$
1/2	0.032740	0.032740
1	0.030157	0.030159
2	0.022065	0.022090
4	0.009119	0.009254

Translation invariance extends each entry to every centre for the corresponding parameter choice. These values support positive detection at the four tested frequency–radius products. The analytic all-radii theorem concerns the polynomial mode.

6 What the readouts recover

Let $\mathcal{M}_1$ assign to a tangent the pair $(DE_B, DA_B/(4G))$ for each ball of the fixed-slice family. Let $\mathcal{M}_2$ assign its canonical energy on those same wedges. The polynomial solution proves

$$\ker \mathcal{M}_1 \;\not\subset\; \{h : \mathcal{M}_2(h) = 0\}.$$

Its positive response also separates it from boundary-trivial pure gauge. The response is quadratic, so it has the same value on h and $-h$. The example establishes detection of a transverse component, while reconstruction of the perturbation from such data requires additional information. In particular, there is no general kernel or stability theorem here for the restriction of $\mathcal{M}_2$ to $\ker \mathcal{M}_1$.

The distinction matters for the source–readout architecture. Selecting additional regions, times or frames can change the linear map. Raising the response order can also change what is detectable while the regions and reference state are held fixed. In this realization the difference is explicit: the charge couples linearly to the energy density and the first area variation cancels by reflection, whereas quadratic bulk curvature and surface deformation contribute to the same ball's canonical energy.

The boundary interpretation requires a compatible CFT state family, differentiable relative entropy, and the second-order area and stress-tensor dictionary. The polynomial mode has acceptable near-boundary falloff on each wedge but grows in the bulk and has unbounded boundary shear in time. The oscillatory mode is bounded at the boundary but spatially homogeneous. The calculations supply gravitational tangents and responses. Constructing finite-norm preparations with controlled second-order completion, and characterizing the residual kernel of the quadratic map, are natural next problems.

7 Conclusion

The same family of boundary balls can have zero first area and energy variations and positive second-order gravitational response. The all-radii polynomial theorem proves this directly, with a positive fixed-surface lower bound and independent covariant-phase-space checks at two radii.

The oscillatory comparison retains the first-order cancellation and yields positive numerical lower-bound evaluations at four parameter values. In the holographic setting, whenever the state and completion hypotheses hold, the relevant boundary observable is the relative-entropy Hessian. The result gives a specific example of information accessible at a higher order of a fixed source–readout map.

Verification

The calculations use symbolic curvature, surface differentiation and exact integration for the polynomial example. A separate symplectic-current implementation evaluates the two specified radii, the black-brane benchmark and gauge controls. The oscillatory comparison uses high-precision quadrature and nested trial spaces. Suite identifiers for these calculations are E3 (benchmark), E4 (controls), E5 and E5b–c (polynomial), and E6 (oscillatory). The derivative convention is tested in E1 and the symplectic and gauge identities in E2.

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