Local Horizon Thermodynamics and Einstein Gravity in a Source–Readout Architecture

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Abstract

We reconstruct Jacobson's local horizon derivation of Einstein gravity in a source–readout architecture. A single prepared null pencil supplies the boost-energy flux, Unruh temperature and area change. In a shrinking-pencil limit, the normalized Clausius residual is the null contraction of a symmetric tensor. Its vanishing on a determining set of directions is equivalent to the Einstein equation under local stress-energy conservation and the stipulated area-entropy and temperature laws. We characterize determining sets in four dimensions, give an explicit nine-direction reconstruction, and identify a four-dimensional kernel for any nondegenerate circle of directions. For a prescribed boost and temperature readout we compute the first correction to the area-based Clausius relation along an equilibrium generator. Its extension to a pencil of finite transverse width requires a separate estimate. A boost-weighted expansion prescription removes the cubic term and yields a signed remainder within the prescribed readout model.

1 Introduction

Jacobson derived the Einstein equation by imposing a local Clausius relation on Rindler horizons through each spacetime point [1]. The heat is the boost-energy flux through a null pencil. The temperature uses the Unruh normalization, and the entropy variation is proportional to the pencil's area change. Null focusing supplies the geometric response. The derivation then turns on the compatibility of three quantities computed for the same local preparation.

We express that compatibility as a condition on the readouts of a common source. This has two consequences. First, the local thermodynamic condition becomes a tensor reconstruction problem with an explicit kernel when the available directions are restricted. Second, the use of one preparation exposes the difference between the leading law and its continuation to finite affine length or transverse width. We prove the leading equivalence in both directions, recover the coupling with units restored, and calculate the first correction in a regime where it is controlled.

The construction is a realization of the Shadow Theory source–readout architecture [2]. It uses the same distinction between source data and observable projections as the transport realization of the spin-two route to Einstein gravity [3]. The local boost normalization is also compatible with the modular convention in the spherical entanglement realization [4, 5]. The thermodynamic and geometric hypotheses used here are stated explicitly. Work is in four dimensions with signature $(-+++)$, $c = k_B = 1$ until units are restored, and $\hbar$ retained.

2 Common source and local preparation

Let U be a connected spacetime region. The index of an aperture is a pair $(p, [k])$, with $p \in U$ and $[k]$ a future null ray at p. An aperture selects a marked local preparation. Its physical realization uses a common history $z = (g, T, \omega, \eta, \mathcal{P})$. Here g is a smooth Lorentzian metric, T a symmetric stress tensor, ω the field state used for the local thermal limit, $\eta > 0$ a constant entropy per unit area, and $\mathcal{P}$ an assignment of horizon data to the apertures. The metric sector admits off-shell data. The matter tensor is treated as a prescribed classical tensor or expectation value, with fluctuations outside the calculation.

Choose a representative k and a spacelike screen complementary to k in $k^{\perp}$. A spacelike two-surface P through p has that screen as its tangent plane. Choose the null normal on the selected side, and extend it by affinely parametrized null geodesics from P. Denote the affine parameter by λ, zero on P, and use the segment $-h \le \lambda \le 0$, $h > 0$. The extension defines a twist-free null hypersurface locally, before caustics. The future oriented measure is $\mathrm{d}\Sigma^b = k^b \mathrm{d}\lambda\, \mathrm{d}A_\lambda$. On the hypersurface the prescribed boost is

$$\chi = -\kappa\lambda k, \qquad \kappa > 0.$$

In flat spacetime take $k = \partial_t - \partial_x$ on the past horizon of the right wedge, $t = -x$, $x > 0$. There $\chi = \kappa(x\partial_t + t\partial_x)$ is the restriction of the exact Killing boost. In the curved construction χ is specified on the null pencil and its tangent-space normalization is inherited from this model.

The initial expansion and shear vanish at the central point. To see that the preparation exists, take Riemann normal coordinates and use a local planar embedding at second order. Its null second fundamental form then vanishes at p. This pointwise condition suffices for the leading law when the transverse domains shrink with centered first moments. Stronger uniform equilibrium conditions can change the higher-order analysis.

Write the three projections of z at a selected aperture as

$$\pi_Q = \int T_{ab}\chi^a \mathrm{d}\Sigma^b, \qquad \pi_T = \frac{\hbar\kappa}{2\pi}, \qquad \pi_S = \eta\big[A(0) - A(-h)\big].$$

The same g, T, η are used at every aperture, and the same boost is used in π_Q, π_T. Evaluation on the realized pencil gives Jacobson's heat, temperature and entropy expressions [1]. This is the recovery map at the level of observables. The leading coefficient of $\pi_Q - \pi_T\pi_S$ is the joint readout used to select histories. Its zero set is imposed as a physical law.

3 Boost normalization and temperature

The Minkowski vacuum restricted to a wedge is thermal in the Kubo–Martin–Schwinger sense for the future boost with inverse rapidity temperature 2π [6]. In the convention $\alpha_s(A) = e^{isH}Ae^{-isH}$, the modular group is the future boost evaluated at rapidity $-2\pi s$. Since the χ-flow parameter advances rapidity at rate κ, the thermal energy scale is $\hbar\kappa/(2\pi)$. The curved-space readout uses this as a local scaling-limit premise. The calculation requires neither a global boost Killing field nor a finite-length correction to the temperature.

For a flat boost orbit at distance ρ, the proper acceleration is $1/\rho$ and $|\chi| = \kappa\rho$. Multiplying the proper temperature by $|\chi|$ gives the displayed boost temperature. This normalization also fixes the flux-to-temperature ratio,

$$\frac{\pi_Q}{\pi_T} = \frac{2\pi}{\hbar}\int_{-h}^{0} (-\lambda) T_{kk}\, \mathrm{d}\lambda\, \mathrm{d}A_\lambda, \qquad T_{kk} = T_{ab}k^a k^b.$$

A positive rescaling of χ multiplies both readouts by the same factor. Under $\lambda \mapsto a\lambda$, $k \mapsto k/a$ with $a > 0$, the integrand and measure together leave the ratio invariant. If the heat and temperature use independently chosen positive normalizations κ_Q, κ_T, the inferred coupling is multiplied by κ_Q/κ_T. Reversing the heat boost while retaining the original positive temperature changes the sign of the inferred coupling. That is a mismatched pairing. A direction-dependent mismatch can give angular data of degree greater than two, for which a symmetric tensor representation need not exist.

4 Leading Clausius coefficient

For an affinely parametrized, twist-free congruence the Raychaudhuri and area equations are

$$\dot{\theta} = -R_{kk} - \tfrac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab}, \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{d}A_\lambda = \theta\,\mathrm{d}A_\lambda.$$

Use the curvature convention in which positive R_{kk} focuses the congruence. With $\theta(0,p) = 0$, the central generator has $\theta(\lambda,p) = -\lambda R_{kk}(p) + O(\lambda^2)$. Let $P_\ell \subset P$ be a centered transverse domain of size ℓ and area A_0. Assume a fixed smooth congruence and bounded local derivatives in its screen coordinates. Centering means that the transverse coordinates have zero first moment with respect to $\mathrm{d}A_0$. Taylor expansion gives

$$\frac{1}{A_0}\int_{P_\ell} \theta(0,y)\,\mathrm{d}A_0(y) = O(\ell^2).$$

A transverse derivative of the initial expansion is allowed here. It can contribute to an uncentered pencil at leading order. With $\ell = O(h)$, integration of the area equation gives

$$\frac{\delta A}{A_0} = \frac{h^2}{2}R_{kk}(p) + o(h^2).$$

Indeed the initial-expansion contribution is $O(h\ell^2)$; variation of the linear focusing coefficient tends to zero with ℓ; and the terms quadratic in expansion and shear vanish at the center of the initial section. The flux expansion on the same domain is

$$\frac{\delta Q}{A_0} = \frac{\kappa h^2}{2}T_{kk}(p) + o(h^2).$$

It follows that the normalized residual has limit

$$C(p,k) = \lim_{h\to 0}\frac{2(\pi_Q - \pi_T\pi_S)}{\kappa A_0 h^2} = T_{kk}(p) - \frac{\hbar\eta}{2\pi}R_{kk}(p). \tag{1}$$

A relative $o(1)$ error in the thermal readout does not change this limit. This is precisely the order at which the local scaling-limit temperature is needed. The area entropy law and vanishing of C are constitutive inputs. The following theorem specifies their consequence.

Theorem 4.1. *Suppose g is C^3, T is C^1 and conserved, and the thermal and entropy readouts above hold at leading order with constant $\eta > 0$. If $C(p,k) = 0$ at every p for all future null directions, or on a determining set as in Section 5, then*

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar\eta}T_{ab}$$

for a constant Λ on U. Conversely this tensor equation implies $C(p,k) = 0$ at every aperture.

Proof. Let $E_{ab} = R_{ab} - 2\pi T_{ab}/(\hbar\eta)$. In an orthonormal frame, a normalized future null vector is $k = (1, n)$, $|n| = 1$. Vanishing of $E(k, k)$ for all n and $-n$ gives $E_{0i} = 0$. Then $E_{ij}n^i n^j = -E_{00}$ for all unit n, giving $E_{ij} = -E_{00}\delta_{ij}$. Thus $E = \phi g$. A determining set has the same implication. Taking the divergence of $R_{ab} - \phi g_{ab} = 2\pi T_{ab}/(\hbar\eta)$ and using the contracted Bianchi identity yields $\mathrm{d}(\phi - R/2) = 0$. Connectedness gives $\phi = R/2 - \Lambda$. Conversely a metric-proportional E has zero null contraction. □

Without conservation the readouts recover the trace-free equation with an undetermined scalar function. A constant vacuum-energy contribution proportional to g has zero null flux and can be absorbed into the cosmological constant. Restoring units with $T_\chi = \hbar c\kappa/(2\pi k_B)$ for geometric κ and physical entropy density η, the curvature coupling is $2\pi k_B/(\hbar c\eta) = 8\pi G/c^4$, hence

$$G = \frac{k_B c^3}{4\hbar\eta}, \qquad \eta = \frac{k_B}{4\ell_P^2}.$$

The equality fixes the coupling in terms of the supplied entropy density.

5 Determining sets of null directions

Aperture completeness is a finite-dimensional question at each point. In the frame above, write

$$E(k, k) = F(n) = A + b_i n^i + Q_{ij} n^i n^j, \qquad Q_{ij} = Q_{ji}, \quad \operatorname{tr} Q = 0.$$

Here $A = E_{00} + \sum_i E_{ii}/3$, $b_i = 2E_{0i}$, and Q is the Euclidean trace-free spatial block. The kernel of $E \mapsto F$ is exactly $\mathbb{R}g$, so F determines the spacetime trace-free tensor. The function space has dimension $1+3+5 = 9$. A set of directions is determining precisely if its evaluation map is injective on this space. Any open set suffices by real analyticity. A change of orthonormal observer rescales the normalized null contraction by a nonvanishing weight and relabels the sphere, preserving this criterion.

An explicit nine-direction set consists of $\pm e_i$ and $(e_i+e_j)/\sqrt{2}$ for $i < j$, where e_i are orthonormal spatial basis vectors. Put $s_i = [F(e_i) + F(-e_i)]/2$. Then

$$b_i = \tfrac{1}{2}[F(e_i) - F(-e_i)], \qquad A = \tfrac{1}{3}\sum_i s_i, \qquad Q_{ii} = s_i - A,$$

and

$$Q_{ij} = F((e_i + e_j)/\sqrt{2}) - \tfrac{1}{2}(s_i + s_j) - \frac{b_i + b_j}{\sqrt{2}}.$$

This proves full rank constructively. Since the evaluation determinant is analytic and nonzero at this tuple, full rank also holds for a generic tuple of nine directions. Eight evaluations leave at least one dimension undetermined. For a nondegenerate circle, rotate axes so $n_3 = c$, $|c| < 1$. Restriction yields trigonometric polynomials of order at most two, and all five Fourier modes are obtained by restricting $1, n_1, n_2, n_1^2 - n_2^2, n_1 n_2$. Thus the rank is five and the kernel has dimension four. It is spanned by $(n_3 - c)\{1, n_1, n_2, n_3\}$. These are pointwise statements. Reconstruction from noisy data depends on the conditioning of the chosen evaluation matrix.

6 The first finite-length correction

Hold the prescribed boost and the leading temperature $\hbar\kappa/(2\pi)$ fixed as readouts. Work first with densities per unit initial area along a central generator, and write R_j, T_j for the zeroth and first

Taylor coefficients of R_{kk}, T_{kk} in λ. Take smooth data sufficient for the Taylor remainders below. With $\theta(0) = \sigma(0) = 0$, Raychaudhuri gives

$$\theta = -R_0\lambda - \tfrac{1}{2}R_1\lambda^2 + O(\lambda^3).$$

The area Jacobian $J(\lambda) = \mathrm{d}A_\lambda/\mathrm{d}A_0$ obeys $J(0) = 1$, $\dot{J} = \theta J$. Thus

$$\frac{\mathrm{d}(\delta A)}{\mathrm{d}A_0} = \frac{R_0 h^2}{2} - \frac{R_1 h^3}{6} + O(h^4), \qquad \frac{\mathrm{d}(\delta Q)}{\mathrm{d}A_0} = \kappa\left(\frac{T_0 h^2}{2} - \frac{T_1 h^3}{3}\right) + O(h^4).$$

The flux uses $J = 1 + O(\lambda^2)$, so its area-factor correction first occurs at fourth order. On an Einstein solution $T_j = \hbar\eta R_j/(2\pi)$ for $j = 0, 1$. The resulting residual is

$$\frac{\mathrm{d}(\delta Q - \pi_T \eta \delta A)}{\mathrm{d}A_0} = -\frac{\hbar\kappa\eta}{12\pi} h^3 \nabla_k R_{kk}(p) + O(h^4). \tag{2}$$

As an explicit example, use the spatially flat FLRW metric with scale factor $a(t) = t$, $t > 0$, and an affine radial null generator with $k^t = 1/t$. The Ricci contraction is $R_{kk} = -2\dot{H}(k^t)^2 = 2/t^4$, where $H = 1/t$. At $t = 1$, $\nabla_k R_{kk} = -8$. Taking the conserved Einstein stress tensor gives a nonzero coefficient in (2). Thus higher-order cancellation is not automatic even on shell.

Transfer to a transverse pencil. For a centered smooth family the initial expansion contributes $O(h\ell^2)$ per unit initial area. Transverse averages of the other Taylor coefficients differ from their central values by $O(\ell^2)$ under the stated smoothness and centering conditions. More explicitly, expand the heat and area expressions in h with coefficients smooth in the transverse coordinates. Averaging the coefficients replaces them by their values at the center plus $O(\ell^2)$, including the coefficient of h in the area change. The residual divided by A_0 therefore has the central expansion (2) with an additional $O(h\ell^2)$ error. Therefore $\ell = o(h)$ is sufficient to preserve the cubic coefficient, while the leading-order regime $\ell = O(h)$ can carry an extra cubic term. This comparison keeps the same readout prescription. The stated quartic remainder is retained if $h\ell^2 = O(h^4)$ in fixed local units. A relative temperature correction of order h, or a quadratic correction in the on-horizon boost, can also change the cubic coefficient.

A distinct weighted prescription. Define

$$\delta S_{\mathrm{w}} = \eta \int_H (-\lambda)(-\dot{\theta})\, \mathrm{d}\lambda\, \mathrm{d}A_\lambda.$$

It agrees at leading order with the plain area entropy on an equilibrium generator. Direct use of Raychaudhuri gives the exact identity for these prescribed readouts,

$$\delta Q - \pi_T \delta S_{\mathrm{w}} = \kappa \int_H (-\lambda)\left[T_{kk} - \frac{\hbar\eta}{2\pi}(R_{kk} + \theta^2/2 + \sigma_{ab}\sigma^{ab})\right] \mathrm{d}\lambda\, \mathrm{d}A_\lambda.$$

On shell the surviving terms are nonpositive. On the central equilibrium generator the residual density starts at order h^4. Writing $S_1 = (\nabla_k \sigma)|_p$ in a parallel screen frame gives the leading term

$$-\frac{\hbar\kappa\eta}{8\pi} h^4 \left(\frac{R_0^2}{2} + S_{1\,ab} S_1^{ab}\right).$$

Indeed the quadratic optical terms have this coefficient of λ^2 before integrating against $-\lambda$, and the area Jacobian is $1 + O(\lambda^2)$. The sign is for the displayed residual convention. This is a weighted expansion functional with a derived identity. Establishing a physical entropy interpretation would require additional input. In particular the construction is separate from the entropy production laws used in nonequilibrium generalizations [7].

7 Preparation, assumptions and verification

The pointwise equilibrium requirement should be distinguished from demanding that the whole null second fundamental form vanish through first transverse order. In normal coordinates at p, cubic changes of the surface embedding vary the first transverse derivative of this form by a totally symmetric three-tensor. Its trace can be prescribed, so vanishing first derivative of the initial expansion is available. The antisymmetric derivative is instead constrained by Codazzi. For a screen basis e_2, e_3, let $C_j = R(k, e_j, e_2, e_3)$ and $\Phi_{ij} = R(k, e_i, k, e_j)$. With the second fundamental form zero at p, the Codazzi obstruction is C, up to the overall convention for the curvature tensor. The screen change $e_i \mapsto e_i + c_i k$ changes C by $\Phi(-c_3, c_2)^\top$. Indeed, the shift of the second slot gives zero by antisymmetry, and the shifts of the third and fourth slots give $c_2 R(k, e_j, k, e_3) + c_3 R(k, e_j, e_2, k) = c_2 \Phi_{j3} - c_3 \Phi_{j2}$. The quadratic shift vanishes by antisymmetry in the last pair. Thus a screen with no Codazzi obstruction exists exactly when C lies in the range of Φ. Once this holds, symmetry in the last two indices of the differentiated second fundamental form and the Codazzi symmetry make it totally symmetric, so the cubic embedding freedom cancels it. Invertible Φ suffices. For a conformally flat algebraic curvature with $R_{kk} = 0$ but a nonzero transverse $R(k, e_i)$, one has $\Phi = 0$ and $C \neq 0$, exhibiting the obstruction. The leading proof uses the weaker pointwise preparation and centered average and applies also in this degenerate case.

The inputs are local Lorentzian geometry and its optical identities, the boost-energy definition of heat, the tangent-space thermal normalization, the area-entropy relation, the leading Clausius condition, and local stress-energy conservation. The source–readout construction identifies which inputs must be shared between apertures and which residual they constrain. It does not require the horizon entropy coefficient to be extracted from the spherical holographic entropy calculation. Similarly, the transport derivation of Einstein gravity starts with different physical premises. Its coupling and equations do not supply the missing thermodynamic inputs in this argument.

The companion research calculations were replayed from a fresh extraction. All 59 results matched the archived reference, including 43 exact symbolic checks, 14 controls that reproduced an expected failure, and two numerical diagnostics. Checks include Raychaudhuri for explicit metrics, affine and boost normalizations, the FLRW residual, Codazzi controls, null-direction ranks, a contracted Bianchi calculation on explicit metrics, and unit restoration. The shrinking-width estimate and explicit nine-direction inversion above are analytic additions to the drafting pass.

8 Conclusion

The leading Clausius residual is a common readout of heat, temperature and area for a prepared local horizon. Its vanishing determines the trace-free curvature response from a sufficient family of null directions. Conservation and the Bianchi identity then complete the tensor equation up to the cosmological constant. The finite-direction reconstruction specifies which apertures suffice and which leave a kernel. At the next order the plain area prescription produces a curvature-gradient residual under the fixed boost and temperature readouts. Transverse averaging and thermal corrections must be controlled before interpreting that residual beyond the generator limit.

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Verification materials. The Zenodo record for this version provides the scripts and reference outputs.

The archive is `Jacobson_Verification_24029166.zip` at the Zenodo record. The archive includes the targeted checks of the correction and instructions for replaying the original suites.

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