# Boundary Ball Readouts and Linearized Einstein Gravity in a Source–Readout Architecture

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**Abstract**

We reconstruct the Einstein-gravity branch of the derivation of Faulkner, Guica, Hartman, Myers and Van Raamsdonk in a source–readout architecture. One off-shell bulk perturbation supplies the first area variation and the asymptotic charge for every boundary ball. Subtracting the two readouts gives a weighted integral of the linearized Einstein residual. We recover this identity by covariant phase space and independently by the scalar-curvature variation on a time-symmetric slice. All centers and radii on one slice determine its Hamiltonian constraint. A finite determining family of Lorentz frames, with all slices in each frame, determines the remaining boundary-index contractions, and the Bianchi and Ward identities complete the local equation. We construct explicit non-gauge perturbations in the fixed-frame kernel, including a compactly supported lapse perturbation and an example with nonzero Ward-compatible boundary data. These results distinguish recovery of the field equation from reconstruction of the metric.

## 1 Introduction

The first law of entanglement for boundary balls constrains the bulk metric in holographic conformal field theory. Faulkner et al. proved that, for first variations about the vacuum, the complete family of ball first laws recovers the linearized gravitational equations about pure AdS [1]. The essential identity expresses the discrepancy between the two boundary-law readouts as an integral of a local bulk residual. The scope of the converse depends on which balls are supplied and what is assumed at the conformal boundary.

We reconstruct the Einstein branch in the source–readout architecture [2]. The vacuum ball and its bulk Killing wedge reuse the normalization of the spherical modular-transport construction [3, 4]. The family of readouts plays a role analogous to the determining null directions in the local horizon construction [5]. The present problem also requires a spatial inversion. The nonlinear transport derivation from the spin-two bootstrap has a different set of input laws and a different endpoint [6].

The contributions are an explicit compatibility calculation for one common off-shell perturbation, an elementary inversion argument on the local class used here, and metric-level examples of what restricted families miss. The holographic dictionary, the Einstein area prescription, and the boundary first law are inputs. We first treat the charge defect as a geometric observable on the off-shell class, then use the dictionary to identify it with the first-law defect. The reconstruction is at first order around pure AdS and at classical bulk order.

## 2 Common source and the ball projections

Use boundary dimension $d \geq 2$, $c = \hbar = 1$, and Poincaré coordinates with

$$\bar{g} = \frac{L^2}{z^2}(\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu), \qquad \Lambda = -\frac{d(d-1)}{2L^2}.$$

Let $\mathcal{E}_{ab} = \delta(G_{ab} + \Lambda g_{ab})$ denote the unscaled Einstein residual. The metric tangent $h$ has radial gauge $h_{za} = 0$, with $h_{\mu\nu} = z^{d-2}H_{\mu\nu}$ and $H$ locally $C^2$ up to $z = 0$. Denote its boundary value by $H^{(0)}$. All integrations are over bounded balls, so this class imposes no global spatial decay. The geometric tests allow arbitrary tangents in this class. When comparing with a CFT state, the state–geometry dictionary and the flat-space Ward conditions

$$\eta^{\mu\nu}H^{(0)}_{\mu\nu} = 0, \qquad \partial^\mu H^{(0)}_{\mu\nu} = 0 \tag{1}$$

are additional inputs. These conditions alone do not supply a CFT state for each off-shell test.

A source history contains the reference state and geometry and their first variations. An aperture is a ball $B$ in a chosen boundary frame, together with its background bulk wedge. All apertures evaluate the same tangent $h$. In rest coordinates $s = t - t_0$, $y = x - x_0$, and $r = |y|$, put

$$\xi_B = \frac{\pi}{R}\left[(R^2 - s^2 - r^2 - z^2)\partial_t - 2s(y^i\partial_i + z\partial_z)\right].$$

Its boundary limit is the ball conformal Killing field. It is Killing in AdS and vanishes at $s = 0$, $r^2 + z^2 = R^2$, the bifurcation hemisphere $\widetilde{B}$. Write $\Sigma_B$ for the upper half-ball bounded by $B$ and $\widetilde{B}$. The factor $2\pi R$ relative to hyperbolic time gives surface gravity $2\pi$. This is the normalization appropriate for the dimensionless modular-energy readout. On the slice the positive lapse of $\xi_B$ is $V = (L/z)\xi^t_B$.

The energy projection for a CFT perturbation is the one-sided ball stress-tensor integral

$$\delta E^{\mathrm{hol}}_B = \frac{\pi}{R}\int_B (R^2 - r^2)\,\delta\langle T_{tt}\rangle\,\mathrm{d}^{d-1}y, \qquad \delta\langle T_{\mu\nu}\rangle = \frac{dL^{d-3}}{16\pi G}H^{(0)}_{\mu\nu}.$$

The local stress-tensor weight comes from the vacuum ball modular flow. The full modular implementer in the vacuum representation includes the complementary action. We use the one-sided energy variation in the first law. In a regulated formulation it is the variation of the expectation of the reference modular Hamiltonian. In a continuum formulation the first law is used for perturbations admitting the relevant first variations, with the relative-entropy derivative vanishing at the reference state. The continuum argument requires no density matrix for the diamond algebra.

The entropy projection at Einstein order is $\delta S_B = \delta A(\widetilde{B})/(4G)$. The first metric variation at fixed embedding is

$$\delta A(\widetilde{B}) = \frac{RL^{d-3}}{2}\int_B z^{2-d}\left(\delta^{ij} - \frac{y^i y^j}{R^2}\right)h_{ij}\,\mathrm{d}^{d-1}y, \qquad z = \sqrt{R^2 - r^2}. \tag{2}$$

Indeed the induced metric in the $y$-chart is $L^2z^{-2}(\delta_{ij} + y_iy_j/z^2)$, with inverse $z^2L^{-2}(\delta^{ij} - y^iy^j/R^2)$ and volume density $RL^{d-1}z^{-d}$. The hemisphere is extremal and the time-slice extrinsic curvature vanishes. To use (2) for the perturbed extremal surface, one must also eliminate its cutoff endpoint term. In the $z$-chart the background radius is $r(z) = \sqrt{R^2 - z^2}$, so the radial canonical momentum is

$$L^{d-1}z^{1-d}r^{d-2}\frac{r'}{\sqrt{1 + (r')^2}} = -\frac{L^{d-1}}{R}r^{d-2}z^{2-d}.$$

The time momentum vanishes on the background. With a fixed $z$ cutoff and regular angular parametrization, uniform radial coordinate displacement $o(z^{d-2})$ suffices for the endpoint to vanish. We assume this falloff for the anchored first variation. It includes the stronger $O(z^{d+1})$ displacement condition used in the construction. In $d = 3$ the criterion reduces to $o(z)$. This estimate is a sufficient condition and allows cancellations when the integrated endpoint term vanishes by other means.

## 3 Off-shell charge identity

For arbitrary admissible geometric tangents define the asymptotic spatial charge

$$\delta E_B^{\mathrm{sp}} = \frac{dL^{d-3}}{16\pi G}\,\frac{\pi}{R}\int_B (R^2 - r^2)\sum_i H_{ii}^{(0)}\,\mathrm{d}^{d-1}y.$$

The difference between this expression and the holographic energy is the weighted boundary trace. The trace Ward identity therefore suffices for their equality. Conversely if they agree on all balls on a slice, the small-radius limit, divided by the integral of the weight, gives pointwise vanishing of the trace there. Boundary conservation will be used at a different step.

**Theorem 3.1.** *For the stated geometric class the defect $D_B = \delta S_B - \delta E_B^{\mathrm{sp}}$ satisfies*

$$D_B = -\frac{1}{8\pi G}\int_{\Sigma_B} \xi_B^t \mathcal{E}_{tt}(L/z)^{d-1}\,\mathrm{d}z\,\mathrm{d}^{d-1}y = -\frac{1}{16\pi G}\int_{\Sigma_B} V\,\delta\mathcal{R}[q]\,\mathrm{d}v_{\bar{q}}, \tag{3}$$

*where $q$ denotes the spatial metric variation on $\Sigma_B$ and $\mathcal{R}$ its scalar curvature. In other frames the same formula uses the time-time contraction in that frame.*

*Proof.* For the Einstein Lagrangian the presymplectic potential and Noether charge are

$$\Theta = \frac{\epsilon_a}{16\pi G}(\bar{\nabla}_b h^{ab} - \bar{\nabla}^a h^b{}_b), \qquad Q_\xi = -\frac{\epsilon_{ab}}{16\pi G}\bar{\nabla}^a\xi^b.$$

Here the contracted volume forms and index raising use the background, with the full metric dependence varied in $Q_\xi$. Set $\chi = \delta Q_\xi - \xi\cdot\Theta$ at fixed $\xi$. The linearized Noether identity about an on-shell background gives

$$\mathrm{d}\chi = -\frac{1}{8\pi G}\xi^a\mathcal{E}_{ab}\epsilon^b.$$

This is the Einstein specialization of the covariant phase-space identity [7, 1]. It follows by varying the Noether current, subtracting the derivative of $\xi\cdot\Theta$, and using $\mathcal{L}_\xi\bar{g} = 0$ to eliminate the symplectic current. The pullback at the bifurcation surface gives $\delta A/(4G)$, since $\xi = 0$ and its normal boost has surface gravity $2\pi$. At the boundary the pullback density is $\xi_B^t\,dL^{d-3}\,\mathrm{tr}_{\mathrm{sp}}\,H^{(0)}/(16\pi G)$, where $\mathrm{tr}_{\mathrm{sp}}\,H^{(0)} = \sum_i H_{ii}^{(0)}$. The radial derivative of $h$ and the variation of the volume form both enter this limit. Stokes' theorem with the charge orientation $\widetilde{B} - B$ gives the first identity.

For a separate slice proof, $\bar{q} = L^2 z^{-2}(\mathrm{d}z^2 + \mathrm{d}y^2)$ is time symmetric, so the Gauss constraint gives $\mathcal{E}_{tt} = (L/z)^2\delta\mathcal{R}/2$. In particular the constraint depends only on $q$. Define

$$U^A = V(D_B q^{AB} - D^A\,\mathrm{tr}\,q) - q^{AB}D_B V + (\mathrm{tr}\,q)D^A V$$

with $D$ the connection of $\bar{q}$. Integration by parts at the differential level gives

$$V\delta\mathcal{R} - D_A U^A = q^{AB}(D_A D_B V - \bar{q}_{AB}\Delta V - V\,\mathrm{Ric}_{AB}).$$

Direct differentiation of $V = (\pi L/Rz)(R^2 - r^2 - z^2)$ yields $D_A D_B V = V\bar{q}_{AB}/L^2$, so the right side vanishes. On the hemisphere $V = 0$ and the outward derivative is $-2\pi$, giving $U \cdot \nu = -2\pi \operatorname{tr}_{\widetilde{B}} q$. On the face $z = \varepsilon$ with outward normal $-z\partial_z/L$, the density tends to $dL^{d-3}\xi^t \sum_i H^{(0)}_{ii}$. To see the coefficient, write $\operatorname{tr} q = L^{-2} z^d \sum_i H_{ii}$. Radial gauge gives $D_B q^{zB} = (z/L^2) \operatorname{tr} q$, and the two leading $V$-terms in the flux yield $(d-1)$ and 1 times the same boundary density. The sum of the two surface fluxes is $16\pi G(\delta E^{\mathrm{sp}}_B - \delta S_B)$, proving the second identity independently of the spacetime form calculation. □

The covariant and slice routes return the same numerical normalization for the same source tangent. In particular the factor in the slice formula depends on the Einstein action normalization and the ball's surface gravity. Using the unit hyperbolic-time generator in only one charge changes the balance.

## 4 From readout data to the local equation

Put $f = (L/z)^{d-1}\mathcal{E}_{tt}$ on a fixed slice. The radial-gauge falloff gives continuous $f$ up to $z = 0$. For example the Gauss constraint expresses it as

$$f = -\frac{L^{d-3}}{2}\left[z\partial_z^2 H_{ii} + (d+1)\partial_z H_{ii} + z(\partial_i\partial_i H_{jj} - \partial_i\partial_j H_{ij})\right].$$

Differentiating $RD_B$ in $R$ removes the weight, since $R\xi^t = \pi(R^2 - r^2 - z^2)$ vanishes at the hemisphere. Zero defect for all radii and centers thus gives zero unweighted integrals of $f$ over all half-balls.

For completeness these integrals are injective on the stated class by a local polynomial argument. Suppose all half-ball integrals of $x^\alpha f$ vanish for $|\alpha| \le n$. Radius differentiation gives their zero integrals on the hemispheres. Center differentiation gives zero hemisphere integrals with an additional factor $(x^i - x^i_0)/R$, the normal velocity. The flat face contributes zero to both differentiation formulas. Combining the two derivatives gives zero hemisphere moments with the factor $x^i$, and integrating over concentric radii gives the half-ball moments at degree $n+1$. Induction starts with degree zero. On the closed upper hemisphere, polynomials in the horizontal coordinates separate points because $z = \sqrt{R^2 - r^2}$. They are dense in the continuous functions there. The finite signed measure $f\,\mathrm{d}A$ therefore vanishes, and continuity gives pointwise vanishing in the interior. The argument uses bounded domains. For balls whose closures lie in an open boundary region it works on the union of their half-balls, since center differentiation and shrinking radii stay within the available family. This is a direct proof of the scalar recovery used in [1]; related hyperbolic integral geometry is used in [8].

**Theorem 4.1.** *In the geometric class above with* (1)*, the ball first law for all balls in all frames is equivalent to $\mathcal{E}_{ab} = 0$. It suffices to use all slices and balls of $d(d+1)/2$ frames whose time vectors have outer squares spanning the symmetric boundary tensors.*

*Proof.* Scalar recovery in each frame gives $\mathcal{E}_{\mu\nu}u^\mu u^\nu = 0$ on every slice of that frame. The slices must cover the common bulk region where the contractions are compared. Polarization then gives $\mathcal{E}_{\mu\nu} = 0$. The span condition is attainable: in an inertial basis take unnormalized future timelike vectors $e_0$, $e_0 \pm v e_i$, and $e_0 + v(e_i + e_j)$ for $i < j$, with $0 < v < 1/\sqrt{2}$. The pair differences recover time-space entries, pair sums recover spatial diagonal entries, and the last set recovers spatial cross entries. Normalizing the vectors does not change their span. Generic tuples also span by the nonvanishing determinant at this example.

The linearized Bianchi identity $\bar{\nabla}^a \mathcal{E}_{ab} = 0$ follows by varying the exact identity at a background with zero residual. For $C^2$ perturbations the differential identity is understood weakly, by local smooth approximation. The radial equations below then give the stated integrated form in distributions, with the continuous residual fixing the pointwise values. With $\mathcal{E}_{\mu\nu} = 0$ its radial equations integrate to

$$\mathcal{E}_{z\mu} = z^{d-1} C_\mu, \qquad \mathcal{E}_{zz} = z^{d-2} C_z - \tfrac{1}{2} z^d \partial_\mu C^\mu.$$

Indeed the $\mu$ equation is $\partial_z \mathcal{E}_{z\mu} - (d-1)\mathcal{E}_{z\mu}/z = 0$. The $z$ equation is $\partial_z \mathcal{E}_{zz} - (d-2)\mathcal{E}_{zz}/z = -\partial^\mu \mathcal{E}_{z\mu}$. Expanding the linearized curvature at the boundary gives

$$C_z = -\frac{d(d-1)}{2L^2} \operatorname{tr}_\eta H^{(0)}, \qquad C_\mu = \frac{d}{2L^2} (\partial^\nu H^{(0)}_{\nu\mu} - \partial_\mu \operatorname{tr}_\eta H^{(0)}).$$

These are the leading radial constraint coefficients in the unscaled Einstein normalization. The Ward conditions set them to zero. Conversely $\mathcal{E} = 0$ sets these leading coefficients to zero, so the two energies agree and (3) gives every ball first law. □

For a fixed background and admissible class the joint defect map factors as $h \mapsto \mathcal{E}[h] \mapsto (D_B)_B$. With the complete family, the second arrow is injective on the image of Ward-compatible perturbations by the theorem applied to a difference. Thus it has a return map on the realized data range, even when the map from metric perturbations to the residual has a kernel. Restricting the balls to one slice instead gives a return map only for its time-time residual. This identifies the quotient associated with each choice of aperture.

The finite count is sufficient for the polarization stage. With fewer frames the algebraic evaluation of a general symmetric tensor has a kernel at a point. Realization of every such kernel by a metric tangent subject to the declared conditions is a separate question. Likewise, the theorem determines an equation residual. Distinct on-shell metrics can have the same zero defects.

## 5 Metric perturbations invisible to restricted families

All radii and centers on one slice give $D_B = 0$ exactly when $\delta\mathcal{R}[q] = 0$ on that slice. The slice proof shows why this does not control temporal derivatives or the other components of $h$. For example a spatial perturbation proportional to $(t - t_0)^2$ can vanish with its first time derivative at $t_0$ while changing the spatial evolution equations there. We give two explicit constructions with a larger invisibility family.

**Compactly supported lapse perturbations.** In $d \geq 3$, fix a spatial point $p$ strictly inside the bulk slice. For the static metric $-N^2 \mathrm{d}t^2 + \bar{q}$ with $N = L/z$, vary only $N$ by a smooth compactly supported $n$ with $n(p) = Dn(p) = 0$ and a nonzero mixed Hessian $D_1 D_2 n(p)$ in an orthonormal horizontal frame. Such a function is a mixed quadratic near $p$ multiplied by a bump equal to one there. The perturbation $h_{tt} = -2Nn$ has radial gauge and vanishes near the conformal boundary. The spatial metric is unchanged at every $t$. Thus every rest-frame ball has zero area variation and zero boundary charge.

For a static lapse the spacetime spatial Ricci tensor is $\mathrm{Ric}_{AB}[\bar{q}] - N^{-1} D_A D_B N$, obtained by substituting the mixed time connection coefficients. At $p$ its off-diagonal variation is $-N^{-1} D_1 D_2 n$. Trace terms and the cosmological term contribute zero to that component, so $\mathcal{E}_{12} \neq 0$. A linearized diffeomorphism of an on-shell background gives zero residual by covariance. The construction is therefore a non-gauge example with compact spatial support and vanishing boundary stress variation. The example remains invisible to time translations in the same frame.

**Ward-compatible nonzero boundary data.** In boundary dimension three use the symmetric off-diagonal perturbation

$$h_{12} = z(t^2 + z^2/7 + 1/3)$$

with other independent components zero. Its boundary tensor is traceless, conserved and nonzero, since its only entries are spatial cross components independent of the spatial coordinates. On a rest slice the induced perturbation is trace free and divergence free on the hyperbolic background, and hence $\delta\mathcal{R} = 0$. Directly in (2) the area integral is odd in $y^1$ and $y^2$ separately. The boundary time-time charge also vanishes. Yet the transverse tensor equation for the profile $H_{12}$ is

$$(\partial_z^2 + 4z^{-1}\partial_z - \partial_t^2)H_{12} = 0,$$

which here has residual $-4/7$. To see the rescaling, the dimensionless metric perturbation inside the Fefferman–Graham conformal factor is $\gamma_{12} = z^2 h_{12}/L^2 = z^3 H_{12}/L^2$. Its transverse, trace-free equation is $(\partial_z^2 - 2z^{-1}\partial_z - \partial_t^2)\gamma_{12} = 0$. Substituting $z^3 H_{12}/L^2$ cancels the term without derivatives and gives the displayed operator on $H_{12}$. The perturbation lies off shell and is non-gauge. A boosted frame with velocity components in both spatial directions couples to the spatial cross residual and can detect it. Ward compatibility here describes geometric boundary data. The example does not assert the existence of a corresponding CFT state.

**Additional restrictions on center and radius.** For AdS in three bulk dimensions, take $h_{xx} = z^4 \cos(kx)$, $k > 0$. Its asymptotic charge vanishes. Equation (2) gives

$$GD(R, x_0) = \frac{\cos(kx_0)}{8LR} \int_{-R}^{R} (R^2 - y^2)^3 \cos(ky)\, \mathrm{d}y = \frac{\cos(kx_0)}{8LR} \sqrt{\pi}\Gamma(4)(2R/k)^{7/2} J_{7/2}(kR).$$

The last identity follows by integrating the cosine series against the polynomial weight, or from the standard integral representation of $J_\nu$. Thus any positive zero at $kR_0$ gives a nonzero perturbation invisible at the one radius $R_0$ for every center. This uses the geometric class allowing nondecaying horizontal profiles. Replacing the cosine by $\sin(k(x - x_0))$ gives an odd integrand and a fixed-center kernel for every radius. These examples concern the defect readout. Even on shell, separate entropy data in one frame can be blind to components. For example the spatial cross profile $H_{12} = t^2 + z^2/5 + 1/3$ solves the displayed wave equation, has the same boundary cross stress, and has zero area variation for all rest-frame balls. Its nonzero boundary stress distinguishes it from a boundary-trivial gauge transformation.

## 6 Verification and scope

The nine verification suites used for this study were replayed from a fresh extraction. All 149 results matched the reference outputs, including 122 exact symbolic checks, 26 reproduced failure controls, and one numerical diagnostic. They cover the Killing normalization, charge identity by two routes, boundary coefficients, scalar inversion identities, and explicit on-shell and off-shell examples. The compactly supported lapse construction and the dimension-dependent anchoring estimate above are analytic additions. The fixed-radius example follows from the exact Bessel formula. Its numerical check illustrates the zero and recovery at another radius.

The main equivalence concerns the Einstein residual in the declared function class and at first order. The readout depends on the aperture and on the shared holographic inputs. The restriction to a time slice leaves an infinite-dimensional kernel at the metric level, including genuine off-shell examples. Additional slices and boosts change the available contractions. Once those contractions span, the radial equations still require boundary information. This separates the roles of scalar inversion, polarization, and the Ward data in recovering the local equations.

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**Verification materials.** The Zenodo record for this version provides the scripts and reference outputs.

The archive is `Faulkner_Verification_24029188.zip` at the Zenodo record. The archive includes the targeted checks of the correction and instructions for replaying the original suites.

## References

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