Modular Transport and Spherical Holographic Entanglement Entropy in a Source–Readout Architecture

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Abstract

We reconstruct the Casini–Huerta–Myers spherical entanglement derivation as a joint source–readout realization. A single vacuum preparation supplies a restricted state on a causal diamond and, through conformal transport, a thermal state on the hyperbolic cylinder. The algebraic recovery fixes the physical inverse temperature $2\pi R$ and requires no continuum density matrix. With the classical holographic dictionary supplied, a bulk coordinate realization recovers the same time normalization and the regulated horizon entropy. We separate localization of the observable algebra from the choice of counting regulator. An exact coefficient formula identifies which analytic cutoff changes preserve the constant in odd dimension, while the even-dimensional logarithmic coefficient is invariant within the stated class. The construction identifies the assumptions carried by each readout and gives explicit checks on their compatibility.

1 Introduction

A global vacuum and a thermal state can describe the same local observables once the algebra and its evolution are specified. Spherical entanglement supplies an exact case. Casini, Huerta and Myers (CHM) map the vacuum on a ball's causal development to a thermal conformal field theory on hyperbolic space and, under holographic assumptions, compute its entropy from a Killing horizon [1]. This argument involves three operations with different domains: restriction of a state, transport of its dynamics, and regulated counting. In particular the first two operate on continuum algebras, while the third needs additional data.

We formulate the chain within the Shadow Theory source–readout architecture [2, 3]. The construction carries one preparation through the local and thermal descriptions and checks that the bulk extension realizes the same boundary transport. It also identifies where two regulators that agree at leading order can differ at the order relevant to entropy. The CFT, modular-theoretic and gravitational inputs are stated as hypotheses. We use units $\hbar = c = k_B = 1$, boundary spacetime dimension $d \geq 2$, ball radius $R > 0$, and Lorentzian signature $(- + \cdots +)$.

2 Source data, apertures and return maps

Let the prepared source datum be

$$z = (\mathcal{A}, U, \Omega;\, g_{\mathrm{bulk}}, \mathcal{L};\, \mathfrak{r}).$$

The first entries give a conformally covariant local net, its vacuum representation and its vacuum vector. The bulk entries are used only for theories with the stipulated classical AdS dual. The separately specified datum $\mathfrak{r}$ specifies a counting prescription, including cutoffs and any boundary data needed to define the regulated functional. Admission of these data is an assumption of the realization. In particular the abstract source architecture does not select a unique net, action or counting prescription.

For each region O an aperture is the inclusion $i_O : \mathcal{A}(O) \hookrightarrow \mathcal{A}_{\text{global}}$ in the vacuum representation. The state readout is $\omega_O = \omega \circ i_O$, with $\omega(A) = \langle \Omega, A\Omega \rangle$. Given a conformal identification $\chi : O \to O'$, write j_χ for the induced algebra isomorphism. The state on the destination is $\omega_{O'} = \omega_O \circ j_\chi^{-1}$. Thus for any finite product,

$$\omega_{O'}(j_\chi(A_1) \cdots j_\chi(A_n)) = \omega_O(A_1 \cdots A_n).$$

The return map is j_χ^{-1} on this transported algebra. Its domain is the image of the localized algebra, so this identity gives recovery of local correlation data. A global extension of the restricted state is separate data.

The dynamical readout is obtained by conjugation with j_χ. In particular the diagram for the modular groups commutes:

$$j_\chi \circ \sigma_s^{\omega_O} = \sigma_s^{\omega_{O'}} \circ j_\chi.$$

Indeed on the cyclic GNS vectors the rule $A\Omega \mapsto j_\chi(A)\Omega'$ preserves the inner product, extends to a unitary, and intertwines the densely defined Tomita maps $A\Omega \mapsto A^*\Omega$. Closure and polar decomposition intertwine the modular operators. The entropy readout has a different domain. It takes a regulated realization as input. The commutative diagram for continuum transport does not by itself identify two independently regulated entropy functionals.

3 Vacuum restriction and geometric thermality

Let $D = \{(t, \boldsymbol{x}) : |t| + r < R\}$, with $r = |\boldsymbol{x}|$. Assume the vacuum is cyclic and separating for the relevant local von Neumann algebras, the net admits the conformal wedge-to-diamond map, and the vacuum wedge satisfies Bisognano–Wichmann [5, 6]. For the right wedge $X^1 > |X^0|$, the future boost sends $X^1 \pm X^0$ to $e^{\pm\eta}(X^1 \pm X^0)$. With Heisenberg evolution $\alpha_t(A) = e^{itH} A e^{-itH}$ and modular convention $\sigma_s = \operatorname{Ad} \Delta^{is}$, the wedge modular group is the boost at rapidity $-2\pi s$.

An explicit wedge map uses $b = -e_1/(2R)$, $D_b = 1 - 2b \cdot X + b^2 X^2$, and

$$x = \frac{X - bX^2}{D_b} + 2R^2 b.$$

On the wedge $4R^2 D_b = 4R^2 + 4RX^1 + X^2 > 0$. The null-tip identities are $(R \pm t)^2 - r^2 = 2R(X^1 \pm X^0)/D_b$. The connected wedge contains $X = 2Re_1$, which maps to the center, so its image lies in D. For the inverse set $Y = x - 2R^2 b$ and use $X = (Y + bY^2)/(1 + 2b \cdot Y + b^2 Y^2)$. The inverse denominator is $((R - x^1)^2 + |\boldsymbol{x}_\perp|^2 - t^2)/(4R^2) > 0$ on D. The null-tip identities now imply $X^1 \pm X^0 > 0$. Differentiating the conformal map gives $D_b^{-2}\eta$ for the metric pullback.

For the second map put $v = \tau/R$. On $\mathbb{R} \times H^{d-1}$, use

$$t = \frac{R \sinh v}{\cosh u + \cosh v}, \qquad r = \frac{R \sinh u}{\cosh u + \cosh v}, \qquad r \pm t = R \tanh\left(\frac{u \pm v}{2}\right).$$

The angular coordinates are unchanged. The inverse follows by taking $\operatorname{artanh}((r \pm t)/R)$ and adding or subtracting. The pullback of the Minkowski metric is

$$(\cosh u + \cosh v)^{-2} \left[-\mathrm{d}\tau^2 + R^2(\mathrm{d}u^2 + \sinh^2 u \, \mathrm{d}\Omega_{d-2}^2)\right].$$

For $d = 2$, the radial description uses two directions from the center, giving the full hyperbolic line. Conformal covariance supplies the net isomorphism, including the Weyl weights in a field description. State transport gives the corresponding smeared correlation functions.

The pushforward of the unit future boost to the diamond is the vector field $R\partial_\tau$, where ∂_τ in Minkowski coordinates is

$$\frac{1}{2R^2}\big[(R^2 - t^2 - r^2)\partial_t - 2tr\partial_r\big].$$

This follows by differentiating the wedge map or the diamond null-coordinate flow. It fixes the stress-tensor charge on $t = 0$: the formal one-sided dimensionless generator has local expression

$$2\pi \int_{r<R} \frac{R^2 - r^2}{2R}\, T_{00}(0, \boldsymbol{x})\, \mathrm{d}^{d-1}x$$

up to scalar and regulator-dependent terms. The continuum modular implementer is the two-sided operator on the vacuum representation.

Theorem 3.1. *With these hypotheses the transported state on the hyperbolic cylinder is Kubo–Martin–Schwinger (KMS) at $\beta = 2\pi R$ for future τ-translations.*

Proof. The two algebra transports give $\sigma_s = \alpha_{-2\pi Rs}$. For analytic elements the modular boundary identity is $\omega(A\sigma_{-i}(B)) = \omega(BA)$. Transporting the modular strip by this rescaling gives the KMS strip of width $2\pi R$ for $\omega(A\alpha_\tau(B))$, with the reversed product on its upper edge. The algebraic proof controls the combined boundary values. It requires no separate analytic continuation of each Weyl factor in a field-coordinate expression. □

If the diamond algebra is type III, the algebra has no intrinsic density matrix defined with a normal semifinite trace. A split inclusion, where available, can supply a type I intermediate algebra for a separated inner diamond. On a type I factor with a faithful reduced density matrix ρ, modular action reduces to $\rho^{is} A \rho^{-is}$. In a cyclic and separating purification with nonzero Schmidt weights p_i, the modular eigenvalues in the Schmidt matrix-unit basis are p_i/p_j. This is the precise connection to finite modular readouts in CSCF [4]. The choice of type I factor and finiteness of its entropy require separate control. The continuum KMS result alone supplies neither a thermal trace nor a finite partition function on the infinite hyperbolic volume.

4 Bulk realization and horizon entropy

Assume the boundary vacuum has a classical AdS dual of radius L, the boundary frame is chosen through the bulk defining function, and the thermal entropy of the dual saddle is its stationary horizon entropy at leading bulk semiclassical order. Write $h = \sqrt{\rho^2 - L^2}$ for $\rho > L$. The hyperbolic chart maps to Poincaré coordinates by

$$Q = \rho\cosh u + h\cosh(\tau/R), \qquad z_P = \frac{RL}{Q}, \quad t = \frac{Rh\sinh(\tau/R)}{Q}, \quad r = \frac{R\rho\sinh u}{Q}.$$

Substitution into $L^2 z_P^{-2}(\mathrm{d}z_P^2 - \mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega_{d-2}^2)$ gives

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{f(\rho)} - f(\rho)\mathrm{d}\widetilde{\tau}^2 + \rho^2\mathrm{d}H_{d-1}^2, \qquad f(\rho) = \rho^2/L^2 - 1, \quad \widetilde{\tau} = L\tau/R.$$

The limit $\rho \to \infty$ recovers the conformal map of the preceding section. The inverse radial relation is

$$\rho^2 - L^2 = \frac{L^2}{4R^2 z_P^2}\big[(R - t)^2 - r^2 - z_P^2\big]\big[(R + t)^2 - r^2 - z_P^2\big].$$

On the exterior connected to the boundary diamond both factors are positive. Its null boundaries meet on $t = 0,\ r^2 + z_P^2 = R^2$. The chart therefore reads a wedge of the same pure AdS geometry. The bulk Killing generator restricts to the same future boundary flow with the $z_P\partial_{z_P}$ term included before restriction. Euclidean smoothness requires period $4\pi/f'(L) = 2\pi L$ in $\widetilde{\tau}$, giving $2\pi R$ in τ. Independently the surface gravity for ∂_τ is $1/R$.

For a local Lagrangian scalar $\mathcal{L}(g, R_{abcd})$ the stationary prescription is [8]

$$S = -2\pi \int_\Sigma \sqrt{\gamma}\, \frac{\partial \mathcal{L}}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd}, \qquad \epsilon_{ab}\epsilon^{ab} = -2.$$

For Einstein gravity the contraction is $-1/(8\pi G)$, giving area divided by $4G$, in agreement with the spherical Ryu–Takayanagi surface [9]. For Gauss–Bonnet with coefficient $\lambda L_0^2/[(d-2)(d-3)]$ inside the Einstein normalization, use $d \geq 4$, $f_\infty = L_0^2/L^2$ and $1 - f_\infty + \lambda f_\infty^2 = 0$. The induced section has scalar curvature $-(d-1)(d-2)/L^2$. The Gauss–Bonnet stationary entropy density is $1/(4G)$ times $1 + 2\lambda L_0^2 \mathcal{R}_\Sigma/[(d-2)(d-3)]$ [10]. Hence

$$a_d^* = \frac{\pi^{d/2} L^{d-1}}{8\pi G \Gamma(d/2)} \left(1 - \frac{2(d-1)\lambda f_\infty}{d-3}\right).$$

The independent curvature contraction agrees with this density. The identification with the definition through the on-shell AdS Lagrangian uses the vacuum equation. The calculation concerns stationary spherical horizons and makes no general claim about entropy functionals for arbitrary entangling surfaces.

5 Counting and matching the cutoffs

At the bifurcation surface $z_P = R/\cosh u$. Thus $z_P = \delta$ gives $\cosh u_{\rm bulk} = R/\delta$. On the boundary at $\tau = 0$, $r = R\tanh(u/2)$, so the sharp boundary radius $r = R - \delta$ gives $e^{-u_{\rm CFT}} = \delta/(2R - \delta)$. Write $\varepsilon = \delta/R$. In the common variable $X = \sinh u_{\max}$ the prescriptions become

$$X_{\rm bulk} = \varepsilon^{-1}\sqrt{1-\varepsilon^2}, \qquad X_{\rm CFT} = \varepsilon^{-1}\frac{2(1-\varepsilon)}{2-\varepsilon}.$$

Both give $e^{-u_{\max}} \sim \varepsilon/2$, with bulk minus CFT equal to $-\varepsilon^2/4 + O(\varepsilon^4)$ in that variable.

The spatial cutoff applied for every τ defines a timelike tube U in the diamond. It contains the entire open segment $r = 0,\ |t| < R$. A point at (t_0, r_0) is spacelike to that segment precisely when $r_0 \geq R + |t_0|$, with the usual open-region convention for causal complements. By inclusion the double causal complement of the tube has interior D. The timelike envelope can also be identified directly. For any $p = (t, \boldsymbol{x}) \in D$, choose two points on the axis inside U, one strictly to its timelike past and one strictly to its timelike future. The two line segments through p form a piecewise timelike curve. Contracting its spatial component linearly to the axis keeps the endpoints fixed and preserves timelikeness. Corners can be smoothed within the open timelike cones. Thus p is in the timelike envelope of U. Conversely a future timelike curve between any two points of U stays in the causally convex diamond. Under the hypotheses of the applicable timelike-tube theorem [7], this gives $\mathcal{A}(U) = \mathcal{A}(D)$. The smaller diamond of radius $R - \delta$ shares the cutoff tube's spatial section at $\tau = 0$, while its full image under conformal transport has finite time extent. The equality of those sections does not identify the two spacetime apertures or specify their regulated traces. Even the smaller diamond's local algebra remains type III under the same net hypothesis.

6 Universal coefficients and a finite-jet criterion

For the constant Wald density, the entropy is

$$S = P_d I_d(X), \quad P_d = \frac{2\pi a_d^*}{\pi^{d/2}}\Gamma(d/2)\,\mathrm{Vol}(S^{d-2}), \qquad I_d(X) = \int_0^X \frac{x^{d-2}}{\sqrt{1+x^2}}\,\mathrm{d}x.$$

In Einstein gravity $P_d = L^{d-1}\,\mathrm{Vol}(S^{d-2})/(4G)$. For $d > 2$ the leading term is $P_d X^{d-2}/(d-2)$, giving the regulated area law. In even $d = 2m$ the integrand expansion has an x^{-1} term with coefficient $(-1)^{m-1}\binom{2m-2}{m-1}/4^{m-1}$. The gamma duplication formula reduces its product with P_d to $(-1)^{m-1}4a_d^*$. The resulting term is logarithmic also in $d = 2$, where $a_2^* = c/12$.

For odd $d = 2n+3$ let $y = \sqrt{1+x^2}$. Then $I_d = \int_1^{\sqrt{1+X^2}} (y^2-1)^n\,\mathrm{d}y$. The upper endpoint has only odd powers and inverse odd powers of X in its expansion. The constant comes from the lower endpoint and equals $-(-1)^n\int_0^1(1-y^2)^n\,\mathrm{d}y$. Evaluating the beta integral and multiplying by P_d gives $(-1)^{(d-1)/2}2\pi a_d^*$.

Theorem 6.1. *Let $X = g(\varepsilon)/\varepsilon$, with g analytic at zero and $g(0) = 1$. In even dimension the logarithmic coefficient of $P_d I_d(X)$ is invariant. In odd dimension the change in the constant from the reference $g = 1$ is*

$$\Delta S_0 = P_d \sum_{k=0}^{(d-3)/2} \frac{b_k}{d-2-2k}[\varepsilon^{d-2-2k}]\,g(\varepsilon)^{d-2-2k}, \qquad b_k = (-1)^k\binom{2k}{k}4^{-k}.$$

Proof. The powers of X in the divergent expansion have exponents $p = d-2-2k > 0$ and coefficients b_k/p, obtained by integrating the large-x expansion. In even dimension $\log X = \log(1/\varepsilon) + \log g$; the second term is analytic and the power terms have Laurent expansions. In odd dimension a term $X^p = \varepsilon^{-p}g^p$ contributes precisely its indicated Taylor coefficient to order zero. The remainder vanishes at large X. □

Preservation at a fixed odd dimension is equivalent to the vanishing of the displayed sum. Evenness of g suffices. For example, $g = 1+\varepsilon^3$ preserves the constant in $d = 3$. Non-even maps can also preserve the constant by cancellation within the relevant jet. In $d = 5$, writing $g = 1 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + O(\varepsilon^4)$, the shift in units of P_5 is $a_3 + 2a_1a_2 + a_1^3/3 - a_1/2$. For $g = 1+\varepsilon+\varepsilon^3/6$ this vanishes. More generally the criterion only uses the jet through order $d-2$. If a single analytic g must preserve the constants in *every* odd dimension with nonzero a_d^*, evenness is also necessary. Indeed if its first odd Taylor coefficient has order j, take $d = j+2$. For exponents $p < j$ the required odd coefficients vanish, while the $p = j$ term contributes exactly $P_d[\varepsilon^j]g$. This yields a nonzero shift.

For the specified sharp CFT map $g'(0) = -1/2$, whereas the bulk map is even. In $d = 3$ the formula gives $-2\pi a_3^*$ for the bulk prescription and $-3\pi a_3^*$ when the sharp CFT cutoff relation is inserted into the *same* horizon integral. This is a comparison of cutoff substitutions in a fixed functional. An independent field-theory entropy calculation would also have to specify boundary contributions and the regulated observable. The result therefore does not identify the second number with a universal CFT entropy.

7 Anomaly check and verification

For even d, CHM's separate CFT route maps the diamond to the static de Sitter patch and continues it to the round Euclidean sphere. For example the map is $t = R\cos\theta\sinh(\tau/R)/(1+\cos\theta\cosh(\tau/R))$

and $r = R\sin\theta/(1+\cos\theta\cosh(\tau/R))$. The Euclidean period is $2\pi R$ by regularity. The integrated Killing energy is finite, as its radial integral is proportional to $\int_0^{\pi/2}\cos\theta\sin^{d-2}\theta\,\mathrm{d}\theta = 1/(d-1)$. The sphere is conformally flat and its normalized Euler integral is two. With the CHM anomaly convention the scale derivative of $\log Z$ is $4(-1)^{d/2-1}A$, giving the same log coefficient when $a_d^* = A$ [1, 11]. The check is independent of the bulk cutoff substitution. In $d = 4$ the hyperbolic energy term alone has log coefficient $-(3a+c)/2$ when written with $\log(R/\delta)$. Recovering the full $-4a$ consequently requires an additional $(c-5a)/2$ in the same convention. A sharp volume truncation without the required boundary contribution is insufficient.

The archived verification checks the conformal maps and their inverses, boost orientation, cutoff relations, bulk Killing generator, surface gravity, Einstein and Gauss–Bonnet densities, entropy coefficients and anomaly arithmetic. It also retains deliberately failing controls, including off-shell action comparisons and cutoff substitutions that change the odd-dimensional constant. A fresh extraction reproduced all 130 recorded outcomes, comprising 106 exact symbolic checks, 13 numerical diagnostics and 11 expected-failure controls. Finite checks supplement the dimension-independent arguments above. The necessity statement across odd dimensions follows from the first nonzero odd Taylor coefficient argument. The fixed-dimension cancellation example was also checked by exact expansion of the original integral.

8 Conclusion

The source–readout construction recovers the spherical CHM chain under its field-theory and holographic hypotheses. The local vacuum and hyperbolic thermal state are related exactly through their observable algebras and geometric evolution. The bulk chart recovers the same normalization and gives the horizon entropy once the classical dual and counting prescription are supplied. The distinction between an algebra aperture and a counting regulator is essential to the recovery claim. The finite-jet criterion isolates the ambiguity of a cutoff substitution and specifies which part of the answer survives it. A natural further test is to compare regulated observables defined by mutual information with the horizon prescription, including the finite odd-dimensional term.

A Geometric checks and normalization

Let $x^\pm = r \pm t$. Translation by $R\eta$ on the hyperbolic cylinder induces

$$x^\pm \longmapsto R\,\frac{x^\pm/R \pm \tanh(\eta/2)}{1 \pm (x^\pm/R)\tanh(\eta/2)}.$$

The inverse takes η to $-\eta$, and the composition law follows from addition of rapidities. The radial expression is interpreted with angular continuation through the origin. In the bulk the corresponding forward generator is

$$\zeta = \frac{(R^2 - t^2 - r^2 - z_P^2)\partial_t - 2t(\boldsymbol{x}\cdot\nabla_{\boldsymbol{x}} + z_P\partial_{z_P})}{2R^2}.$$

Its Lie derivative of the flat metric in Poincaré coordinates is $-2t/R^2$ times that metric. Its derivative on the warp factor is $2t/R^2$ times the warp factor, proving the Killing identity. Near the horizon take $\rho = L + \xi^2/(2L)$. The leading Euclidean radial and time metric is $\mathrm{d}\xi^2 + \xi^2\mathrm{d}\widetilde{\tau}_E^2/L^2$, fixing the smooth period.

For clarity about conventions in [1], the conformal parameter in the wedge map above sends the right wedge to the diamond. The parameter of Δ^{is} runs opposite to forward geometrical time

with our Heisenberg convention. The bulk time rescaling is $\widetilde{\tau} = L\tau/R$. In the cylinder variant of CHM, for cap opening θ_0 the first form of (2.36) reads

$$\Omega^2 = \frac{\sin^2\theta_0}{(\cosh u + \cos\theta_0 \cosh v)^2 + \sin^2\theta_0 \sinh^2 v}, \qquad v = \tau/R_{\rm hyp},$$

where $R_{\rm hyp}$ denotes the hyperbolic time scale in that map. These formula choices preserve the physical temperature and the entropy coefficients.

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