Second-Order AdS_4 Gravitational Boundary-Ball Readouts: Lorentzian Timelike Extremal Surfaces, Non-Identifiability, Localized Wavepackets, and Quadratic Reconstruction Stability

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Abstract

In Poincaré AdS_4, every rest-frame spherical boundary ball $B(R,\mathbf{x}_0)$ on a fixed boundary time slice $t=0$ annihilates a large infinite-dimensional subspace $\ker M_1$ of on-shell linearized vacuum metric perturbations at first order in both asymptotic modular energy and extremal-surface area, yet registers strictly positive Hollands–Wald canonical energy $\mathcal{E}_B(h)$ at second order. We resolve the open structure, identifiability, symmetry-breaking lifting, localization, and quantitative stability of this second-order gravitational readout map $M_2|_{\ker M_1}$.

First, we derive the off-shell linear charge–area prefactor $+\frac{1}{8\pi}$ directly from the Einstein–Hilbert covariant phase space and prove that spatial and timelike normal perturbations of the totally geodesic hemisphere $\widetilde{B}\cong\mathbb{H}^2$ decouple identically in the Lorentzian Hubeny–Rangamani–Takayanagi (HRT) second area variation $A''_{\mathrm{ext}}[h]$ and share the exact same $\mathbb{H}^2$ Jacobi operator $\mathcal{L}_{m=2}$, whose closed-form Green's function yields exact all-degree bilinear canonical energy matrices $\alpha_{\mathrm{mon}}(i,j)>0$ and $\gamma_{\mathrm{mon}}(i,j)>0$. In the time-even sector, the surface and bulk terms combine into a single manifestly positive one-line integral identity requiring no surface minimization; in the time-odd sector, Lorentzian extremality *maximizes* area in the timelike direction and subtracts a nonlocal operator contribution governed by a Hilbert–Schmidt integral operator T that we diagonalize completely analytically on an invariant polynomial flag, proving $\mathrm{spec}(T)=\{0\}\cup\{\frac{2}{(2n+4)(2n+5)}:n\in\mathbb{N}_0\}$ and sharp operator norm $\|T\|_{\mathrm{op}}=1/10$ (attained uniquely by O_0 on the wedge interval $[0,R]$), whence $\mathcal{E}_B(h_{\mathrm{odd}})\geq\frac{9}{10}(-I[h_{\mathrm{odd}}])>0$. We independently verify $\mathcal{E}_B(O_0)=\pi R^8/1400$ and $\mathcal{E}_B(O_1)=89\pi R^{12}/1401400$ by three distinct routes: the charge–area identity, direct covariant phase-space evaluation of both the surface-fixed symplectic pairing and $\int_{\widetilde{B}}\rho(h,V)$ (verifying exact $O(\varepsilon_z)$ vanishing at $z=\varepsilon_z\to 0$), and explicit closed-form Fefferman–Graham second-order vacuum completions k solving $D\mathcal{F}[k]+D^2\mathcal{F}[h,h]=0$ on the entire Poincaré patch.

Second, we prove sharp non-identifiability theorems for all rest-frame spherical balls at $t=0$: an explicit single-polarization counterpair on a three-mode subspace of $\ker M_1$ has identical canonical energy for all $R>0$ and all $\mathbf{x}_0\in\mathbb{R}^2$ while one member is time-symmetric and the other carries nonzero kinetic energy; across both polarizations, $O(2)_E\times O(2)_O$

*Prepared as a self-contained mathematical physics study extending the AdS_4 boundary-ball readout programme of Rodgers (2026) and the canonical-energy / quantum Fisher information framework of Hollands–Wald (2013) and Lashkari–Van Raamsdonk (2016). Complete symbolic and numerical verification scripts accompany this manuscript in the companion reproducibility package.

symmetry blinds all rest-frame balls at all times to linear polarization angle and the parity-odd polarization chirality $\mathcal{H}[h] = H_+(0,0)\dot{H}_\times(0,0) - H_\times(0,0)\dot{H}_+(0,0)$ (Stokes V). We compute the exact lifting produced by three minimal symmetry-breaking extensions: (i) time shifts $t = t_0$, which separate the single-polarization counterpair linearly everywhere except a unique first-derivative blind-spot radius $R_* = \sqrt{45/11}$ (resolved at $O(t_0^2)$); (ii) static $t = 0$ boundary multipoles $r(\varphi) = R(1 + \epsilon\cos m\varphi)$, where ellipses ($m = 2$) lift time-even shear in DA_{B_ϵ} ($DA_{B_\epsilon}[E_0^+] = \frac{3\pi R^3}{8}\epsilon$), time-odd modes have vanishing DA_{B_ϵ} and local energy-density smears for all shapes, and quadrupoles ($m = 4$) lift time-odd polarization at second order via $\partial_\epsilon A''_{\mathrm{ext},B_\epsilon^{(4)}}[O_0]|_{\epsilon=0} = -\frac{\pi R^8}{1200}(b_+^2 - b_\times^2)$ (independent of any $\pi/4$-rotation-invariant completion k); and (iii) Lorentz-boosted, time-translated balls $B^{(v)}(R, \mathbf{x}_0, t_c)$, whose homogeneous first-order readouts are independent of spatial center $\mathbf{x}_0$ (annihilating O_0 at $t_c = 0$) yet recover all six modes of $\mathrm{span}\{E_0^\pm, O_0^\pm, \widetilde{E}_1^\pm\}$ and polarization chirality at first order from an explicit six-ball measurement design with exact rank-6 certificate and order-optimal $O(v^{-2})$, $O(v^{-2}\tau^{-1})$ conditioning rates, while at $t_c = 0$ inside $\ker M_1^{(v)}$ an exact relativistic cancellation at $O(v^2)$ is followed by exact quartic separation $\Delta\mathcal{E}_{B^{(v)}(R,\mathbf{x}_0,0)} = \frac{4\pi R^8}{315}\frac{v^4}{(1-v^2)^2}$ at $O(v^4)$.

Third, formulating general positive-semidefinite quadratic identification and exact ℓ_∞ minimax optimal recovery (error $\varepsilon L_{\mathcal{D}}(Q)$ under $\|\eta\|_\infty \le \varepsilon$), we establish a sharp dimensional phase transition along the time-even tower V_n^{even}: at $n = 2$, a symbolic sum-of-squares certificate proves universal exactness of linear/SDP lifting ($L_{\mathrm{pure}}(Q) \equiv L_{\mathrm{SDP}}(Q) = L_{\mathrm{lin}}(Q)$); for every $n \ge 3$, radial monomial degeneracy collapses the SDP/mixed-state relaxation ($L_{\mathrm{SDP}} = \infty$ even with all $R > 0$) while pure-state recovery remains globally identifiable and Lipschitz stable from $2n - 1$ radii. At $n = 3$, a three-line Intermediate Value Theorem argument proves a sharp zero-or-infinite dichotomy for all $m \ge 5$ radii (with an outward-rounded full-cube certified upper bound $L_{\mathrm{pure}}(e_1 e_1^\top) \le 1.1718 \times 10^7$), and we prove both linear/SDP Chebyshev bounds and an unconditional pure-state lower bound $(R_*/R_{\max})^{4n+2}$ showing exponential noise amplification in bulk-depth extrapolation.

Finally, we prove a sharp C^5-regular Axial Potential Theorem constructing globally smooth, rationally decaying, finite-energy localized AdS_4 graviton wavepackets strictly inside $\ker M_1$ for all balls at all radii, centers, and times, compute their exact closed-form wedge and finite global canonical energies, establish both an exact rate of change of modular-weighted wedge canonical energy and a holomorphic finite-time separation identity, and prove a constructive, PSD-minimal three-ball reconstruction theorem for the fixed-width localized family $uh_{\mathrm{even}}^{(a)} + wh_{\mathrm{odd}}^{(a)}$ with certified rank-one minimax stability constants scaling as a^{12} and $O(a^{13}/|t_c|)$ small-shift amplification. Extending to the full axisymmetric radial axial sector, we prove an exact manifestly positive one-dimensional real-space canonical-energy identity for every centered ball $B(R, 0, t)$ (yielding unconditional classical positivity, exact $R \to 0^+$ boundary-trace limit $\frac{8\pi}{105}p(t,0)^2$, and uniform global-energy domination $\mathcal{E}_{B(R,0,t)} \le \frac{363}{175}\pi R\,\mathcal{E}_{\partial_t}$), derive the exact polarized two-width holomorphic master kernel $\mathcal{K}_R(\alpha,\beta)$ and elementary frequency-space kernels $\widehat{\mathcal{K}}_R^+, \widehat{\mathcal{H}}_R$ via two independent routes, and settle the infinite-dimensional single-radius vs. multi-radius inverse problem completely. As our headline reconstruction theorem, at *any single fixed centered radius* $R_0 > 0$ (with zero restriction on R_0 relative to the width support), exact canonical-energy data $t \mapsto \mathcal{E}_{R_0,t}(h[\mu])$ on any nonempty open time interval $I \subset \mathbb{R}$ determine *every* perturbation $h[\mu]$ in the compact positive width-measure class $\mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$ ($0 < a_{\min} \le a_{\max} < \infty$) unconditionally up to one overall sign—both via large-t boundary dominance in the polarized real-space identity and via a constructive strictly triangular exterior Laurent moment recursion—resolving arbitrary unknown-cardinality interlaced finite superpositions ($N \ge 1$) and arbitrary overlapping or singular continuous measures simultaneously (alongside direct complex slit-endpoint recovery of single unknown-width packets via the order-12 leading singularity $\frac{75\pi}{4004}c^2$ and of finite-superposition widths and squared amplitudes). By sharp contrast, the *unrestricted lifted* map collapses analytically for any finite set of K radii once $N \ge 14K$ via indefinite boundary-vs-bulk cancellation at large t (illustrated by high-precision 100-digit numerical evidence at $N = 15, R = 1$), whereas multi-radius centered data on any open rectangle $(R_1, R_2)\times I$ restore complete infinite-dimensional injectivity on unrestricted lifted symmetric measures $\Gamma \in \mathcal{M}_{\mathrm{sym}}(K \times K)$ via a direct real-space small-R chain and Stone–Weierstrass (as well as an independent small-R Fourier–Laplace physical proof). Finally, we establish quan-

titative stability bounds separating noiseless uniqueness from noisy finite-window recovery: unconditional local two-point and compact-global Lipschitz stability on the sign quotient $K_{\mathrm{par}}/\langle S_{-1}\rangle$ for separated finite-N superpositions even at one fixed radius $R_0 > 0$ on a continuous window $[-T, T]$ (kept distinct from finite discrete-sample designs, whose rank and global injectivity hypotheses are stated separately; with Schur-complement $O(\eta/\epsilon^2)$ small-amplitude width blow-up, exact nearby-width global-energy Gram conditioning $\frac{16a^2}{13\delta^2}$, and proved $\Theta(\delta^2)$ lifted singular-value collapse on $L^2([-T,T])/L^\infty([-T,T]))$ contrasted with an unconditional exponential/logarithmic ill-conditioning barrier governed by the exact Bernstein parameter $\rho_B = (\sqrt{a_{\max}} + \sqrt{a_{\min}})/(\sqrt{a_{\max}} - \sqrt{a_{\min}})$, yielding a sharp minimax L^2 error floor of 1 on the unit ball and an $\Omega((\log(1/\eta))^{-2s})$ lower bound on H^s.

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1 Introduction

In the AdS/CFT correspondence, the entanglement entropy S_B and modular energy $E_B = \langle K_B \rangle$ of spherical boundary balls $B(R, \mathbf{x}_0)$ encode the bulk spacetime geometry via the Hubeny–Rangamani–Takayanagi (HRT) extremal-surface area formula [1, 2] and the Hollands–Wald covariant phase-space formalism [3–5]. At *first order* in a metric perturbation h around pure Poincaré AdS_{d+1}, the modular first law $\delta S_B = \delta E_B$ is equivalent to the linearized vacuum Einstein equations in the bulk [5]. Yet even on shell, restricting observations to rest-frame spherical boundary balls $B(R, \mathbf{x}_0)$ on a single boundary time slice $t = 0$ leaves a large infinite-dimensional linear kernel $\ker M_1$: every perturbation whose boundary energy density vanishes on $t = 0$ and whose spatial shear averages to zero over every hemisphere $\widetilde{B}(R, \mathbf{x}_0)$ produces $DE_B[h] = \frac{1}{4} DA_B[h] = 0$ across *all* radii $R > 0$ and *all* centers $\mathbf{x}_0 \in \mathbb{R}^{d-1}$ [5, 6].

At *second order*, the fundamental observable associated to a ball B is the Hollands–Wald canonical energy $\mathcal{E}_B(h) = W_{\Sigma_B}(\gamma, \mathcal{L}_{\xi_B}\gamma)$ [4], which in the holographic dual is conjectured to equal the relative-entropy Hessian (quantum Fisher information) $\delta^{(2)} S(\rho_B \| \sigma_B) = \delta^{(2)} E_B - \delta^{(2)} S_B$ for coherent/semiclassical boundary states [7]. For Poincaré AdS_4 ($d = 3$), Rodgers [8] recently showed that the charge–area identity eliminates the second-order metric perturbation k from $\mathcal{E}_B(h)$ completely, established strict positivity $\mathcal{E}_B(h) > 0$ on explicit time-even polynomial and oscillatory spatially homogeneous transverse-traceless (TT) gravitational waves inside $\ker M_1$, and highlighted the central open inverse-problem questions for the quadratic readout map $M_2|_{\ker M_1}$: its full Lorentzian time-odd structure, its injectivity (identifiability) across radii and polarizations, the minimal geometric extensions needed to lift blind spots, its quantitative stability under measurement noise, and whether the same phenomena hold for localized, finite-energy bulk excitations.

The present paper develops a complete, self-contained mathematical resolution of these questions in Poincaré AdS_4. Every theorem is proved analytically from standard four-dimensional Lorentzian general relativity, covariant phase space, extremal-surface geometry, and finite-dimensional conic optimization, and is backed by an independent symbolic and numerical reproducibility suite (`verify_01`–`verify_06`).

1.1 Summary of Main Results

Our results organize into five main contributions, summarized with their verification status in Table 1:

1. **Off-shell charge–area derivation, Lorentzian HRT spectral diagonalization, Hollands–Wald audit, and vacuum completions (Sections 2–3).** Starting from the Einstein–Hilbert action and Lee–Wald symplectic conventions, we derive the off-shell linear charge–area prefactor $+\frac{1}{8\pi}$ and sign directly (cross-checked on an explicit off-shell low mode $v_{xx} = v_{yy} = z^3$; Theorem 2.1). On the spatially homogeneous TT space $V_{\mathrm{hom}} \subset \ker M_1$, we prove that spatial and timelike normal displacements of the hemisphere $\widetilde{B} \cong \mathbb{H}^2$ decouple identically in the Lorentzian HRT second area variation $A''_{\mathrm{ext}}[h]$ and share the exact same $\mathbb{H}^2$ Jacobi operator $\mathcal{L}_{m=2}$, whose closed-form Green's function (Lemma 3.1) determines exact all-degree bilinear canonical energy matrices $\alpha_{\mathrm{mon}}(i,j) > 0$ and $\gamma_{\mathrm{mon}}(i,j) > 0$ (Theorem 3.2). In the time-even sector, bulk and surface terms collapse into a single manifestly positive one-line integral identity (3.17); in the time-odd sector, Lorentzian signature causes timelike extremization to *maximize* area ($A''_{\mathrm{ext}}[h_{\mathrm{odd}}] > 0$), subtracting a nonlocal Hilbert–Schmidt operator T that we diagonalize completely analytically on the invariant polynomial flag $\psi_0 \cdot \mathrm{span}\{1, y^2, \ldots, y^{2n}\}$, proving $\mathrm{spec}(T) = \{0\} \cup \{\frac{2}{(2n+4)(2n+5)} : n \in \mathbb{N}_0\}$ and sharp operator norm $\|T\|_{\mathrm{op}} = \frac{1}{10}$ (attained uniquely by O_0 on the wedge interval $[0, R]$). This proves $\mathcal{E}_B(h_{\mathrm{odd}}) \geq \frac{9}{10}(-I[h_{\mathrm{odd}}]) > 0$ for every nonzero wedge datum $H_1|_{[0,R]}$. We confirm $\mathcal{E}_B(O_0) = \frac{\pi R^8}{1400}$ and $\mathcal{E}_B(O_1) = \frac{89\pi R^{12}}{1401400}$ by three

independent derivations: (a) the charge–area identity, (b) direct covariant phase-space evaluation of both $W_{\Sigma_B}(\gamma, \mathcal{L}_{\xi_B}\gamma)$ and $\int_{\widetilde{B}} \rho(h, V)$ with an explicit conformal-boundary corner audit verifying exact $O(\varepsilon_z)$ vanishing at $z = \varepsilon_z \to 0$ (Section 3.4), and (c) explicit closed-form Fefferman–Graham second-order vacuum completions k solving all ten equations $D\mathcal{F}[k] + D^2\mathcal{F}[h, h] = 0$ on the entire Poincaré patch with a free mass parameter c_M that cancels identically in $\mathcal{E}_B$ (Proposition 3.4).

2. **Rest-frame non-identifiability, symmetry-breaking lifting, and constructive six-ball boosted recovery (Section 4).** Positivity of $\mathcal{E}_B$ does not imply injectivity of $M_2|_{\ker M_1}$. On $t = 0$ rest-frame balls we exhibit an explicit single-polarization counterpair $(h^{(1)}, h^{(2)})$ on $\mathrm{span}\{E_0^\times, \widetilde{E}_1^\times, O_0^\times\}$—one purely static, one carrying kinetic energy—whose canonical energies agree identically for all $R > 0$ and all $\mathbf{x}_0 \in \mathbb{R}^2$, establish $O(2)_E \times O(2)_O$ polarization blindness at $t = 0$, and prove by spatial reflection that *all* rest-frame balls at *all* times $t_0 \in \mathbb{R}$ are permanently blind to the parity-odd polarization chirality $\mathcal{H}[h] = H_+(0,0)\dot{H}_\times(0,0) - H_\times(0,0)\dot{H}_+(0,0)$ (Stokes V: $aE_0^+ \pm bO_0^\times$; Theorem 4.1). We then compute the exact lifting produced by the three minimal geometric extensions:

   - *Time shifts* ($t = t_0$) linearly separate $(h^{(1)}, h^{(2)})$ everywhere except a unique first-derivative blind-spot radius $R_* = \sqrt{45/11}$ (resolved at $O(t_0^2)$ with $\frac{\mathrm{d}^2}{\mathrm{d}t_0^2}\Delta\mathcal{E}|_{t_0=0} = \frac{2\pi R^6}{35}$; Proposition 4.2), while remaining blind to polarization chirality $\mathcal{H}$ for all t_0.
   - *Static $t = 0$ boundary multipoles* $r(\varphi) = R(1 + \epsilon\cos m\varphi)$ lift time-even modes linearly in extremal-surface area via ellipses ($m = 2$: $DA_{B_\epsilon}[E_0^+] = \frac{3\pi R^3}{8}\epsilon$), give vanishing DA_{B_ϵ} and local energy-density smears $D\mathcal{T}_{B_\epsilon, w}$ on all time-odd modes for every $t = 0$ shape, and lift time-odd polarization $b_+O_0^+ + b_\times O_0^\times$ at second order in extremal area via quadrupoles ($m = 4$): $\partial_\epsilon A''_{\mathrm{ext}, B_\epsilon^{(4)}}[O_0]|_{\epsilon=0} = -\frac{\pi R^8}{1200}(b_+^2 - b_\times^2)$, independently of any $\pi/4$-rotation-invariant completion k such as the $SO(2)$-invariant completion of Proposition 3.4 (Theorem 4.3).
   - *Lorentz-boosted, time-translated balls* $B^{(v)}(R, \mathbf{x}_0, t_c)$ have homogeneous first-order readouts independent of spatial center $\mathbf{x}_0$—annihilating O_0 identically at $t_c = 0$ for all $\mathbf{x}_0$—yet lift all six modes of $V_6 = \mathrm{span}\{E_0^\pm, O_0^\pm, \widetilde{E}_1^\pm\}$ and polarization chirality $\mathcal{H}$ already at *first order* in $M_1^{(v)}$ once $t_c \neq 0$ (Theorem 4.4). We construct an explicit six-ball recovery design with exact rank-6 determinant and order-optimal $O(v^{-2})$, $O(v^{-2}\tau^{-1})$ conditioning rates (Theorem 4.5). Meanwhile, at $t_c = 0$ along $\alpha = 0$ (where $V_{\mathrm{hom}}^\times \subset \ker M_1^{(v)}$ for all $|v| < 1$), we prove an exact all-v relativistic second-order formula exhibiting an exact cancellation at $O(v^2)$ followed by exact quartic lifting $\mathcal{E}_{B^{(v)}}(h^{(2)}) - \mathcal{E}_{B^{(v)}}(h^{(1)}) = \frac{4\pi R^8 v^4}{315(1-v^2)^2} > 0$ for every $\mathbf{x}_0 \in \mathbb{R}^2$ (Theorem 4.6).

3. **PSD-quadratic stability, dimensional phase transition at $n = 3$, and exponential depth ill-conditioning (Section 5).** Formulating exact worst-case ℓ_∞ minimax optimal recovery (error $\varepsilon L_{\mathcal{D}}(Q)$ under $\|\eta\|_\infty \le \varepsilon$) for general PSD quadratic readouts (Theorem 5.1), we prove $L_{\mathrm{SDP}} \equiv L_{\mathrm{lin}}$, characterize exact zero-gap attainment under $\ker\mathcal{A} \cap \mathcal{V}_{\le 2} = \{0\}$, and establish a sharp dimensional phase transition along the time-even tower V_n^{even}: at $n = 2$, a symbolic sum-of-squares determinant certificate proves universal zero pure-vs-SDP gap ($L_{\mathrm{pure}} \equiv L_{\mathrm{SDP}}$; Theorem 5.3); for every $n \ge 3$, radial monomial degeneracy ($2n - 1 < \binom{n+1}{2}$) permanently collapses SDP relaxation ($L_{\mathrm{SDP}} = \infty$ even with all $R > 0$) while pure-state recovery remains globally identifiable and Lipschitz stable from any $2n - 1$ radii (Theorem 5.5). At $n = 3$, an Intermediate Value Theorem argument on the inertia-$(2, 1)$ kernel matrix K_3 proves a sharp zero-or-infinite dichotomy for all $m \ge 5$ radii (with a full-cube $[-1, 1]^5$ certified finite upper bound for $L_{\mathrm{pure}}(e_1 e_1^\top)$; Theorem 5.6),

while Theorem 5.8 establishes both continuum Chebyshev bounds for linear/SDP extrapolation and an unconditional pure-state lower bound $L^{\mathrm{cont}}_{\mathrm{pure}}(A_n(R_*)) \geq (R_*/R_{\max})^{4n+2}$ for all $n \geq 1$.

4. **Finite-energy localized AdS_4 graviton wavepackets strictly inside $\ker M_1$, holomorphic time-shift separation, and constructive three-ball reconstruction (Section 6).** We prove a sharp C^5-regular Axial Potential Theorem (Theorem 6.1, with explicit sharpness tests in Remark 6.2) mapping any boundary-smooth solution $\Phi \in C^5(\mathbb{R}\times[0,\infty)\times\mathbb{R}^2)$ of $(\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 - \frac{4}{z}\partial_z)\Phi = 0$ to a normalizable Fefferman–Graham TT vacuum graviton $h[\Phi]$ strictly inside $\ker M_1$ for all balls at all radii, centers, and times. Instantiating Φ with the real-analytic complexified AdS_4 inversion $\Phi_a = (-(t-ia)^2+r^2+z^2)^{-3}$ yields explicit globally smooth, $O(r^{-9})$-decaying wavepackets $h^{(a)}_{\mathrm{odd}}$ and $h^{(a)}_{\mathrm{even}}$ with closed-form strictly positive wedge canonical energies, purely rational difference, and identical finite global energy $\mathcal{E}_{\partial_t} = \frac{675\pi}{1024\,a^{13}}$ (verified against direct bulk symplectic pairing to 2.7×10^{-14}; Theorem 6.3). For the time-reversed pair $h^{(\pm)}_{\mathrm{loc}} = h^{(a)}_{\mathrm{even}} \pm h^{(a)}_{\mathrm{odd}}$, we prove exact all-balls non-identifiability at $t=0$, compute the exact instantaneous rate of change of the modular-weighted wedge canonical energy $\dot{\mathcal{E}}_{B(R,0)}[h^{(\pm)}_{\mathrm{loc}}] = \mp\frac{12288\pi\hat{R}^8(171\hat{R}^4-143\hat{R}^2+286)}{1001\,a^{13}(1+\hat{R}^2)^{13}}$ (giving blind-spot-free small-shift separation at every centered radius), and prove an exact holomorphic finite-time separation identity across all $t_0 \in \mathbb{R}$ (Theorem 6.4). Finally, for the fixed-width two-parameter localized family $h(u,w) = uh^{(a)}_{\mathrm{even}} + wh^{(a)}_{\mathrm{odd}}$, we prove a constructive, PSD-minimal three-ball reconstruction theorem recovering every state $(u,w)\in\mathbb{R}^2$ up to global sign via linear inversion of (u^2,w^2,uw) on $(R_1,0) = (\frac{1}{2}a,0)$, $(R_2,0) = (a,0)$, and $(R_3,t_c) = (\frac{1}{2}a,\frac{1}{2}a)$, with exact closed-form entries $(2E^{(1)}_{eo}(\frac{1}{2},\frac{1}{2}) = -\frac{27\pi}{1760})$, an exact PSD minimality proof ($m \geq 3$) with direct positive-definiteness certificates for the three chosen matrices, certified exact rank-one minimax stability constants scaling as a^{12}, and exact $O(a^{13}/|t_c|)$ mixed-entry blow-up as $t_c \to 0$ (Theorem 6.5).

5. **Infinite-dimensional second-order reconstruction: single-radius physical uniqueness, multi-radius lifted injectivity, and Bernstein-rate stability (Section 7).** Passing to arbitrary radial axisymmetric axial potentials $\Phi(t,\varrho)$ and the compact positive width-measure class $\mathcal{M}_{\mathbb{C}}([a_{\min},a_{\max}])$ ($0 < a_{\min} \leq a_{\max} < \infty$; $h[\mu] = \mathrm{Re}\int g_a\,\mathrm{d}\mu(a) \in \ker M_1$, zero gauge redundancy, finite global Bergman energy $\frac{1}{2}\iint \frac{10800\pi}{(a+b)^{13}}\,\mathrm{d}\mu(a)\,\mathrm{d}\overline{\mu(b)}$; Proposition 7.1), we prove an exact manifestly positive one-dimensional real-space canonical-energy identity (7.11) for *every* centered ball $B(R,0,t)$—yielding unconditional classical positivity, exact $R \to 0^+$ limit $\lim_{R\to0^+} R^{-8}\mathcal{E}_{B(R,0,t)} = \frac{8\pi}{105}p(t,0)^2$, and uniform Hardy domination $\mathcal{E}_{B(R,0,t)} \leq \frac{363}{175}\pi R\,\mathcal{E}_{\partial_t}$ (Theorem 7.2). We derive the exact polarized two-width holomorphic master kernel $\mathcal{K}_R(\alpha,\beta)$ governing both complex-bilinear and Hermitian pairings (verified identically by bulk symplectic pairing, polarized charge–area with surface term, and 64-node GL quadrature) together with exact elementary frequency-space kernels $\widehat{\mathcal{K}}^+_R, \widehat{\mathcal{H}}_R$ and diagonal Taylor positivity $P(n) > 0$ ($\forall n \geq 9$; Theorem 7.4). As our headline reconstruction result (Theorem 7.7), at *any fixed centered radius* $R_0 > 0$ (with zero restriction on R_0 relative to $[a_{\min},a_{\max}]$), exact canonical-energy data $t \mapsto \mathcal{E}_{R_0,t}(h[\mu])$ on any nonempty open time interval $I \subset \mathbb{R}$ determine *every* perturbation $h[\mu]$ in the compact positive width-measure class $\mathcal{M}_{\mathbb{C}}([a_{\min},a_{\max}])$ unconditionally up to one overall sign—both via large-t boundary dominance in polarized (7.11) and via a constructive strictly triangular exterior Laurent moment recursion—resolving every unknown-N interlaced finite superposition and every overlapping or singular continuous measure simultaneously (alongside direct slit-endpoint recovery of single unknown-width packets via the leading order-12 singularity $\frac{75\pi}{4004}c^2$ in Theorem 7.5 and of finite-superposition widths and c_j^2 in Theorem 7.6). By sharp contrast, any finite set of K radii collapses the *unrestricted lifted* map analytically once $N \geq 14K$

via indefinite boundary-vs-bulk cancellation at large t (illustrated by high-precision 100-digit numerical evidence at $N = 15, R = 1$; Theorem 7.6(ii) and Remark 7.8), whereas centered balls on any open rectangle $(R_1, R_2) \times I$ restore complete infinite-dimensional injectivity on unrestricted lifted symmetric measures $\Gamma \in \mathcal{M}_{\mathrm{sym}}(K \times K)$ via a direct real-space small-R chain and Stone–Weierstrass (together with an independent small-R Fourier–Laplace physical proof; Theorem 7.9). Finally, Theorem 7.11 quantifies stability across small amplitudes (Schur-complement $O(\eta/\epsilon^2)$ width blow-up accounting for amplitude absorption), nearby widths ($\kappa_{\mathrm{glob}} = \frac{16a^2}{13\delta^2}$ and proved $\Theta(\delta^2)$ lifted singular-value collapse on $L^2([-T,T])/L^\infty([-T,T])$), unconditional local two-point and compact-global Lipschitz stability on the sign quotient $K_{\mathrm{par}}/\langle S_{-1}\rangle$ for finite-N separated superpositions even at one fixed radius $R_0 > 0$ on a continuous window $[-T,T]$ (kept distinct from finite discrete-sample designs, whose rank and global injectivity hypotheses are stated separately), and continuous densities (an unconditional non-asymptotic Legendre–Bergman exponential decay bound with Bernstein rate $\rho_B = (\sqrt{a_{\max}} + \sqrt{a_{\min}})/(\sqrt{a_{\max}} - \sqrt{a_{\min}})$, yielding a sharp unit-ball minimax L^2 error floor of 1 via the orthonormal Legendre tail and an $\Omega((\log(1/\eta))^{-2s})$ logarithmic minimax lower bound on H^s).

1.2 Outline of the Paper

Section 2 sets up the Poincaré AdS_4 geometry, modular Killing vectors, covariant phase space, charge–area identity, and polynomial/oscillatory bases of V_{hom}, while carefully separating classical canonical energy from CFT relative-entropy Hessian hypotheses. Section 3 proves the exact Lorentzian HRT decoupling, $\mathbb{H}^2$ Green's function, one-line time-even integral identity, sharp time-odd spectral theorem ($\|T\|_{\mathrm{op}} = 1/10$), direct Hollands–Wald corner audit, and closed-form Fefferman–Graham completions. Section 4 establishes the rest-frame non-identifiability theorems and derives the exact lifting laws for time shifts, $m = 2/m = 4$ boundary multipoles, and Lorentz-boosted balls. Section 5 develops the general PSD-quadratic optimal recovery theory, the $n = 2$ vs. $n \geq 3$ SDP phase transition, the $n = 3$ zero-or-infinite dichotomy, and exponential radial extrapolation bounds. Section 6 proves the Axial Potential Theorem and computes exact closed-form wedge and finite global energies for localized wavepackets in $\ker M_1$. Section 7 proves the exact real-space positivity identity for all centered balls, derives the polarized two-width master kernel and frequency-space kernels, proves our headline unconditional single-radius infinite-dimensional physical rank-one reconstruction theorem (determining every perturbation in the compact positive width-measure class $\mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$ up to one overall sign at any single fixed centered radius $R_0 > 0$ from any nonempty open time interval I), contrasts this with finite-K-radius lifted collapse and multi-radius lifted measure injectivity on $(R_1, R_2) \times I$, and establishes exact quantitative stability (distinguishing continuous-window sign-quotient Lipschitz stability from conditional discrete-sample designs) and Bernstein-rate exponential/logarithmic ill-conditioning bounds (including the sharp unit-ball minimax error floor of 1). Section 8 situates these findings within the holographic bulk reconstruction, integral-geometry, shape-deformation, and phase-retrieval literatures and states all remaining open problems explicitly. Technical formulas and certificates are collected in Appendices A–C.

2 Geometric Setup, Covariant Phase Space, and the Fixed-Slice Linear Kernel

2.1 Poincaré AdS_4 geometry and boundary balls

Throughout this paper we work in four-dimensional Poincaré Anti-de Sitter space AdS_4 ($d = 3$ boundary spacetime dimensions) in units $G = \ell = 1$ with Lorentzian signature $(-,+,+,+)$ and

Table 1: Overview of main theorems, attribution, and verification receipts in the companion reproducibility suite.

Result	Status	Scope / Relationship	Reproducibility receipt
Thm. 2.1 & 3.2: $+\frac{1}{8\pi}$ off-shell derivation, all-k $\alpha_{\mathrm{mon}}, \gamma_{\mathrm{mon}}$, one-line even integral, analytic spec(T)	**Proved** (exact)	Extends [8] to all degrees, analytic odd spectrum & full off-shell normalization	`verify_01` (off-shell $v_{xx} = v_{yy} = z^3$, exact flag $k = 0..4$, 120-node GL)
Sec. 3.4 & Prop. 3.4: Direct HW eq. (2.9)/(2.10), $O(\varepsilon_z)$ corner audit, FG completions k	**Proved** (exact)	Independent 3-route audit of [4, 8] on V_{hom}	`verify_01` (pointwise $d\chi = \omega$, exact c_M-cancellation)
Thm. 4.1 & Prop. 4.2: All-radii counterpair, $O(2)_E \times O(2)_O$, all-times $\mathcal{H}$ blindness, $R_* = \sqrt{45/11}$	**Proved** (exact)	Resolves fixed-frame kernel / identifiability question of [8]	`verify_02` (exact cubic $\Delta\mathcal{E}(t_0)$ in SymPy)
Thm. 4.3: $m = 2$ ellipse (DA) & $m = 4$ quadrupole $(A''_{\mathrm{ext}}, \delta^{(2)}S)$ lifting on O_0	**Proved** (exact)	Extends shape-deformation HEE [9, 10] to 2nd-order timelike HRT	`verify_02` (exact SymPy + $D^2\mathcal{F}$ isotropy + 64-node GL)
Thm. 4.4–4.6: Time-translated boosted $M_1^{(v)}$, 6-ball rank-6 design, all-v $M_2^{(v)}$ ($O(v^2)$ cancel, $O(v^4)$ lift)	**Proved** (exact)	Constructive 6-mode recovery & relativistic lifting inside $\ker M_1^{(v)}$	`verify_02` ((x_0, t_c) regression, 6-ball $\det/L_i$, exact symbolic v)
Thm. 5.1–5.8: Exact εL optimal recovery, $n = 2$ SOS, $n \geq 3$ SDP collapse, $n = 3$ IVT, linear/pure depth LBs	**Proved** (exact; $n \geq 4$ finite-gap open)	Applies optimal recovery [11, 12] to holographic M_2	`verify_03` (exact SOS, full-cube cubic UB, IVT witnesses, pure LB)
Thm. 6.1–6.5: C^5 Axial Potential Thm, localized $h^{(a)}_{\mathrm{odd/even}}$, exact $\mathcal{E}_B$, $\mathcal{E}_{\partial_t}$, holomorphic separation, minimal 3-ball reconstruction & stability	**Proved** (exact; 2nd-order k_{loc} open)	Extends [8] to finite-energy localized packets, time-shift lifting & constructive 3-ball recovery	`verify_04` (exact SymPy, holomorphic Q_R, direct W_{Σ_B} GL64, `mpmath.iv` 3-ball cert.)
Prop. 7.1 & Thm. 7.2–7.11: 1D real-space positivity, two-width $\mathcal{K}_R$, unknown-width $N = 1$, finite-N lifted collapse, unconditional 1-R ∞-dim rank-1 uniqueness & moment recursion, multi-R lifted injectivity, Bernstein stability	**Proved** (exact; non-axisymmetric Φ open)	Resolves ∞-dim second-order reconstruction in $\ker M_1$ on axisymmetric radial axial sector (1-R physical vs. multi-R lifted)	`verify_05`–`06` (1D IBP, Route 1=Route 2=GL64, Darboux chains, $P(n) > 0$, $N = 15$ witness & large-t bnd/bulk cancel, $N = 6$ interlaced & $R_0 = 10$ continuum checks, $R_0 = 5$ moment recovery, jet rank-10, Legendre bound)

cosmological constant $\Lambda = -d(d-1)/(2\ell^2) = -3$. In Poincaré coordinates $x^a = (t, z, x, y)$ with $z > 0$ and $\mathbf{x} = (x, y) \in \mathbb{R}^2$, the background metric and vacuum Einstein residual are

$$\bar{g} = \frac{1}{z^2}\big(-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}x^2 + \mathrm{d}y^2\big), \qquad \mathcal{F}_{ab}(g) := G_{ab}(g) - 3g_{ab}, \tag{2.1}$$

so that $\mathcal{F}_{ab}(\bar{g}) = 0$ with $\bar{R}_{ab} = -3\bar{g}_{ab}$ and $\bar{R} = -12$.

Let $B = B(R, \mathbf{x}_0) \subset \{t = 0, z = 0\}$ be the rest-frame circular boundary ball of radius $R > 0$ and center $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$. Writing $r := |\mathbf{x} - \mathbf{x}_0|$, its bulk Ryu–Takayanagi / Hubeny–Rangamani–Takayanagi (HRT) bifurcation surface in pure AdS_4 is the totally geodesic hemisphere

$$\widetilde{B} = \big\{(t, z, \mathbf{x}) : t = 0,\ z > 0,\ z^2 + r^2 = R^2\big\}, \tag{2.2}$$

enclosing the constant-time bulk half-ball $\Sigma_B = \{t = 0,\, z > 0,\, z^2 + r^2 \le R^2\}$ inside the AdS–Rindler wedge $D_{\mathrm{bulk}}(B)$. The reference Killing vector field generating the wedge modular flow with surface gravity $\kappa = 2\pi$ is [2, 5, 7]

$$\xi_B = \frac{\pi}{R}\Big[\big(R^2 - t^2 - z^2 - r^2\big)\partial_t - 2t\big(z\partial_z + (\mathbf{x} - \mathbf{x}_0)\cdot\nabla_{\mathbf{x}}\big)\Big]. \tag{2.3}$$

On the $t = 0$ slice, $\xi_B = \xi_B^t \partial_t$ is future-directed timelike in the interior of Σ_B, vanishes identically on $\widetilde{B}$, and limits at $z = 0$ to the boundary conformal Killing vector generating the one-sided vacuum modular Hamiltonian

$$K_B = \frac{\pi}{R}\int_{r<R} (R^2 - r^2)\, T_{tt}\, \mathrm{d}^2x + \text{const.} \tag{2.4}$$

In spherical coordinates $(\varrho, \theta, \varphi)$ centered at $\mathbf{x}_0$ on the $t = 0$ hyperbolic slice $\mathbb{H}^3$,

$$z = \varrho\cos\theta, \qquad x - x_0 = \varrho\sin\theta\cos\varphi, \qquad y - y_0 = \varrho\sin\theta\sin\varphi, \tag{2.5}$$

with $\theta \in [0, \pi/2)$ and $\varphi \in [0, 2\pi)$, the induced background metric on Σ_B is

$$\bar{g}|_{\Sigma_B} = (\varrho\cos\theta)^{-2}\big(\mathrm{d}\varrho^2 + \varrho^2\mathrm{d}\theta^2 + \varrho^2\sin^2\theta\,\mathrm{d}\varphi^2\big).$$

The hemisphere $\widetilde{B}$ sits at $\varrho = R$, with induced metric $\bar{q}_{AB} = \mathrm{diag}(\cos^{-2}\theta, \sin^2\theta\cos^{-2}\theta)$ and area element $\sqrt{\det\bar{q}} = \sin\theta/\cos^2\theta$, isometric to $\mathbb{H}^2$. It is convenient to introduce the dimensionless height variable $y := \cos\theta = z/R \in (0, 1]$, where $y \to 0^+$ approaches the boundary anchor ∂B and $y = 1$ is the bulk apex $\theta = 0$.

2.2 Admissible families, covariant phase space, and the charge–area identity

Consider a smooth one-parameter family of metrics $g_\lambda = \bar{g} + \lambda h + \frac{1}{2}\lambda^2 k + o(\lambda^2)$ with fixed flat conformal boundary metric, normalizable Fefferman–Graham falloff $h_{\mu\nu} = zH_{\mu\nu}$ ($H_{\mu\nu}$ smooth up to $z = 0$, $h_{za} = 0$ in radial gauge), and vacuum Einstein equations through second order:

$$D\mathcal{F}[h] = 0, \qquad D\mathcal{F}[k] + D^2\mathcal{F}[h, h] = 0. \tag{2.6}$$

The linearized boundary stress tensor associated with $h_{\mu\nu} = zH_{\mu\nu}$ is $\delta T_{\mu\nu} = \frac{3}{16\pi}H^{(0)}_{\mu\nu}$, where $H^{(0)}_{\mu\nu} := H_{\mu\nu}|_{z=0}$ satisfies the flat-space Ward identities $\eta^{\mu\nu}H^{(0)}_{\mu\nu} = 0$ and $\partial^\mu H^{(0)}_{\mu\nu} = 0$ on shell [5, 6].

For the Einstein–Hilbert Lagrangian $L = (R - 2\Lambda)\boldsymbol{\epsilon}/(16\pi)$, the Lee–Wald presymplectic potential, Noether charge 2-form, linearized charge variation, and symplectic current 3-form are [3, 4, 7]

$$\Theta(g; v) = \frac{1}{16\pi}\big(\nabla_b v^{ab} - \nabla^a v^b{}_b\big)\epsilon_a, \qquad Q_X = -\frac{1}{16\pi}\nabla^a X^b \epsilon_{ab}, \tag{2.7}$$

$$\chi(v; X) = \delta_v Q_X - \iota_X\Theta(\bar{g}; v), \qquad \omega(\bar{g}; v, w) = \delta_v\Theta(g; w)\big|_{\bar{g}} - \delta_w\Theta(g; v)\big|_{\bar{g}}. \tag{2.8}$$

We orient the bulk symplectic form $W_{\Sigma_B}(v,w) := \int_{\Sigma_B} \omega(\bar{g}; v, w)$ so that in coordinates on the $t = 0$ slice it integrates the densitized future time component $\sqrt{-\bar{g}}\,\omega^t(\bar{g}; v, w)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$, matching the sign of the gravitational modular energy E_B^{grav}. Setting $F_B(g_\lambda) := E_B^{\mathrm{grav}}(g_\lambda) - \frac{1}{4}A_B(g_\lambda)$ (with vacuum constants subtracted and A_B evaluated on the boundary-anchored HRT extremal surface $\widetilde{B}_\lambda$), the Hollands–Wald–Lashkari–Van Raamsdonk canonical-energy theorem states [4, 7, 8]:

Theorem 2.1 (Covariant phase-space and charge–area representations)**.** *Let $g_\lambda = \bar{g} + \lambda h + \frac{1}{2}\lambda^2 k + o(\lambda^2)$ satisfy (2.6) with fixed conformal boundary metric, fixed entangling sphere ∂B, and sufficient regularity up to $\partial\Sigma_B$. Then $F_B'(0) = 0$ and the second variation equals the wedge canonical energy:*

$$F_B''(0) = \mathcal{E}_B(h) = W_{\Sigma_B}\big(\gamma, \mathcal{L}_{\xi_B}\gamma\big), \tag{2.9}$$

where $\gamma = h + \mathcal{L}_V\bar{g}$ is in surface-fixed Hollands–Wald gauge on $\widetilde{B}$ (vanishing linearized extremality displacement, vanishing mixed normal–tangential components, and boost-invariant normal block on $\widetilde{B}$) with V yielding no asymptotic boundary contribution at $z \to 0$. Equivalently, in the boundary-preserving representative h,

$$\mathcal{E}_B(h) = W_{\Sigma_B}\big(h, \mathcal{L}_{\xi_B}h\big) + \int_{\widetilde{B}} \rho(h, V), \qquad \rho(h,V) := \chi\big(h + \mathcal{L}_V\bar{g};\, [\xi_B, V]\big) - \chi\big(\mathcal{L}_{\xi_B}h;\, V\big). \tag{2.10}$$

Moreover, eliminating the second-order metric k via the off-shell linear charge–area identity

$$DF_B[v] = DE_B^{\mathrm{grav}}[v] - \frac{1}{4}DA_B[v] = +\frac{1}{8\pi}\int_{\Sigma_B} \xi_B^t\, D\mathcal{F}_{tt}[v]\,\frac{\mathrm{d}z\,\mathrm{d}^2x}{z^2} \tag{2.11}$$

and (2.6) yields

$$\mathcal{E}_B(h) = -\frac{1}{4}A''_{\mathrm{ext}}[h] - I[h], \qquad I[h] := \frac{1}{8\pi}\int_{\Sigma_B} \xi_B^t\, D^2\mathcal{F}_{tt}[h,h]\,\frac{\mathrm{d}z\,\mathrm{d}^2x}{z^2}, \tag{2.12}$$

where $A''_{\mathrm{ext}}[h] := \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}A(\widetilde{B}_\lambda; \bar{g} + \lambda h)\Big|_{\lambda=0}$ is evaluated along the affine metric path $\bar{g} + \lambda h$ including the exact first-order spatial and timelike displacement of the extremal surface $\widetilde{B}_\lambda$.

Derivation of the off-shell prefactor and sign in (2.11)–(2.12). Fix the spacetime volume 4-form $\boldsymbol{\epsilon} := \sqrt{-g}\,\mathrm{d}t \wedge \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y$ and its contractions $\epsilon_a := \iota_{\partial_a}\boldsymbol{\epsilon}$ and $\epsilon_{ab} := \iota_{\partial_b}\iota_{\partial_a}\boldsymbol{\epsilon}$, so that $\mathrm{d}(V^a\epsilon_a) = (\nabla_a V^a)\boldsymbol{\epsilon}$, $\mathrm{d}(\frac{1}{2}Q^{ab}\epsilon_{ab}) = (\nabla_b Q^{ab})\epsilon_a$, and $\iota_X(V^a\epsilon_a) = V^aX^b\epsilon_{ab} = \frac{1}{2}(V^aX^b - V^bX^a)\epsilon_{ab}$. For $L = \frac{1}{16\pi}(R - 2\Lambda)\boldsymbol{\epsilon}$ with $\Lambda = -3$, the metric variation is $\delta L = E^{ab}\delta g_{ab}\boldsymbol{\epsilon} + \mathrm{d}\Theta(g; \delta g)$ with $E^{ab} = -\frac{1}{16\pi}\mathcal{F}^{ab}(g)$ and $\Theta(g; v) = \frac{1}{16\pi}(\nabla_b v^{ab} - \nabla^a v^b{}_b)\epsilon_a$. With $Q_X = -\frac{1}{16\pi}\nabla^a X^b\epsilon_{ab} = \frac{1}{2}Q_X^{ab}\epsilon_{ab}$ ($Q_X^{ab} = -\frac{1}{16\pi}(\nabla^a X^b - \nabla^b X^a)$), the off-shell Noether current $J_X = \Theta(g; \mathcal{L}_X g) - \iota_X L = \mathrm{d}Q_X + C_X$ has constraint 3-form

$$C_X = -2E^a{}_b X^b\epsilon_a = +\frac{1}{8\pi}\mathcal{F}^a{}_b X^b\epsilon_a$$

(verified directly from $\Theta^a(\mathcal{L}_X g) - (\iota_X L)^a - \nabla_b Q_X^{ab} = \frac{1}{16\pi}[2R^a{}_b X^b - (R - 2\Lambda)X^a] = \frac{1}{8\pi}\mathcal{F}^a{}_b X^b$). Taking the metric variation δ_v at the vacuum Killing background $\bar{g}$ (where $E(\bar{g}) = 0$ and $\mathcal{L}_{\xi_B}\bar{g} = 0$), we have $\delta_v L = \mathrm{d}\Theta(\bar{g}; v)$ and hence, by Cartan's magic formula $\iota_{\xi_B}\mathrm{d}\Theta = \mathcal{L}_{\xi_B}\Theta - \mathrm{d}\iota_{\xi_B}\Theta$ together with $\omega(\bar{g}; v, \mathcal{L}_{\xi_B}\bar{g}) = \delta_v\Theta(\mathcal{L}_{\xi_B}g) - \mathcal{L}_{\xi_B}\Theta(\bar{g}; v) = 0$,

$$\delta_v J_{\xi_B} = \delta_v\Theta(\mathcal{L}_{\xi_B}g) - \iota_{\xi_B}\mathrm{d}\Theta(\bar{g}; v) = \mathrm{d}\big(\iota_{\xi_B}\Theta(\bar{g}; v)\big).$$

Comparing with $\delta_v(\mathrm{d}Q_{\xi_B} + C_{\xi_B}) = \mathrm{d}(\delta_v Q_{\xi_B}) + \delta_v C_{\xi_B}$ yields the exact off-shell Iyer–Wald differential-form identity

$$\mathrm{d}\chi(v; \xi_B) = \mathrm{d}\big(\delta_v Q_{\xi_B} - \iota_{\xi_B}\Theta(\bar{g}; v)\big) = -\delta_v C_{\xi_B} = -\frac{1}{8\pi}D\mathcal{F}^a{}_b[v]\,\xi_B^b\,\epsilon_a.$$

Orient the constant-time half-ball $\Sigma_B = \{t = 0,\ z \geq 0,\ z^2 + r^2 \leq R^2\}$ positively by $\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y$ (consistent with W_{Σ_B} above), so that $\epsilon_t|_{\Sigma_B} = +\sqrt{-\bar{g}}\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y = +z^{-4}\,\mathrm{d}z\,\mathrm{d}^2x$ and $\mathrm{d}\chi|_{\Sigma_B} = \partial_i(\sqrt{-\bar{g}}\,\chi^{ti})\,\mathrm{d}z\,\mathrm{d}^2x$. Using $\bar{g}^{tt} = -z^2$, the minus sign in $\mathrm{d}\chi = -\delta_v C_{\xi_B}$ cancels the metric signature minus sign identically:

$$-\delta_v C_{\xi_B}\Big|_{\Sigma_B} = -\frac{1}{8\pi} D\mathcal{F}^t{}_t[v]\,\xi_B^t\,\epsilon_t\Big|_{\Sigma_B} = -\frac{1}{8\pi}\big(-z^2 D\mathcal{F}_{tt}[v]\big)\xi_B^t\big(+z^{-4}\,\mathrm{d}z\,\mathrm{d}^2x\big) = +\frac{1}{8\pi}\xi_B^t\,D\mathcal{F}_{tt}[v]\,\frac{\mathrm{d}z\,\mathrm{d}^2x}{z^2}.$$

By Stokes's theorem on $\Sigma_B \subset \{z \geq 0\}$ (equivalently the Euclidean divergence theorem with outward unit normals $-\partial_z$ on $B = \{z = 0\}$ and $+\partial_\varrho$ on $\widetilde{B} = \{\varrho = R\}$),

$$\int_{\Sigma_B} \mathrm{d}\chi(v;\xi_B) = \int_{\partial\Sigma_B} \chi(v;\xi_B) = \int_B \big(-\sqrt{-\bar{g}}\,\chi^{tz}\big)\,\mathrm{d}^2x + \int_{\widetilde{B}} \big(n_\varrho \cdot \sqrt{-\bar{g}}\,\chi^t\big)\,\mathrm{d}S_\varrho.$$

Orienting both B and $\widetilde{B}$ toward the conformal boundary $z = 0$ (so B carries its outward boundary orientation $-\partial_z \mapsto \mathrm{d}x \wedge \mathrm{d}y$ while $\widetilde{B}$ carries the inward normal $-\partial_\varrho$, giving $\partial\Sigma_B = B \cup (-\widetilde{B})$), any normalizable Fefferman–Graham perturbation $v_{\mu\nu} = zH_{\mu\nu}$ ($v_{za} = 0$) with traceless boundary data $\eta^{\mu\nu}H^{(0)}_{\mu\nu} = 0$ satisfies $\int_B \chi(v;\xi_B) = \int_B(-\sqrt{-\bar{g}}\,\chi^{tz})\,\mathrm{d}^2x = \frac{3}{16\pi}\int_B \xi_B^t(H^{(0)}_{xx} + H^{(0)}_{yy})\,\mathrm{d}^2x = \frac{3}{16\pi}\int_B \xi_B^t H^{(0)}_{tt}\,\mathrm{d}^2x = DE_B^{\mathrm{grav}}[v]$ and $\int_{\widetilde{B}} \chi(v;\xi_B) = \int_{\widetilde{B}}(-n_\varrho \cdot \sqrt{-\bar{g}}\,\chi^t)\,\mathrm{d}S_\varrho = +\frac{1}{4}DA_B[v]$ (since $\kappa = 2\pi$ gives $\frac{\kappa}{8\pi} = \frac{1}{4}$), so $\int_{\partial\Sigma_B}\chi = \int_B \chi - \int_{\widetilde{B}}\chi = DE_B^{\mathrm{grav}}[v] - \frac{1}{4}DA_B[v]$ yields (2.11) with exact prefactor $+\frac{1}{8\pi}$. Applying (2.11) to $v = k$ in $F_B''(0) = DF_B[k] - \frac{1}{4}A''_{\mathrm{ext}}[h]$ and substituting $D\mathcal{F}_{tt}[k] = -D^2\mathcal{F}_{tt}[h,h]$ from (2.6) yields (2.12) (noting that every completion k of Proposition 3.4 satisfies $\eta^{\mu\nu}k^{(0)}_{\mu\nu} = 0$ identically). As explicit low-mode checks directly evaluating $\sqrt{-\bar{g}}\,\chi^{ti}$ and its boundary fluxes (`verify_01`): (a) on the on-shell planar AdS–Schwarzschild mass perturbation ($v_{tt} = 2z$, $v_{xx} = v_{yy} = z$, $D\mathcal{F}_{tt} = 0$), $\int_B(-\sqrt{-\bar{g}}\,\chi^{tz})\,\mathrm{d}^2x = +\frac{3\pi R^3}{16} = DE_B^{\mathrm{grav}}[v]$ and $\int_{\widetilde{B}}(n_\varrho \cdot \sqrt{-\bar{g}}\,\chi^t)\,\mathrm{d}S_\varrho = -\frac{3\pi R^3}{16} = -\frac{1}{4}DA_B[v]$, summing to 0; and (b) on the off-shell spatial trace mode $v_{xx} = v_{yy} = z^3$ ($v_{tt} = 0$), direct 4D evaluation gives $D\mathcal{F}_{tt}[v] = -10z^3$, $\partial_i(\sqrt{-\bar{g}}\,\chi^{ti}) = -\delta_v C^t_{\xi_B}\sqrt{-\bar{g}} = \frac{5z(z^2+r^2-R^2)}{4R}$, outward flux $0 = DE_B^{\mathrm{grav}}[v]$ at B, and outward flux $-\frac{5\pi R^5}{48} = -\frac{1}{4}DA_B[v]$ at $\widetilde{B}$, confirming $DF_B[v] = -\frac{5\pi R^5}{48} = +\frac{1}{8\pi}\int_{\Sigma_B}\xi_B^t(-10z^3)z^{-2}\,\mathrm{d}z\,\mathrm{d}^2x$ identically. The same $+\frac{1}{8\pi}$ normalization governs (3.2), reconciles identically with the direct covariant phase-space evaluation (2.9)–(2.10) in Section 3.4, and matches $DF_B[k] = -I[h]$ on the explicit completions of Section 3.5. □

Remark 2.2 (Classical canonical energy versus CFT relative-entropy Hessian). Equations (2.9)–(2.12) are identities of classical vacuum Einstein gravity: the quadratic form $\mathcal{E}_B(h)$ is well-defined on any linearized vacuum tangent h solving $D\mathcal{F}[h] = 0$ on the wedge Σ_B. Identifying $\mathcal{E}_B(h)$ with the Hessian $F_B''(0)$ of an actual one-parameter vacuum family additionally requires a second-order completion k solving (2.6) (which we construct explicitly in Section 3.5), while identifying $F_B''(0)$ with the boundary CFT quantum Fisher information (second derivative of relative entropy $D(\rho_B(\lambda)\|\rho_B(0))$) further requires the holographic dictionary and an admissible family of finite-norm CFT states [7, 8]. We maintain this separation rigorously throughout.

2.3 The fixed-slice linear kernel $\ker M_1$ and spatially homogeneous TT modes

Define the first-order and second-order boundary-ball readout maps on the fixed $t = 0$ rest-frame family $\mathcal{B}_0 := \{B(R,\mathbf{x}_0) : R > 0,\ \mathbf{x}_0 \in \mathbb{R}^2\}$:

$$M_1(h) := \big\{(DE_B[h],\ \tfrac{1}{4}DA_B[h])\big\}_{B\in\mathcal{B}_0}, \qquad M_2(h) := \big\{\mathcal{E}_B(h)\big\}_{B\in\mathcal{B}_0}. \tag{2.13}$$

As established in [5, 6], the vanishing of the first-law defect $DF_B[h] = DE_B[h] - \frac{1}{4}DA_B[h]$ for all $B \in \mathcal{B}_0$ recovers the slice Hamiltonian constraint $D\mathcal{F}_{tt}[h]|_{t=0} = 0$, but does *not* recover the

full perturbation h; indeed, $\ker M_1$ contains infinite-dimensional families of non-gauge on-shell vacuum solutions.

A fundamental model subspace inside $\ker M_1$ is the space $V_{\rm hom}$ of spatially homogeneous transverse-traceless (TT) vacuum perturbations:

$$h_{ij}(t,z) = z\begin{pmatrix} H_+(t,z) & H_\times(t,z) \\ H_\times(t,z) & -H_+(t,z)\end{pmatrix}, \qquad h_{ta} = h_{za} = 0, \tag{2.14}$$

where each polarization amplitude $H_\alpha(t,z)$ ($\alpha \in \{+,\times\}$) is smooth up to $z=0$ and solves the linearized vacuum equation

$$\left(\partial_t^2 - \partial_z^2 - \frac{4}{z}\partial_z\right) H_\alpha(t,z) = 0. \tag{2.15}$$

For every $h \in V_{\rm hom}$, we have $H^{(0)}_{tt} = 0 \implies DE_B[h] = 0$, while in the first-order area formula [6]

$$DA_B[h] = \frac{R}{2}\int_{r<R}\frac{1}{z}\left(\delta^{ij} - \frac{(x^i-x_0^i)(x^j-x_0^j)}{R^2}\right) h_{ij}\,\mathrm{d}^2x, \qquad z = \sqrt{R^2-r^2}, \tag{2.16}$$

the contraction produces $\cos(2\varphi)H_+ + \sin(2\varphi)H_\times$, whose azimuthal average over $[0,2\pi)$ vanishes identically. Hence $V_{\rm hom} \subset \ker M_1$.

Expanding (2.15) in regular Frobenius series around $z=0$ splits $V_{\rm hom}$ into time-even ($H_t|_{t=0} = 0$) and time-odd ($H|_{t=0} = 0$) sectors. For each single polarization, the regular polynomial vacuum tower is spanned by the time-even basis $\{E_0, \widetilde{E}_1, \widetilde{E}_2, \dots\}$ and time-odd basis $\{O_0, O_1, O_2, \dots\}$ defined recursively by

$$\widetilde{E}_k(t,z) = \sum_{j=0}^{k} c_{k,j}\, t^{2(k-j)} z^{2j}, \qquad c_{k,0} = 1, \quad c_{k,j} = \frac{(2k-2j+2)(2k-2j+1)}{2j(2j+3)}\, c_{k,j-1}, \tag{2.17}$$

$$O_k(t,z) = \sum_{j=0}^{k} d_{k,j}\, t^{2(k-j)+1} z^{2j}, \quad d_{k,0} = 1, \quad d_{k,j} = \frac{(2k-2j+3)(2k-2j+2)}{2j(2j+3)}\, d_{k,j-1}. \tag{2.18}$$

Explicitly, the lowest modes are

$$\begin{aligned} &E_0 = 1, \quad \widetilde{E}_1 = t^2 + \tfrac{1}{5}z^2, \quad \widetilde{E}_2 = t^4 + \tfrac{6}{5}t^2z^2 + \tfrac{3}{35}z^4, \quad \widetilde{E}_3 = t^6 + 3t^4z^2 + \tfrac{9}{7}t^2z^4 + \tfrac{1}{21}z^6, \\ &O_0 = t, \quad O_1 = t^3 + \tfrac{3}{5}tz^2, \quad O_2 = t^5 + 2t^3z^2 + \tfrac{3}{7}tz^4. \end{aligned} \tag{2.19}$$

In this notation, the benchmark mode of Rodgers [8] is $E_{\rm Rodgers} = \widetilde{E}_1 + \frac{1}{3}E_0$. The bounded oscillatory vacuum modes at frequency $\omega > 0$ are

$$h_c(t,z) = z\cos(\omega t)\, j(\omega z), \qquad h_s(t,z) = z\sin(\omega t)\, j(\omega z), \qquad j(u) := 3\,\frac{\sin u - u\cos u}{u^3}, \tag{2.20}$$

which solve $j'' + \frac{4}{u}j' + j = 0$ with $j(0) = 1$. In Section 6 we generalize far beyond $V_{\rm hom}$ to construct an infinite-dimensional class of spatially localized, finite-energy AdS_4 graviton wavepackets that remain strictly inside $\ker M_1$ for all balls at all times.

3 Lorentzian Timelike HRT Extrema, Exact $\mathbb{H}^2$ Green's Function, Hollands–Wald Verification, and Second-Order Completions

In this section we completely solve the second-order canonical energy functional $\mathcal{E}_B(h)$ on $V_{\rm hom}$ for arbitrary time-even and time-odd modes, verify the time-odd canonical energy by three completely independent gravitational derivations, and construct exact closed-form Fefferman–Graham second-order vacuum completions.

3.1 Exact 4D Einstein expansion and background divergence-freeness

Consider a single-polarization homogeneous TT perturbation $h_{xy} = h_{yx} = zH(t,z)$ (by $SO(2)$ rotational invariance in the (x,y) plane, any fixed linear polarization yields the same scalar invariants; the full two-polarization $(H_+, H_\times)$ structure follows directly by polarization and rotational covariance). Direct fourth-dimensional expansion of $\mathcal{F}_{ab}(\bar{g}+\lambda h) = G_{ab} - 3g_{ab}$ through $O(\lambda^2)$ gives $D\mathcal{F}_{xy}[h] = \frac{z^2}{2}(zH_{tt} - zH_{zz} - 4H_z)$ with all other first-order components vanishing, whose nonvanishing second-order components (listed in full in Appendix A) have time-time component

$$D^2\mathcal{F}_{tt}[h,h] = \frac{z^4}{2}\Big(4z^2HH_{zz} + 3z^2H_z^2 + 34zHH_z + 27H^2 - z^2H_t^2\Big). \tag{3.1}$$

Observe that time-even data $(H|_{t=0})$ and time-odd data $(H_t|_{t=0})$ decouple identically in (3.1), and the time-odd bulk energy density $-D^2\mathcal{F}_{tt}[h_{\rm odd}, h_{\rm odd}]|_{t=0} = +\frac{1}{2}z^6H_t(0,z)^2$ is strictly positive. Direct covariant differentiation confirms that on shell $(D\mathcal{F}_{xy}[h] = 0)$, the entire quadratic source is identically background divergence-free: $\bar{\nabla}^a D^2\mathcal{F}_{ab}[h,h] \equiv 0$ for all $b \in \{t,z,x,y\}$.

Integrating (3.1) over Σ_B using $\int_{r<\sqrt{R^2-z^2}} \xi_B^t \, \mathrm{d}^2x = \frac{\pi^2}{2R}(R^2-z^2)^2$ yields the bulk contribution $I[h] = I[h_{\rm even}] + I[h_{\rm odd}]$ in (2.12), with

$$-I[h_{\rm odd}] = \frac{\pi}{32R}\int_0^R (R^2-z^2)^2 z^4 \, H_t(0,z)^2 \, \mathrm{d}z > 0. \tag{3.2}$$

3.2 Even–odd decoupling and the shared $\mathbb{H}^2$ Jacobi Green's function

For a general perturbation $h = h_{\rm even} + h_{\rm odd} \in V_{\rm hom}$ with $h_{xy} = zH(t,z)$, the hemisphere $\varrho = R, t = 0$ is perturbed in both normal directions:

$$\varrho(\theta,\varphi) = R + \lambda w(\theta,\varphi) + O(\lambda^2), \qquad t(\theta,\varphi) = \lambda\tau(\theta,\varphi) + O(\lambda^2), \tag{3.3}$$

anchored at ∂B ($w, \tau = o(z)$ as $z \to 0$). Expanding the induced 2-metric $q_{AB}(\lambda)$ on the perturbed surface through $O(\lambda^2)$ shows that only the $m = 2$ azimuthal harmonic $\sin(2\varphi)$ is linearly forced; setting $w(\theta,\varphi) = f(\theta)\sin(2\varphi)$ and $\tau(\theta,\varphi) = g(\theta)\sin(2\varphi)$ and integrating $\partial_\lambda^2\sqrt{\det q}\big|_{\lambda=0}$ over $\varphi \in [0, 2\pi)$ yields exact even–odd decoupling:

$$A''[h; f, g] = \int_0^1 L_{\rm even}\big(y; f, f', H_0\big)\,\mathrm{d}y + \int_0^1 L_{\rm odd}\big(y; g, g', H_1\big)\,\mathrm{d}y, \qquad L_{\rm cross} \equiv 0, \tag{3.4}$$

where $y = \cos\theta = z/R \in (0,1]$, $H_0(z) := H(0,z)$, $H_1(z) := H_t(0,z)$, and the time-odd area-Hessian Lagrangian density is

$$L_{\rm odd}(y) = -\frac{\pi}{R^2}\left[\frac{1-y^2}{y^2}g'(y)^2 + \frac{4}{y^2(1-y^2)}g(y)^2\right] - \pi R^3 y(1-y^2)H_1(Ry)\,g(y). \tag{3.5}$$

The critical Lorentzian sign structure is transparent in (3.5): because $\bar{g}_{tt} = -z^{-2} < 0$, the timelike shape Hessian enters with a *negative* sign,

$$Q_{\rm tm}[g] := \frac{\pi}{R^2}\int_0^1 \left[\frac{1-y^2}{y^2}g'(y)^2 + \frac{4}{y^2(1-y^2)}g(y)^2\right]\mathrm{d}y > 0, \tag{3.6}$$

so Lorentzian extremality *maximizes* the area in the timelike direction:

$$A''_{\rm ext}[h_{\rm odd}] = \max_g\big\{-Q_{\rm tm}[g] + 2M_{\rm tm}[g]\big\} = +Q_{\rm tm}[g_*] > 0. \tag{3.7}$$

Consequently, in the charge–area identity (2.12), the timelike surface contribution $-\frac{1}{4}A''_{\rm ext}[h_{\rm odd}] = -\frac{1}{4}Q_{\rm tm}[g_*] < 0$ *subtracts* from the positive bulk kinetic integral $-I[h_{\rm odd}] > 0$, in sharp contrast to the time-even reflection-symmetric case where spatial minimization *adds* $+\frac{1}{4}Q_{\rm sp}[f_*] > 0$ [8].

Because $\widetilde{B} \cong \mathbb{H}^2$ is totally geodesic in pure AdS_4 in both normal directions, the Euler–Lagrange equations for both the spatial displacement $f(y)$ and the timelike displacement $g(y)$ share the *identical* Sturm–Liouville Jacobi operator

$$\mathcal{L}_{m=2}[u](y) := \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1-y^2}{y^2}u'(y)\right) - \frac{4}{y^2(1-y^2)}u(y), \tag{3.8}$$

subject to pole regularity $u(1) = 0$ and anchor condition $u(y) = O(y^3)$ as $y \to 0^+$.

Lemma 3.1 (Exact closed-form Green's function of $\mathcal{L}_{m=2}$)**.** *The homogeneous equation $\mathcal{L}_{m=2}[u] = 0$ on $(0,1)$ has the two exact linearly independent solutions*

$$\psi_{\mathrm{pole}}(y) = \frac{1-y}{1+y}(1+2y), \qquad \psi_{\mathrm{anch}}(y) = \frac{y^3}{1-y^2}, \tag{3.9}$$

satisfying $\psi_{\mathrm{pole}}(1) = 0$, $\psi_{\mathrm{pole}}(0) = 1$, $\psi_{\mathrm{anch}}(y) = O(y^3)$ as $y \to 0^+$, and constant Sturm–Liouville Wronskian

$$\frac{1-y^2}{y^2}\big(\psi_{\mathrm{anch}}\psi'_{\mathrm{pole}} - \psi'_{\mathrm{anch}}\psi_{\mathrm{pole}}\big) \equiv -3. \tag{3.10}$$

Consequently, for any continuous forcing F with $F(y) = O(y)$ as $y \to 0^+$, the unique anchored regular solution of $\mathcal{L}_{m=2}[u] = -F$ is given in closed form by

$$u(y) = \int_0^1 G(y,y')F(y')\,\mathrm{d}y', \qquad G(y,y') := \frac{1}{3}\,\psi_{\mathrm{anch}}\big(\min(y,y')\big)\,\psi_{\mathrm{pole}}\big(\max(y,y')\big) > 0. \tag{3.11}$$

Proof. Direct differentiation confirms $\mathcal{L}_{m=2}[\psi_{\mathrm{pole}}] = 0$ (which coincides with the $m = 2$ linearized minimal-surface multipole profile of [9, 13]). Because $\mathcal{L}_{m=2}$ is invariant under $y \mapsto -y$, $\psi_{\mathrm{pole}}(-y)$ is also a solution, and $\frac{1}{4}(\psi_{\mathrm{pole}}(y) - \psi_{\mathrm{pole}}(-y)) = y^3/(1-y^2) = \psi_{\mathrm{anch}}(y)$. Computing the Wronskian gives (3.10), so the standard Sturm–Liouville jump condition yields $\mathcal{L}_{m=2}[G(\cdot,y')] = -\delta(\cdot - y')$ with the stated boundary asymptotics. □

Applying Lemma 3.1 to the monomial basis y^{2k} yields an exact algebraic identity for all integers $k \geq 0$: defining

$$P_k(y) := y^3(1-y^2)\sum_{m=0}^{k}(m+1)y^{2m} = y^3\,\frac{1-(k+2)y^{2k+2}+(k+1)y^{2k+4}}{1-y^2}, \tag{3.12}$$

direct symbolic differentiation verifies for all integers $k \geq 0$ that

$$\mathcal{L}_{m=2}[P_k](y) = -2(k+1)(k+2)(2k+5)\,y^{2k+1}(1-y^2). \tag{3.13}$$

For a time-odd forcing $H_1(z) = z^{2k}$, the Euler–Lagrange equation is $\mathcal{L}_{m=2}[g] = +\frac{1}{2}R^{2k+5}y^{2k+1}(1-y^2)$; for a time-even forcing $H_0(z) = z^{2k}$, the spatial Euler–Lagrange equation is $\mathcal{L}_{m=2}[f] = +\frac{2k+5}{2}R^{2k+4}y^{2k+1}(1-y^2)$. Thus (3.13) immediately yields the exact HRT displacements for the entire polynomial vacuum tower:

$$f_*^{(k)}(y) = -\frac{R^{2k+4}}{4(k+1)(k+2)}P_k(y), \qquad g_*^{(k)}(y) = -\frac{R^{2k+5}}{4(k+1)(k+2)(2k+5)}P_k(y). \tag{3.14}$$

3.3 Closed-form bilinear energy matrices, exact one-line even identity, and sharp spectral theorem for time-odd modes

Evaluating (2.12) on monomial pairs (z^{2i}, z^{2j}) using (3.14) yields closed-form bilinear canonical-energy matrices for all integers $i, j \geq 0$.

Theorem 3.2 (All-degree canonical energy formulas, one-line even identity, and sharp time-odd spectral positivity)**.** *Let $h \in V_{\mathrm{hom}}$ have single polarization $h_{xy}(t,z) = zH(t,z)$ with smooth even Cauchy data $H_0(z) = H(0,z)$ and $H_1(z) = H_t(0,z)$. Then $\mathcal{E}_{B(R)}(h) = \mathcal{E}_{B(R)}(h_{\mathrm{even}}) + \mathcal{E}_{B(R)}(h_{\mathrm{odd}})$ satisfies:*

(i) ***Exact all-(i,j) monomial bilinear forms.*** *For monomials z^{2i}, z^{2j} $(i,j \geq 0)$, the bilinear canonical energies are $\pi R^{6+2(i+j)}\alpha_{\mathrm{mon}}(i,j)$ (time-even) and $\pi R^{8+2(i+j)}\gamma_{\mathrm{mon}}(i,j)$ (time-odd), where*

$$\alpha_{\mathrm{mon}}(i,j) = \frac{8ij + 6i + 6j + 9}{8\,(2i+2j+3)(2i+2j+5)(2i+2j+7)} > 0, \tag{3.15}$$

$$\gamma_{\mathrm{mon}}(i,j) = \frac{8ij + 18i + 18j + 45}{8\,(2i+5)(2j+5)(2i+2j+5)(2i+2j+7)(2i+2j+9)} > 0. \tag{3.16}$$

(ii) ***Exact one-line time-even integral identity.*** *For* every *regular time-even vacuum tangent $H_0(z) = \sum_{k=0}^{\infty} c_k z^{2k}$ (including bounded oscillatory modes), after incorporating the exact spatial HRT surface minimization w_*, the canonical energy collapses to the single manifestly positive integral*

$$\mathcal{E}_{B(R)}(h_{\mathrm{even}}) = \frac{\pi}{64R}\int_0^R (R^2 - z^2)^2 z^2 \Big[\big(3H_0(z) + zH_0'(z)\big)^2 + z^2 H_0'(z)^2\Big]\,\mathrm{d}z > 0. \tag{3.17}$$

In particular, every finite-dimensional time-even Gram matrix $[\alpha_{\mathrm{mon}}(i,j)]_{0\leq i,j\leq n-1}$ is strictly positive definite.

(iii) ***Sharp time-odd spectral positivity theorem.*** *Writing $\psi(y) := y^2(1-y^2)H_1(Ry) \in L^2([0,1])$, the time-odd canonical energy depends only on the wedge Cauchy velocity $H_1|_{[0,R]}$ and takes the exact operator form*

$$\mathcal{E}_{B(R)}(h_{\mathrm{odd}}) = \frac{\pi R^8}{32}\,\big\langle \psi,\, (I - T)\psi \big\rangle_{L^2([0,1])}, \tag{3.18}$$

where T is the Hilbert–Schmidt (hence compact), self-adjoint, strictly positive integral operator on $L^2([0,1])$ with kernel

$$T(y,y') = \frac{2\,G(y,y')}{y\,y'} = \frac{2}{3}\,\frac{y_<^2}{1-y_<^2}\,\frac{(1-y_>)(1+2y_>)}{y_>(1+y_>)}, \qquad y_< := \min(y,y'),\ y_> := \max(y,y'). \tag{3.19}$$

Setting $\psi_0(y) := y^2(1-y^2)$, the nested finite-dimensional subspaces $W_n := \psi_0 \cdot \mathrm{span}\{1, y^2, \ldots, y^{2n}\}$ $(n \geq 0)$ form a T-invariant flag with exact upper-triangular action

$$\begin{aligned}\big(T(\psi_0 y^{2n})\big)(y) &= \frac{2}{(n+1)(2n+4)(2n+5)}\,\psi_0(y)\sum_{m=0}^{n}(m+1)y^{2m} \\ &= \frac{2}{(2n+4)(2n+5)}\,\psi_0(y)y^{2n} + W_{n-1}.\end{aligned} \tag{3.20}$$

Its orthogonalized polynomial basis $\{\phi_n\}_{n=0}^{\infty}$ ($W_n \ominus W_{n-1} = \mathrm{span}\{\phi_n\}$) is a complete orthonormal eigenbasis of T in $L^2([0,1])$, giving the exact spectrum

$$\mathrm{spec}(T) = \{0\} \cup \left\{\lambda_n := \frac{2}{(2n+4)(2n+5)} : n = 0,1,2,\ldots\right\}, \qquad (T\psi_0)(y) \equiv \frac{1}{10}\,\psi_0(y), \tag{3.21}$$

with every nonzero eigenvalue simple and strictly decreasing from $\|T\|_{\mathrm{op}} = \lambda_0 = \frac{1}{10}$. Consequently, for every *time-odd Cauchy datum H_1 whose restriction to the wedge interval*

$[0,R]$ *satisfies* $\psi \in L^2([0,1]) \setminus \{0\}$ *(in particular every* $H_1 \in C([0,R])$ *not identically zero on* $[0,R]$*),*

$$\frac{\frac{1}{4}A''_{\mathrm{ext}}[h_{\mathrm{odd}}]}{-I[h_{\mathrm{odd}}]} \le \frac{1}{10}, \qquad \textit{so} \qquad \mathcal{E}_{B(R)}(h_{\mathrm{odd}}) \ge \frac{9}{10}\big(-I[h_{\mathrm{odd}}]\big) > 0, \tag{3.22}$$

with equality if and only if $H_1(z)$ *is constant almost everywhere on the wedge interval* $[0,R]$ *(extending for analytic homogeneous modes to* $H_1 \equiv \mathrm{const}$*, i.e.,* O_0*).*

Proof. (i) For $H_1 = z^{2i}, z^{2j}$, (3.2) gives $-I(z^{2i}, z^{2j}) = \frac{\pi R^{8+2(i+j)}}{32}\int_0^1 (1-y^2)^2 y^{4+2i+2j}\mathrm{d}y = \frac{\pi R^{8+2(i+j)}}{4(2s+5)(2s+7)(2s+9)}$ with $s = i+j$, while $\frac{1}{4}Q_{\mathrm{tm}}(g_*^{(i)}, g_*^{(j)}) = \frac{\pi R^{8+2s}}{32(i+1)(i+2)(2i+5)}\int_0^1 y^{2j+1}(1-y^2)P_i(y)\mathrm{d}y$. Since $(1-y^2)$ cancels in the integrand (3.12), elementary term-by-term integration and simplification yield (3.16). For the time-even sector, combining the exact bulk integral from (3.1), the fixed-hemisphere metric Hessian A_{gg}, and $+\frac{1}{4}Q_{\mathrm{sp}}(f_*^{(i)}, f_*^{(j)})$ cancels the factors $(2i+5)(2j+5)$ identically and yields (3.15).

(ii) Observe that $8ij + 6i + 6j + 9 = (2i+3)(2j+3) + 4ij$, while $\int_0^1 (1-y^2)^2 y^{2i+2j+2}\mathrm{d}y = \frac{8}{(2s+3)(2s+5)(2s+7)}$. Therefore

$$\pi R^{6+2s}\alpha_{\mathrm{mon}}(i,j) = \frac{\pi}{64R}\int_0^R (R^2-z^2)^2 z^2\Big[(3z^{2i}+2iz^{2i})(3z^{2j}+2jz^{2j}) + (2iz^{2i})(2jz^{2j})\Big]\mathrm{d}z,$$

which coincides with the polarization of (3.17) on every monomial pair (z^{2i}, z^{2j}). By Weierstrass density of even polynomials in $C^1([0,R])$ (every regular solution of (2.15) is even in z), (3.17) holds for every regular time-even mode.

(iii) Rewriting $-I[h_{\mathrm{odd}}] = \frac{\pi R^8}{32}\|\psi\|^2_{L^2([0,1])}$ (from (3.2), inheriting the $\frac{1}{8\pi}$ prefactor of (2.12)) and substituting (3.11) into $\frac{1}{4}Q_{\mathrm{tm}}[g_*] = \frac{\pi R^8}{16}\iint y(1-y^2)y'(1-y'^2)G(y,y')H_1(Ry)H_1(Ry')\,\mathrm{d}y\,\mathrm{d}y'$ cancels $y(1-y^2)$ against $\psi(y)/y$ to yield (3.18)–(3.19), with $T(y,y') = 2\,G(y,y')/(yy')$. Notice $0 < T(y,y') \le 2\,y_<^2/y_>$ on $0 < y_< \le y_> < 1$ (since $\frac{1-y_>}{1-y_<^2} \le \frac{1}{1+y_<} \le 1$ and $\frac{1+2y_>}{3(1+y_>)} \le \frac{1}{2}$), so $\iint_{(0,1)^2} T(y,y')^2\,\mathrm{d}y\,\mathrm{d}y' \le 8\int_0^1\int_0^{y_>}\frac{y_<^4}{y_>^2}\,\mathrm{d}y_<\,\mathrm{d}y_> = \frac{2}{5} < \infty$: thus T is a symmetric Hilbert–Schmidt (hence compact, self-adjoint) operator on $L^2([0,1])$.

To diagonalize T analytically, evaluate $u_n(y) := \frac{1}{2}y\big(T(\psi_0 y^{2n})\big)(y) = \int_0^1 G(y,y')\,y'^{2n+1}(1-y'^2)\,\mathrm{d}y'$ directly on the polynomial-flag basis elements $f_n(y) := \psi_0(y)y^{2n} = y^{2n+2}(1-y^2)$ $(n \ge 0)$. Because $F_n(y') := f_n(y')/y' = y'^{2n+1}(1-y'^2)$ is continuous on $[0,1]$ and vanishes as $O(y')$ at $y' = 0$, the explicit Green's function (3.11) yields $u_n(y) = O(y^3)$ as $y \to 0^+$ (note that this $O(y^3)$ bound relies on the vanishing of $f_n(y)/y$ at $y = 0$ and does not hold for general $f \in L^2([0,1])$, e.g. $f \equiv 1$ produces a $y^3\log(1/y)$ term) and $u_n(1) = 0$, so u_n is the unique boundary-regular solution of $\mathcal{L}_{m=2}[u_n] = -y^{2n+1}(1-y^2)$. By (3.13), this unique solution is $u_n(y) = \frac{P_n(y)}{2(n+1)(n+2)(2n+5)}$ (as can also be verified directly by Sturm–Liouville integration by parts of $\int_0^1 G(y,y')\mathcal{L}_{m=2}[P_n](y')\,\mathrm{d}y'$, where all boundary terms vanish at $y' = 0, 1$), whence

$$\big(T(\psi_0 y^{2n})\big)(y) = \frac{2u_n(y)}{y} = \frac{P_n(y)/y}{(n+1)(n+2)(2n+5)} = \frac{2}{(n+1)(2n+4)(2n+5)}\,\psi_0(y)\sum_{m=0}^{n}(m+1)y^{2m},$$

proving (3.20) for every integer $n \ge 0$ (independently checked by direct integration of (3.19) for $n = 0, \dots, 4$ in `verify_01`). In particular, $T(W_n) \subset W_n$ for every $n \ge 0$, with leading diagonal coefficient $\lambda_n = \frac{2}{(2n+4)(2n+5)}$. Because T is self-adjoint, each 1-dimensional orthogonal complement $W_n \ominus W_{n-1}$ $(W_{-1} := \{0\})$ is T-invariant and therefore spanned by a normalized eigenfunction ϕ_n of T with exact eigenvalue $\lambda_n = \frac{2}{(2n+4)(2n+5)}$.

Finally, $\bigcup_{n\ge 0} W_n = \psi_0 \cdot \mathbb{R}[y^2]$ is dense in $L^2([0,1])$: for any $f \in C_c^\infty((0,1))$ (a dense subset of $L^2([0,1])$), the quotient f/ψ_0 lies in $C_c^\infty((0,1)) \subset C([0,1])$; since y^2 separates points on

$[0,1]$, the Stone–Weierstrass theorem yields polynomials $p_k(y^2)$ with $\|p_k - f/\psi_0\|_{L^\infty([0,1])} \to 0$, and hence $\|\psi_0 p_k - f\|_{L^2([0,1])} \le \|\psi_0\|_{L^2([0,1])}\|p_k - f/\psi_0\|_{L^\infty([0,1])} \to 0$. Therefore $\{\phi_n\}_{n=0}^\infty$ is a complete orthonormal basis of $L^2([0,1])$, establishing (3.21) completely analytically (with $\phi_0 \propto \psi_0$, $\lambda_0 = \frac{1}{10}$, and $\lambda_n < \frac{1}{10}$ for all $n \ge 1$). Expanding any $\psi \in L^2([0,1])$ as $\psi = \sum_{n=0}^\infty c_n\phi_n$ gives $\langle\psi, T\psi\rangle = \sum_{n=0}^\infty \lambda_n|c_n|^2 \le \frac{1}{10}\|\psi\|^2$, with equality iff $c_n = 0$ for all $n \ge 1$, i.e. $\psi \propto \psi_0$, which is equivalent to $H_1(z)$ being constant a.e. on the wedge data interval $[0, R]$. □

Remark 3.3 (Structural origin of the even–odd asymmetry). Notice why the time-even energy (3.17) reduces to a local radial integral while the time-odd energy (3.18) remains genuinely nonlocal. In the time-even sector, the spatial HRT forcing $F^{(k)}_{\text{even}} \propto (2k+5)y^{2k+1}(1-y^2)$ carries the exact factor $(2k+5)$ that cancels the denominator of $\mathcal{L}^{-1}_{m=2}$ in (3.13); in the time-odd sector, the timelike HRT forcing $F^{(k)}_{\text{odd}} \propto y^{2k+1}(1-y^2)$ lacks that factor, leaving the irreducible Hilbert–Schmidt operator T. Nyström discretization of (3.19) at 120 and 240 Gauss–Legendre nodes confirms $O(N^{-2})$ convergence across the $y = y'$ Green's kink to the exact spectrum $\lambda_n = 2/[(2n+4)(2n+5)]$: for $n = 0, \dots, 4$, the 240-node eigenvalues are $(0.100011, 0.047630, 0.027789, 0.018192, 0.012831)$ against exact $(0.1, 1/21, 1/36, 1/55, 1/78)$, with $\|T\|_{\text{HS}} \approx 0.117484$.

Table 2 collects the exact closed-form HRT displacements, bulk/surface splits, and canonical energies for the lowest modes of the polynomial vacuum tower (2.19), obtained by scaling (3.15)–(3.16) by the on-shell coefficients $c_{k,k}$ and $d_{k,k}$. (We also verified the 4×4 even matrix $[\alpha_{ij}]_{0\le i,j\le3}$ and 3×3 odd matrix $[\gamma_{ij}]_{0\le i,j\le2}$ directly via Lemma 3.1 in `verify_01_timelike_hrt_and_hw.py`, confirming all leading principal minors are strictly positive.)

Table 2: Exact closed-form HRT surface displacements ($w_* = f_*(y)\sin2\varphi$ or $\tau_* = g_*(y)\sin2\varphi$, $y = \cos\theta = z/R$), bulk terms $-I$, surface terms $-\frac{1}{4}A''_{\text{ext}}$, and canonical energies $\mathcal{E}_{B(R)}$ for the lowest polynomial vacuum modes in $\ker M_1$. Notice that for O_0, neglecting the timelike HRT shift τ_* overestimates $\mathcal{E}_B(O_0)$ by the exact factor 10/9.

Mode	Exact HRT shift (f_* or g_*)	Bulk term $-I$	Surface $-\frac{1}{4}A''_{\text{ext}}$	Exact $\mathcal{E}_{B(R)}$
$E_0\,(H=1)$	$-\frac{R^4}{8}(1-y^2)y^3$	$-\frac{9\pi R^6}{140}$	$+\frac{3\pi R^6}{40}$	$\frac{3\pi R^6}{280}$
$\widetilde{E}_1\,(t^2+\frac{z^2}{5})$	$-\frac{R^6}{120}(1-y^2)y^3(2y^2+1)$	$-\frac{23\pi R^{10}}{13860}$	$+\frac{259\pi R^{10}}{138600}$	$\frac{29\pi R^{10}}{138600}$
$\widetilde{E}_2\,(t^4+\frac{6}{5}t^2z^2+\frac{3}{35}z^4)$	$-\frac{R^8}{560}(1-y^2)y^3(3y^4+2y^2+1)$	$-\frac{209\pi R^{14}}{2162160}$	$+\frac{269\pi R^{14}}{2162160}$	$\frac{3\pi R^{14}}{107800}$
$\widetilde{E}_3$	$-\frac{R^{10}}{1680}(1-y^2)y^3\sum_{m=0}^3(m+1)y^{2m}$	—	—	$\frac{13\pi R^{18}}{1899240}$
$O_0\,(H=t)$	$-\frac{R^5}{40}(1-y^2)y^3 = -\frac{1}{20}xyz^3$	$+\frac{\pi R^8}{1260}$	$-\frac{\pi R^8}{12600}$ (-10%)	$\frac{\pi R^8}{1400}$
$O_1\,(t^3+\frac{3}{5}tz^2)$	$-\frac{R^7}{280}(1-y^2)y^3(2y^2+1)$	$+\frac{\pi R^{12}}{14300}$	$-\frac{9\pi R^{12}}{1401400}$ $(-\frac{9}{98})$	$\frac{89\pi R^{12}}{1401400}$
$O_2\,(t^5+2t^3z^2+\frac{3}{7}tz^4)$	$-\frac{R^9}{1008}(1-y^2)y^3(3y^4+2y^2+1)$	$+\frac{3\pi R^{16}}{216580}$	$-\frac{\pi R^{16}}{899640}$ $(-\frac{13}{162})$	$\frac{149\pi R^{16}}{11695320}$

For later reference in Sections 4 and 5, the exact 4×4 time-even dimensionless coefficient matrix $[\alpha_{ij}]$ (defined by $\mathcal{E}_{B(R)}(\widetilde{E}_i, \widetilde{E}_j) = \pi\alpha_{ij}R^{6+2(i+j)}$) and 3×3 time-odd matrix $[\gamma_{ij}]$ ($\mathcal{E}_{B(R)}(O_i, O_j) = \pi\gamma_{ij}R^{8+2(i+j)}$) are

$$\alpha^{(4)} = \begin{pmatrix} \frac{3}{280} & \frac{1}{840} & \frac{1}{3080} & \frac{1}{8008} \\ \frac{1}{840} & \frac{29}{138600} & \frac{43}{600600} & \frac{19}{600600} \\ \frac{1}{3080} & \frac{43}{600600} & \frac{3}{107800} & \frac{29}{2165800} \\ \frac{1}{8008} & \frac{19}{600600} & \frac{29}{2165800} & \frac{13}{1899240} \end{pmatrix}, \qquad \gamma^{(3)} = \begin{pmatrix} \frac{1}{1400} & \frac{3}{15400} & \frac{3}{40040} \\ \frac{3}{15400} & \frac{89}{1401400} & \frac{23}{840840} \\ \frac{3}{40040} & \frac{23}{840840} & \frac{149}{11695320} \end{pmatrix}. \tag{3.23}$$

For $E_{\text{Rodgers}} = \widetilde{E}_1 + \frac{1}{3}E_0$, (3.23) reproduces Rodgers's Theorem 4.1 [8], $\mathcal{E}_B(E_{\text{Rodgers}}) = \frac{\pi R^6}{138600}(29R^4 + 110R^2 + 165)$, identically.

3.4 Rigorous Hollands–Wald gauge audit, corner asymptotics, and direct eq. (2.10) evaluation

Because the timelike HRT subtraction $-\frac{\pi R^8}{12600}$ for O_0 carries the opposite sign to the spatial reflection-symmetric case, it is essential to verify $\mathcal{E}_B(O_0) = \frac{\pi R^8}{1400}$ directly from the covariant phase-space formalism (2.9)–(2.10), with full scrutiny of all Hollands–Wald gauge conditions and $z \to 0$ corner/asymptotic-boundary behavior.

Observe that in Cartesian coordinates, the exact timelike HRT displacements of O_0 and O_1 collapse to pure polynomials:

$$\tau_*^{(O_0)} = -\frac{1}{20}xyz^3, \qquad \tau_*^{(O_1)} = -\frac{1}{140}xyz^3\big(2z^2 + R^2\big). \tag{3.24}$$

Define the surface-fixing diffeomorphism generator

$$V^a = \big(\tau_*,\ t\partial_z\tau_*,\ t\partial_x\tau_*,\ t\partial_y\tau_*\big), \qquad \gamma_{ab} := v_{ab} + (\mathcal{L}_V\bar{g})_{ab}, \tag{3.25}$$

where $v_{xy} = zH(t,z)$ is the boundary-preserving representative. For O_0 ($\tau_* = -\frac{1}{20}xyz^3$), direct symbolic computation in `verify_01_timelike_hrt_and_hw.py` establishes all required surface and asymptotic properties:

(1) **Vanishing on $t = 0$ and mixed components:** $\gamma_{ab}|_{t=0} \equiv 0$ identically everywhere on Σ_B, and $\gamma_{0i}(t,z,x,y) \equiv 0$ for all $t \in \mathbb{R}$ and $i \in \{z,x,y\}$. In particular, all mixed normal–tangential components vanish on $\widetilde{B}$.

(2) **Linearized timelike extremality on $\widetilde{B}$:** $\bar{q}^{AB}\partial_t\gamma_{AB}\big|_{t=0,\,\varrho=R} \equiv 0$, confirming that the perturbed HRT surface remains fixed at $t = 0, \varrho = R$ in the γ gauge.

(3) **Boost invariance on $\widetilde{B}$:** $(\mathcal{L}_{\xi_B}\gamma)_{ab}\big|_{t=0,\,\varrho=R} \equiv 0$ for the entire 4×4 tensor on $\widetilde{B}$, guaranteeing boost invariance of the normal block and vanishing of mixed components under modular flow.

(4) **Direct surface-fixed symplectic pairing (2.9):** Evaluating $\omega(\bar{g};\gamma,\mathcal{L}_{\xi_B}\gamma)$ directly from (2.7)–(2.8) gives

$$W_{\Sigma_B}\big(v,\mathcal{L}_{\xi_B}v\big) = \frac{\pi R^8}{1260}, \qquad W_{\Sigma_B}\big(\gamma,\mathcal{L}_{\xi_B}\gamma\big) = \frac{\pi R^8}{1400}, \tag{3.26}$$

matching Table 2 to exact zero. Adding a $\widetilde{B}$-vanishing gauge shift $\frac{1}{7}(R^2 - z^2 - x^2 - y^2)xyz^3$ to V^t leaves $W_{\Sigma_B}(\gamma,\mathcal{L}_{\xi_B}\gamma) = \frac{\pi R^8}{1400}$ invariant, whereas flipping the sign $V^t = -\tau_*$ (violating extremality) gives $\frac{13\pi R^8}{12600} \neq \frac{\pi R^8}{1400}$. For O_1, the same construction (3.25) gives $W_{\Sigma_B}(\gamma_{O_1},\mathcal{L}_{\xi_B}\gamma_{O_1}) = \frac{89\pi R^{12}}{1401400}$, again matching Table 2 to exact zero.

(5) **Direct evaluation of the boundary 2-form $\rho(h,V)$ (2.10) and $z \to 0$ corner audit:** A subtle regularity check is required for (3.25) because $V^z = t\partial_z\tau_* = -\frac{3}{20}txyz^2 = O(tz^2)$ induces $\partial_t(\mathcal{L}_V\bar{g})_{zz}|_{t=0} = O(z^{-1})$ near $z = 0$. To verify rigorously that this slower radial falloff introduces *no* spurious conformal-boundary or corner term at $z = \varepsilon_z \to 0$, we compute the densitized 2-form bivector $\rho^{0i}(v,V) := \sqrt{-\bar{g}}\big(\chi^{0i}(\gamma;[\xi_B,V]) - \chi^{0i}(\mathcal{L}_{\xi_B}v;V)\big)$ directly from (2.8). Pointwise on Σ_B, it satisfies the exact Stokes identity

$$\partial_z\rho^{0z} + \partial_x\rho^{0x} + \partial_y\rho^{0y} \equiv \sqrt{-\bar{g}}\,\Big[\omega^t(\gamma,\mathcal{L}_{\xi_B}\gamma) - \omega^t(v,\mathcal{L}_{\xi_B}v)\Big]. \tag{3.27}$$

The vertical flux component is

$$\rho^{0z}(v,V) = -\frac{3z}{3200R}\Big[6x^2y^2(x^2+y^2-R^2) + z^2\big(x^4+y^4-R^2(x^2+y^2)\big) + z^4(x^2+y^2)\Big], \tag{3.28}$$

which vanishes pointwise as $O(z^1)$ at $z \to 0$ (slower than $O(z^4)$ precisely because of $V^z = O(tz^2)$, yet still strictly vanishing!). Integrating over the regulated boundary disk $B_{\varepsilon_z} = \{t = 0,\, z = \varepsilon_z,\, r \le \sqrt{R^2 - \varepsilon_z^2}\}$ yields

$$\int_{B_{\varepsilon_z}} (-\rho^{0z})\, \mathrm{d}x\, \mathrm{d}y = -\frac{3\pi\, \varepsilon_z (R^2 - \varepsilon_z^2)^3 (R^2 + 7\varepsilon_z^2)}{51200 R} = O(\varepsilon_z) \xrightarrow{\varepsilon_z \to 0} 0, \tag{3.29}$$

while integrating $\rho^{0i} n_i$ directly over the bifurcation hemisphere $\widetilde{B}$ yields

$$\int_{\widetilde{B}} \rho(v, V) = -\frac{\pi R^8}{12600}, \qquad W_{\Sigma_B}(v, \mathcal{L}_{\xi_B} v) + \int_{\widetilde{B}} \rho(v, V) = \frac{\pi R^8}{1260} - \frac{\pi R^8}{12600} = \frac{\pi R^8}{1400}. \tag{3.30}$$

This closes all corner and gauge-falloff loopholes for the Hollands–Wald verification of timelike HRT canonical energy.

3.5 Explicit Fefferman–Graham second-order vacuum completions

Equation (2.12) evaluates $\mathcal{E}_B(h)$ without constructing the second-order metric k; Rodgers [8] left open whether an explicit completion k solving (2.6) exists on the wedge. For any spatially homogeneous TT tangent $h_{xy}(t, z) = zH(t, z)$ (or arbitrary superposition of both polarizations $(H_+, H_\times)$, since $D^2\mathcal{F}_{ab}[h, h]$ depends only on the $SO(2)$-invariant combination $H_+^2 + H_\times^2$), the quadratic source $D^2\mathcal{F}_{ab}[h, h]$ depends solely on (t, z) and has only tt, tz, zz, and $xx = yy$ components. Taking a diagonal Fefferman–Graham ansatz

$$k_{ab}(t, z) = \frac{1}{z^2}\, \mathrm{diag}\big(A(t, z),\, 0,\, B(t, z),\, B(t, z)\big), \tag{3.31}$$

the linearized Einstein equations $D\mathcal{F}_{ab}[k] + D^2\mathcal{F}_{ab}[h, h] = 0$ decouple triangularly:

$$-\partial_z^2 B + \frac{2}{z}\partial_z B = -D^2\mathcal{F}_{tt}[h, h], \qquad -\partial_t \partial_z B = -D^2\mathcal{F}_{tz}[h, h], \tag{3.32}$$

$$\partial_t^2 B + \frac{1}{z}\big(\partial_z A - 2\partial_z B\big) = -D^2\mathcal{F}_{zz}[h, h], \qquad D\mathcal{F}_{xx}[k] + D^2\mathcal{F}_{xx}[h, h] = 0, \tag{3.33}$$

where the fourth equation holds automatically once the first three are solved by background divergence-freeness $\bar{\nabla}^a D^2\mathcal{F}_{ab} = 0$. Since $D^2\mathcal{F}_{ab}[h, h] = O(z^4)$ near $z = 0$, integrating $\partial_z(z^{-2}\partial_z B) = z^{-2} D^2\mathcal{F}_{tt}$ produces $B(t, z) = c_M z^3 + O(z^6)$ and $A(t, z) = 2c_M z^3 + O(z^6)$ with *zero* non-normalizable boundary metric deformation, *zero* logarithmic terms, and a single free constant $c_M \in \mathbb{R}$ parameterizing the planar black-brane mass density ($H_{tt}^{(0)} = 2c_M$, $H_{xx}^{(0)} = H_{yy}^{(0)} = c_M$, satisfying the trace and conservation Ward identities $-H_{tt}^{(0)} + 2H_{xx}^{(0)} = 0$ and $\partial_t H_{tt}^{(0)} = 0$).

Proposition 3.4 (Exact closed-form second-order vacuum completions on Poincaré AdS_4)**.** *Every polynomial tangent $h \in \mathrm{span}\{E_0, \widetilde{E}_1, O_0, O_1\} \subset V_{\mathrm{hom}}$ admits an exact, closed-form, globally smooth (on the Poincaré patch) Fefferman–Graham second-order vacuum completion (3.31) solving all ten equations $D\mathcal{F}_{ab}[k] + D^2\mathcal{F}_{ab}[h, h] \equiv 0$, with fixed flat conformal boundary metric ($A, B = O(z^3)$). Explicitly:*

$$E_0: \qquad A = \frac{z^6}{4} + 2c_M z^3, \qquad B = \frac{3z^6}{4} + c_M z^3, \tag{3.34}$$

$$\widetilde{E}_1: \qquad A = \frac{z^6(70t^4 + 3z^4)}{280} + 2c_M z^3, \qquad B = \frac{z^6(1050t^4 + 385t^2 z^2 + 46z^4)}{1400} + c_M z^3, \tag{3.35}$$

$$O_0: \qquad A = \frac{z^6(10t^2 - z^2)}{40} + 2c_M z^3, \qquad B = \frac{z^6(60t^2 - z^2)}{80} + c_M z^3, \tag{3.36}$$

$$O_1: \qquad A = \frac{z^6(150t^6 + 45t^4 z^2 + 27t^2 z^4 - 2z^6)}{600} + 2c_M z^3,$$
$$B = \frac{z^6(900t^6 + 1035t^4 z^2 + 324t^2 z^4 - 2z^6)}{1200} + c_M z^3, \tag{3.37}$$

with analogous closed-form polynomial completions for E_{Rodgers} *and for any general linear combination* $a_0E_0 + a_1\widetilde{E}_1 + b_0O_0$*. Moreover, evaluating* $F_B''(0) = DE_B[k] - \frac{1}{4}(DA_B[k] + A_{\mathrm{ext}}''[h])$ *directly on the explicit vacuum family* $g_\lambda = \bar{g} + \lambda h + \frac{1}{2}\lambda^2 k$ *cancels the arbitrary mass parameter* c_M *identically and reproduces every entry of Table 2 to exact zero.*

Proposition 3.4 provides a *third completely independent verification* of our canonical energy formulas and settles wedge-local (and Poincaré-patch) classical vacuum completion through second order for polynomial modes. We emphasize, however, that these homogeneous completions grow as $z \to \infty$ and $t \to \infty$ and therefore do not yield finite-energy global CFT states; finite-energy localized wavepackets are constructed in Section 6.

3.6 Bounded oscillatory modes: exact one-line and smooth Green's quadrature

For the bounded oscillatory modes $h_c = z\cos(\omega t)j(\omega z)$ and $h_s = z\sin(\omega t)j(\omega z)$ in (2.20), the exact one-line time-even integral (3.17) removes the need for any finite-dimensional variational trial basis, while for the time-odd mode h_s the triangular change of variables $y' = ys$ ($s \in [0,1]$) in (3.18)–(3.19) removes the $y = y'$ Green's kink and yields a C^∞ integrand. Table 3 reports the resulting 60- vs. 120-node Gauss–Legendre values at $R = 1$, which converge to $< 3 \times 10^{-17}$ and agree with all published entries of Rodgers [8, §5] within its six-decimal rounding ($< 3.5 \times 10^{-7}$). Notice that the timelike HRT subtraction ratio $\frac{1}{4}A_{\mathrm{ext}}''/(-I)$ approaches the sharp upper bound $1/10$ as $\omega R \to 0$ (where $j(\omega z) \to 1$ approaches O_0) and decreases monotonically as ωR grows, in full agreement with Theorem 3.2(iii). For known $\omega = 1$, measuring two radii ($R_1 = 1, R_2 = 2$) separates the two quadrature amplitudes (c^2, s^2) in $ch_c + sh_s$ with invertible 2×2 response matrix ($\det = 9.169 \times 10^{-3}$, condition number $\kappa = 235.55$).

Table 3: High-precision numerical canonical energies at $R = 1$ for the bounded oscillatory modes $h_c = z\cos(\omega t)j(\omega z)$ (via the exact one-line identity (3.17)) and $h_s = z\sin(\omega t)j(\omega z)$ (via smooth triangular Green's quadrature of (3.18)). Maximum 60- vs. 120-node Gauss–Legendre discrepancy is $< 3 \times 10^{-17}$; all h_c values match [8, §5] to six-decimal rounding.

ωR	$\mathcal{E}_B(h_c)$	Rodgers [8] ($N = 3$)	$\mathcal{E}_B(h_s)$	$\frac{1}{4}A_{\mathrm{ext}}''/(-I)$	$\mathcal{E}_B(h_s)/\mathcal{E}_B(h_c)$
0.5	0.032740344472	0.032740	0.000548395514	0.099999	0.016750
1.0	0.030159094080	0.030159	0.002049183990	0.099980	0.067946
2.0	0.022090305920	0.022090	0.006251233376	0.099649	0.282985
4.0	0.009253729312	0.009254	0.008888803776	0.092891	0.960564

4 Rest-Frame Non-Identifiability and Symmetry-Breaking Lifting

While Section 3 shows that every nonzero perturbation in $V_{\mathrm{hom}} \subset \ker M_1$ has strictly positive second-order canonical energy $\mathcal{E}_B(h) > 0$, positivity alone does not imply *identifiability*: distinct physical perturbations can share the exact same canonical energy across every rest-frame spherical boundary ball $B(R, \mathbf{x}_0)$ on the $t = 0$ slice. In this section we prove exact all-radii, all-centers non-identifiability theorems for rest-frame balls—both within a single polarization and across both graviton polarizations—and compute the exact lifting produced by the three minimal symmetry-breaking extensions of the measurement family: boundary time shifts, deformed $t = 0$ regions (ellipses and quadrupoles), and Lorentz-boosted, time-translated balls, culminating in an explicit six-ball constructive recovery theorem with sharp conditioning bounds.

4.1 Fixed-Time Non-Identifiability and All-Times Polarization-Chirality Blindness

To formalize the parity-odd cross-polarization observable of homogeneous gravitational waves, define on V_{hom} the boundary-shear *polarization-chirality* (Stokes-V) pseudoscalar at $t = 0$:

$$\mathcal{H}[h] := H_+(0,0)\,\partial_t H_\times(0,0) - H_\times(0,0)\,\partial_t H_+(0,0). \tag{4.1}$$

Notice that $\mathcal{H}[h]$ is strictly invariant under $SO(2)$ spatial rotations in the (x, y) plane (which rotate $(H_+, H_\times)$ by 2θ), flips sign under spatial reflection $\mathcal{R}_y : y \mapsto -y$ (which maps $(H_+, H_\times) \mapsto (+H_+, -H_\times)$), and coincides with the circular standing-wave helicity $\pm\omega A^2$ on the $O(t)$ truncation of a monochromatic circularly polarized wave $A\big(h_c^+ \pm h_s^\times\big) = A\big(E_0^+ \pm \omega O_0^\times\big) + O(t^2)$.

Theorem 4.1 (All-Radii Non-Identifiability of Rest-Frame Balls at $t = 0$)**.** *Let $\mathcal{B}_{t=0} := \{B(R, \mathbf{x}_0) : R > 0,\ \mathbf{x}_0 \in \mathbb{R}^2\}$ be the family of all rest-frame spherical boundary balls on the $t = 0$ slice.*

(i) ***Single-polarization static-vs-kinetic counterpair:*** *Inside the three-mode single-polarization subspace* $\mathrm{span}\{E_0^\times, \widetilde{E}_1^\times, O_0^\times\} \subset V_{\mathrm{hom}} \subset \ker M_1$*, define*

$$h^{(1)} := E_0^\times + \widetilde{E}_1^\times, \qquad h^{(2)} := E_0^\times - \widetilde{E}_1^\times + b_{0*}\,O_0^\times, \qquad b_{0*} := \frac{2\sqrt{15}}{3}. \tag{4.2}$$

Then $h^{(1)}$ is strictly time-symmetric ($\partial_t h^{(1)}|_{t=0} = 0$, zero kinetic energy) while $h^{(2)}$ carries nonzero bulk kinetic energy, yet for every *radius $R > 0$ and* every *center $\mathbf{x}_0 \in \mathbb{R}^2$,*

$$\mathcal{E}_{B(R,\mathbf{x}_0)}\big(h^{(1)}\big) \;=\; \mathcal{E}_{B(R,\mathbf{x}_0)}\big(h^{(2)}\big) \;=\; \frac{\pi R^6}{138600}\big(29R^4 + 330R^2 + 1485\big). \tag{4.3}$$

(ii) ***Dual-polarization $O(2)_E \times O(2)_O$ degeneracy:*** *On the full two-polarization homogeneous space $V_{\mathrm{hom}} = V_{\mathrm{hom}}^{\mathrm{even},+} \oplus V_{\mathrm{hom}}^{\mathrm{even},\times} \oplus V_{\mathrm{hom}}^{\mathrm{odd},+} \oplus V_{\mathrm{hom}}^{\mathrm{odd},\times}$, the canonical energy of every rest-frame ball at $t = 0$ decomposes into four decoupled squares,*

$$\mathcal{E}_{B(R,\mathbf{x}_0)}(h) = \mathcal{E}_B(h^+_{\mathrm{even}}) + \mathcal{E}_B(h^\times_{\mathrm{even}}) + \mathcal{E}_B(h^+_{\mathrm{odd}}) + \mathcal{E}_B(h^\times_{\mathrm{odd}}), \tag{4.4}$$

and is therefore invariant under the continuous symmetry group $O(2)_E \times O(2)_O$ rotating $(h^+_{\mathrm{even}}, h^\times_{\mathrm{even}})$ and $(h^+_{\mathrm{odd}}, h^\times_{\mathrm{odd}})$ independently. In particular, rest-frame $t = 0$ balls cannot distinguish linear polarization orientation or relative phase between time-even and time-odd polarizations.

(iii) ***Permanent all-times polarization-chirality blindness:*** *Consider the opposite-chirality pair ($\mathcal{H}[h^{(\pm)}] = \pm ab$)*

$$h^{(\pm)} := a\,E_0^+ \pm b\,O_0^\times \qquad (a, b \in \mathbb{R} \setminus \{0\}). \tag{4.5}$$

For every *radius $R > 0$,* every *center $\mathbf{x}_0 \in \mathbb{R}^2$, and* every *boundary time slice $t_0 \in \mathbb{R}$,*

$$\mathcal{E}_{B(R,\mathbf{x}_0,t_0)}\big(h^{(+)}\big) \;=\; \mathcal{E}_{B(R,\mathbf{x}_0,t_0)}\big(h^{(-)}\big). \tag{4.6}$$

Thus no collection of rest-frame spherical boundary balls at any combination of radii, centers, and times can ever detect the parity-odd polarization chirality $\mathcal{H}[h]$.

Proof. (i) By spatial translation invariance of V_{hom}, $\mathcal{E}_{B(R,\mathbf{x}_0)}$ is independent of $\mathbf{x}_0$. By even–odd decoupling (3.4) and Table 2,

$$\begin{aligned}
\mathcal{E}_B(h^{(1)}) &= \mathcal{E}_B(E_0) + 2\,\mathcal{E}_B(E_0, \widetilde{E}_1) + \mathcal{E}_B(\widetilde{E}_1),\\
\mathcal{E}_B(h^{(2)}) &= \mathcal{E}_B(E_0) - 2\,\mathcal{E}_B(E_0, \widetilde{E}_1) + \mathcal{E}_B(\widetilde{E}_1) + b_{0*}^2\,\mathcal{E}_B(O_0).
\end{aligned}$$

Their difference is $b_{0*}^2\mathcal{E}_B(O_0) - 4\mathcal{E}_B(E_0, \widetilde{E}_1) = b_{0*}^2 \frac{\pi R^8}{1400} - 4\frac{\pi R^8}{840} = \left(\frac{b_{0*}^2}{1400} - \frac{1}{210}\right)\pi R^8$, which vanishes identically for all $R > 0$ precisely when $b_{0*}^2 = \frac{1400}{210} = \frac{20}{3}$, i.e. $b_{0*} = \frac{2\sqrt{15}}{3}$. Evaluating either expression gives (4.3).

(ii) Under spatial rotation by θ in the (x, y) plane, the symmetric traceless shear tensor $(H_+, H_\times)$ rotates by angle 2θ, while the background hemisphere $\widetilde{B}$ and Killing field ξ_B are $SO(2)$-invariant. Combined with even–odd decoupling (3.4) at $t = 0$, $SO(2)$ invariance forces the quadratic form on each parity sector to be a multiple of $H_+^2 + H_\times^2$ with zero $+\times$ cross-term, yielding (4.4) and its $O(2)_E \times O(2)_O$ invariance.

(iii) Consider the spatial reflection isometry $\mathcal{R}_y : (t, z, x, y) \mapsto (t, z, x, -y)$ of Poincaré AdS_4. On homogeneous shear components, $\mathcal{R}_y^*(h_{xx}\mathrm{d}x^2 + h_{yy}\mathrm{d}y^2 + 2h_{xy}\mathrm{d}x\,\mathrm{d}y) = h_{xx}\mathrm{d}x^2 + h_{yy}\mathrm{d}y^2 - 2h_{xy}\mathrm{d}x\,\mathrm{d}y$, so $\mathcal{R}_y^*$ maps $(H_+, H_\times) \mapsto (+H_+, -H_\times)$ and hence maps $h^{(+)} \mapsto h^{(-)}$ identically for all $t \in \mathbb{R}$. Because $\mathcal{R}_y$ maps every rest-frame spherical ball $B(R, (x_0, y_0), t_0)$ isometrically onto $B(R, (x_0, -y_0), t_0)$ while preserving the modular Killing field ξ_B, diffeomorphism invariance and spatial translation invariance of $h^{(\pm)}$ yield $\mathcal{E}_{B(R,(x_0,y_0),t_0)}(h^{(+)}) = \mathcal{E}_{B(R,(x_0,-y_0),t_0)}(h^{(-)}) = \mathcal{E}_{B(R,(x_0,y_0),t_0)}(h^{(-)})$. □

4.2 Extension I: Boundary Time Shifts $t = t_0$ and the First-Derivative Blind-Spot Radius

Evaluating the canonical energy of a fixed homogeneous solution $h(t, z)$ on a time-shifted boundary ball $B(R, \mathbf{x}_0, t_0)$ is equivalent to evaluating $\mathcal{E}_{B(R,0,0)}$ on the time-translated perturbation $h_{t_0}(t, z) := h(t + t_0, z)$. On the counterpair (4.2), the profiles are $H^{(1)}(t, z) = 1 + t^2 + z^2/5$ and $H^{(2)}(t, z) = 1 - t^2 - z^2/5 + b_{0*}t$, so shifting $t \mapsto t + t_0$ generates exact mixtures of $E_0, \widetilde{E}_1, O_0$:

$$h_{t_0}^{(1)} = (1 + t_0^2)E_0^\times + \widetilde{E}_1^\times + 2t_0\, O_0^\times, \tag{4.7}$$

$$h_{t_0}^{(2)} = (1 + b_{0*}t_0 - t_0^2)E_0^\times - \widetilde{E}_1^\times + (b_{0*} - 2t_0)\, O_0^\times. \tag{4.8}$$

Proposition 4.2 (Time-Shift Lifting and Exact Blind-Spot Radius)**.** *For the single-polarization counterpair (4.2), the canonical-energy difference at arbitrary boundary time $t_0 \in \mathbb{R}$ is the exact cubic polynomial*

$$\mathcal{E}_{B(R,\mathbf{x}_0,t_0)}\big(h^{(2)}\big) - \mathcal{E}_{B(R,\mathbf{x}_0,t_0)}\big(h^{(1)}\big) = -\frac{\sqrt{15}\,\pi R^6}{3150}\big(11R^2 - 45\big)\, t_0 + \frac{\pi R^6}{35}\, t_0^2 - \frac{\sqrt{15}\,\pi R^6}{70}\, t_0^3. \tag{4.9}$$

In particular, $\frac{\mathrm{d}}{\mathrm{d}t_0}\mathcal{E}_{B(R,\mathbf{x}_0,t_0)}(h^{(1)})\Big|_{t_0=0} = 0$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\mathcal{E}_{B(R,\mathbf{x}_0,t_0)}\big(h^{(2)}\big)\Big|_{t_0=0} = -\frac{\sqrt{15}\,\pi R^6}{3150}\big(11R^2 - 45\big). \tag{4.10}$$

Thus first-order time shifts linearly separate $h^{(1)}$ and $h^{(2)}$ at every radius $R \neq R_*$, where

$$R_* := \sqrt{\frac{45}{11}} \approx 2.022600 \tag{4.11}$$

is a unique exact first-derivative blind-spot radius. At $R = R_*$, separation is achieved at second order in t_0 via the globally positive second derivative $\frac{\mathrm{d}^2}{\mathrm{d}t_0^2}\Delta\mathcal{E}\Big|_{t_0=0} = \frac{2\pi R^6}{35}$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t_0^2}\Big[\mathcal{E}_{B(R_*,\mathbf{x}_0,t_0)}\big(h^{(2)}\big) - \mathcal{E}_{B(R_*,\mathbf{x}_0,t_0)}\big(h^{(1)}\big)\Big]\Big|_{t_0=0} = \frac{36450\,\pi}{9317} \approx 12.2903 > 0. \tag{4.12}$$

However, by Theorem 4.1(iii), time shifts leave the opposite-chirality pair $h^{(\pm)} = aE_0^+ \pm bO_0^\times$ completely unlifted for all $t_0 \in \mathbb{R}$.

Proof. Substituting $h^{(1)}_{t_0}$ and $h^{(2)}_{t_0}$ into the exact bilinear form of Table 2 (`verify_02`) gives

$$\mathcal{E}_B(h^{(1)}_{t_0}) = \pi\Big[\alpha_{00}(1+t_0^2)^2R^6 + 2\alpha_{01}(1+t_0^2)R^8 + \alpha_{11}R^{10} + 4\gamma_{00}t_0^2R^8\Big],$$
$$\mathcal{E}_B(h^{(2)}_{t_0}) = \pi\Big[\alpha_{00}(1+b_{0*}t_0-t_0^2)^2R^6 - 2\alpha_{01}(1+b_{0*}t_0-t_0^2)R^8 + \alpha_{11}R^{10} + \gamma_{00}(b_{0*}-2t_0)^2R^8\Big].$$

Expanding the difference term-by-term, the t_0^0 term vanishes by $b_{0*}^2\gamma_{00} = 4\alpha_{01}$, both the $\alpha_{01}t_0^2$ terms ($+2\alpha_{01}t_0^2 - 2\alpha_{01}t_0^2 = 0$) and the $\gamma_{00}t_0^2$ terms ($4\gamma_{00}t_0^2 - 4\gamma_{00}t_0^2 = 0$) cancel identically, and the t_0^4 terms ($\alpha_{00}t_0^4 - \alpha_{00}t_0^4 = 0$) cancel identically as well, leaving

$$\mathcal{E}_B(h^{(2)}_{t_0}) - \mathcal{E}_B(h^{(1)}_{t_0}) = 2b_{0*}\pi\big(\alpha_{00}R^6 - (\alpha_{01}+2\gamma_{00})R^8\big)t_0 + (b_{0*}^2-4)\pi\alpha_{00}R^6t_0^2 - 2b_{0*}\pi\alpha_{00}R^6t_0^3.$$

Inserting $\alpha_{00} = \frac{3}{280}$, $\alpha_{01} = \frac{1}{840}$, $\gamma_{00} = \frac{1}{1400}$, and $b_{0*} = \frac{2\sqrt{15}}{3}$ ($b_{0*}^2 - 4 = \frac{8}{3}$) yields (4.9). In particular, the $O(t_0)$ coefficient vanishes uniquely at $R_* = \sqrt{45/11}$, while $\frac{\mathrm{d}^2}{\mathrm{d}t_0^2}\Delta\mathcal{E}|_{t_0=0} = 2(b_{0*}^2-4)\pi\alpha_{00}R^6 = \frac{2\pi R^6}{35}$ evaluates at $R = R_*$ to $\frac{2\pi}{35}(45/11)^3 = \frac{36450\pi}{9317}$. □

4.3 Extension II: Deformed $t = 0$ Regions ($m = 2$ Ellipses and $m = 4$ Quadrupoles)

We next keep the boundary time slice fixed at $t = 0$ and deform the circular boundary ball $\partial B_0 = \{r = R\}$ to a smooth m-fold multipole region

$$\partial B_\epsilon^{(m)}: \quad r(\varphi) = R\big(1 + \epsilon\cos(m\varphi)\big), \qquad m \in \{2, 4\}, \tag{4.13}$$

with $|\epsilon| \ll 1$. For a non-spherical region $B_\epsilon^{(m)}$ on $t = 0$, there is no local conformal Killing field ξ_{B_ϵ} and the vacuum modular Hamiltonian K_{B_ϵ} is nonlocal in general. To keep every observable precisely defined within classical gravity, we measure on $B_\epsilon^{(m)}$:

- the first-order extremal-surface area variation $DA_{B_\epsilon^{(m)}}[h]$ on the pure-AdS$_4$ minimal surface $\widetilde{B}_\epsilon^{(m)}$;
- any smooth local boundary energy-density smear $D\mathcal{T}_{B_\epsilon^{(m)},w}[h] := \int_{B_\epsilon^{(m)}} w(\mathbf{x})\, T^{(1)}_{tt}(0,\mathbf{x})\,\mathrm{d}^2x$; and
- at second order, the fixed-boundary extremal-surface area Hessian $A''_{\mathrm{ext},B_\epsilon^{(m)}}[h]$ along the affine metric path $\bar{g} + \lambda h$, together with the total second-order extremal area $\delta^{(2)}A_{B_\epsilon^{(m)}}[g_\lambda] = A''_{\mathrm{ext},B_\epsilon^{(m)}}[h] + DA_{B_\epsilon^{(m)}}[k]$ of a vacuum completion $g_\lambda = \bar{g} + \lambda h + \frac{1}{2}\lambda^2 k + o(\lambda^2)$ (which under the HRT dictionary corresponds to the second-order entanglement entropy $\delta^{(2)}S_{B_\epsilon^{(m)}} = \frac{1}{4}\delta^{(2)}A_{B_\epsilon^{(m)}}$, distinct from canonical energy).

In pure Poincaré AdS$_4$ ($t = 0$, isometric to $\mathbb{H}^3$), the minimal surface $\widetilde{B}_\epsilon^{(m)}$ anchored on $\partial B_\epsilon^{(m)}$ is given to $O(\epsilon)$ in spherical coordinates ($r = \varrho\sin\theta$, $z = \varrho\cos\theta$) by

$$\varrho_\epsilon(\theta,\varphi) = R\Big[1 + \epsilon\,\psi_m(\cos\theta)\cos(m\varphi)\Big] + O(\epsilon^2), \tag{4.14}$$

where $\psi_m(y)$ solves the exact $\mathbb{H}^2$ Jacobi equation $\mathcal{L}_m[\psi_m] = 0$ (eq. (3.8)) with $\psi_m(0) = 1$ and pole regularity at $y = 1$. Its unique closed-form solution for every integer $m \geq 1$ is [9, 10]

$$\psi_m(y) = (1+my)\left(\frac{1-y}{1+y}\right)^{m/2} = \frac{(1+my)(1-y)^m}{(1-y^2)^{m/2}}. \tag{4.15}$$

Explicitly, $\psi_2(y) = \frac{(1+2y)(1-y)}{1+y}$ ($m = 2$) and $\psi_4(y) = \frac{(1+4y)(1-y)^2}{(1+y)^2}$ ($m = 4$).

Theorem 4.3 (Multipole Lifting: First-Order $m = 2$ on Even Modes and Second-Order $m = 4$ Area Response on O_0)**.** *Consider the boundary multipole deformation $\partial B_\epsilon^{(m)}$ ($r(\varphi) = R(1 + \epsilon \cos m\varphi)$) at $t = 0$.*

(i) ***Elliptical ($m = 2$) first-order area lifting of time-even shear:*** *For every time-even mode $\widetilde{E}_k^+ \in V_{\rm hom}^{\rm even,+}$ ($\widetilde{E}_0^+ := E_0^+$), the first-order extremal-surface area variation is lifted linearly in ϵ:*

$$\begin{aligned} DA_{B_\epsilon^{(2)}}[E_0^+] &= \frac{3\pi R^3}{8}\,\epsilon + O(\epsilon^2), \qquad DA_{B_\epsilon^{(2)}}[\widetilde{E}_1^+] = \frac{\pi R^5}{40}\,\epsilon + O(\epsilon^2), \\ DA_{B_\epsilon^{(2)}}[\widetilde{E}_2^+] &= \frac{3\pi R^7}{560}\,\epsilon + O(\epsilon^2). \end{aligned} \tag{4.16}$$

The $\times$-polarized modes $\widetilde{E}_k^\times$ vanish at $O(\epsilon)$ on $\cos(2\varphi)$ ellipses and are lifted identically by $\pi/4$-rotated ellipses $r(\varphi) = R(1 + \epsilon \sin 2\varphi)$.

(ii) ***First-order persistence of time-odd modes on all $t = 0$ shapes:*** *Every time-odd perturbation $h_{\rm odd} \in V_{\rm hom}^{\rm odd}$ satisfies $h_{{\rm odd},ab}|_{t=0} \equiv 0$, so both the first-order extremal-surface area variation $DA_{B_\epsilon}[h_{\rm odd}] = 0$ and every local $t = 0$ energy-density smear $D\mathcal{T}_{B_\epsilon,w}[h_{\rm odd}] = 0$ vanish identically for* every *smooth boundary region $B_\epsilon \subset \{t = 0\}$. (If the holographic first law $\delta\langle K_{B_\epsilon}\rangle = \frac{1}{4} DA_{B_\epsilon}$ applies to B_ϵ, the first-order modular-energy variation likewise vanishes by $t \mapsto -t$ parity.)*

(iii) ***Quadrupolar ($m = 4$) second-order timelike HRT area lifting of O_0:*** *Let $h = b_+ O_0^+ + b_\times O_0^\times$ ($b_+, b_\times \in \mathbb{R}$). On an elliptical deformation ($m = 2$), the $O(\epsilon)$ shift of the second-order timelike HRT area Hessian vanishes identically: $\partial_\epsilon A''_{{\rm ext},B_\epsilon^{(2)}}[h]|_{\epsilon=0} = 0$. On a quadrupolar deformation ($m = 4$), the $m = 2$ timelike normal displacement $\tau_* \propto b_+ \cos 2\varphi + b_\times \sin 2\varphi$ couples resonantly to $\cos(4\varphi)$, yielding*

$$\frac{\partial}{\partial\epsilon} A''_{{\rm ext},B_\epsilon^{(4)}}\big[b_+ O_0^+ + b_\times O_0^\times\big]\Big|_{\epsilon=0} = -\frac{\pi R^8}{1200}\big(b_+^2 - b_\times^2\big). \tag{4.17}$$

Moreover, the second-order Einstein source $D^2\mathcal{F}_{ab}[h,h]$ is strictly $SO(2)$-invariant (proportional to $b_+^2 + b_\times^2$), so the explicit axially symmetric Fefferman–Graham completion $k = (b_+^2 + b_\times^2)k[O_0]$ of Proposition 3.4 is $SO(2)$-invariant. More generally, invariance of the $t = 0$ spatial metric $k_{ij}|_{t=0}$ under the $\pi/4$ rotation $\varphi \mapsto \varphi + \pi/4$ is a sufficient *condition for $\partial_\epsilon DA_{B_\epsilon^{(4)}}[k]|_{\epsilon=0} = 0$, under which the total second-order extremal area and HRT entanglement entropy satisfy*

$$\frac{\partial}{\partial\epsilon}\delta^{(2)} A_{B_\epsilon^{(4)}}\Big|_{\epsilon=0} = -\frac{\pi R^8}{1200}\big(b_+^2 - b_\times^2\big), \qquad \frac{\partial}{\partial\epsilon}\delta^{(2)} S_{B_\epsilon^{(4)}}\Big|_{\epsilon=0} = -\frac{\pi R^8}{4800}\big(b_+^2 - b_\times^2\big), \tag{4.18}$$

whereas adding a non-axisymmetric (hence spatially inhomogeneous) solution $k_{\rm lin}$ of the linearized vacuum equation $D\mathcal{F}[k_{\rm lin}] = 0$ that breaks $\pi/4$-rotation invariance in the $m = 4$ channel can shift $\partial_\epsilon DA_{B_\epsilon^{(4)}}[k]|_{\epsilon=0}$.

Proof. (i) Expanding the induced 2-metric $q_\epsilon = q_0 + \epsilon q_1 + \lambda\delta q_\epsilon$ on the $O(\epsilon)$ minimal surface (4.14) ($m = 2$) and evaluating $DA_{B_\epsilon^{(2)}}[h] = \frac{1}{2}\int_{\widetilde{B}_\epsilon} \sqrt{\det q_\epsilon}\, q_\epsilon^{AB} h_{AB}$ to $O(\epsilon\lambda)$ isolates the $\cos^2(2\varphi)$ channel and evaluates directly to (4.16) (`verify_02`).

(ii) Since $h_{{\rm odd},ab}|_{t=0} \equiv 0$, both the induced metric perturbation on any $t = 0$ minimal surface $\widetilde{B}_\epsilon$ and $T_{tt}^{(1)}|_{t=0}$ vanish identically, giving $DA_{B_\epsilon}[h_{\rm odd}] = D\mathcal{T}_{B_\epsilon,w}[h_{\rm odd}] = 0$.

(iii) For $h = b_+ O_0^+ + b_\times O_0^\times$ ($h_{xx} = -h_{yy} = ztb_+$, $h_{xy} = ztb_\times$), perturbing $t(\theta,\varphi) = \lambda\tau(\theta,\varphi)$ on $\varrho = \varrho_\epsilon(\theta,\varphi)$ yields a quadratic area functional $A''_{B_\epsilon^{(m)}}[h;\tau]$. Because the $\epsilon = 0$ timelike

extremum $\tau_*(\theta,\varphi) = g_*(y)(b_+\cos 2\varphi + b_\times \sin 2\varphi)$ ($g_*(y) = -\frac{R^5}{40}y^3(1-y^2)$, eq. (3.14)) satisfies the $\epsilon = 0$ Euler–Lagrange equation $\delta A''_{\epsilon=0}/\delta\tau = 0$, the *envelope theorem* (Hellmann–Feynman stationarity) guarantees that the $O(\epsilon)$ shift τ_1 drops out identically from $\partial_\epsilon A''_{\mathrm{ext},B_\epsilon^{(m)}}|_{\epsilon=0}$:

$$\frac{\partial}{\partial\epsilon}A''_{\mathrm{ext},B_\epsilon^{(m)}}[h]\Big|_{\epsilon=0} = \frac{\partial}{\partial\epsilon}A''_{B_\epsilon^{(m)}}[h;\tau_*]\Big|_{\epsilon=0} = \int_0^1 \mathrm{d}y \int_0^{2\pi} \mathrm{d}\varphi\, \mathcal{I}_m\big(y,\varphi;\,\psi_m(y),\psi'_m(y),g_*(y),g'_*(y)\big). \tag{4.19}$$

Since $\tau_*^2 \propto (b_+\cos 2\varphi + b_\times \sin 2\varphi)^2 = \frac{1}{2}(b_+^2+b_\times^2) + \frac{1}{2}(b_+^2-b_\times^2)\cos(4\varphi) + b_+b_\times\sin(4\varphi)$, only $m=4$ (not $m=2$) couples to $\cos(m\varphi)$, giving $\partial_\epsilon A''_{\mathrm{ext},B_\epsilon^{(2)}}|_{\epsilon=0} = 0$ immediately. For $m=4$, substituting $\psi_4(y) = (1+4y)(1-y)^2/(1+y)^2$ and $g_*(y) = -\frac{R^5}{40}y^3(1-y^2)$ into (4.19) (`verify_02`, cross-checked to 10^{-16} by 64-node Gauss–Legendre quadrature) cancels all $(1+y)$ poles and integrates to $-\frac{\pi R^8}{1200}(b_+^2-b_\times^2)$, proving (4.17). Direct 4D expansion (`verify_02`) confirms every nonvanishing component of $D^2\mathcal{F}_{ab}[b_+O_0^+ + b_\times O_0^\times]$ is strictly proportional to $b_+^2+b_\times^2$ with $D^2\mathcal{F}_{xx} = D^2\mathcal{F}_{yy}$ (as follows from $M^2 = (b_+^2+b_\times^2)I_2$ for $M = \begin{pmatrix} b_+ & b_\times \\ b_\times & -b_+ \end{pmatrix}$), so $k = (b_+^2+b_\times^2)k[O_0]$ is $SO(2)$-invariant. Under $\varphi \mapsto \varphi+\pi/4$, $\cos(4\varphi) \mapsto -\cos(4\varphi)$ (i.e. $\epsilon \mapsto -\epsilon$); hence whenever $k_{ij}|_{t=0}$ is $\pi/4$-rotation invariant, $DA_{B_\epsilon^{(4)}}[k]$ is an even function of ϵ, forcing $\partial_\epsilon DA_{B_\epsilon^{(4)}}[k]|_{\epsilon=0} = 0$ and yielding (4.18). □

Notice that static $t=0$ shape deformations preserve $t \mapsto -t$ even–odd decoupling, so no combination of static $t=0$ regions can measure the polarization chirality $\mathcal{H}[h]$ of (4.5).

4.4 Extension III: Boosted, Time-Translated Balls, Constructive Six-Mode Recovery, and Quartic Lifting Inside $\ker M_1^{(v)}$

Finally, consider a spherical boundary ball $B^{(v)}(R,\mathbf{x}_0,t_c)$ of proper radius $R>0$, boost velocity $\mathbf{v} = v(\cos\alpha,\sin\alpha)$ ($|v|<1$, $\gamma = (1-v^2)^{-1/2}$), centered at an *arbitrary* boundary spacetime event $(t_c,\mathbf{x}_0)\in\mathbb{R}\times\mathbb{R}^2$, where the spatial center $\mathbf{x}_0$ and center time t_c are varied independently:

$$t - t_c = \gamma(t' + v x'_\parallel), \qquad x_\parallel - x_{0,\parallel} = \gamma(x'_\parallel + v t'), \qquad x_\perp - x_{0,\perp} = x'_\perp, \qquad z = z', \tag{4.20}$$

with $x_\parallel = x\cos\alpha + y\sin\alpha$, $x_\perp = -x\sin\alpha + y\cos\alpha$, and the ball occupying $t'=0$, $x'^2_\parallel + x'^2_\perp \le R^2$ in its own rest frame. Because the Poincaré metric $\bar g = z^{-2}(-\mathrm{d}t^2+\mathrm{d}z^2+\mathrm{d}x^2+\mathrm{d}y^2)$ is invariant under the boundary Poincaré transformation (4.20), $B^{(v)}(R,\mathbf{x}_0,t_c)$ possesses an exact modular Killing field $\xi_{B^{(v)}}$ and totally geodesic extremal hemisphere $\widetilde{B}^{(v)} = \{t'=0,\, z'^2+x'^2+y'^2 = R^2\}$. Evaluating the first- and second-order readouts on $B^{(v)}(R,\mathbf{x}_0,t_c)$ reduces to pulling back h to (t',z',x',y') via (4.20).

Theorem 4.4 (Exact First-Order Boosted Readouts $M_1^{(v)}$ and Time-Center Requirement)**.** *Let $B^{(v)}(R,\mathbf{x}_0,t_c)$ be a boosted boundary ball defined by (4.20) with independent spatial center $\mathbf{x}_0\in\mathbb{R}^2$ and center time $t_c\in\mathbb{R}$. On V_{hom}, the first-order boosted modular energy $DE_{B^{(v)}}[h] = \frac{1}{4}DA_{B^{(v)}}[h]$ is* identically independent of the spatial center $\mathbf{x}_0$ *and takes the exact closed form (setting $P_\alpha := \cos(2\alpha)$ for $+$ polarization and $P_\alpha := \sin(2\alpha)$ for $\times$ polarization):*

$$DE_{B^{(v)}(R,\mathbf{x}_0,t_c)}[E_0] = \frac{3\pi R^3}{32}\frac{v^2}{1-v^2}\,P_\alpha, \tag{4.21}$$

$$DE_{B^{(v)}(R,\mathbf{x}_0,t_c)}[O_0] = \frac{3\pi R^3}{32}\frac{v^2}{1-v^2}\,t_c\,P_\alpha, \tag{4.22}$$

$$DE_{B^{(v)}(R,\mathbf{x}_0,t_c)}[\widetilde{E}_1] = \frac{3\pi R^3}{32}\frac{v^2}{1-v^2}\left(t_c^2 + \frac{v^2R^2}{6(1-v^2)}\right)P_\alpha. \tag{4.23}$$

Consequently:

(i) ***Vanishing of O_0 at $t_c = 0$ for all spatial centers:*** *Whenever $t_c = 0$, the pullback $t = \gamma v x'_\parallel$ on $t' = 0$ is odd in $x'_\parallel$, so $DE_{B^{(v)}(R,\mathbf{x}_0,0)}[O_0] \equiv 0$ for* every *spatial center $\mathbf{x}_0 \in \mathbb{R}^2$, every radius $R > 0$, and every boost $\mathbf{v}$. Hence spatial center shifts alone at $t_c = 0$ cannot lift O_0 or separate the opposite-chirality pair (4.5).*

(ii) ***Exact single-ball separation of polarization chirality and counterpair blind locus:*** *For the opposite-chirality pair $h^{(\pm)} = aE_0^+ \pm bO_0^\times$ of (4.5),*

$$DE_{B^{(v)}(R,\mathbf{x}_0,t_c)}\big[h^{(+)}\big] - DE_{B^{(v)}(R,\mathbf{x}_0,t_c)}\big[h^{(-)}\big] \;=\; \frac{3\pi R^3}{16}\frac{v^2}{1-v^2}\,t_c\,b\,\sin(2\alpha), \tag{4.24}$$

so a single boosted ball with $v \neq 0$, $\sin(2\alpha) \neq 0$, and $t_c \neq 0$ (e.g. $\alpha = \pi/8$, giving $DE_{B^{(v)}}[h^{(\pm)}] = \frac{3\pi R^3 v^2}{32\sqrt{2}(1-v^2)}(a \pm bt_c)$) separates $h^{(+)}$ from $h^{(-)}$ and lifts the parity-odd chirality $\mathcal{H}[h^{(\pm)}] = \pm ab$ at first order *in $M_1^{(v)}$. By contrast, for the single-polarization counterpair $(h^{(1)}, h^{(2)})$ of (4.2) ($h^{(1)} = E_0^\times + \widetilde{E}_1^\times$, $h^{(2)} = E_0^\times - \widetilde{E}_1^\times + b_{0*}O_0^\times$, $b_{0*} = \frac{2\sqrt{15}}{3}$), letting $\mu := \frac{3\pi R^3 v^2}{32(1-v^2)}$ and $\sigma := \frac{v^2R^2}{6(1-v^2)}$ gives*

$$DE_{B^{(v)}(R,\mathbf{x}_0,t_c)}\big[h^{(2)}\big] - DE_{B^{(v)}(R,\mathbf{x}_0,t_c)}\big[h^{(1)}\big] \;=\; \mu\,\sin(2\alpha)\Big[b_{0*}t_c - 2\big(t_c^2 + \sigma\big)\Big]. \tag{4.25}$$

Hence a single boosted ball separates $(h^{(1)}, h^{(2)})$ in $M_1^{(v)}$ if and only if $v \neq 0$, $\sin(2\alpha) \neq 0$, and (R, v, t_c) avoids the exact single-ball blind locus $2(t_c^2 + \sigma) = b_{0*}t_c$ (when $v = 0$, the prefactor μ vanishes identically and separation fails for all R, t_c, α; for $v \neq 0$ and $\sin(2\alpha) \neq 0$, the blind locus has real solutions $t_c = \frac{b_{0*} \pm \sqrt{b_{0*}^2 - 16\sigma}}{4} > 0$ iff $\sigma \leq \frac{5}{12}$, and never occurs for $t_c \leq 0$—including $t_c = 0$—or whenever $\sigma > \frac{5}{12}$). Unconditional separation of $(h^{(1)}, h^{(2)})$ for all parameters follows from the full-rank six-ball design in Theorem 4.5.

(iii) ***Boost-axis kernel at $\alpha = 0$:*** *For any boost along the x-axis ($\alpha = 0$), $\sin(2\alpha) = 0$, so the entire $\times$-polarized subspace $V_{\text{hom}}^\times$ remains strictly inside $\ker M_1^{(v)}$ for all $v \in (-1, 1)$, all $R > 0$, all $\mathbf{x}_0 \in \mathbb{R}^2$, and all $t_c \in \mathbb{R}$.*

Proof. Under (4.20) with $\alpha = 0$ (general α follows by rotating $(H_+, H_\times)$ by 2α), the boundary stress tensor $T_{\mu\nu}^{(1)} = \frac{3}{16\pi}\lim_{z\to 0}(h_{\mu\nu}/z)$ transforms on $t' = 0$ to

$$T'^{(1)}_{t't'}\big|_{t'=0} = \gamma^2 v^2 T_{xx}^{(1)}(t, 0) = \frac{3}{16\pi}\gamma^2 v^2 H_+^{(0)}\big(t_c + \gamma v x'_\parallel\big),$$

which is manifestly independent of $\mathbf{x}_0$ because $h \in V_{\text{hom}}$ is spatially homogeneous (while $T'^{(1)}_{t't'} \equiv 0$ for $H_\times$). Integrating $DE_{B^{(v)}} = \frac{\pi}{R}\int_{r'<R}(R^2 - r'^2)T'^{(1)}_{t't'}\,\mathrm{d}^2x'$ using $\int_{r'<R}(R^2 - r'^2)\,\mathrm{d}^2x' = \frac{\pi R^4}{2}$, $\int_{r'<R}(R^2 - r'^2)x'_\parallel\,\mathrm{d}^2x' = 0$, and $\int_{r'<R}(R^2 - r'^2)(x'_\parallel)^2\,\mathrm{d}^2x' = \frac{\pi R^6}{12}$ directly yields (4.21)–(4.23). As an independent bulk cross-check (`verify_02`), evaluating the hemisphere area formula $\frac{1}{4}DA_{B^{(v)}}[h]$ directly from the pulled-back bulk metric (including the $z'^2/5$ term in $\widetilde{E}_1$) reproduces (4.21)–(4.23) identically for all three modes, and an explicit regression sweep confirms complete invariance under $(x_{0,\parallel}, x_{0,\perp})$ at fixed t_c. Statements (i)–(iii) follow immediately. □

Theorem 4.4 leads directly to a constructive, minimal-size first-order measurement theorem recovering the full six-dimensional subspace $V_6 := \operatorname{span}\{E_0^+, O_0^+, \widetilde{E}_1^+, E_0^\times, O_0^\times, \widetilde{E}_1^\times\} \subset \ker M_1$, together with exact conditioning bounds as $v \to 0$ or $t_c \to 0$.

Theorem 4.5 (Constructive Six-Ball Boosted Recovery and Sharp Conditioning Rates)**.** *Write a general element $h \in V_6$ as*

$$h = a_0^+E_0^+ + b_0^+O_0^+ + a_1^+\widetilde{E}_1^+ + a_0^\times E_0^\times + b_0^\times O_0^\times + a_1^\times\widetilde{E}_1^\times,$$

and fix a proper radius $R > 0$, boost speed $v \in (0, 1)$, and time step $\tau > 0$. Set $\mu := \frac{3\pi R^3 v^2}{32(1-v^2)}$ and $\sigma := \frac{v^2R^2}{6(1-v^2)}$.

(i) ***Exact six-ball recovery design (Design A):*** *Measure* $y_\alpha(t_c) := DE_{B^{(v)}(R,\mathbf{x}_0,t_c)}[h]$ *on the six boosted balls* $(t_c,\alpha) \in \{-\tau,0,+\tau\}\times\{0,\pi/4\}$ *(with arbitrary spatial centers* $\mathbf{x}_0 \in \mathbb{R}^2$*). The* 6×6 *readout matrix is block diagonal,* $\mathcal{M}_6 = \mathrm{diag}(D_{\mathrm{time}}, D_{\mathrm{time}})$*, with*

$$D_{\mathrm{time}} = \mu \begin{pmatrix} 1 & -\tau & \tau^2+\sigma \\ 1 & 0 & \sigma \\ 1 & +\tau & \tau^2+\sigma \end{pmatrix}, \qquad \det(\mathcal{M}_6) = (\det D_{\mathrm{time}})^2 = 4\,\mu^6\tau^6 > 0. \tag{4.26}$$

Hence $\mathrm{rank}(\mathcal{M}_6) = 6$*, and for each polarization* $\alpha \in \{0,\pi/4\}$ *(corresponding to* $+,\times$*) the six coefficients are reconstructed uniquely in closed form by*

$$a_0^\alpha = \frac{y_\alpha(0)}{\mu} - \sigma\, a_1^\alpha, \qquad b_0^\alpha = \frac{y_\alpha(\tau) - y_\alpha(-\tau)}{2\mu\tau}, \qquad a_1^\alpha = \frac{y_\alpha(\tau) - 2y_\alpha(0) + y_\alpha(-\tau)}{2\mu\tau^2}. \tag{4.27}$$

(ii) ***Exact*** ℓ_∞ ***minimax noise amplification and*** $v,\tau\to 0$ ***deterioration:*** *Under worst-case measurement noise* $\|\eta\|_\infty \le \varepsilon$ *on the six readouts, the exact minimax estimation errors* $\varepsilon L(\cdot)$ *of the six coordinates are governed by the* ℓ_1 *row norms of* D_{time}^{-1}*:*

$$L(a_0^\alpha) = \frac{32(1-v^2)}{3\pi R^3 v^2} + \frac{32}{9\pi R\,\tau^2}, \qquad L(b_0^\alpha) = \frac{32(1-v^2)}{3\pi R^3 v^2\tau}, \qquad L(a_1^\alpha) = \frac{64(1-v^2)}{3\pi R^3 v^2\tau^2}. \tag{4.28}$$

Thus inversion deteriorates as $O(v^{-2})$ *for* a_0*,* $O(v^{-2}\tau^{-1})$ *for* b_0*, and* $O(v^{-2}\tau^{-2})$ *for* a_1 *as* $v\to 0$ *or* $\tau\to 0$*.*

(iii) ***Decoupling*** (a_0,a_1) ***from*** τ ***(Design B) and universal lower bounds:*** *Replacing the three* (R,t_c) *settings by* $\{(R_1,0),(R_2,0),(R_1,\tau)\}$ *(*$0<R_1<R_2$*, at* $\alpha\in\{0,\pi/4\}$*) again has full rank 6 (block determinant* $\frac{9\pi^3 R_1^6 R_2^3 v^8\tau(R_1^2-R_2^2)}{65536(1-v^2)^4} \neq 0$*) and recovers* a_0^α, a_1^α *from the* $t_c = 0$ *pair alone (with* τ*-independent conditioning* $O(v^{-2})$ *and* $O(v^{-4})$*), while* b_0^α *still requires* $t_c \neq 0$*. Conversely, for* any *finite collection of boosted balls* $\{B^{(v_j)}(R_j,\mathbf{x}_{0,j},t_{c,j})\}$ *with* $R_j \le R_{\max}$*,* $|v_j| \le v_{\max} < 1$*, and* $|t_{c,j}| \le \tau_{\max}$*, testing the single modes* $h = E_0^+$ *and* $h = O_0^+$ *yields the universal minimax lower bounds*

$$L_{\mathrm{any}}(a_0^\pm) \;\ge\; \frac{32(1-v_{\max}^2)}{3\pi R_{\max}^3 v_{\max}^2}, \qquad L_{\mathrm{any}}(b_0^\pm) \;\ge\; \frac{32(1-v_{\max}^2)}{3\pi R_{\max}^3 v_{\max}^2\tau_{\max}}, \tag{4.29}$$

proving that the v^{-2} *and* $v^{-2}\tau^{-1}$ *deterioration rates of Design A are asymptotically order-optimal.*

Proof. (i) Evaluating (4.21)–(4.23) at $\alpha = 0$ ($\cos 2\alpha = 1, \sin 2\alpha = 0$) and $\alpha = \pi/4$ ($\cos 2\alpha = 0, \sin 2\alpha = 1$) decouples the $+$ and $\times$ sectors into identical 3×3 blocks D_{time}. Subtracting row 2 from rows 1 and 3 gives $\det D_{\mathrm{time}} = \mu^3 \det \begin{pmatrix} 0 & -\tau & \tau^2 \\ 1 & 0 & \sigma \\ 0 & \tau & \tau^2 \end{pmatrix} = 2\mu^3\tau^3 > 0$, and direct inversion verifies (4.27) (`verify_02`).

(ii) Since $\mathcal{M}_6$ is invertible, the minimax worst-case ℓ_∞ error for recovering any linear coordinate $e_i^\top x$ from $y = \mathcal{M}_6 x + \eta$ ($\|\eta\|_\infty \le \varepsilon$) is $\varepsilon\|e_i^\top \mathcal{M}_6^{-1}\|_1$ [11]. Reading the rows of $D_{\mathrm{time}}^{-1} = \frac{1}{2\mu\tau^2}\begin{pmatrix} -\sigma & 2(\sigma+\tau^2) & -\sigma \\ -\tau & 0 & \tau \\ 1 & -2 & 1 \end{pmatrix}$ directly gives $\|e_1^\top D_{\mathrm{time}}^{-1}\|_1 = \frac{2\sigma+\tau^2}{\mu\tau^2} = \frac{1}{\mu} + \frac{2\sigma}{\mu\tau^2}$ (with $\frac{2\sigma}{\mu} = \frac{32}{9\pi R}$), $\|e_2^\top D_{\mathrm{time}}^{-1}\|_1 = \frac{1}{\mu\tau}$, and $\|e_3^\top D_{\mathrm{time}}^{-1}\|_1 = \frac{2}{\mu\tau^2}$, proving (4.28).

(iii) The determinant and inverse of Design B are verified directly in `verify_02`. For any linear readout family bounded by $(R_{\max}, v_{\max}, \tau_{\max})$, comparing $h = E_0^+$ against 0 gives $\max_j |DE_{B_j}[E_0^+]| \le \frac{3\pi R_{\max}^3 v_{\max}^2}{32(1-v_{\max}^2)}$, and comparing $h = O_0^+$ against 0 gives $\max_j |DE_{B_j}[O_0^+]| \le \frac{3\pi R_{\max}^3 v_{\max}^2\tau_{\max}}{32(1-v_{\max}^2)}$; dividing 1 by these maxima yields (4.29). □

Finally, Theorem 4.4(iii) still poses a sharp purely second-order question: when the $\times$-polarized counterpair $(h^{(1)}, h^{(2)})$ of (4.2) is probed by boosted balls along the x-axis at $t_c = 0$ ($\alpha = 0$, arbitrary spatial center $\mathbf{x}_0 \in \mathbb{R}^2$), both perturbations remain strictly inside $\ker M_1^{(v)}$ for *all* $v \in (-1, 1)$! Pulling back $h_{xy} = zH(t, z)$ into the boosted frame at $t_c = 0$ gives

$$h'_{x'y'} = \gamma z'\, H(\gamma(t' + vx'), z'), \qquad h'_{t'y'} = \gamma v z'\, H(\gamma(t' + vx'), z'), \tag{4.30}$$

independent of $\mathbf{x}_0$. Solving the coupled spatial and timelike $\mathbb{H}^2$ Jacobi equations across all excited azimuthal harmonics $m \in \{1, 2, 3, 4\}$ yields an exact all-v relativistic theorem that survives completely unchanged from the centered $t_c = 0$ calculation (and holds for every spatial center $\mathbf{x}_0 \in \mathbb{R}^2$).

Theorem 4.6 (Exact All-v Boosted Canonical Energies at $t_c = 0$, $O(v^2)$ Cancellation, and Quartic Lifting Inside $\ker M_1^{(v)}$)**.** *For boosted balls $B^{(v)}(R, \mathbf{x}_0, 0)$ along the x-axis at center time $t_c = 0$ ($\alpha = 0$, arbitrary $\mathbf{x}_0 \in \mathbb{R}^2$, where $V_{\mathrm{hom}}^{\times} \subset \ker M_1^{(v)}$ for all $|v| < 1$), the exact second-order canonical energies on $\mathrm{span}\{E_0^{\times}, \widetilde{E}_1^{\times}, O_0^{\times}\}$ are rational functions of v^2 independent of $\mathbf{x}_0$:*

$$\mathcal{E}_{B^{(v)}(R,\mathbf{x}_0,0)}(E_0^{\times}) = \frac{3\pi R^6}{280}\frac{1+v^2}{1-v^2}, \tag{4.31}$$

$$\mathcal{E}_{B^{(v)}(R,\mathbf{x}_0,0)}(E_0^{\times}, \widetilde{E}_1^{\times}) = \frac{\pi R^8}{840}\frac{1+v^2}{(1-v^2)^2}, \tag{4.32}$$

$$\mathcal{E}_{B^{(v)}(R,\mathbf{x}_0,0)}(\widetilde{E}_1^{\times}) = \frac{\pi R^{10}}{138600}\frac{29 + 29v^2 - 16v^4 + 80v^6}{(1-v^2)^3}, \tag{4.33}$$

$$\mathcal{E}_{B^{(v)}(R,\mathbf{x}_0,0)}(O_0^{\times}) = \frac{\pi R^8}{4200}\frac{3 + 3v^2 + 8v^4}{(1-v^2)^2}. \tag{4.34}$$

Consequently, on the single-polarization counterpair (4.2) ($h^{(1)} = E_0^{\times} + \widetilde{E}_1^{\times}$, $h^{(2)} = E_0^{\times} - \widetilde{E}_1^{\times} + \frac{2\sqrt{15}}{3} O_0^{\times}$), the $O(v^2)$ relativistic corrections cancel identically *between $h^{(1)}$ and $h^{(2)}$, while exact quartic relativistic lifting separates them for every $v \neq 0$:*

$$\mathcal{E}_{B^{(v)}(R,\mathbf{x}_0,0)}\big(h^{(2)}\big) - \mathcal{E}_{B^{(v)}(R,\mathbf{x}_0,0)}\big(h^{(1)}\big) \;=\; \frac{4\pi R^8}{315}\frac{v^4}{(1-v^2)^2} \;>\; 0 \qquad \forall\, v \in (-1,1)\setminus\{0\}. \tag{4.35}$$

Proof. By spatial translation invariance of V_{hom}, we set $\mathbf{x}_0 = 0$ without loss of generality. At $t_c = 0$ and $\alpha = 0$, the reflection $(t', x') \mapsto (-t', -x')$ preserves $B^{(v)}(R, 0, 0)$ and $\xi_{B^{(v)}}$, leaves $E_0^{\times}$ and $\widetilde{E}_1^{\times}$ even, and reverses the sign of $O_0^{\times}$, so $O_0^{\times}$ remains decoupled from $\mathrm{span}\{E_0^{\times}, \widetilde{E}_1^{\times}\}$ for all $v \in (-1, 1)$. Substituting (4.30) into the 4D Einstein residual $D^2\mathcal{F}_{t't'}[h', h']$ and the exact Lorentzian second area variation on $\widetilde{B}^{(v)}$ (solved across all active harmonics $m \in \{1, 2, 3, 4\}$ via the closed-form $\mathbb{H}^2$ Jacobi Green's functions in `verify_02`) yields (4.31)–(4.34), with all four $O(v^0)$ and $O(v^2)$ coefficients independently cross-checked against an explicit perturbative $O(v^2)$ expansion. With $b_{0*}^2 = 20/3$,

$$\begin{aligned}
\mathcal{E}_{B^{(v)}}(h^{(2)}) - \mathcal{E}_{B^{(v)}}(h^{(1)}) &= \frac{20}{3}\,\mathcal{E}_{B^{(v)}}(O_0^{\times}) - 4\,\mathcal{E}_{B^{(v)}}(E_0^{\times}, \widetilde{E}_1^{\times}) \\
&= \frac{\pi R^8}{(1-v^2)^2}\left[\frac{20}{3}\cdot\frac{3 + 3v^2 + 8v^4}{4200} - 4\cdot\frac{1+v^2}{840}\right] \\
&= \frac{\pi R^8}{(1-v^2)^2}\left[\frac{3 + 3v^2 + 8v^4 - 3 - 3v^2}{630}\right] = \frac{4\pi R^8}{315}\frac{v^4}{(1-v^2)^2},
\end{aligned}$$

exhibiting directly why both $O(v^0)$ and $O(v^2)$ vanish identically while the $8v^4$ term in $\mathcal{E}_{B^{(v)}}(O_0^{\times})$ yields (4.35). □

Table 4: Comparison of the three minimal symmetry-breaking extensions of rest-frame $t = 0$ boundary-ball readouts on $V_{\mathrm{hom}} \subset \ker M_1$. For spatially homogeneous modes, all boosted readouts are independent of the spatial center $\mathbf{x}_0$; lifting O_0 and polarization chirality at first order requires boosting *and* shifting the center time $t_c \neq 0$.

Extension	Counterpair $(h^{(1)}, h^{(2)})$	Even polarizations $(E_0^+, E_0^\times)$	Odd polarizations $(O_0^+, O_0^\times)$	Chirality $\mathcal{H}[h^{(\pm)}]$ $(aE_0^+ \pm bO_0^\times)$
Rest-frame $t = 0$ balls	Blind $(\forall R, \mathbf{x}_0)$	Blind $(O(2)_E)$	Blind $(O(2)_O)$	Blind
Time shifts $t = t_0$ (rest-frame)	Lifted at $O(t_0)$ (blind at $R = \sqrt{45/11}$, lifted $O(t_0^2)$)	Blind $(O(2)_E)$	Blind $(O(2)_O)$	**Blind for all** $t_0 \in \mathbb{R}$
Static $t = 0$ multipoles $(m = 2, 4)$	Lifted in DA_{B_ϵ} $(O(\epsilon),\ m = 2)$	Lifted in DA_{B_ϵ} $(O(\epsilon),\ m = 2)$	$DA, D\mathcal{T} = 0\ \forall$ shapes; lifted in A''_{ext} $(O(\epsilon),\ m = 4)$	Blind $(t \mapsto -t$ decoupling)
Boosted balls $(v \neq 0,\ t_c \neq 0$; 2 angles, 6-ball design)	Lifted in $M_1^{(v)}$ (6-ball full rank; single ball with $v \neq 0, \sin 2\alpha \neq 0$ blind on $2(t_c^2 + \sigma) = b_{0*}t_c)$	Lifted in $M_1^{(v)}$ $(O(v^2))$	**Lifted in** $M_1^{(v)}$ **(**$O(v^2 t_c)$**)**	**Lifted in** $M_1^{(v)}$ **(**$O(v^2 t_c)$**)**
Boosted balls at $t_c = 0$ $(\alpha = 0$, $h_\times \in \ker M_1^{(v)})$	$O(v^2)$ cancels; lifted in $M_2^{(v)}$ at $O(v^4)$ $(\forall \mathbf{x}_0)$	Lifted in $M_1^{(v)}$ for h_+ $(O(v^2))$	$O_0 \in \ker M_1^{(v)}$ $\forall \mathbf{x}_0, \alpha$ at $t_c = 0$	Blind at $t_c = 0$ $(t', x' \mapsto -t', -x')$

Table 4 summarizes the exact symmetry-breaking hierarchy. Time shifts break $t \mapsto -t$ symmetry but preserve spatial parity $\mathcal{R}_y$, leaving polarization chirality $\mathcal{H}$ permanently invisible. Static $t = 0$ multipole deformations break $SO(2)$ isotropy (lifting E_k in DA_{B_ϵ} via $m = 2$ and O_0 in $A''_{\mathrm{ext},B_\epsilon}$ via $m = 4$) while preserving $t \mapsto -t$ decoupling. Boosted balls at $t_c = 0$ break $SO(2)$ and lift E_k in $M_1^{(v)}$ (and lift the $\ker M_1^{(v)}$ counterpair at $O(v^4)$ in $M_2^{(v)}$), yet remain odd-integrand blind to O_0 and $\mathcal{H}$ at first order for every spatial center $\mathbf{x}_0$. Only combining Lorentz boosts ($v \neq 0$) with boundary time shifts ($t_c \neq 0$) breaks all three symmetries simultaneously, enabling complete first-order recovery of all six modes in V_6 from just six explicit balls (Theorem 4.5).

5 Positive-Semidefinite Quadratic Identification and Optimal Recovery Stability

Having characterized the exact canonical-energy bilinears on $V_{\mathrm{hom}} \subset \ker M_1$ and the blind spots of rest-frame balls in Sections 3–4, we now turn to the quantitative inverse problem inside an identifiable subspace: given finitely many noisy second-order canonical-energy readouts $y_k = \mathcal{E}_{B_k}(h) + \eta_k$ $(k = 1, \ldots, m)$, when can a quadratic functional $\varphi(h) = \langle h, Qh \rangle$ (such as a mode amplitude squared or the canonical energy at an unobserved radius R_*) be reconstructed uniquely, when does convex semidefinite lifting preserve the exact worst-case noise amplification, and how rapidly does stability degrade with the number of modes n and the extrapolation ratio $R_*/R_{\max}$?

5.1 General PSD-Quadratic Identification and Minimax Optimal Recovery

Let $V \cong \mathbb{R}^n$ be a finite-dimensional real subspace of $\ker M_1$ equipped with m symmetric positive-semidefinite (PSD) quadratic measurement operators $A_1, \dots, A_m \in \mathrm{Sym}^2(\mathbb{R}^n)$, $A_k \succeq 0$. Define the quadratic readout map and its lifted linear operator on $\mathrm{Sym}^2(\mathbb{R}^n)$ by

$$M_2(v) := \big(v^\top A_1 v, \dots, v^\top A_m v\big) \in \mathbb{R}^m, \qquad \mathcal{A}(X) := \big(\mathrm{Tr}(A_1 X), \dots, \mathrm{Tr}(A_m X)\big) \in \mathbb{R}^m, \quad (5.1)$$

so that $M_2(v) = \mathcal{A}(vv^\top)$ restricts $\mathcal{A}$ to the rank-one PSD cone $\mathcal{P}_1 := \{vv^\top : v \in \mathbb{R}^n\} \subset \mathcal{P}_+ := \{X \in \mathrm{Sym}^2(\mathbb{R}^n) : X \succeq 0\}$. We say that M_2 *identifies pure states up to global sign* on $\mathbb{R}^n$ if $M_2(u) = M_2(v) \implies u = \pm v$ (equivalently, $\mathcal{A}(uu^\top - vv^\top) = 0 \implies uu^\top = vv^\top$), and that $\mathcal{A}$ *identifies mixed states* if $\ker \mathcal{A} \cap (\mathcal{P}_+ - \mathcal{P}_+) = \{0\}$.

Given a quadratic target functional $\varphi_Q(v) := v^\top Q v = \mathrm{Tr}(Q\, vv^\top)$ ($Q \in \mathrm{Sym}^2(\mathbb{R}^n)$) and worst-case ℓ_∞ measurement noise $\|\eta\|_\infty \le \varepsilon$ ($y = \mathcal{A}(X) + \eta$), two feasible states $X, Y \in \mathcal{D}$ can produce the same noisy observation y if and only if $\|\mathcal{A}(X - Y)\|_\infty \le 2\varepsilon$ (choosing $y = \frac{1}{2}(\mathcal{A}(X) + \mathcal{A}(Y))$). By the classical optimal-recovery theorem [11, 12], the minimax worst-case estimation error over all estimators $\widehat{\varphi} : \mathbb{R}^m \to \mathbb{R}$ on a domain $\mathcal{D} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ equals the half-diameter of the feasible target interval at separation 2ε:

$$E_{\mathrm{opt}}(\mathcal{D}; Q, \varepsilon) := \inf_{\widehat{\varphi}:\mathbb{R}^m \to \mathbb{R}} \sup_{\substack{X \in \mathcal{D} \\ \|\eta\|_\infty \le \varepsilon}} \big|\widehat{\varphi}(\mathcal{A}(X) + \eta) - \mathrm{Tr}(QX)\big| = \frac{1}{2} \sup_{\substack{X, Y \in \mathcal{D} \\ \|\mathcal{A}(X-Y)\|_\infty \le 2\varepsilon}} \big|\mathrm{Tr}(Q(X - Y))\big|. \quad (5.2)$$

Because $\mathcal{P}_1$, $\mathcal{P}_+$, and $\mathrm{Sym}^2(\mathbb{R}^n)$ are all positively homogeneous cones, rescaling (X, Y) by $(2\varepsilon)^{-1}$ cancels the factor of $\frac{1}{2}$ against 2ε identically, so the exact minimax optimal recovery error is strictly linear in ε,

$$E_{\mathrm{opt}}(\mathcal{D}; Q, \varepsilon) \;=\; \varepsilon\, L_{\mathcal{D}}(Q) \qquad \big(\mathcal{D} \in \{\mathcal{P}_1, \mathcal{P}_+, \mathrm{Sym}^2(\mathbb{R}^n)\}\big), \quad (5.3)$$

with exact worst-case Lipschitz amplification constants

$$L_{\mathrm{pure}}(Q) := \sup_{\substack{u, v \in \mathbb{R}^n \\ uu^\top \neq vv^\top}} \frac{|u^\top Q u - v^\top Q v|}{\|\mathcal{A}(uu^\top - vv^\top)\|_\infty}, \quad (5.4)$$

$$L_{\mathrm{SDP}}(Q) := \sup_{\substack{X, Y \succeq 0 \\ X \neq Y}} \frac{|\,\mathrm{Tr}(Q(X - Y))|}{\|\mathcal{A}(X - Y)\|_\infty}, \quad (5.5)$$

$$L_{\mathrm{lin}}(Q) := \sup_{\substack{\Delta \in \mathrm{Sym}^2(\mathbb{R}^n) \\ \Delta \neq 0}} \frac{|\,\mathrm{Tr}(Q\Delta)|}{\|\mathcal{A}(\Delta)\|_\infty} = \inf\Big\{\|c\|_1 : c \in \mathbb{R}^m,\ \sum_{k=1}^m c_k A_k = Q\Big\}, \quad (5.6)$$

with the convention $\inf \emptyset = +\infty$ and $1/0 = +\infty$. In particular, for any unbiased linear estimator $\widehat{\varphi}_c(y) = c^\top y$ ($\sum_k c_k A_k = Q$), the worst-case error $|\widehat{\varphi}_c(\mathcal{A}(X) + \eta) - \mathrm{Tr}(QX)| = |c^\top \eta| \le \|c\|_1 \varepsilon$ achieves the exact bound $\varepsilon L_{\mathrm{lin}}(Q)$ upon minimizing $\|c\|_1$, confirming (5.3).

Theorem 5.1 (PSD-Quadratic Identification, Exactness of Conic Lifting, and ℓ_∞ Minimax Stability)**.** *Let $A_1, \dots, A_m \in \mathrm{Sym}^2(\mathbb{R}^n)$ be symmetric matrices and $Q \in \mathrm{Sym}^2(\mathbb{R}^n)$. Then:*

(i) ***Identity of SDP and unrestricted linear constants:*** *Every symmetric matrix $\Delta \in \mathrm{Sym}^2(\mathbb{R}^n)$ admits a Jordan decomposition $\Delta = \Delta_+ - \Delta_-$ with $\Delta_\pm \succeq 0$. Consequently, $\mathcal{P}_+ - \mathcal{P}_+ = \mathrm{Sym}^2(\mathbb{R}^n)$ and*

$$L_{\mathrm{pure}}(Q) \;\le\; L_{\mathrm{SDP}}(Q) \;=\; L_{\mathrm{lin}}(Q) \;=\; \inf\Big\{\|c\|_1 : \sum_{k=1}^m c_k A_k = Q\Big\}. \quad (5.7)$$

In particular, $L_{\mathrm{SDP}}(Q) < \infty$ if and only if $Q \in \mathrm{span}\{A_1, \dots, A_m\} = (\ker \mathcal{A})^\perp$, and mixed-state identifiability holds for all targets Q if and only if $\mathrm{rank}\, \mathcal{A} = \binom{n+1}{2}$.

(ii) ***Pure-state secant characterization:*** *Let $\mathcal{V}_{\le 2}$ be the cone of symmetric matrices $\Delta \in \mathrm{Sym}^2(\mathbb{R}^n)$ with inertia $(n_+(\Delta), n_-(\Delta))$ satisfying $n_+(\Delta) \le 1$ and $n_-(\Delta) \le 1$ (at most one strictly positive and at most one strictly negative eigenvalue). Then $\mathcal{P}_1 - \mathcal{P}_1 = \mathcal{V}_{\le 2}$, so*

$$L_{\mathrm{pure}}(Q) = \sup_{\Delta \in \mathcal{V}_{\le 2} \setminus \{0\}} \frac{|\operatorname{Tr}(Q\Delta)|}{\|\mathcal{A}(\Delta)\|_\infty}. \tag{5.8}$$

Moreover, M_2 identifies pure states up to global sign on $\mathbb{R}^n$ if and only if $\ker \mathcal{A} \cap \mathcal{V}_{\le 2} = \{0\}$ (equivalently, every nonzero matrix $\Delta \in \ker \mathcal{A}$ satisfies $n_+(\Delta) \ge 2$ or $n_-(\Delta) \ge 2$, i.e. has at least two strictly positive or at least two strictly negative eigenvalues), in which case $L_{\mathrm{pure}}(Q) < \infty$ for every *$Q \in \mathrm{Sym}^2(\mathbb{R}^n)$ by compactness of the unit sphere in $\mathcal{V}_{\le 2}$.*

(iii) ***Special rank-one case (complementary-kernel phase retrieval):*** *If each $A_k = \phi_k \phi_k^\top$ has rank one, then $\Delta = uu^\top - vv^\top \in \ker \mathcal{A}$ iff $\langle \phi_k, u+v\rangle\langle \phi_k, u-v\rangle = 0$ for all k, recovering the classical complementary-kernel condition [14]: pure-state identifiability holds iff for every partition $I \sqcup I^c = \{1, \dots, m\}$, either $\operatorname{span}\{\phi_k : k \in I\} = \mathbb{R}^n$ or $\operatorname{span}\{\phi_k : k \in I^c\} = \mathbb{R}^n$.*

(iv) ***Zero-gap attainment criterion:*** *Let $Q \in \operatorname{span}\{A_1, \dots, A_m\}$ (so $L_{\mathrm{lin}}(Q) < \infty$) and consider the linear primal problem*

$$L_{\mathrm{lin}}(Q) = \max\big\{\operatorname{Tr}(Q\Delta) : \Delta \in \mathrm{Sym}^2(\mathbb{R}^n),\ \|\mathcal{A}(\Delta)\|_\infty \le 1\big\}. \tag{5.9}$$

If (5.9) attains its maximum at some $\Delta_* \in \mathcal{V}_{\le 2}$, then $L_{\mathrm{pure}}(Q) = L_{\mathrm{SDP}}(Q) = L_{\mathrm{lin}}(Q)$. Conversely, under pure-state identifiability ($\ker \mathcal{A} \cap \mathcal{V}_{\le 2} = \{0\}$), the feasible cone slice $P \cap \mathcal{V}_{\le 2}$ (where $P := \{\Delta : \|\mathcal{A}(\Delta)\|_\infty \le 1\}$) is compact, so equality $L_{\mathrm{pure}}(Q) = L_{\mathrm{SDP}}(Q) = L_{\mathrm{lin}}(Q)$ holds if and only if *some maximizer of (5.9) lies in $\mathcal{V}_{\le 2}$.*

Proof. (i) For any $\Delta \in \mathrm{Sym}^2(\mathbb{R}^n)$ with spectral decomposition $\Delta = \sum_i \lambda_i e_i e_i^\top$, setting $\Delta_+ = \sum_{\lambda_i > 0} \lambda_i e_i e_i^\top \succeq 0$ and $\Delta_- = \sum_{\lambda_i < 0} (-\lambda_i) e_i e_i^\top \succeq 0$ yields $\Delta = \Delta_+ - \Delta_-$, so $\mathcal{P}_+ - \mathcal{P}_+ = \mathrm{Sym}^2(\mathbb{R}^n)$ and $L_{\mathrm{SDP}}(Q) = L_{\mathrm{lin}}(Q)$. The second equality in (5.6) is standard ℓ_1–ℓ_∞ linear-programming duality (or Sion's minimax theorem on the compact unit ball $\{\|\mathcal{A}(\Delta)\|_\infty \le 1\}$ modulo $\ker \mathcal{A}$): the polar of $\{\Delta : \|\mathcal{A}(\Delta)\|_\infty \le 1\}$ is $\{\sum_k c_k A_k : \|c\|_1 \le 1\}$, and by the Hahn–Banach theorem $L_{\mathrm{lin}}(Q) < \infty$ iff $\ker \mathcal{A} \subset \ker \operatorname{Tr}(Q\,\cdot)$ iff $Q \in (\ker \mathcal{A})^\perp = \operatorname{span}\{A_1, \dots, A_m\}$.

(ii) If $\Delta = uu^\top - vv^\top$, then Δ maps $\mathbb{R}^n$ into $\operatorname{span}\{u, v\}$ so $\operatorname{rank} \Delta \le 2$; if $\operatorname{rank} \Delta = 2$, then Δ restricted to $\operatorname{span}\{u, v\}$ has determinant $-\|u\|^2\|v\|^2 \sin^2\theta < 0$, so its two nonzero eigenvalues have opposite signs. Conversely, any symmetric Δ with eigenvalues $\lambda_1 \ge 0 \ge \lambda_2$ (all others zero) and orthonormal eigenvectors e_1, e_2 satisfies $\Delta = (\sqrt{\lambda_1} e_1)(\sqrt{\lambda_1} e_1)^\top - (\sqrt{-\lambda_2} e_2)(\sqrt{-\lambda_2} e_2)^\top \in \mathcal{P}_1 - \mathcal{P}_1$. Thus $\mathcal{P}_1 - \mathcal{P}_1 = \mathcal{V}_{\le 2}$, proving (5.8). Because $\mathcal{V}_{\le 2}$ is a closed cone in $\mathrm{Sym}^2(\mathbb{R}^n)$, its intersection with the Frobenius unit sphere $S := \{\Delta \in \mathcal{V}_{\le 2} : \|\Delta\|_F = 1\}$ is compact. The continuous function $\Delta \mapsto \|\mathcal{A}(\Delta)\|_\infty$ vanishes on S iff $\ker \mathcal{A} \cap \mathcal{V}_{\le 2} \ne \{0\}$; if it has no zero on S, it attains a strictly positive minimum $\delta_0 > 0$, whence $L_{\mathrm{pure}}(Q) \le \|Q\|_F / \delta_0 < \infty$ for every Q.

(iii) Immediate from $u^\top(\phi_k\phi_k^\top)u - v^\top(\phi_k\phi_k^\top)v = \langle \phi_k, u+v\rangle\langle \phi_k, u-v\rangle$.

(iv) Because $\mathcal{V}_{\le 2} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ and $Q \in (\ker \mathcal{A})^\perp$, any $\Delta_* \in P \cap \mathcal{V}_{\le 2}$ attaining $\operatorname{Tr}(Q\Delta_*) = L_{\mathrm{lin}}(Q)$ immediately satisfies $L_{\mathrm{lin}}(Q) = \operatorname{Tr}(Q\Delta_*) \le L_{\mathrm{pure}}(Q) \le L_{\mathrm{lin}}(Q)$, proving sufficiency unconditionally. Conversely, suppose $\ker \mathcal{A} \cap \mathcal{V}_{\le 2} = \{0\}$. By part (ii), there exists $\delta_0 > 0$ such that $\|\mathcal{A}(\Delta)\|_\infty \ge \delta_0 \|\Delta\|_F$ for all $\Delta \in \mathcal{V}_{\le 2}$. Hence every $\Delta \in P \cap \mathcal{V}_{\le 2}$ obeys $\|\Delta\|_F \le 1/\delta_0$; since $P \cap \mathcal{V}_{\le 2}$ is also closed, it is *compact*. Therefore the continuous linear functional $\Delta \mapsto \operatorname{Tr}(Q\Delta)$ attains its supremum $L_{\mathrm{pure}}(Q)$ at some $\Delta_* \in P \cap \mathcal{V}_{\le 2}$, so $L_{\mathrm{pure}}(Q) = L_{\mathrm{lin}}(Q)$ holds iff that maximizer also maximizes (5.9). (Both $n = 2$ in Theorem 5.3 and $n = 3$ in Theorem 5.6 satisfy $\ker \mathcal{A} \cap \mathcal{V}_{\le 2} = \{0\}$ and in fact invoke only the unconditional sufficiency direction.) □

Remark 5.2 (Full-rank gravitational operators vs. rank-one phase retrieval). In gravitational boundary-ball reconstruction, the canonical-energy matrix $A_n(R) \in \mathrm{Sym}^2(\mathbb{R}^n)$ on the time-even polynomial tower $V_n^{\mathrm{even}} = \operatorname{span}\{E_0, \widetilde{E}_1, \dots, \widetilde{E}_{n-1}\}$ is strictly *positive definite* (full rank

n; Theorem 3.2), not rank one. As we prove next, this produces a sharp dimensional phase transition between $n = 2$ (where SDP lifting is universally exact) and $n \geq 3$ (where radial monomial degeneracy collapses the SDP relaxation even with infinitely many radii while pure-state identification remains globally Lipschitz stable).

5.2 Universal Exactness of Linear/SDP Lifting at $n = 2$

Restrict first to the two-mode time-even subspace $V_2^{\text{even}} = \text{span}\{E_0, \widetilde{E}_1\}$ (single polarization). By Theorem 3.2, the canonical-energy matrix at radius $R > 0$ is

$$A_2(R) = \pi R^6 \begin{pmatrix} \frac{3}{280} & \frac{1}{840}R^2 \\ \frac{1}{840}R^2 & \frac{29}{138600}R^4 \end{pmatrix} \succ 0. \tag{5.10}$$

Since $\dim \text{Sym}^2(\mathbb{R}^2) = 3$, any three distinct radii $0 < R_1 < R_2 < R_3$ determine an invertible 3×3 Vandermonde linear map $\mathcal{A} : \text{Sym}^2(\mathbb{R}^2) \to \mathbb{R}^3$, so every $Q \in \text{Sym}^2(\mathbb{R}^2)$ has a unique linear estimator $c = \mathcal{A}^{-\top} Q$ with $L_{\text{lin}}(Q) = L_{\text{SDP}}(Q) = \|c\|_1 < \infty$.

A priori, restricting to pure states $\Delta = uu^\top - vv^\top \in \mathcal{V}_{\leq 2}$ could strictly reduce the worst-case noise amplification below $L_{\text{lin}}(Q)$ if the unique extremal perturbation $\Delta_* = \mathcal{A}^{-1}(s_*)$ ($s_* = \text{sgn}(c) \in \{\pm 1\}^3$) were positive or negative definite ($\det \Delta_* > 0$). Remarkably, for the gravitational matrix (5.10) this *never* happens at any sign pattern $s \in \{\pm 1\}^3$ or any radii $0 < R_1 < R_2 < R_3$.

Theorem 5.3 (Universal Zero Pure-vs-SDP Gap at $n = 2$)**.** *For every ordered triple of boundary-ball radii $0 < R_1 < R_2 < R_3$ and every target quadratic form $Q \in \text{Sym}^2(\mathbb{R}^2)$ on V_2^{even},*

$$L_{\text{pure}}(Q) \;=\; L_{\text{SDP}}(Q) \;=\; L_{\text{lin}}(Q) \;=\; \big\|\mathcal{A}(R_1, R_2, R_3)^{-\top} Q\big\|_1. \tag{5.11}$$

Hence the unique linear estimator $c^\top M_2(h)$ achieves the exact nonlinear pure-state minimax ℓ_∞ error bound simultaneously for all targets and all three-radius designs.

Proof. Since $\dim \mathbb{R}^2 = 2$, $\Delta \in \mathcal{V}_{\leq 2}$ holds iff $\det \Delta \leq 0$. By Theorem 5.1(iv) and sign-reversal symmetry $\det(-\Delta) = \det(\Delta)$, it suffices to prove $\det\big(\mathcal{A}^{-1}(s)\big) < 0$ for all four inequivalent vertices $s \in \{(1,1,1), (1,1,-1), (1,-1,1), (-1,1,1)\}$ of the ℓ_∞ unit cube $[-1,1]^3$ and all $0 < R_1 < R_2 < R_3$.

Let $x_k := R_k^2 > 0$ ($k = 1, 2, 3$). Solving $\text{Tr}(A_2(R_k)\Delta) = s_k$ ($k = 1, 2, 3$) for $\Delta = \begin{pmatrix} \Delta_{00} & \Delta_{01} \\ \Delta_{01} & \Delta_{11} \end{pmatrix}$ gives the exact Vandermonde inverse

$$\Delta_{00} = \frac{280}{3\pi} \sum_{k=1}^{3} \frac{s_k}{x_k^3} \frac{\prod_{j \neq k} x_j}{\prod_{j \neq k}(x_j - x_k)}, \qquad \Delta_{01} = -\frac{420}{\pi} \sum_{k=1}^{3} \frac{s_k}{x_k^3} \frac{\sum_{j \neq k} x_j}{\prod_{j \neq k}(x_j - x_k)},$$
$$\Delta_{11} = \frac{138600}{29\pi} \sum_{k=1}^{3} \frac{s_k}{x_k^3} \frac{1}{\prod_{j \neq k}(x_j - x_k)}. \tag{5.12}$$

Case 1: Alternating and single-minus vertices. Write $P(x) := \text{Tr}(A_2(\sqrt{x})\Delta)/(\pi x^3) = \frac{3}{280}\Delta_{00} + \frac{1}{420}\Delta_{01} x + \frac{29}{138600}\Delta_{11} x^2$. If Δ were definite (say $\Delta \succ 0$), then $P(x) = \text{Tr}(\widetilde{A}_2(x)\Delta) > 0$ for all $x > 0$ because $\widetilde{A}_2(x) \succ 0$. Consequently, $s_k = \pi x_k^3 P(x_k)$ must have the *same* sign for all $k = 1, 2, 3$. This immediately rules out all three mixed-sign vertices $s \in \{(1,1,-1), (1,-1,1), (-1,1,1)\}$ with no calculation at all: any Δ with $s_i s_j < 0$ is automatically indefinite ($\det \Delta < 0$)!

Case 2: The all-ones vertex $s = (1,1,1)$. Here $s_k > 0$ does not rule out definiteness by sign alone; one must use the specific gravitational ratios in (5.10). Direct symbolic algebra (Appendix B and `verify_03`) cancels all Vandermonde differences identically between numerator

and denominator, yielding

$$\det \Delta(1,1,1) = \frac{N_{111}(x_1,x_2,x_3)}{29\,\pi^2 x_1^6 x_2^6 x_3^6}, \tag{5.13}$$

where $N_{111}(x_1,x_2,x_3)$ is a symmetric homogeneous degree-10 polynomial in $x_1,x_2,x_3>0$ that decomposes into three manifestly negative sum-of-squares blocks plus 27 remaining monomials $c_{abc}x_1^a x_2^b x_3^c$, every one of whose integer coefficients satisfies $c_{abc}\in[-27{,}400{,}800,\,-4{,}821{,}600]<0$ (explicit formula in Appendix B). Hence $\det\Delta(1,1,1)<0$ for every $x_1,x_2,x_3>0$. Combining Cases 1 and 2, every vertex of $[-1,1]^3$ maps into $\mathcal{V}_{\le 2}$, completing the proof. □

Remark 5.4 (Sharpness of the gravitational ratio). Case 2 is a genuine gravitational property rather than a tautology for arbitrary PSD matrices: for a general quadratic family $A(R)=\pi x^3\begin{pmatrix} a & bx \\ bx & cx^2\end{pmatrix}\succ 0$ ($x=R^2$), solving $\mathrm{Tr}(A(R_k)\Delta)=1$ requires $a\Delta_{00}+2b\Delta_{01}x+c\Delta_{11}x^2$ to interpolate the strictly convex function $(\pi x^3)^{-1}$ at three positive points $x_1<x_2<x_3$. In the confluent limit $x_1,x_2,x_3\to u$, the quadratic Taylor interpolant of $(\pi x^3)^{-1}$ is $\frac{1}{\pi}(10u^{-3}-15u^{-4}x+6u^{-5}x^2)$, so $a\Delta_{00}=10/(\pi u^3)$, $2b\Delta_{01}=-15/(\pi u^4)$, and $c\Delta_{11}=6/(\pi u^5)$, whence $\det\Delta=\frac{1}{\pi^2u^8}\big(\frac{60}{ac}-\frac{225}{4b^2}\big)<0$ if and only if $b^2/(ac)<15/16$ (and similarly across separated triples, $\det\Delta(1,1,1)$ becomes positive if $b^2/(ac)$ approaches 1). For AdS_4 gravity, the cross-coupling ratio $\alpha_{01}^2/(\alpha_{00}\alpha_{11})=(1/840)^2/[(3/280)(29/138600)]=55/87\approx 0.6322<15/16$ sits well inside the strictly indefinite regime for every triple $0<R_1<R_2<R_3$.

5.3 Dimensional Phase Transition at $n\ge 3$: All-n Pure-State Identifiability vs. SDP Collapse

The situation changes dramatically as soon as $n\ge 3$ radial modes are included in $V_n^{\mathrm{even}}=\mathrm{span}\{E_0,\widetilde{E}_1,\dots,\widetilde{E}_{n-1}\}$. By Theorem 3.2, the canonical-energy matrix at radius $R>0$ is

$$\big[A_n(R)\big]_{ij}=\pi\,\alpha_{ij}\,R^{6+2i+2j},\qquad 0\le i,j\le n-1, \tag{5.14}$$

with $\alpha_{ij}>0$ for every pair (i,j) (exact rational closed form in Theorem 3.2). Because the radial exponent $6+2(i+j)$ depends *only on the index sum* $i+j$, the entire infinite family $\{A_n(R):R>0\}$ spans a subspace of $\mathrm{Sym}^2(\mathbb{R}^n)$ of dimension at most $2n-1$, namely

$$\mathcal{E}_{B(R)}(a)=\pi\sum_{s=0}^{2n-2}C_s(a)\,R^{6+2s},\qquad C_s(a):=\sum_{\substack{0\le i,j\le n-1\\ i+j=s}}\alpha_{ij}\,a_ia_j. \tag{5.15}$$

For $n=2$, $2n-1=3=\binom{n+1}{2}$, so no dimensional deficit occurs. For every $n\ge 3$, however,

$$2n-1\;<\;\binom{n+1}{2},\qquad \dim\ker\mathcal{A}_{\mathrm{all}\,R}=\binom{n+1}{2}-(2n-1)=\binom{n-1}{2}\ge 1. \tag{5.16}$$

This simple dimensional inequality produces a sharp, structural separation between pure-state recovery and mixed-state/SDP relaxation across the entire tower $n\ge 3$.

Theorem 5.5 (All-n Pure-State Identifiability vs. SDP Relaxation Collapse for $n\ge 3$)**.** *Let $V_n^{\mathrm{even}}=\mathrm{span}\{E_0,\widetilde{E}_1,\dots,\widetilde{E}_{n-1}\}$ with $n\ge 3$, and let $0<R_1<\cdots<R_m$ be any collection of $m\ge 2n-1$ distinct boundary-ball radii. Then:*

(i) ***Permanent SDP / mixed-state collapse:*** *Even if canonical energy is observed at* every *radius $R\in(0,\infty)$, the lifted linear map $\mathcal{A}:\mathrm{Sym}^2(\mathbb{R}^n)\to\mathbb{R}^m$ has $\mathrm{rank}\,\mathcal{A}=2n-1$ and $\dim\ker\mathcal{A}=\binom{n-1}{2}\ge 1$. Consequently, for every target $Q\notin\mathrm{span}\{A_n(R):R>0\}$ (in particular, for every interior mode amplitude $Q=e_ke_k^{\top}$ with $1\le k\le n-2$),*

$$L_{\mathrm{SDP}}(Q)\;=\;L_{\mathrm{lin}}(Q)\;=\;+\infty. \tag{5.17}$$

There exist explicit distinct mixed states $X_{\mathrm{mix}} \succeq 0$ *and pure states* $Y_{\mathrm{pure}} = vv^\top \succeq 0$ *with* $\mathcal{E}_B(X_{\mathrm{mix}}) \equiv \mathcal{E}_B(Y_{\mathrm{pure}})$ *for all* $R > 0$ *yet* $\mathrm{Tr}(QX_{\mathrm{mix}}) \neq \mathrm{Tr}(QY_{\mathrm{pure}})$.

(ii) ***Global pure-state identifiability and finite Lipschitz stability:*** *In stark contrast, the rank-one quadratic map* $M_2 : \mathbb{R}^n/\{\pm 1\} \to \mathbb{R}^m$ *is* globally injective *for every* $m \geq 2n-1$ *distinct radii, and therefore*

$$L_{\mathrm{pure}}(Q) \;<\; \infty \qquad \textit{for every } Q \in \mathrm{Sym}^2(\mathbb{R}^n). \tag{5.18}$$

Proof. (i) The $m \geq 2n-1$ distinct radii determine the $2n-1$ Hankel-sum functionals $X \mapsto \sum_{i+j=s} \alpha_{ij} X_{ij}$ $(s = 0, \ldots, 2n-2)$ via an invertible $(2n-1) \times (2n-1)$ Vandermonde block, since every anti-diagonal has $\alpha_{ij} > 0$. Thus $\operatorname{rank} \mathcal{A} = 2n-1$ independently of $m \geq 2n-1$, leaving $\dim \ker \mathcal{A} = \binom{n+1}{2} - (2n-1) = \binom{n-1}{2} \geq 1$. For any interior index $1 \leq k \leq n-2$, the matrix $K^{(k)} \in \mathrm{Sym}^2(\mathbb{R}^n)$ supported on $(k-1, k+1)$, $(k+1, k-1)$, and (k,k) with entries $K^{(k)}_{k-1,k+1} = K^{(k)}_{k+1,k-1} = -\alpha_{kk}$ and $K^{(k)}_{kk} = 2\alpha_{k-1,k+1}$ satisfies $\sum_{i+j=s} \alpha_{ij} K^{(k)}_{ij} = 0$ for all s (the only nontrivial sum is $s = 2k$: $2\alpha_{k-1,k+1}(-\alpha_{kk}) + \alpha_{kk}(2\alpha_{k-1,k+1}) = 0$), so $K^{(k)} \in \ker \mathcal{A}$ while $\mathrm{Tr}(e_k e_k^\top K^{(k)}) = 2\alpha_{k-1,k+1} > 0$. By Theorem 5.1(i), $L_{\mathrm{SDP}}(e_k e_k^\top) = L_{\mathrm{lin}}(e_k e_k^\top) = +\infty$. Decomposing $K^{(k)} = X_{\mathrm{mix}} - Y_{\mathrm{pure}}$ with $Y_{\mathrm{pure}} = \alpha_{kk}(e_{k-1} + e_{k+1})(e_{k-1} + e_{k+1})^\top$ and $X_{\mathrm{mix}} = \alpha_{kk}(e_{k-1}e_{k-1}^\top + e_{k+1}e_{k+1}^\top) + 2\alpha_{k-1,k+1} e_k e_k^\top \succeq 0$ gives the explicit all-radii mixed-vs-pure counterpair!

(ii) Suppose $M_2(a) = M_2(b)$ for $a, b \in \mathbb{R}^n$ at $m \geq 2n-1$ distinct positive radii. By Vandermonde invertibility of the radial monomials R^{6+2s} $(s = 0, \ldots, 2n-2)$, $C_s(a) = C_s(b)$ for every $s = 0, 1, \ldots, 2n-2$. We prove $a = \pm b$ by strong induction on the first index $k_0 := \min\{k : a_k \neq 0 \text{ or } b_k \neq 0\}$ (if no such index exists, $a = b = 0$):

- For $s = 2k_0$: since $a_i = b_i = 0$ for all $i < k_0$, the only nonzero term in C_{2k_0} is $i = j = k_0$, yielding $\alpha_{k_0 k_0} a_{k_0}^2 = \alpha_{k_0 k_0} b_{k_0}^2$. Because $\alpha_{k_0 k_0} > 0$, we have $b_{k_0} = \pm a_{k_0} \neq 0$. Replacing b by $-b$ if necessary, we may assume $b_{k_0} = a_{k_0} \neq 0$.
- Assume inductively that $b_i = a_i$ for all $i < r$ (where $k_0 < r \leq n-1$), and consider the equation $C_{k_0+r}(a) = C_{k_0+r}(b)$ (valid since $k_0 + r \leq 2n-2$):

$$2\alpha_{k_0 r} a_{k_0} a_r + \sum_{\substack{k_0 < i,j < r \\ i+j = k_0+r}} \alpha_{ij} a_i a_j \;=\; 2\alpha_{k_0 r} b_{k_0} b_r + \sum_{\substack{k_0 < i,j < r \\ i+j = k_0+r}} \alpha_{ij} b_i b_j. \tag{5.19}$$

  Every term in the two sums involves indices strictly less than r, hence cancels identically by the induction hypothesis. Since $b_{k_0} = a_{k_0} \neq 0$ and $\alpha_{k_0 r} > 0$ (Theorem 3.2), dividing by $2\alpha_{k_0 r} a_{k_0}$ yields $b_r = a_r$.

By induction, $a = b$ (up to the global sign fixed at k_0). Therefore $\ker \mathcal{A} \cap \mathcal{V}_{\leq 2} = \{0\}$, and Theorem 5.1(ii) immediately implies $L_{\mathrm{pure}}(Q) < \infty$ for every $Q \in \mathrm{Sym}^2(\mathbb{R}^n)$. □

5.4 Sharp Zero-or-Infinite Dichotomy at $n = 3$

At the threshold dimension $n = 3$ ($V_3^{\mathrm{even}} = \mathrm{span}\{E_0, \widetilde{E}_1, \widetilde{E}_2\}$), Theorem 3.2 gives the exact 3×3 gravitational matrix

$$\alpha^{(3)} = \begin{pmatrix} \frac{3}{280} & \frac{1}{840} & \frac{1}{3080} \\ \frac{1}{840} & \frac{29}{138600} & \frac{43}{600600} \\ \frac{1}{3080} & \frac{43}{600600} & \frac{3}{107800} \end{pmatrix} \succ 0, \tag{5.20}$$

and the all-radii kernel is one-dimensional: $\ker \mathcal{A} = \operatorname{span}\{K_3\}$, where (scaling by 138600×2 to clear fractions)

$$K_3 = \begin{pmatrix} 0 & 0 & -29 \\ 0 & 90 & 0 \\ -29 & 0 & 0 \end{pmatrix}, \qquad \operatorname{spec}(K_3) = \{-29,\ +29,\ +90\}. \tag{5.21}$$

Observe the critical algebraic fact: K_3 has *inertia* $(2, 1)$ (two strictly positive eigenvalues and one strictly negative eigenvalue), so $K_3 \notin \mathcal{V}_{\le 2}$—confirming directly why $\ker \mathcal{A} \cap \mathcal{V}_{\le 2} = \{0\}$! Remarkably, this inertia $(2, 1)$ also drives a complete classification of the pure-vs-SDP stability gap at $n = 3$ for *all* targets Q and *all* designs with $m \ge 5$ radii.

Theorem 5.6 (Sharp Zero-or-Infinite Pure-vs-SDP Dichotomy at $n = 3$)**.** *Let $V_3^{\mathrm{even}} = \operatorname{span}\{E_0, \widetilde{E}_1, \widetilde{E}_2\}$, and let $0 < R_1 < \cdots < R_m$ ($m \ge 5$) be distinct positive radii. Every target $Q \in \operatorname{Sym}^2(\mathbb{R}^3)$ falls into exactly one of two exhaustive types:*

(i) ***Type I** ($\operatorname{Tr}(QK_3) \neq 0$**, infinite gap):***

$$L_{\mathrm{pure}}(Q) < \infty \qquad \text{and} \qquad L_{\mathrm{SDP}}(Q) = L_{\mathrm{lin}}(Q) = +\infty. \tag{5.22}$$

For example, for the interior mode amplitude $Q = e_1 e_1^\top$ ($\operatorname{Tr}(QK_3) = 90 \neq 0$), the explicit mixed state $X_{\mathrm{mix}} = \operatorname{diag}(29, 90, 29)$ and pure state $Y_{\mathrm{pure}} = 29(e_0 + e_2)(e_0 + e_2)^\top$ satisfy $\mathcal{E}_B(X_{\mathrm{mix}}) \equiv \mathcal{E}_B(Y_{\mathrm{pure}})$ for all $R > 0$ while $\operatorname{Tr}(e_1 e_1^\top (X_{\mathrm{mix}} - Y_{\mathrm{pure}})) = 90$. On the five-radius design $R_{\mathrm{obs}} = \{0.5, 0.75, 1.0, 1.25, 1.5\}$, exact pure-state perturbation bounds certify

$$1.3793 \times 10^6 \ \le\ L_{\mathrm{pure}}(e_1 e_1^\top) \ \le\ 1.1718 \times 10^7 \ <\ \infty. \tag{5.23}$$

(ii) ***Type II** ($\operatorname{Tr}(QK_3) = 0$**, exact zero gap):***

$$L_{\mathrm{pure}}(Q) \ =\ L_{\mathrm{SDP}}(Q) \ =\ L_{\mathrm{lin}}(Q) \ <\ \infty. \tag{5.24}$$

In particular, for every boundary-ball extrapolation/interpolation target $Q = A_3(R_*)$ ($R_* > 0$) and for the endpoint mode amplitudes $Q = e_0 e_0^\top$ and $Q = e_2 e_2^\top$, the linear/SDP estimator attains the exact nonlinear pure-state minimax optimum with zero gap.

Proof. (i) Since $\ker \mathcal{A} = \operatorname{span}\{K_3\}$, $\operatorname{Tr}(QK_3) \neq 0 \iff Q \notin (\ker \mathcal{A})^\perp = \operatorname{span}\{A_3(R_k)\}$, so $L_{\mathrm{SDP}}(Q) = L_{\mathrm{lin}}(Q) = +\infty$ by Theorem 5.1(i), whereas $L_{\mathrm{pure}}(Q) < \infty$ by Theorem 5.5(ii). To certify the explicit finite upper bound in (5.23) uniformly over the *entire* measurement cube $y = \mathcal{A}(\Delta) \in [-1, 1]^5$, let V_5 be the 5×5 Vandermonde matrix $[V_5]_{jk} = R_j^{6+2k}$ ($1 \le j \le 5$, $0 \le k \le 4$) mapping the five radial monomial coefficients m_k of $\operatorname{Tr}(A_3(R)\Delta) = \sum_{k=0}^4 m_k R^{6+2k}$ to $y \in [-1, 1]^5$. For *every* $y \in [-1, 1]^5$ (not merely vertices), Hölder's inequality gives the exact uniform bound $|m_k| = |(V_5^{-1} y)_k| \le \|(V_5^{-1})_{k,\cdot}\|_1 =: M_k$, which bounds all matrix entries except Δ_{11} uniformly across $[-1, 1]^5$:

$$|\Delta_{00}| \le B_{00} := \frac{M_0}{\pi \alpha_{00}}, \qquad |\Delta_{01}| \le B_{01} := \frac{M_1}{2\pi \alpha_{01}}, \qquad |\widetilde{m}_2| \le B_{m_2} := \frac{M_2}{2\pi \alpha_{02}},$$
$$|\Delta_{12}| \le B_{12} := \frac{M_3}{2\pi \alpha_{12}}, \qquad |\Delta_{22}| \le B_{22} := \frac{M_4}{\pi \alpha_{22}},$$

where $\Delta_{02} = \widetilde{m}_2 - \kappa \Delta_{11}$ with $\kappa := \frac{\alpha_{11}}{2\alpha_{02}} = \frac{29}{90}$. Every $\Delta \in \mathcal{V}_{\le 2}$ has $\operatorname{rank} \Delta \le 2$, hence $\det \Delta = 0$; substituting $\Delta_{02} = \widetilde{m}_2 - \kappa \Delta_{11}$ and dividing by $-\kappa^2$ yields the exact monic cubic (`verify_03`)

$$\Delta_{11}^3 + p_2 \Delta_{11}^2 + p_1 \Delta_{11} + p_0 = 0,$$

with $p_2 = -\frac{2\widetilde{m}_2}{\kappa}$, $p_1 = \frac{\widetilde{m}_2^2 + 2\kappa \Delta_{01}\Delta_{12} - \Delta_{00}\Delta_{22}}{\kappa^2}$, and $p_0 = \frac{\Delta_{00}\Delta_{12}^2 + \Delta_{22}\Delta_{01}^2 - 2\widetilde{m}_2 \Delta_{01}\Delta_{12}}{\kappa^2}$. By the triangle inequality, throughout the entire cube $[-1, 1]^5$, $|p_2| \le p_{2,\max} := 2B_{m_2}/\kappa$, $|p_1| \le p_{1,\max} := (B_{m_2}^2 +$

$2\kappa B_{01}B_{12} + B_{00}B_{22})/\kappa^2$, and $|p_0| \le p_{0,\max} := (B_{00}B_{12}^2 + B_{22}B_{01}^2 + 2B_{m_2}B_{01}B_{12})/\kappa^2$. Because any root x of a monic cubic $x^3+p_2x^2+p_1x+p_0 = 0$ satisfies $|x| \le \max\big(3|p_2|,\ \sqrt{3|p_1|},\ (3|p_0|)^{1/3}\big)$ (otherwise $|p_2x^2 + p_1x + p_0| < \frac{1}{3}|x|^3 + \frac{1}{3}|x|^3 + \frac{1}{3}|x|^3 = |x|^3$), every $\Delta \in P \cap \mathcal{V}_{\le 2}$ obeys $|\Delta_{11}| \le \max\big(3p_{2,\max},\ \sqrt{3p_{1,\max}},\ (3p_{0,\max})^{1/3}\big) = 11{,}717{,}038.408\ldots < 1.1718 \times 10^7$ (evaluated in exact rational arithmetic in `verify_03`), certifying the outward-rounded upper bound (5.23) on the full cube $[-1,1]^5$.

(ii) Suppose $\mathrm{Tr}(QK_3) = 0$. Then $Q \in (\ker \mathcal{A})^\perp$, so $L_{\mathrm{lin}}(Q) < \infty$ is attained at some $\Delta_* \in \mathrm{Sym}^2(\mathbb{R}^3)$ with $\|\mathcal{A}(\Delta_*)\|_\infty \le 1$. Consider the one-parameter affine line inside the feasible polytope

$$\Delta(\tau) := \Delta_* + \tau K_3, \qquad \tau \in \mathbb{R}. \tag{5.25}$$

Because $K_3 \in \ker \mathcal{A}$ and $\mathrm{Tr}(QK_3) = 0$, every point on this line satisfies both $\mathcal{A}(\Delta(\tau)) = \mathcal{A}(\Delta_*)$ (hence $\|\mathcal{A}(\Delta(\tau))\|_\infty \le 1$) and $\mathrm{Tr}(Q\Delta(\tau)) = \mathrm{Tr}(Q\Delta_*) = L_{\mathrm{lin}}(Q)$! Now inspect the three ordered eigenvalues $\lambda_1(\tau) \le \lambda_2(\tau) \le \lambda_3(\tau)$ of $\Delta(\tau)$ as $\tau \to \pm\infty$:

- As $\tau \to +\infty$, $\Delta(\tau)/\tau \to K_3$, which has eigenvalues $(-29, +29, +90)$; hence $\lambda_2(\tau) \sim +29\tau \to +\infty$.
- As $\tau \to -\infty$, $\Delta(\tau)/|\tau| \to -K_3$, which has eigenvalues $(-90, -29, +29)$; hence $\lambda_2(\tau) \sim -29|\tau| \to -\infty$.

By continuity of the middle eigenvalue $\lambda_2(\tau)$ and the Intermediate Value Theorem, there exists $\tau_0 \in \mathbb{R}$ such that $\lambda_2(\tau_0) = 0$! At $\tau = \tau_0$, the ordered eigenvalues of $\Delta(\tau_0)$ satisfy $\lambda_1(\tau_0) \le 0 = \lambda_2(\tau_0) \le \lambda_3(\tau_0)$, which means $\Delta(\tau_0)$ has at most one positive and at most one negative eigenvalue—i.e. $\Delta(\tau_0) \in \mathcal{V}_{\le 2}$! By Theorem 5.1(iv), $L_{\mathrm{pure}}(Q) = L_{\mathrm{lin}}(Q) = L_{\mathrm{SDP}}(Q)$. □

Remark 5.7 (A cautionary lesson for randomized non-convex searches, and the $n \ge 4$ frontier). Notice how deceptive naive numerical sampling can be on Theorem 5.6(ii): on $R_{\mathrm{obs}} = \{0.5, 0.75, 1.0, 1.25, 1.5\}$, 10^5 random rank-two restarts recover only 14% of L_{lin} for the interpolation target $A_3(1.125)$ (2.54 vs. 17.44) and 55% for extrapolation $A_3(2.0)$ (2.13×10^5 vs. 3.85×10^5), because the exact $\mathcal{V}_{\le 2}$ maximizer $\Delta(\tau_0)$ lies at $\tau_0 \approx 1.56\times10^4$ far along the kernel line! Solving the cubic $\det(\Delta_*+\tau K_3) = 0$ explicitly yields exact pure-state pairs (u_*, v_*) matching L_{lin} to 12 significant digits (Table 5). For $n \ge 4$, the kernel dimension $\dim\ker\mathcal{A} = \binom{n-1}{2}$ matches the real codimension $\binom{n-1}{2}$ of the rank-≤ 2 symmetric determinantal variety in $\mathrm{Sym}^2(\mathbb{R}^n)$ (verified at $n = 4$: $\dim \mathrm{Sym}^2 = 10$, $\mathrm{rank}\,\mathcal{A} = 7$, $\dim\ker\mathcal{A} = 3$). Whether the affine subspace $\Delta_* + \ker\mathcal{A}$ always intersects the signature-$(1,1)$ component $\mathcal{V}_{\le 2}$ over $\mathbb{R}$ for $n \ge 4$, or whether a strict finite gap $L_{\mathrm{pure}}(Q) < L_{\mathrm{SDP}}(Q) < \infty$ first emerges at $n = 4$, remains an open problem in real algebraic geometry.

Table 5: Exact $n = 3$ dichotomy verification on $V_3^{\mathrm{even}} = \mathrm{span}\{E_0, \widetilde{E}_1, \widetilde{E}_2\}$ with five observation radii $R_{\mathrm{obs}} = \{0.5, 0.75, 1.0, 1.25, 1.5\}$ (`verify_03`). For all Type II targets ($\mathrm{Tr}(QK_3) = 0$), solving $\det(\Delta_* + \tau K_3) = 0$ constructs explicit pure states attaining L_{lin} to 10^{-11} relative precision.

Target Q	Type	$L_{\mathrm{SDP}} = L_{\mathrm{lin}}$	L_{pure} (exact IVT / certified)	Gap
Mode-1 amplitude $e_1e_1^\top$	I ($\mathrm{Tr} = 90$)	$+\infty$	$[1.3793 \times 10^6,\ 1.1718 \times 10^7]$	∞
Mode-0 amplitude $e_0e_0^\top$	II ($\mathrm{Tr} = 0$)	6,816.1660	6,816.1660 (0.99999999999)	0
Mode-2 amplitude $e_2e_2^\top$	II ($\mathrm{Tr} = 0$)	1,524,219.0357	1,524,219.0357 (0.99999999999)	0
Interpolate $A_3(1.125)$	II ($\mathrm{Tr} = 0$)	17.4355	17.4355 (0.99999999999)	0
Extrapolate $A_3(2.0)$	II ($\mathrm{Tr} = 0$)	385,332.4464	385,332.4464 (0.99999999999)	0

5.5 Universal Lagrange Scaling and Exponential Bulk-Depth Ill-Conditioning

Finally, we quantify how the minimax noise amplification $L_*(n, R_*)$ for predicting the canonical energy $\mathcal{E}_{B(R_*)}$ at an unobserved radius R_* from $m = 2n-1$ observed radii $\{R_1, \dots, R_{2n-1}\} \subset [R_{\min}, R_{\max}]$ scales with the modal truncation order n and the bulk-depth extrapolation ratio $R_*/R_{\max}$.

Theorem 5.8 (Universal Lagrange Identity, Linear/SDP Chebyshev Bound, and Pure-State Exponential Lower Bound)**.** *Consider V_n^{even} ($n \geq 1$) and any minimal set of $m = 2n-1$ distinct positive radii $R_1 < \cdots < R_{2n-1}$. Let $x_k := R_k^2$ and $x_* := R_*^2$. Then:*

(i) ***Exact universal Lagrange formula (linear/SDP recovery):*** *The linear/SDP optimal recovery constant for predicting the unobserved canonical energy $\mathcal{E}_{B(R_*)}(h)$ ($Q = A_n(R_*)$) is completely independent of the gravitational coefficients α_{ij} and is given in closed form by*

$$L_{\text{SDP}}\big(A_n(R_*)\big) = L_{\text{lin}}\big(A_n(R_*)\big) = \sum_{k=1}^{2n-1} \left(\frac{R_*}{R_k}\right)^6 |\ell_k(x_*)|, \qquad \ell_k(x) := \prod_{\substack{1\leq j\leq 2n-1 \\ j\neq k}} \frac{x - x_j}{x_k - x_j}. \tag{5.26}$$

For $n \in \{1, 2, 3\}$, Theorems 5.3 and 5.6(ii) guarantee that $L_{\text{pure}}(A_n(R_*)) = L_{\text{lin}}(A_n(R_*))$ identically.

(ii) ***Continuum Chebyshev exponential lower bound (linear/SDP recovery):*** *Even if canonical energy is observed at* every *radius in the continuum interval $R \in [R_{\min}, R_{\max}]$ ($x_{\min} = R_{\min}^2$, $x_{\max} = R_{\max}^2$) and extrapolated to $R_* > R_{\max}$, the linear/SDP stability constant obeys*

$$L_{\text{SDP}}^{\text{cont}}\big(A_n(R_*)\big) = L_{\text{lin}}^{\text{cont}}\big(A_n(R_*)\big) \;\geq\; \left(\frac{R_*}{R_{\max}}\right)^6 T_{2n-2}(\xi_*), \qquad \xi_* := \frac{2R_*^2 - R_{\max}^2 - R_{\min}^2}{R_{\max}^2 - R_{\min}^2} > 1, \tag{5.27}$$

where $T_d(\xi) = \cosh(d \operatorname{arccosh} \xi) \sim \frac{1}{2}\big(\xi + \sqrt{\xi^2 - 1}\big)^d$ is the degree-d Chebyshev polynomial of the first kind. Thus linear/SDP bulk-depth extrapolation amplifies worst-case noise at least exponentially in n, with asymptotic base $\beta(\xi_*) = \big(\xi_* + \sqrt{\xi_*^2 - 1}\big)^2$ per added mode (and the same bound holds for $L_{\text{pure}}^{\text{cont}}$ at $n \in \{1, 2, 3\}$).

(iii) ***Unconditional pure-state exponential lower bound for all $n \geq 1$:*** *Independently of whether $L_{\text{pure}} = L_{\text{SDP}}$ for $n \geq 4$, comparing $u = e_{n-1}$ against $v = 0$ ($\Delta = e_{n-1}e_{n-1}^\top \in \mathcal{V}_{\leq 2}$) yields for every $n \geq 1$ and $R_* > R_{\max}$ (even from continuum observations on $[R_{\min}, R_{\max}]$):*

$$L_{\text{pure}}^{\text{cont}}\big(A_n(R_*)\big) \;\geq\; \left(\frac{R_*}{R_{\max}}\right)^{4n+2}. \tag{5.28}$$

Consequently, under worst-case noise $\|\eta\|_\infty \leq \varepsilon$ across $[R_{\min}, R_{\max}]$, the minimax prediction error for the unobserved readout $\mathcal{E}_{B(R_*)}(h)$ is $\varepsilon L_{\text{SDP}}(A_n(R_*))$ for linear/SDP estimators and $\varepsilon L_{\text{pure}}(A_n(R_*))$ for arbitrary nonlinear pure-state estimators, both growing at least exponentially in the modal cutoff n for every $R_* > R_{\max}$ (with guaranteed base at least $(R_*/R_{\max})^4 > 1$ per mode for all nonlinear estimators, and $\beta(\xi_*)$ for linear/SDP estimators).

Proof. (i) Write $A_n(R) = \pi R^6 \sum_{s=0}^{2n-2} (R^2)^s H_s$, where the $2n-1$ anti-diagonal matrices H_s ($[H_s]_{ij} = \alpha_{ij}\delta_{i+j,s}$) are linearly independent because $\alpha_{ij} > 0$. Expanding $A_n(R_*) = \sum_{k=1}^{2n-1} c_k A_n(R_k)$ equates coefficients of each H_s ($s = 0, \dots, 2n-2$), yielding $x_*^s = \sum_{k=1}^{2n-1} c_k (R_k/R_*)^6 x_k^s$. By uniqueness of Lagrange interpolation of degree $2n-2$, $c_k = (R_*/R_k)^6 \ell_k(x_*)$, and (5.6) yields (5.26).

(ii) On the continuum $[R_{\min}, R_{\max}]$, any bounded signed Borel measure μ reproducing $A_n(R_*)$ satisfies $\int_{x_{\min}}^{x_{\max}} x^s\, x^3\, \mathrm{d}\mu(x) = x_*^{s+3}$ for $s = 0, \dots, 2n-2$, and $L_{\mathrm{lin}}^{\mathrm{cont}}(A_n(R_*)) = \inf \|\mu\|_{\mathrm{TV}}$. By duality, $L_{\mathrm{lin}}^{\mathrm{cont}}(A_n(R_*)) = \sup\{p(x_*)x_*^3 : \deg p \le 2n-2,\ \sup_{x\in[x_{\min},x_{\max}]} x^3|p(x)| \le 1\}$. Choosing the test polynomial $p(x) := x_{\max}^{-3}\, T_{2n-2}\big(\frac{2x-x_{\max}-x_{\min}}{x_{\max}-x_{\min}}\big)$ satisfies $x^3|p(x)| \le (x/x_{\max})^3 \le 1$ throughout $[x_{\min}, x_{\max}]$, while $x_*^3 p(x_*) = (R_*/R_{\max})^6 T_{2n-2}(\xi_*)$, proving (5.27). (For $n = 3$, the IVT argument of Theorem 5.6(ii) applies verbatim on the continuum feasible set $\{\Delta : \sup_{R\in[R_{\min},R_{\max}]} |\operatorname{Tr}(A_3(R)\Delta)| \le 1\}$ because $\operatorname{Tr}(A_3(R)K_3) \equiv 0$ for all $R > 0$, so $L_{\mathrm{pure}}^{\mathrm{cont}}(A_3(R_*)) = L_{\mathrm{lin}}^{\mathrm{cont}}(A_3(R_*))$.)

(iii) Setting $u = e_{n-1}$ and $v = 0$ gives $\Delta = e_{n-1}e_{n-1}^\top \in \mathcal{V}_{\le 2}$ with $\operatorname{Tr}(A_n(R)\Delta) = \pi\alpha_{n-1,n-1}R^{6+4(n-1)} = \pi\alpha_{n-1,n-1}R^{4n+2}$ strictly increasing in $R > 0$ (since $\alpha_{n-1,n-1} > 0$). Therefore

$$L_{\mathrm{pure}}^{\mathrm{cont}}\big(A_n(R_*)\big) \;\ge\; \frac{\operatorname{Tr}(A_n(R_*)\, e_{n-1}e_{n-1}^\top)}{\sup_{R\in[R_{\min},R_{\max}]} \operatorname{Tr}(A_n(R)\, e_{n-1}e_{n-1}^\top)} \;=\; \frac{\pi\alpha_{n-1,n-1}R_*^{4n+2}}{\pi\alpha_{n-1,n-1}R_{\max}^{4n+2}} \;=\; \left(\frac{R_*}{R_{\max}}\right)^{4n+2},$$

proving (5.28). □

Table 6: Exact universal Lagrange linear/SDP stability constant $L_{\mathrm{lin}}(A_n(R_*))$ on $[R_{\min}, R_{\max}] = [0.5, 1.5]$ for uniform vs. Chebyshev nodes in R^2 ($m = 2n-1$), compared with the linear/SDP continuum lower bound (5.27) and the unconditional pure-state lower bound (5.28) (`verify_03`). At $R_* = 2.0$, linear/SDP growth is $\beta(2.75) \approx 28.21\times$ per mode and pure-state guaranteed growth is $(4/3)^4 \approx 3.16\times$ per mode.

n	m	$L_{\mathrm{unif}}(1.25)$	$L_{\mathrm{Cheb}}(1.25)$	$L_{\mathrm{unif}}(2.0)$	$L_{\mathrm{Cheb}}(2.0)$	Lin. Cont. LB (2.0)	Pure LB (2.0)
1	1	0.3349	0.3349	5.6187	5.6187	5.6187	5.6187
2	3	66.71	28.06	1.4809×10^4	1.0100×10^4	79.36	17.76
3	5	1.000†	28.18	3.8533×10^5	1.7322×10^5	2.2364×10^3	56.12
4	7	44.54	25.76	1.4274×10^7	4.4489×10^6	6.3099×10^4	177.4
5	9	1.000†	15.71	6.4757×10^8	1.2406×10^8	1.7803×10^6	560.6
6	11	76.91	1.973	3.2956×10^{10}	3.4961×10^9	5.0231×10^7	1.772×10^3

†At $n = 3$ and $n = 5$, $R_* = 1.25$ coincides with a uniform node ($1.25 \in \{R_k\}$), so $L_{\mathrm{unif}}(1.25) = 1$.

Table 6 demonstrates the stark physical dichotomy between *interpolation* ($R_* \in [R_{\min}, R_{\max}]$, where Chebyshev-spaced radii keep $L_{\mathrm{Cheb}}(1.25) \le 28.2$ across all $n \le 6$) and *bulk-depth extrapolation* ($R_* > R_{\max}$, where even observing the entire continuum $[0.5, 1.5]$ forces noise amplification to grow by a factor of 28.21 per added mode at $R_* = 2.0$ and 238.25 per added mode at $R_* = 3.0$, reaching $L_{\mathrm{Cheb}}(2.0) = 3.50 \times 10^9$ and $L_{\mathrm{Cheb}}(3.0) = 2.16 \times 10^{15}$ by $n = 6$). In holographic terms, quadratic boundary-ball readouts reconstruct deep-bulk geometry stably *inside* the entanglement shadow covered by the observed ball radii, while extrapolating deeper into the bulk ($z > R_{\max}$) from finite-radius balls is severely exponentially ill-conditioned—analogous to analytic continuation from a UV cutoff in holographic renormalization and Euclidean reconstruction [7, 15].

6 Finite-Energy Localized AdS_4 Graviton Wavepackets Strictly Inside $\ker M_1$

All solutions analyzed in Sections 3–5—as well as the polynomial and oscillatory benchmarks of Rodgers [8]—lie in the spatially homogeneous subspace V_{hom}, whose perturbations do not decay as $|\mathbf{x}| \to \infty$ and therefore carry infinite global energy on the Poincaré patch. A natural physical question is whether the phenomena established above—strict membership in $\ker M_1$, positivity of $\mathcal{E}_B$, and all-balls non-identifiability at $t = 0$—are artifacts of non-normalizable homogeneous

waves, or whether they persist for *globally smooth, spatially localized, finite-energy* AdS_4 graviton wavepackets.

In this section we answer this question affirmatively by proving a general *Axial Potential Theorem* that constructs an infinite-dimensional family of exact Fefferman–Graham transverse-traceless (TT) vacuum gravitons lying strictly inside $\ker M_1$ for *every* rest-frame spherical boundary ball at all radii, centers, and times. Applying this construction to a complexified AdS_4 inversion solution yields explicit closed-form, globally smooth, rationally decaying ($O(r^{-9})$) wavepackets with exact analytical wedge canonical energies and equal finite global energy $\mathcal{E}_{\partial_t} = \frac{675\pi}{1024\, a^{13}} > 0$.

6.1 General Axial Potential Theorem in Fefferman–Graham TT Gauge

On Poincaré AdS_4, $\bar{g} = z^{-2}(-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}x^2 + \mathrm{d}y^2)$, consider a smooth scalar potential $\Phi(t, z, x, y)$ satisfying the massive-like scalar wave equation (the same radial operator governing homogeneous TT gravitons in (2.15), now with full 2D boundary Laplacian $\Delta_{\mathbf{x}} := \partial_x^2 + \partial_y^2$):

$$\left(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 - \frac{4}{z}\partial_z\right)\Phi(t, z, x, y) = 0. \tag{6.1}$$

From Φ we construct an axial (odd-parity / Regge–Wheeler-type) symmetric metric perturbation $h[\Phi]_{ab}$ in radial Fefferman–Graham gauge ($h_{zz} = h_{z\mu} = 0$) with vanishing energy component ($h_{tt} = 0$) by

$$\begin{aligned} h_{tx} &= z\,\partial_y \Delta_{\mathbf{x}} \Phi, & h_{ty} &= -z\,\partial_x \Delta_{\mathbf{x}} \Phi, \\ h_{xx} &= -h_{yy} = 2z\,\partial_t \partial_x \partial_y \Phi, & h_{xy} &= z\,\partial_t\big(\partial_y^2 - \partial_x^2\big)\Phi, \end{aligned} \tag{6.2}$$

with all remaining components $h_{tt} = h_{zz} = h_{zt} = h_{zx} = h_{zy} = 0$.

Theorem 6.1 (General Axial Potential Theorem Inside $\ker M_1$ for All Balls and All Times)**.** *Let $\Phi \in C^5\big(\mathbb{R} \times [0, \infty) \times \mathbb{R}^2\big)$ be a solution of (6.1) that is five-times continuously differentiable up to the conformal boundary $z = 0$ (thereby selecting the normalizable Frobenius branch $\Phi = \Phi^{(0)}(t, \mathbf{x}) + O(z^2)$ and excluding the singular z^{-3} branch), and let $h[\Phi]$ be defined by (6.2). Then:*

(i) ***Exact 4D normalizable vacuum TT solution:*** *$h[\Phi]_{\mu\nu} = zH_{\mu\nu}$ has $H_{\mu\nu} \in C^2(\mathbb{R} \times [0, \infty) \times \mathbb{R}^2)$ smooth up to $z = 0$ (preserving the flat conformal boundary metric identically), is traceless ($\bar{g}^{ab} h_{ab} = 0$) and transverse ($\bar{\nabla}^a h_{ab} = 0$), and satisfies* all ten *linearized 4D vacuum Einstein equations $D\mathcal{F}_{ab}[h[\Phi]] = 0$ identically on $\mathbb{R} \times (0, \infty) \times \mathbb{R}^2$.*

(ii) ***Exact pointwise total angular-derivative identity:*** *For* every *center $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$,* every *radius $R > 0$, and* every *time $t_0 \in \mathbb{R}$, introduce polar coordinates $x - x_0 = r\cos\varphi$, $y - y_0 = r\sin\varphi$ around $\mathbf{x}_0$. Then the induced trace of $h[\Phi]$ on the hemisphere $\widetilde{B}(R, \mathbf{x}_0, t_0) = \{t = t_0,\ z^2 + r^2 = R^2\}$ is an exact total azimuthal derivative of a C^2 2π-periodic function:*

$$\left(\delta^{ij} - \frac{(x^i - x_0^i)(x^j - x_0^j)}{R^2}\right) h_{ij}(t_0, z, x, y) \;=\; -\frac{2z}{R^2}\,\frac{\partial}{\partial\varphi}\Big[(r\partial_r - 1)\,\partial_t \Phi(t_0, z, x, y)\Big]. \tag{6.3}$$

(iii) ***Universal membership in*** $\ker M_1$***:*** *Consequently, for* every *rest-frame spherical boundary ball $B(R, \mathbf{x}_0, t_0)$ at all $R > 0$, $\mathbf{x}_0 \in \mathbb{R}^2$, and $t_0 \in \mathbb{R}$,*

$$DE_{B(R,\mathbf{x}_0,t_0)}\big[h[\Phi]\big] = 0 \qquad \textit{and} \qquad DA_{B(R,\mathbf{x}_0,t_0)}\big[h[\Phi]\big] = 0. \tag{6.4}$$

Proof. (i) Because $\Phi \in C^5(\mathbb{R} \times [0, \infty) \times \mathbb{R}^2)$, the third-derivative boundary coefficients $H_{\mu\nu} = h[\Phi]_{\mu\nu}/z$ in (6.2) lie in $C^2(\mathbb{R} \times [0, \infty) \times \mathbb{R}^2)$ (giving normalizable Fefferman–Graham falloff $h_{\mu\nu} = O(z)$ and fixed flat conformal boundary metric), and all second derivatives entering $D\mathcal{F}_{ab}[h[\Phi]]$

exist continuously on $z > 0$. Direct inspection of (6.2) gives $\bar{g}^{ab}h_{ab} = z^2(h_{xx} + h_{yy}) = 0$, $\partial_x h_{tx} + \partial_y h_{ty} = z(\partial_x\partial_y - \partial_y\partial_x)\Delta_{\mathbf{x}}\Phi = 0$, and

$$-\partial_t h_{tx} + \partial_x h_{xx} + \partial_y h_{xy} = z\partial_t\big[-\partial_y\Delta_{\mathbf{x}} + 2\partial_x^2\partial_y + \partial_y(\partial_y^2 - \partial_x^2)\big]\Phi = 0,$$

and similarly $-\partial_t h_{ty} + \partial_x h_{xy} + \partial_y h_{yy} = 0$, proving $\bar{\nabla}^a h_{ab} = 0$. Since the wave operator (6.1) commutes with all boundary derivatives $(\partial_t, \partial_x, \partial_y)$, every component $h_{\mu\nu} = zH_{\mu\nu}$ inherits $(\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 - \frac{4}{z}\partial_z)H_{\mu\nu} = 0$, which is equivalent to $D\mathcal{F}_{ab}[h] = 0$ in Fefferman–Graham TT gauge (independently verified for all 10 components of $D\mathcal{F}_{ab}[h[\Phi]]$ directly from the 4D Ricci tensor for arbitrary Φ in `verify_04`).

(ii) Writing $x - x_0 = r\cos\varphi$, $y - y_0 = r\sin\varphi$, since $h_{xx} = -h_{yy}$, the left-hand side of (6.3) equals

$$-\frac{r^2}{R^2}\Big[\cos(2\varphi)\,h_{xx} + \sin(2\varphi)\,h_{xy}\Big] = -\frac{zr^2}{R^2}\Big[2\cos(2\varphi)\,\partial_x\partial_y + \sin(2\varphi)(\partial_y^2 - \partial_x^2)\Big]\partial_t\Phi.$$

Transforming the differential operator into polar coordinates (r, φ) around $\mathbf{x}_0$ yields the identity

$$r^2\big[2\cos(2\varphi)\,\partial_x\partial_y + \sin(2\varphi)(\partial_y^2 - \partial_x^2)\big] \equiv 2\,\partial_\varphi(r\partial_r - 1)$$

(verified symbolically in `verify_04`), proving (6.3).

(iii) Since $h_{tt} \equiv 0$ and $H_{\mu\nu}$ is smooth up to $z = 0$, the linearized boundary energy density $T_{tt}^{(1)} = \frac{3}{16\pi}\lim_{z\to 0}(h_{tt}/z) \equiv 0$ everywhere, so $DE_{B(R,\mathbf{x}_0,t_0)}[h[\Phi]] = 0$. Integrating (6.3) over $\varphi \in [0, 2\pi]$ at fixed $z = \sqrt{R^2 - r^2}$ annihilates the total ∂_φ-derivative of the C^2 2π-periodic function $(r\partial_r - 1)\partial_t\Phi$ uniformly up to $z \to 0^+$ (with $h_{ij}/z = H_{ij}$ bounded on $\widetilde{B}$), whence $DA_{B(R,\mathbf{x}_0,t_0)}[h[\Phi]] = \frac{1}{2}\int_{\widetilde{B}}\sqrt{\det q_0}\,q_0^{AB}h_{AB} = 0$ identically. □

Remark 6.2 (Sharpness of boundary regularity and non-gauge criterion). Both hypotheses in Theorem 6.1—regularity up to $z = 0$ and nonvanishing of the boundary third derivatives—are sharp (`verify_04`):

(a) *Necessity of $z \to 0$ regularity:* The non-normalizable Frobenius solution $\Phi_{\mathrm{bad}}(t, z, x, y) := txy\,z^{-3}$ solves (6.1) identically on $z > 0$, yet produces $h[\Phi_{\mathrm{bad}}]_{xx} = -h[\Phi_{\mathrm{bad}}]_{yy} = 2z^{-2}$, a leading $O(z^{-2})$ conformal boundary-metric deformation violating fixed-boundary Fefferman–Graham falloff. Requiring $\Phi \in C^5(\mathbb{R} \times [0, \infty) \times \mathbb{R}^2)$ eliminates the z^{-3} branch cleanly.

(b) *Kernel of $\Phi \mapsto h[\Phi]$ and physical non-gauge criterion:* Nonzero solutions of (6.1) can satisfy $h[\Phi] \equiv 0$ identically—for instance, any spatially homogeneous solution $\Phi(t, z)$ (such as $t^2 + z^2/5$) or any static boundary-harmonic polynomial $\Phi(x, y)$ (such as $x^2 - y^2$). Conversely, $h[\Phi]$ is a genuinely physical, non-gauge radiative gravitational wave whenever its linearized Fefferman–Graham boundary stress tensor $T_{\mu\nu}^{(1)} = \frac{3}{16\pi}\lim_{z\to 0}(h[\Phi]_{\mu\nu}/z)$ does not vanish identically—i.e. whenever at least one of the four third-order boundary derivatives $\big(\partial_y\Delta_{\mathbf{x}}\Phi,\ -\partial_x\Delta_{\mathbf{x}}\Phi,\ 2\partial_t\partial_x\partial_y\Phi,\ \partial_t(\partial_y^2 - \partial_x^2)\Phi\big)\big|_{z=0}$ is nonzero—since $T_{\mu\nu}^{(1)}$ is gauge invariant under boundary-preserving diffeomorphisms.

6.2 Explicit Globally Smooth, Rationally Decaying Finite-Energy Wavepackets

To instantiate Theorem 6.1 with an explicit, globally smooth, localized, finite-energy bulk graviton wavepacket, observe that the bulk-to-boundary scalar propagator of conformal dimension $\Delta = 3$ in AdS_4—complexified in time by a width parameter $a > 0$ to remove the light-cone singularity—provides an elementary closed-form solution of (6.1):

$$\Phi_a(t, z, x, y) := \frac{1}{\big(-(t - ia)^2 + x^2 + y^2 + z^2\big)^3}, \qquad a > 0. \tag{6.5}$$

Indeed, setting $\sigma := -(t-ia)^2 + r^2 + z^2$ (with $r^2 = x^2 + y^2$), one verifies directly (`verify_04`) that

$$\left(\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 - \frac{4}{z}\partial_z\right)\sigma^{-3} = 12\sigma^{-5}\big[4\big(-(t-ia)^2 + r^2 + z^2\big) - 4\sigma\big] \equiv 0.$$

Moreover, at every real spacetime point $(t, z, x, y) \in \mathbb{R} \times [0,\infty) \times \mathbb{R}^2$, the imaginary part of σ is $2at$, so σ can vanish only if $t = 0$ *and* $a^2 + r^2 + z^2 = 0$, which is impossible for $a > 0$. Thus Φ_a is globally real-analytic on $\mathbb{R} \times [0,\infty) \times \mathbb{R}^2$ (in particular $\Phi_a \in C^\infty(\mathbb{R} \times [0,\infty) \times \mathbb{R}^2)$ with purely even powers of z around $z = 0$), verifying all regularity hypotheses of Theorem 6.1!

Taking real and imaginary parts of $h[\Phi_a]$ defines a conjugate pair of real, globally smooth, localized AdS_4 graviton wavepackets strictly inside $\ker M_1$:

$$h^{(a)}_{\mathrm{odd}} := \mathrm{Re}\big(h[\Phi_a]\big), \qquad h^{(a)}_{\mathrm{even}} := \mathrm{Im}\big(h[\Phi_a]\big). \tag{6.6}$$

At $t = 0$, setting $S := a^2 + x^2 + y^2 + z^2 \geq a^2 > 0$, their exact Cauchy data are purely rational functions (`verify_04`):

- **Time-odd localized wavepacket** $h^{(a)}_{\mathrm{odd}}$ ($h_{\mathrm{odd},ij}|_{t=0} = 0$, $\partial_t h_{\mathrm{odd},ti}|_{t=0} = 0$):

$$\begin{aligned} h^{(a)}_{\mathrm{odd},tx}\big|_{t=0} &= -\frac{96\,yz\,(3r^2 - 2a^2 - 2z^2)}{S^6}, & h^{(a)}_{\mathrm{odd},ty}\big|_{t=0} &= \frac{96\,xz\,(3r^2 - 2a^2 - 2z^2)}{S^6}, \\ \partial_t h^{(a)}_{\mathrm{odd},xx}\big|_{t=0} &= \frac{960\,xyz\,(r^2 + z^2 - 11a^2)}{S^7}, & \partial_t h^{(a)}_{\mathrm{odd},xy}\big|_{t=0} &= -\frac{480\,z(x^2 - y^2)(r^2 + z^2 - 11a^2)}{S^7}. \end{aligned} \tag{6.7}$$

- **Time-even localized wavepacket** $h^{(a)}_{\mathrm{even}}$ ($h_{\mathrm{even},ti}|_{t=0} = 0$, $\partial_t h_{\mathrm{even},ij}|_{t=0} = 0$):

$$\begin{aligned} h^{(a)}_{\mathrm{even},xx}\big|_{t=0} &= -\frac{960\,axyz}{S^6}, & h^{(a)}_{\mathrm{even},xy}\big|_{t=0} &= \frac{480\,az(x^2 - y^2)}{S^6}, \\ \partial_t h^{(a)}_{\mathrm{even},tx}\big|_{t=0} &= \frac{1920\,ayz(2r^2 - a^2 - z^2)}{S^7}, & \partial_t h^{(a)}_{\mathrm{even},ty}\big|_{t=0} &= -\frac{1920\,axz(2r^2 - a^2 - z^2)}{S^7}. \end{aligned} \tag{6.8}$$

Both wavepackets are manifestly non-gauge, with rapidly decaying smooth boundary stress tensors in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$:

$$T^{(1)}_{tx}\big[h^{(a)}_{\mathrm{odd}}\big]\Big|_{t=0} = -\frac{18\,y(3r^2 - 2a^2)}{\pi(a^2 + r^2)^6} = O(r^{-9}), \qquad T^{(1)}_{xx}\big[h^{(a)}_{\mathrm{even}}\big]\Big|_{t=0} = -\frac{180\,axy}{\pi(a^2 + r^2)^6} = O(r^{-10}). \tag{6.9}$$

6.3 Exact Closed-Form Wedge and Global Canonical Energies

Remarkably, because Φ_a is $SO(2)$-invariant in (x, y), on every *centered* spherical boundary ball $B(R, 0)$ at $t = 0$ the wavepackets (6.7)–(6.8) satisfy *pointwise* vanishing of the extremal-surface trace:

$$\mathrm{Tr}_{\widetilde{B}}\big(h^{(a)}_{\mathrm{even}}|_{t=0}\big) \equiv 0, \qquad \mathrm{Tr}_{\widetilde{B}}\big(\partial_t h^{(a)}_{\mathrm{odd}}|_{t=0}\big) \equiv 0, \qquad h^{(a)}_{\mathrm{odd},0\varrho}\big|_{t=0} \equiv 0, \qquad h^{(a)}_{\mathrm{even},\varrho\varrho}\big|_{t=0} \equiv 0, \tag{6.10}$$

while the azimuthal shift $h^{(a)}_{\mathrm{odd},0\varphi}|_{t=0}$ is independent of φ (hence divergence-free along $\widetilde{B}$). Consequently, for *every* radius $R > 0$, the centered hemisphere $\widetilde{B}(R, 0)$ remains *strictly extremal through first order* ($w_* \equiv 0$ and $\tau_* \equiv 0$) for both $h^{(a)}_{\mathrm{odd}}$ and $h^{(a)}_{\mathrm{even}}$!

Moreover, on the hemisphere $\varrho^2 = r^2 + z^2 = R^2$, the denominator $S = a^2 + \varrho^2 = a^2 + R^2$ is *constant* along $\widetilde{B}(R, 0)$! This makes both the fixed-hemisphere second area variation $A''_{\mathrm{ext}} = A_{gg}$ and the bulk Einstein integral $-I(R)$ computable in completely closed analytical form!

Theorem 6.3 (Exact Wedge Canonical Energies, Equal Global Energy, and Positivity of Localized Wavepackets)**.** *Let* $h^{(a)}_{\mathrm{odd}}$ *and* $h^{(a)}_{\mathrm{even}}$ *(*$a > 0$*) be the globally smooth, localized* AdS_4 *graviton wavepackets* (6.6) *in* $\ker M_1$*. By Poincaré scaling covariance,* $\mathcal{E}_{B(R,0)}(h^{(a)}) = a^{-12}\,\mathcal{E}_{B(R/a,0)}(h^{(1)})$*, and for* $a = 1$ *(setting* $u = z/\varrho \in [0,1]$*):*

(i) ***Exact extremal-surface second area variations on centered hemispheres*** $\widetilde{B}(R,0)$***:***

$$A''_{\mathrm{ext}}\big[h^{(1)}_{\mathrm{odd}}\big] = 0, \qquad A''_{\mathrm{ext}}\big[h^{(1)}_{\mathrm{even}}\big] = -\frac{409600\,\pi\,R^{10}}{77\,(1+R^2)^{12}}. \tag{6.11}$$

(ii) ***Exact radial Einstein energy densities:*** *Integrating* $D^2\mathcal{F}_{tt}[h,h]/u^2$ *over the polar cosine* $u \in [0,1]$ *at fixed hemisphere radius* $\varrho = \sqrt{r^2+z^2}$ *yields*

$$\int_0^1 \frac{D^2\mathcal{F}_{tt}\big[h^{(1)}_{\mathrm{odd}}\big]}{u^2}\,\mathrm{d}u = -\frac{12288\,\varrho^6\big(748\varrho^8 - 478\varrho^6 + 2893\varrho^4 - 1188\varrho^2 + 693\big)}{385\,(1+\varrho^2)^{14}}, \tag{6.12}$$

$$\int_0^1 \frac{D^2\mathcal{F}_{tt}\big[h^{(1)}_{\mathrm{even}}\big]}{u^2}\,\mathrm{d}u = \frac{61440\,\varrho^8\big(575\varrho^4 - 1034\varrho^2 + 231\big)}{77\,(1+\varrho^2)^{14}}. \tag{6.13}$$

(iii) ***Exact closed-form wedge canonical energies and universal rational difference:*** *For every* $R > 0$*, both* $\mathcal{E}_{B(R,0)}(h^{(1)}_{\mathrm{odd}})$ *and* $\mathcal{E}_{B(R,0)}(h^{(1)}_{\mathrm{even}})$ *are strictly positive, share the exact same transcendental term, and take the closed form*

$$\mathcal{E}_{B(R,0)}\big(h^{(1)}_{\mathrm{odd/even}}\big) = \frac{135\pi(5R^2-3)}{512R}\arctan(R) + \frac{3\pi\,Q_{\mathrm{odd/even}}(R^2)}{2562560\,(1+R^2)^{12}}, \tag{6.14}$$

where $Q_{\mathrm{odd}}(s)$ *and* $Q_{\mathrm{even}}(s)$ *are explicit degree-*12 *integer polynomials (recorded verbatim in* `verify_04_results.json`*) with small-*R *expansions*

$$\mathcal{E}_{B(R,0)}\big(h^{(1)}_{\mathrm{odd}}\big) = \frac{6144\pi}{35}R^8 - \frac{12288\pi}{7}R^{10} + \frac{10309632\pi}{1001}R^{12} + O(R^{14}), \tag{6.15}$$

$$\mathcal{E}_{B(R,0)}\big(h^{(1)}_{\mathrm{even}}\big) = \frac{30720\pi}{77}R^{10} - \frac{368640\pi}{91}R^{12} + O(R^{14}). \tag{6.16}$$

Remarkably, all $\arctan(R)$ *terms cancel identically in their difference, leaving the strictly positive pure rational function*

$$\mathcal{E}_{B(R,0)}\big(h^{(1)}_{\mathrm{odd}}\big) - \mathcal{E}_{B(R,0)}\big(h^{(1)}_{\mathrm{even}}\big) \;=\; \frac{6144\,\pi\,R^8\big(68R^4 - 39R^2 + 143\big)}{5005\,(1+R^2)^{12}} \;>\; 0 \qquad \forall\, R > 0. \tag{6.17}$$

(iv) ***Exact finite global Poincaré energy and direct symplectic verification:*** *Taking the* $R \to \infty$ *limit* $\mathcal{E}_{\partial_t}(h) = \lim_{R\to\infty} \frac{\mathcal{E}_{B(R,0)}(h)}{\pi R}$ *(since* $\xi_{B(R,0)}/(\pi R) \to \partial_t$*) gives exact* equal, strictly positive, finite *global energies for both wavepackets:*

$$\mathcal{E}_{\partial_t}\big(h^{(a)}_{\mathrm{odd}}\big) \;=\; \mathcal{E}_{\partial_t}\big(h^{(a)}_{\mathrm{even}}\big) \;=\; \frac{675\,\pi}{1024\,a^{13}} \;\approx\; \frac{2.0708740636}{a^{13}} \;>\; 0. \tag{6.18}$$

Furthermore, direct evaluation of the Hollands–Wald bulk symplectic pairing $W_{\Sigma_B}(h, \mathcal{L}_{\xi_B}h) = \int_{\Sigma_B}\sqrt{-\bar{g}}\,\omega^t(\bar{g}; h, \mathcal{L}_{\xi_B}h)\,\mathrm{d}z\,\mathrm{d}^2x$ *directly in Fefferman–Graham gauge (without any tangential gauge shift despite* $h_{\mathrm{odd},t\varphi} \neq 0$ *and* $h_{\mathrm{even},\varrho\varphi} \neq 0$*) agrees with* (6.14) *to* 2.7×10^{-14} *at* $R = 1$ *(*`verify_04`*).*

Proof. All expressions in (6.11)–(6.18) are derived and verified symbolically in `verify_04`. Specifically, on $\widetilde{B}(R,0)$ we have $h_{\mathrm{odd},ij}|_{t=0} = 0$ and $\tau_* = 0$, giving $A''_{\mathrm{ext}}[h^{(1)}_{\mathrm{odd}}] = 0$; for $h^{(1)}_{\mathrm{even}}$, $w_* = 0$ and only $h_{\theta\varphi} = R^2 \sin\theta\cos\theta\, h_{\mathrm{even},xy}|_{y=0}$ is nonzero on $\widetilde{B}(R,0)$, so $A''_{\mathrm{ext}}[h^{(1)}_{\mathrm{even}}] = -2\pi R^4 \int_0^1 u^4\big(480R^3u(1-u^2)/(1+R^2)^6\big)^2\,\mathrm{d}u = -\frac{409600\pi R^{10}}{77(1+R^2)^{12}}$. For the bulk integral, $SO(2)$ equivariance implies $D^2\mathcal{F}_{tt}[h,h]|_{t=0}$ depends only on (r,z); evaluating along the ray $y=0$ (after taking all x,y,t derivatives) and integrating in spherical coordinates (ϱ,u) with $S = 1+\varrho^2$ constant in u gives (6.12)–(6.13) immediately. The remaining ϱ-integral reduces via partial fractions in $w = 1+\varrho^2$ to the exact recurrence $J_1(R) = \arctan(R)$, $J_{m+1}(R) = \frac{R}{2m(1+R^2)^m} + \frac{2m-1}{2m}J_m(R)$, yielding (6.14)–(6.18) (independently cross-checked to $< 3\times10^{-14}$ by 48-node 2D Gauss–Legendre quadrature of both the bulk integral $-I(R)$ and the direct covariant phase-space symplectic flux $W_{\Sigma_B}(h,\mathcal{L}_{\xi_B}h)$). In (6.17), the quadratic $68s^2 - 39s + 143$ has discriminant $(-39)^2 - 4(68)(143) = -37375 < 0$, proving strict positivity for every $R > 0$. □

Table 7: Wedge canonical energy $\mathcal{E}_{B(R,0)}$ and normalized ratio $\mathcal{E}_{B(R,0)}/(\pi R) \to \mathcal{E}_{\partial_t} = \frac{675\pi}{1024} \approx 2.0708740636$ for the localized finite-energy wavepackets $h^{(1)}_{\mathrm{odd}}$ and $h^{(1)}_{\mathrm{even}}$ strictly inside $\ker M_1$ (`verify_04`).

R	$\mathcal{E}_{B(R,0)}(h^{(1)}_{\mathrm{odd}})$	$\mathcal{E}_{B(R,0)}(h^{(1)}_{\mathrm{odd}})/(\pi R)$	$\mathcal{E}_{B(R,0)}(h^{(1)}_{\mathrm{even}})$	$\mathcal{E}_{B(R,0)}(h^{(1)}_{\mathrm{even}})/(\pi R)$
0.25	4.65395375×10^{-3}	0.0059255980	6.50275218×10^{-4}	0.0008279561
0.50	2.82408534×10^{-1}	0.1797868565	1.40064400×10^{-1}	0.0891677663
1.00	3.18141646×10^{0}	1.0126763112	3.01947222×10^{0}	0.9611278597
2.00	1.10667733×10^{1}	1.7613316734	1.10624261×10^{1}	1.7606398018
5.00	3.17485133×10^{1}	2.0211731312	3.17485126×10^{1}	2.0211730893
20.00	1.29921680×10^{2}	2.0677677526	1.29921680×10^{2}	2.0677677526
100.00	6.50545239×10^{2}	2.0707498112	6.50545239×10^{2}	2.0707498112

6.4 Fixed-Time Non-Identifiability, Modular-Weighted Energy Rate, and Time-Shift Separation

Next, the localized wavepacket pair (6.6) demonstrates that fixed-time all-balls non-identifiability persists for globally smooth, finite-energy bulk excitations—and admits both an exact small-shift separation law at every centered radius and an exact holomorphic finite-time separation formula across all times $t_0 \in \mathbb{R}$.

Theorem 6.4 (Localized Fixed-Time Non-Identifiability, Modular-Weighted Energy Rate, and Time-Shift Separation)**.** *Define the time-reversed pair of globally smooth, finite-energy, localized graviton wavepackets*

$$h^{(\pm)}_{\mathrm{loc}} := h^{(a)}_{\mathrm{even}} \pm h^{(a)}_{\mathrm{odd}} \;\in\; \ker M_1, \tag{6.19}$$

which satisfy $h^{(-)}_{\mathrm{loc}}(t,z,\mathbf{x}) = h^{(+)}_{\mathrm{loc}}(-t,z,\mathbf{x})$ *and share the identical finite global Poincaré energy* $\mathcal{E}_{\partial_t}(h^{(\pm)}_{\mathrm{loc}}) = \frac{675\pi}{512\,a^{13}}$. *Then:*

(i) ***Exact all-balls non-identifiability at*** $t=0$***:*** *For* every *radius* $R>0$ *and* every *center* $\mathbf{x}_0 \in \mathbb{R}^2$ *on the* $t=0$ *slice,*

$$\mathcal{E}_{B(R,\mathbf{x}_0,0)}\big(h^{(+)}_{\mathrm{loc}}\big) \;=\; \mathcal{E}_{B(R,\mathbf{x}_0,0)}\big(h^{(-)}_{\mathrm{loc}}\big). \tag{6.20}$$

(ii) ***Exact rate of change of modular-weighted wedge canonical energy and small-shift separation at all radii:*** *On every centered ball* $B(R,0,t_0)$ *(*$R>0$, $t_0\in\mathbb{R}$*),* $SO(2)$*-equivariance preserves* $w_* \equiv 0$ *and* $\tau_* \equiv 0$ *for all* $t_0 \in \mathbb{R}$*. At* $t_0 = 0$*, the exact*

instantaneous rate of change of the modular-weighted wedge canonical energy (combining the Bianchi bulk radial momentum-flux integral and the extremal-surface second area derivative $-\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t_0}A''_{\mathrm{ext},B(R,0,t_0)}|_{t_0=0}$) takes the purely rational closed form (setting $\hat{R} := R/a$)

$$\dot{\mathcal{E}}_{B(R,0)}\big[h^{(\pm)}_{\mathrm{loc}}\big] := \frac{\mathrm{d}}{\mathrm{d}t_0}\mathcal{E}_{B(R,0,t_0)}\big(h^{(\pm)}_{\mathrm{loc}}\big)\Big|_{t_0=0} = \mp\,\frac{12288\,\pi\,\hat{R}^8\big(171\hat{R}^4-143\hat{R}^2+286\big)}{1001\,a^{13}\big(1+\hat{R}^2\big)^{13}}. \tag{6.21}$$

Because $171s^2-143s+286>0$ for all $s\in\mathbb{R}$ (discriminant $-175{,}175<0$), $\dot{\mathcal{E}}_{B(R,0)}[h^{(+)}_{\mathrm{loc}}]<0$ and $\dot{\mathcal{E}}_{B(R,0)}[h^{(-)}_{\mathrm{loc}}]>0$ for every $R>0$; in this precise modular-weighted wedge-energy sense, $h^{(+)}_{\mathrm{loc}}$ is strictly outward-flowing ($\dot{\mathcal{E}}_{B(R,0)}<0$) and $h^{(-)}_{\mathrm{loc}}$ is strictly inward-flowing ($\dot{\mathcal{E}}_{B(R,0)}>0$) across every centered wedge $\Sigma_{B(R,0)}$ at $t=0$. Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\Big[\mathcal{E}_{B(R,0,t_0)}\big(h^{(-)}_{\mathrm{loc}}\big)-\mathcal{E}_{B(R,0,t_0)}\big(h^{(+)}_{\mathrm{loc}}\big)\Big]\Big|_{t_0=0} = \frac{24576\,\pi\,\hat{R}^8\big(171\hat{R}^4-143\hat{R}^2+286\big)}{1001\,a^{13}\big(1+\hat{R}^2\big)^{13}} \;>\; 0, \tag{6.22}$$

so for every fixed centered radius $R>0$, sufficiently small nonzero time shifts t_0 separate $h^{(-)}_{\mathrm{loc}}$ from $h^{(+)}_{\mathrm{loc}}$ linearly with zero blind-spot radii.

(iii) ***Exact finite-time holomorphic separation identity and numerical corroboration:*** *Extending the centered canonical energy $\mathbb{C}$-bilinearly to $h^{(a)}_c := h[\Phi_a] = h^{(a)}_{\mathrm{odd}} + i\,h^{(a)}_{\mathrm{even}}$ and continuing (6.17) holomorphically to $\alpha = a+it_0$ in the right half-plane $\{\mathrm{Re}\,\alpha>0\}$ yields the exact finite-time identity valid for* all *$R>0$, $a>0$, and $t_0\in\mathbb{R}$:*

$$\begin{aligned}&\mathcal{E}_{B(R,0,t_0)}\big(h^{(-)}_{\mathrm{loc}}\big)-\mathcal{E}_{B(R,0,t_0)}\big(h^{(+)}_{\mathrm{loc}}\big)\\ &\qquad = -2\,\mathrm{Im}\left[\frac{6144\,\pi\,R^8\big(68R^4-39R^2(a+it_0)^2+143(a+it_0)^4\big)}{5005\,\big((a+it_0)^2+R^2\big)^{12}}\right].\end{aligned} \tag{6.23}$$

In particular, for $a=1$, $R=1$, $\mathbf{x}_0=0$, (6.23) gives exact rational-π separations $+0.254565065106\ldots>0$ at $t_0=\frac{1}{10}$ and $-0.298511156745\ldots<0$ at $t_0=\frac{1}{2}$ (showing a finite-time sign crossover from wavepacket dispersion). As independent numerical corroboration, direct evaluation of the Hollands–Wald bulk symplectic pairing $W_{\Sigma_{B(1,0,t_0)}}(h,\mathcal{L}_{\xi_B}h)$ via 64-node 2D Gauss–Legendre quadrature (`verify_04`, with 48-vs-64-node differences $<10^{-11}$) confirms (6.21) at $R=1$ to 1.4×10^{-9} relative error (exact value $-\frac{471\pi}{1001}\approx-1.4782119279$) and agrees with (6.23) to $<10^{-15}$:

$$\begin{aligned}
t_0=0.1:\quad &\mathcal{E}_{B(1,0,0.1)}\big(h^{(+)}_{\mathrm{loc}}\big)\approx 5.975281155328,\quad \mathcal{E}_{B(1,0,0.1)}\big(h^{(-)}_{\mathrm{loc}}\big)\approx 6.229846220434,\\
&\mathcal{E}_{B(1,0,0.1)}\big(h^{(-)}_{\mathrm{loc}}\big)-\mathcal{E}_{B(1,0,0.1)}\big(h^{(+)}_{\mathrm{loc}}\big)\approx +0.254565065106>0;\\
t_0=0.5:\quad &\mathcal{E}_{B(1,0,0.5)}\big(h^{(+)}_{\mathrm{loc}}\big)\approx 4.149244508691,\quad \mathcal{E}_{B(1,0,0.5)}\big(h^{(-)}_{\mathrm{loc}}\big)\approx 3.850733351947,\\
&\mathcal{E}_{B(1,0,0.5)}\big(h^{(-)}_{\mathrm{loc}}\big)-\mathcal{E}_{B(1,0,0.5)}\big(h^{(+)}_{\mathrm{loc}}\big)\approx -0.298511156745<0.
\end{aligned} \tag{6.24}$$

Proof. (i) Under the time-reflection isometry $\mathcal{T}: t\mapsto -t$ of Poincaré AdS_4, every rest-frame ball $B(R,\mathbf{x}_0,0)$ is mapped onto itself while $\mathcal{T}_*\xi_B=-\xi_B$. Since the canonical energy $\mathcal{E}_B(h)=W_{\Sigma_B}(h,\mathcal{L}_{\xi_B}h)$ is invariant under $\mathcal{T}$ (reversing both the orientation of ξ_B and the future normal to Σ_B), $\mathcal{E}_{B(R,\mathbf{x}_0,0)}(\mathcal{T}^*h)=\mathcal{E}_{B(R,\mathbf{x}_0,0)}(h)$ for every h, every $R>0$, and every $\mathbf{x}_0\in\mathbb{R}^2$. Since $\mathcal{T}^*h^{(a)}_{\mathrm{even}}=+h^{(a)}_{\mathrm{even}}$ and $\mathcal{T}^*h^{(a)}_{\mathrm{odd}}=-h^{(a)}_{\mathrm{odd}}$, we have $\mathcal{T}^*h^{(+)}_{\mathrm{loc}}=h^{(-)}_{\mathrm{loc}}$, proving (6.20).

(ii) By Poincaré scaling covariance, $\mathcal{E}_{B(R,0,t_0)}(h^{(a)})=a^{-12}\mathcal{E}_{B(R/a,0,t_0/a)}(h^{(1)})$, so it suffices to compute at $a=1$. Because $\Phi_1(t,z,r)$ is $SO(2)$-invariant for *all* $t\in\mathbb{R}$, on every centered hemisphere $\widetilde{B}(R,0,t_0)$ the pulled-back components satisfy $h_{t\varrho}=0$, $h_{\varrho\varrho}=0$, $\mathrm{Tr}_{\widetilde{B}}(h)=0$,

$\mathrm{Tr}_{\widetilde{B}}(\partial_t h) = 0$, and $\partial_\varphi h_{t\varphi} = 0$ identically for every $t_0 \in \mathbb{R}$; hence $w_* \equiv 0$ and $\tau_* \equiv 0$ at all t_0, and $A''_{\mathrm{ext},B(R,0,t_0)}[h^{(+)}_{\mathrm{loc}}] = -2\pi R^4 \int_0^1 u^4 \big(h^{(+)}_{\mathrm{loc},xy}(t_0, Ru, R\sqrt{1-u^2}, 0)\big)^2 \,\mathrm{d}u$. Differentiating at $t_0 = 0$ yields the exact surface derivative (`verify_04`)

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\left(-\frac{1}{4}A''_{\mathrm{ext},B(R,0,t_0)}\big[h^{(+)}_{\mathrm{loc}}\big]\right)\Big|_{t_0=0} = -\frac{204800\,\pi\,R^{10}\,(R^2-11)}{77\,(1+R^2)^{13}}.$$

For the bulk term $-I_{B(R,0,t_0)}[h^{(+)}_{\mathrm{loc}}]$, the background Bianchi identity $\bar\nabla^a D^2\mathcal{F}_{at} = 0$ reduces on Poincaré AdS_4 to the exact local conservation law $\partial_t(z^{-2}D^2\mathcal{F}_{tt}) = \partial_z(z^{-2}D^2\mathcal{F}_{zt}) + \partial_x(z^{-2}D^2\mathcal{F}_{xt}) + \partial_y(z^{-2}D^2\mathcal{F}_{yt})$. Integrating against $\xi^t_B = \frac{\pi}{R}(R^2-\varrho^2)$ over Σ_B and integrating by parts (with vanishing boundary terms since $\xi^t_B|_{\widetilde{B}} = 0$ and $z^{-2}D^2\mathcal{F}_{zt} = O(z^3) \to 0$) gives

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\big(-I_{B(R,0,t_0)}[h^{(+)}_{\mathrm{loc}}]\big)\Big|_{t_0=0} = -\frac{1}{4R}\int_{\Sigma_B}\frac{z\,D^2\mathcal{F}_{zt} + x\,D^2\mathcal{F}_{xt} + y\,D^2\mathcal{F}_{yt}}{z^2}\,\mathrm{d}z\,\mathrm{d}^2x.$$

Direct 4D symbolic evaluation of $D^2\mathcal{F}_{zt}$ and $D^2\mathcal{F}_{xt}$ at $t = 0, y = 0$ (`verify_04`) yields the radial flux density

$$\int_0^1 \frac{z\,D^2\mathcal{F}_{zt} + x\,D^2\mathcal{F}_{xt}}{u^2}\,\mathrm{d}u = \frac{8192\,\varrho^8\big(137\varrho^6 - 7892\varrho^4 + 4565\varrho^2 + 594\big)}{77\,(1+\varrho^2)^{14}},$$

whose global integral over $\varrho \in [0,\infty)$ vanishes identically (confirming global energy conservation $\dot{\mathcal{E}}_{\partial_t} = 0$). Integrating over $\varrho \in [0,R]$ via the exact $J_m(R)$ recurrence cancels all $\arctan(R)$ terms identically and combines with the surface derivative to produce (6.21) for $h^{(+)}_{\mathrm{loc}}$; time-reversal $t_0 \mapsto -t_0$ reverses the sign for $h^{(-)}_{\mathrm{loc}}$, establishing (6.22).

(iii) For any complex parameter α in the connected open right half-plane $\Omega := \{\alpha \in \mathbb{C} : \mathrm{Re}\,\alpha > 0\}$, define $\Phi_\alpha(\tilde t, z, \mathbf{x}) := (-(\tilde t - i\alpha)^2 + r^2 + z^2)^{-3}$. At $\tilde t = 0$, the denominator $\alpha^2 + \varrho^2$ vanishes only at $\alpha = \pm i\varrho \in i\mathbb{R}$, so $\alpha^2 + \varrho^2 \neq 0$ uniformly on $\Omega \times \Sigma_{B(R,0,0)}$. Because $w_* \equiv 0$ and $\tau_* \equiv 0$ on $\widetilde{B}(R,0,t_0)$ for every real t_0 and every real combination of $h^{(a)}_{\mathrm{even}}, h^{(a)}_{\mathrm{odd}}$, the $\mathbb{C}$-bilinear canonical-energy functional

$$Q_R(\alpha) := -\frac{1}{4}A_{gg,\widetilde{B}(R,0,0)}\big[h[\Phi_\alpha], h[\Phi_\alpha]\big] - \frac{1}{8\pi}\int_{\Sigma_{B(R,0,0)}} \xi^t_B\, D^2\mathcal{F}_{tt}\big[h[\Phi_\alpha], h[\Phi_\alpha]\big]\,\frac{\mathrm{d}z\,\mathrm{d}^2x}{z^2}$$

has an integrand continuous on the compact half-ball $\Sigma_{B(R,0,0)}$ (with $D^2\mathcal{F}_{tt}/z^2 = O(z^2)$ uniformly on compact subsets of Ω) and holomorphic in $\alpha \in \Omega$, hence $Q_R(\alpha)$ is holomorphic on Ω. On the positive real axis $\alpha \in (0,\infty) \subset \Omega$, Theorem 6.3 (eq. (6.17) with scaling α^{-12}) gives

$$Q_R(\alpha) = E^{(\alpha)}_{oo}(R,0) - E^{(\alpha)}_{ee}(R,0) = \frac{6144\,\pi\,R^8\big(68R^4 - 39R^2\alpha^2 + 143\alpha^4\big)}{5005\,\big(\alpha^2+R^2\big)^{12}}. \tag{6.25}$$

By the Identity Theorem for holomorphic functions on Ω, (6.25) holds identically for all $\alpha \in \Omega$. Setting $\alpha = a + it_0 \in \Omega$ uses the time-shift identity $\Phi_a(\tilde t + t_0, z, \mathbf{x}) = (-(\tilde t - i(a+it_0))^2 + r^2 + z^2)^{-3} = \Phi_{a+it_0}(\tilde t, z, \mathbf{x})$, so $h[\Phi_{a+it_0}](\tilde t) = h^{(a)}_{\mathrm{odd}}(\tilde t + t_0) + i\,h^{(a)}_{\mathrm{even}}(\tilde t + t_0)$ and hence by time-translation invariance

$$Q_R(a+it_0) = \Big[\mathcal{E}_{B(R,0,t_0)}\big(h^{(a)}_{\mathrm{odd}}\big) - \mathcal{E}_{B(R,0,t_0)}\big(h^{(a)}_{\mathrm{even}}\big)\Big] + 2i\,\mathcal{E}_{B(R,0,t_0)}\big(h^{(a)}_{\mathrm{even}}, h^{(a)}_{\mathrm{odd}}\big). \tag{6.26}$$

Since $\mathcal{E}(h^{(-)}_{\mathrm{loc}}) - \mathcal{E}(h^{(+)}_{\mathrm{loc}}) = -4\,\mathcal{E}(h^{(a)}_{\mathrm{even}}, h^{(a)}_{\mathrm{odd}}) = -2\,\mathrm{Im}\,Q_R(a+it_0)$, this proves (6.23) identically (and differentiating (6.23) at $t_0 = 0$ independently reproduces (6.22) in `verify_04`). The numerical values (6.24) are obtained by direct 64-node 2D Gauss–Legendre evaluation of $W_{\Sigma_{B(1,0,t_0)}}(h^{(\pm)}_{\mathrm{loc}}, \mathcal{L}_{\xi_B} h^{(\pm)}_{\mathrm{loc}})$ (`verify_04`), corroborating both (6.21) (via central differences to 1.4×10^{-9}) and (6.23) (to $< 10^{-15}$). □

6.5 Constructive Three-Ball Reconstruction of Localized Wavepackets, Minimality, and Stability

Fix the packet width $a > 0$ (held fixed and known throughout this subsection) and consider the two-dimensional real subspace of globally smooth, finite-energy localized graviton wavepackets strictly inside $\ker M_1$:

$$V_{\mathrm{loc}}^{(a)} := \Big\{ h(u,w) := u\, h_{\mathrm{even}}^{(a)} + w\, h_{\mathrm{odd}}^{(a)} : (u,w) \in \mathbb{R}^2 \Big\} \;\subset\; \ker M_1. \tag{6.27}$$

For any centered boundary ball $B(R,0,t_c)$ ($R > 0$, $t_c \in \mathbb{R}$), the canonical energy on $V_{\mathrm{loc}}^{(a)}$ is a quadratic form in (u,w) and therefore a linear functional of the lifted rank-one coordinate vector $z(u,w) := (u^2,\, w^2,\, uw)^{\top} \in \mathbb{R}^3$:

$$\mathcal{E}_{B(R,0,t_c)}\big(u\, h_{\mathrm{even}}^{(a)} + w\, h_{\mathrm{odd}}^{(a)}\big) = E_{ee}^{(a)}(R,t_c)\, u^2 + E_{oo}^{(a)}(R,t_c)\, w^2 + 2\, E_{eo}^{(a)}(R,t_c)\, uw = \mathrm{Tr}\big(A^{(a)}(R,t_c)\, X\big), \tag{6.28}$$

where $X = \left(\begin{smallmatrix} z_1 & z_3 \\ z_3 & z_2 \end{smallmatrix}\right) \in \mathrm{Sym}^2(\mathbb{R}^2)$ (with $X = \binom{u}{w}\binom{u}{w}^{\top}$ on physical pure states) and $A^{(a)}(R,t_c) := \left(\begin{smallmatrix} E_{ee}^{(a)} & E_{eo}^{(a)} \\ E_{eo}^{(a)} & E_{oo}^{(a)} \end{smallmatrix}\right)$. By polarization, the mixed coefficient $E_{eo}^{(a)}(R,t_c) := \mathcal{E}_{B(R,0,t_c)}(h_{\mathrm{even}}^{(a)}, h_{\mathrm{odd}}^{(a)})$ satisfies

$$2\, E_{eo}^{(a)}(R,t_c) = \frac{1}{2}\Big[\mathcal{E}_{B(R,0,t_c)}\big(h_{\mathrm{even}}^{(a)} + h_{\mathrm{odd}}^{(a)}\big) - \mathcal{E}_{B(R,0,t_c)}\big(h_{\mathrm{even}}^{(a)} - h_{\mathrm{odd}}^{(a)}\big)\Big] = \mathrm{Im}\; Q_R(a + it_c), \tag{6.29}$$

with exact Poincaré scaling $E_{\alpha\beta}^{(a)}(R,t_c) = a^{-12} E_{\alpha\beta}^{(1)}(R/a,\, t_c/a)$ ($\alpha,\beta \in \{e,o\}$).

We now prove that *three* centered boundary balls—two at $t_c = 0$ and one at a nonzero time shift—are both *necessary* (among PSD scalar quadratic readouts) and *sufficient* to reconstruct every state $h(u,w) \in V_{\mathrm{loc}}^{(a)}$ (including boundary states with $u = 0$ or $w = 0$) uniquely up to overall sign, and we exhibit a completely explicit, analytically certified three-ball design with exact inverse and exact rank-one minimax stability constants.

Theorem 6.5 (Constructive Three-Ball Reconstruction, Minimality, and Minimax Stability for Fixed-Width Localized Wavepackets)**.** *Fix the wavepacket width $a > 0$ and consider the two-dimensional localized family $V_{\mathrm{loc}}^{(a)} = \{u\, h_{\mathrm{even}}^{(a)} + w\, h_{\mathrm{odd}}^{(a)} : (u,w) \in \mathbb{R}^2\} \subset \ker M_1$.*

(i) ***Existence and explicit exact three-ball measurement design:*** *Select two centered $t_c = 0$ balls $B(R_1,0,0)$, $B(R_2,0,0)$ ($0 < R_1 < R_2$) and one time-shifted centered ball $B(R_3,0,t_c)$ ($R_3 > 0$, $t_c \neq 0$). By time-reflection invariance ($E_{eo}^{(a)}(R,0) \equiv 0$), the lifted 3×3 measurement matrix mapping $z = (u^2, w^2, uw)^{\top}$ to $y = \big(\mathcal{E}_{B(R_1,0,0)}, \mathcal{E}_{B(R_2,0,0)}, \mathcal{E}_{B(R_3,0,t_c)}\big)^{\top}$ is block lower-triangular:*

$$M_3^{(a)}(R_1,R_2;\, R_3,t_c) = \begin{pmatrix} M_{2\times 2}^{(a)} & 0 \\ r^{(a)\top} & c_3^{(a)} \end{pmatrix} := \begin{pmatrix} E_{ee}^{(a)}(R_1,0) & E_{oo}^{(a)}(R_1,0) & 0 \\ E_{ee}^{(a)}(R_2,0) & E_{oo}^{(a)}(R_2,0) & 0 \\ E_{ee}^{(a)}(R_3,t_c) & E_{oo}^{(a)}(R_3,t_c) & 2E_{eo}^{(a)}(R_3,t_c) \end{pmatrix}. \tag{6.30}$$

Because $E_{ee}^{(1)}(\hat{R})/E_{oo}^{(1)}(\hat{R}) \sim \frac{25}{11}\hat{R}^2 \to 0$ as $\hat{R} \to 0^+$ and $\to 1$ as $\hat{R} \to \infty$, pairs (R_1,R_2) with $\det M_{2\times 2}^{(a)} \neq 0$ exist; and by (6.21) and (6.29), $\partial_{t_c}(2E_{eo}^{(a)}(R_3,t_c))\big|_{t_c=0} = \dot{\mathcal{E}}_{B(R_3,0)}[h_{\mathrm{loc}}^{(+)}] < 0$, so $c_3^{(a)} \neq 0$ for every sufficiently small $t_c \neq 0$ at any $R_3 > 0$. Concretely, for the explicit scaled design

$$(R_1,t_1) = \big(\tfrac{1}{2}a,\, 0\big), \qquad (R_2,t_2) = (a,\, 0), \qquad (R_3,t_3) = \big(\tfrac{1}{2}a,\, \tfrac{1}{2}a\big), \tag{6.31}$$

every entry of $M_3^{(1)} = a^{12} M_3^{(a)}$ *evaluates in exact elementary closed form (*`verify_04`*):*

$$\begin{aligned}
E_{ee}^{(1)}\big(\tfrac{1}{2},0\big) &= -\frac{945\pi}{1024}\arctan\!\left(\frac{1}{2}\right) + \frac{337{,}810{,}189{,}533\,\pi}{715{,}000{,}000{,}000},\\
E_{oo}^{(1)}\big(\tfrac{1}{2},0\big) &= -\frac{945\pi}{1024}\arctan\!\left(\frac{1}{2}\right) + \frac{12{,}957{,}227{,}999{,}799\,\pi}{25{,}025{,}000{,}000{,}000},\\
E_{ee}^{(1)}(1,0) &= \frac{135\pi^2}{1024} + \frac{1095\pi}{2002}, \qquad E_{oo}^{(1)}(1,0) = \frac{135\pi^2}{1024} + \frac{5991\pi}{10010},\\
E_{ee}^{(1)}\big(\tfrac{1}{2},\tfrac{1}{2}\big) &= -\frac{405\pi^2}{2048} + \frac{203{,}787\,\pi}{320{,}320}, \qquad E_{oo}^{(1)}\big(\tfrac{1}{2},\tfrac{1}{2}\big) = -\frac{405\pi^2}{2048} + \frac{102{,}681\,\pi}{160{,}160},
\end{aligned} \tag{6.32}$$

with exact mixed coefficient

$$c_3^{(1)} = 2\,E_{eo}^{(1)}\big(\tfrac{1}{2},\tfrac{1}{2}\big) = -\frac{27\pi}{1760} \;\approx\; -0.048194887299 \;\neq\; 0, \tag{6.33}$$

and exact sign-certified 2×2 *determinant (a sum of three strictly negative terms)*

$$\frac{\det M_{2\times 2}^{(1)}\big(\tfrac{1}{2},1\big)}{\pi^2} = -\frac{3483}{73216}\arctan\!\left(\frac{1}{2}\right) - \frac{5{,}308{,}416}{888{,}671{,}875}\,\pi - \frac{411{,}898{,}871{,}817}{963{,}462{,}500{,}000{,}000} \;<\; 0 \tag{6.34}$$

($\det M_{2\times 2}^{(1)} \approx -0.407121536521$*, whence* $\det M_3^{(1)} = c_3^{(1)} \det M_{2\times 2}^{(1)} \approx +0.019621176570 > 0$*). Independent* 64*-node 2D Gauss–Legendre evaluation of the direct Hollands–Wald bulk symplectic pairing* $W_{\Sigma_{B(\frac{1}{2},0,\frac{1}{2})}}$ *corroborates all three shifted-ball entries* (6.32)–(6.33) *to* $< 1.4\times 10^{-16}$*.*

(ii) ***Exact recovery of*** (u,w) ***up to global sign (all states, including*** $u=0$ ***or*** $w=0$***):*** *Whenever* $\det M_{2\times 2}^{(a)} \neq 0$ *and* $c_3^{(a)} \neq 0$*, the matrix* (6.30) *is invertible with exact block inverse*

$$\big(M_3^{(a)}\big)^{-1} = \begin{pmatrix} \big(M_{2\times 2}^{(a)}\big)^{-1} & 0 \\ -\dfrac{1}{c_3^{(a)}}\, r^{(a)\top}\big(M_{2\times 2}^{(a)}\big)^{-1} & \dfrac{1}{c_3^{(a)}} \end{pmatrix}, \tag{6.35}$$

recovering $(z_1,z_2,z_3)^\top = (M_3^{(a)})^{-1} y$ *uniquely (first solving* $M_{2\times 2}^{(a)}(z_1,z_2)^\top = (y_1,y_2)^\top$*, then subtracting the diagonal shifted-ball contributions via* $z_3 = (y_3 - E_{ee}^{(a)}(R_3,t_c)z_1 - E_{oo}^{(a)}(R_3,t_c)z_2)/c_3^{(a)}$*). Every physical state* $(u,w)\in\mathbb{R}^2$ *is then recovered uniquely up to global sign* $\pm(u,w)$*: if* $z_1 = 0$*, then* $(u,w) = (0,\pm\sqrt{z_2})$*; if* $z_2 = 0$*, then* $(u,w) = (\pm\sqrt{z_1},0)$*; if* $z_1 > 0$ *and* $z_2 > 0$*, choosing* $u = +\sqrt{z_1}$ *fixes the overall sign and yields* $w = z_3/\sqrt{z_1} = \operatorname{sgn}(z_3)\sqrt{z_2}$*.*

(iii) ***Minimality among PSD scalar quadratic readouts*** *(*$m \geq 3$***):*** *No two positive-semidefinite scalar quadratic readouts* $A_1, A_2 \in \operatorname{Sym}^2(\mathbb{R}^2)$ *(*$A_1, A_2 \succeq 0$*) can identify* $(u,w)\in\mathbb{R}^2$ *up to global sign; hence three readouts are strictly minimal within the class of PSD scalar quadratic readouts. For the chosen three-ball design* (6.31)*, all three physical readout matrices carry direct, self-contained positive-definiteness certificates:* $A^{(a)}(R,0) = \operatorname{diag}(E_{ee}^{(a)}(R,0), E_{oo}^{(a)}(R,0)) \succ 0$ *on every centered* $t_c = 0$ *ball by Theorem 6.3, and* $A^{(a)}(\frac{1}{2}a,\frac{1}{2}a) \succ 0$ *by certified positivity of* $E_{ee}^{(1)}(\frac{1}{2},\frac{1}{2}) > 0$*,* $E_{oo}^{(1)}(\frac{1}{2},\frac{1}{2}) > 0$*, and* $\det A^{(1)}(\frac{1}{2},\frac{1}{2}) \approx 2.345871\times 10^{-3} > 0$*. Consequently, three balls are minimal among any collection of boundary-ball canonical-energy readouts on* $V_{\mathrm{loc}}^{(a)}$ *that are positive semidefinite (for instance under the holographic relative-entropy identification [7]; we do not supply a separate classical proof of positivity for arbitrary off-center or arbitrary-time balls on the full classical sector here).*

(iv) ***Exact ℓ_∞ minimax stability, rank-one zero-gap certification, $1/|t_c|$ amplification, and a-scaling:*** *Under additive measurement noise $y = M_3^{(a)} z + \eta$ with $\|\eta\|_\infty \le \varepsilon$, the exact coordinatewise minimax recovery errors for $z_i \in \{u^2, w^2, uw\}$ over unrestricted symmetric matrices $\mathrm{Sym}^2(\mathbb{R}^2)$ (equivalently, over lifted PSD matrices $\mathcal{P}_+$ by Theorem 5.1(i)) are*

$$E_{\mathrm{opt}}\big(\mathrm{Sym}^2(\mathbb{R}^2);\, z_i,\, \varepsilon\big) = \varepsilon\, L_{\mathrm{lin}}(z_i), \qquad L_{\mathrm{lin}}(z_i) = \big\| e_i^\top (M_3^{(a)})^{-1} \big\|_1 \qquad (i = 1, 2, 3), \tag{6.36}$$

and these provide guaranteed worst-case reconstruction error bounds for the linear estimator $(M_3^{(a)})^{-1} y$ on the physical rank-one pure-state domain $\mathcal{P}_1$. Holding dimensionless ratios $(\hat{R}_1, \hat{R}_2, \hat{R}_3, \hat{t}_c) = (R_1/a, R_2/a, R_3/a, t_c/a)$ fixed, $M_3^{(a)} = a^{-12} M_3^{(1)}$ implies $(M_3^{(a)})^{-1} = a^{12} (M_3^{(1)})^{-1}$, so all three amplification constants scale identically as a^{12}. For the explicit design (6.31), 50-digit interval arithmetic (`mpmath.iv` in `verify_04`) certifies both the exact constants

$$L(u^2) = 8.5080858732\, a^{12}, \qquad L(w^2) = 7.7606717882\, a^{12}, \qquad L(uw) = 22.9690755881\, a^{12}, \tag{6.37}$$

and that $\det\big((M_3^{(1)})^{-1} s\big) < -393.68 < 0$ at all four inequivalent vertices $s \in \{(1,1,1), (1,1,-1), (1,-1,1), (-1,1,1)\}$, proving via Theorem 5.1(iv) that exact rank-one minimax equality

$$L_{\mathrm{pure}}(Q) = L_{\mathrm{SDP}}(Q) = L_{\mathrm{lin}}(Q) = \big\| (M_3^{(a)})^{-\top} q \big\|_1$$

holds for every $Q \in \mathrm{Sym}^2(\mathbb{R}^2)$ on the design (6.31). Finally, as $t_c \to 0$ at fixed (R_1, R_2, R_3) and fixed $a > 0$, the mixed-entry amplification diverges as $1/|t_c|$:

$$L_{\mathrm{lin}}(uw;\, t_c) = \frac{1 + \big\| r^{(a)}(R_3, t_c)^\top (M_{2\times 2}^{(a)})^{-1} \big\|_1}{\big| 2E_{eo}^{(a)}(R_3, t_c) \big|} = \frac{1 + \big\| r^{(a)}(R_3, 0)^\top (M_{2\times 2}^{(a)})^{-1} \big\|_1}{\big| \dot{\mathcal{E}}_{B(R_3,0)}[h_{\mathrm{loc}}^{(+)}] \big|\, |t_c|} + O(|t_c|), \tag{6.38}$$

which whenever $R_3 \in \{R_1, R_2\}$ (so $r^{(a)}(R_3, 0)^\top (M_{2\times 2}^{(a)})^{-1} = e_j^\top$ has ℓ_1-norm 1) collapses to the exact closed-form rate

$$L_{\mathrm{lin}}(uw;\, t_c) = \frac{1001\, a^{13} (1 + \hat{R}_3^2)^{13}}{6144\, \pi\, \hat{R}_3^8 \big(171 \hat{R}_3^4 - 143 \hat{R}_3^2 + 286\big)\, |t_c|} + O(|t_c|), \tag{6.39}$$

evaluating to $\approx 0.92547914\, a^{13}/|t_c|$ at $\hat{R}_3 = \frac{1}{2}$ (and $\approx 1.35298597\, a^{13}/|t_c|$ at $\hat{R}_3 = 1$).

Proof. (i) For $t_c = 0$, the time-reflection argument of Theorem 6.4(i) gives $\mathcal{E}_{B(R,0,0)}(h_{\mathrm{even}}^{(a)} + h_{\mathrm{odd}}^{(a)}) = \mathcal{E}_{B(R,0,0)}(h_{\mathrm{even}}^{(a)} - h_{\mathrm{odd}}^{(a)})$, so $E_{eo}^{(a)}(R, 0) \equiv 0$ by (6.29), establishing the block-triangular structure (6.30). From (6.15)–(6.16), $E_{ee}^{(1)}(\hat{R})/E_{oo}^{(1)}(\hat{R}) = \frac{25}{11}\hat{R}^2 + O(\hat{R}^4) \to 0$ as $\hat{R} \to 0^+$, while (6.18) gives $E_{ee}^{(1)}(\hat{R})/E_{oo}^{(1)}(\hat{R}) \to 1$ as $\hat{R} \to \infty$; hence the ratio is non-constant, guaranteeing distinct radii (R_1, R_2) with $\det M_{2\times 2}^{(a)} \neq 0$. Differentiating (6.29) at $t_c = 0$ and applying (6.22) yields $\partial_{t_c}(2E_{eo}^{(a)}(R_3, t_c))\big|_{t_c=0} = \dot{\mathcal{E}}_{B(R_3,0)}[h_{\mathrm{loc}}^{(+)}] < 0$, so $c_3^{(a)} \neq 0$ for sufficiently small $t_c \neq 0$.

For the concrete design (6.31) at $a = 1$, evaluating (6.14) at $\hat{R} = \frac{1}{2}$ and $\hat{R} = 1$ (where $\arctan(1) = \frac{\pi}{4}$) yields the first two rows of (6.32). Using $\det M_{2\times 2}^{(1)}(\frac{1}{2}, 1) = E_{ee}^{(1)}(\frac{1}{2}, 0)\,[E_{oo}^{(1)}(1,0) - E_{ee}^{(1)}(1,0)] - E_{ee}^{(1)}(1,0)\,[E_{oo}^{(1)}(\frac{1}{2},0) - E_{ee}^{(1)}(\frac{1}{2},0)]$ with the rational differences (6.17) yields (6.34) verbatim (`verify_04`); because all three terms in (6.34) are strictly negative and $\pi, \arctan(\frac{1}{2}) > 0$, $\det M_{2\times 2}^{(1)}(\frac{1}{2}, 1) < 0$ is certified with zero numerical approximation. For the shifted ball $(\hat{R}_3, \hat{t}_c) = (\frac{1}{2}, \frac{1}{2})$, let $\mathcal{E}_{R,t_c}(\cdot,\cdot)$ denote the real-symmetric bilinear polarization of the real quadratic canonical energy $\mathcal{E}_{B(R,0,t_c)}(\cdot)$, satisfying the real polarization identity

$$\mathcal{E}_{R,t_c}(h_1, h_2) = \frac{1}{4}\Big[\mathcal{E}_{B(R,0,t_c)}(h_1 + h_2) - \mathcal{E}_{B(R,0,t_c)}(h_1 - h_2)\Big],$$

and extend $\mathcal{E}_{R,t_c}(\cdot,\cdot)$ $\mathbb{C}$-bilinearly to complex perturbations. Setting $g_a := h_c^{(a)} = h[\Phi_a] = h_{\text{odd}}^{(a)} + i\, h_{\text{even}}^{(a)}$ and its explicit complex conjugate $\overline{g_a} = (h_c^{(a)})^* = h_{\text{odd}}^{(a)} - i\, h_{\text{even}}^{(a)}$, the *complex-bilinear self-pairing* of Theorem 6.4 (eqs. (6.25)–(6.26)) yields the diagonal difference and imaginary mixed term:

$$\begin{aligned} Q_R(a+it_c) &= \mathcal{E}_{R,t_c}(g_a,\, g_a) = \mathcal{E}_{R,t_c}\big(h_{\text{odd}}^{(a)} + i\, h_{\text{even}}^{(a)},\; h_{\text{odd}}^{(a)} + i\, h_{\text{even}}^{(a)}\big) \\ &= \big[E_{oo}^{(a)}(R,t_c) - E_{ee}^{(a)}(R,t_c)\big] + 2i\, E_{eo}^{(a)}(R,t_c), \end{aligned}$$

whose imaginary part matches the real polarization identity $2E_{eo}^{(a)} = \frac{1}{2}[\mathcal{E}(h_{\text{even}}^{(a)}+h_{\text{odd}}^{(a)}) - \mathcal{E}(h_{\text{even}}^{(a)} - h_{\text{odd}}^{(a)})]$ (eq. (6.29)) and the sign $\mathcal{E}(h_{\text{loc}}^{(-)}) - \mathcal{E}(h_{\text{loc}}^{(+)}) = -4E_{eo}^{(a)} = -2\,\mathrm{Im}\, Q_R(a+it_c)$ in (6.23). By contrast, explicitly conjugating the second slot defines the *Hermitian (sesquilinear) pairing*, which gives the sum of the diagonal energies:

$$\mathcal{E}_{R,t_c}(g_a,\, \overline{g_a}) = \mathcal{E}_{R,t_c}\big(h_{\text{odd}}^{(a)} + i\, h_{\text{even}}^{(a)},\; h_{\text{odd}}^{(a)} - i\, h_{\text{even}}^{(a)}\big) = E_{oo}^{(a)}(R,t_c) + E_{ee}^{(a)}(R,t_c),$$

where the imaginary cross-terms $+i\mathcal{E}_{R,t_c}(h_{\text{even}}^{(a)}, h_{\text{odd}}^{(a)}) - i\mathcal{E}_{R,t_c}(h_{\text{odd}}^{(a)}, h_{\text{even}}^{(a)}) = 0$ cancel identically by symmetry of $\mathcal{E}_{R,t_c}$ (equivalently, $W_{\Sigma_B}(h_1, \mathcal{L}_{\xi_B} h_2) - W_{\Sigma_B}(h_2, \mathcal{L}_{\xi_B} h_1) = \int_{\partial\Sigma_B} \iota_{\xi_B}\omega(h_1,h_2) = 0$ because $\xi_B|_{\widetilde{B}} = 0$ and $\omega^z = O(z^3) \to 0$ on B, so the Hermitian bulk symplectic pairing $W_{\Sigma_B}(g_a, \mathcal{L}_{\xi_B}\overline{g_a})$ is strictly real and equals $E_{oo}^{(a)} + E_{ee}^{(a)}$). Substituting $a = 1$, $\alpha = 1 + \frac{i}{2}$, and $R = \frac{1}{2}$ into (6.25) uses $\alpha^2 + R^2 = (\frac{3}{4} + i) + \frac{1}{4} = 1 + i$ with $(1+i)^{12} = (2i)^6 = -64$, yielding

$$Q_{1/2}\big(1 + \tfrac{i}{2}\big) = \frac{45\pi}{9152} - \frac{27\pi}{1760}\, i,$$

so $E_{oo}^{(1)}(\frac{1}{2},\frac{1}{2}) - E_{ee}^{(1)}(\frac{1}{2},\frac{1}{2}) = \frac{45\pi}{9152}$ and $2E_{eo}^{(1)}(\frac{1}{2},\frac{1}{2}) = -\frac{27\pi}{1760}$ (eq. (6.33)). Evaluating the Hermitian pairing $W_{\Sigma_{B(\frac{1}{2},0,\frac{1}{2})}}(g_1, \mathcal{L}_{\xi_B}\overline{g_1}) = E_{oo}^{(1)}(\frac{1}{2},\frac{1}{2}) + E_{ee}^{(1)}(\frac{1}{2},\frac{1}{2})$ by exact polynomial Laurent partial fractions in ϱ^2 around the conjugate poles $-(1 \pm \frac{i}{2})^2$ (`verify_04`) gives $E_{oo}^{(1)}(\frac{1}{2},\frac{1}{2}) + E_{ee}^{(1)}(\frac{1}{2},\frac{1}{2}) = -\frac{405\pi^2}{1024} + \frac{31473\pi}{24640}$ (with strictly vanishing imaginary part, exact cancellation of all logarithmic terms, and $\frac{1}{2}[\arctan(1) + \arctan(0)] = \frac{\pi}{8}$). Taking half the sum and half the difference,

$$E_{oo}^{(1)}\big(\tfrac{1}{2},\tfrac{1}{2}\big) = \frac{1}{2}\Big[\big(E_{oo}^{(1)} + E_{ee}^{(1)}\big) + \big(E_{oo}^{(1)} - E_{ee}^{(1)}\big)\Big], \qquad E_{ee}^{(1)}\big(\tfrac{1}{2},\tfrac{1}{2}\big) = \frac{1}{2}\Big[\big(E_{oo}^{(1)} + E_{ee}^{(1)}\big) - \big(E_{oo}^{(1)} - E_{ee}^{(1)}\big)\Big],$$

yields the third row of (6.32). All three shifted-ball values are independently corroborated to $< 1.4 \times 10^{-16}$ by 64-node 2D Gauss–Legendre evaluation of $W_{\Sigma_{B(\frac{1}{2},0,\frac{1}{2})}}$, which also directly confirms the symmetry $W_{\Sigma_B}(h_{\text{even}}^{(1)}, \mathcal{L}_{\xi_B} h_{\text{odd}}^{(1)}) = W_{\Sigma_B}(h_{\text{odd}}^{(1)}, \mathcal{L}_{\xi_B} h_{\text{even}}^{(1)})$.

(ii) Block-matrix multiplication immediately verifies (6.35). Since $(z_1, z_2, z_3) = (u^2, w^2, uw)$ satisfies $z_1, z_2 \geq 0$ and $z_3^2 = z_1 z_2$, the case split ($z_1 = 0$, $z_2 = 0$, or $z_1, z_2 > 0$) uniquely determines (u, w) up to the global sign ± 1 on all of $\mathbb{R}^2$.

(iii) Let $A_1, A_2 \in \mathrm{Sym}^2(\mathbb{R}^2)$ satisfy $A_1, A_2 \succeq 0$, and let $M_2(v) = (v^\top A_1 v, v^\top A_2 v)$ and $\mathcal{A}_2(X) = (\mathrm{Tr}(A_1 X), \mathrm{Tr}(A_2 X))$. If $\ker(A_1 + A_2) \neq \{0\}$, pick nonzero $v \in \ker(A_1 + A_2)$; since $v^\top A_k v \geq 0$ ($k = 1, 2$) and $v^\top(A_1 + A_2)v = 0$, we obtain $M_2(v) = (0,0) = M_2(0)$ with $v \neq \pm 0$. Otherwise $A_1 + A_2 \succ 0$ is strictly positive definite. Since $\dim \mathrm{Sym}^2(\mathbb{R}^2) = 3 > 2$, there exists a nonzero $\Delta \in \ker \mathcal{A}_2 \subset \mathrm{Sym}^2(\mathbb{R}^2)$, which obeys $\mathrm{Tr}((A_1 + A_2)\Delta) = 0$. If $\Delta \succeq 0$, then $\mathrm{Tr}((A_1+A_2)^{1/2}\Delta(A_1+A_2)^{1/2}) = 0$ forces $\Delta = 0$, a contradiction; similarly $\Delta \preceq 0$ is impossible. Hence Δ is strictly indefinite, with eigenvalues $\lambda_1 > 0 > \lambda_2$ and orthonormal eigenvectors $e_1, e_2 \in \mathbb{R}^2$. Setting $a := \sqrt{\lambda_1}\, e_1$ and $b := \sqrt{-\lambda_2}\, e_2$ gives $\Delta = aa^\top - bb^\top$ with $a \perp b$ both nonzero (so $a \neq \pm b$) and $M_2(a) - M_2(b) = \mathcal{A}_2(\Delta) = 0$. Thus no two PSD quadratic readouts identify (u, w) up to sign. (Note that the PSD hypothesis is genuine: the indefinite pair $A_1 = \mathrm{diag}(1, -1)$, $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ realizes complex squaring $z \mapsto z^2$ and identifies $\mathbb{R}^2/\{\pm 1\}$, whereas every readout

matrix in a PSD measurement family—including all three matrices of our explicit design (6.31)—satisfies $A_k \succeq 0$.) For our explicit design (6.31), $A^{(1)}(\frac{1}{2},0), A^{(1)}(1,0) \succ 0$ by Theorem 6.3, and 50-digit interval arithmetic in `verify_04` certifies $E^{(1)}_{ee}(\frac{1}{2},\frac{1}{2}) > 0$, $E^{(1)}_{oo}(\frac{1}{2},\frac{1}{2}) > 0$, and $\det A^{(1)}(\frac{1}{2},\frac{1}{2}) = E_{ee}E_{oo} - E^2_{eo} \in [2.3458711129 \times 10^{-3}, 2.3458711130 \times 10^{-3}] > 0$, proving $A_3 \succ 0$ directly.

(iv) By Theorem 5.1(i) and (5.3), the exact minimax ℓ_∞ recovery error for the linear functional $z_i = e_i^\top z$ on $\mathrm{Sym}^2(\mathbb{R}^2)$ (and on $\mathcal{P}_+$) is $\varepsilon\|e_i^\top (M_3^{(a)})^{-1}\|_1$, and since $(M_3^{(a)})^{-1}y$ is unbiased on $\mathcal{P}_1 \subset \mathrm{Sym}^2(\mathbb{R}^2)$, $\varepsilon\|e_i^\top (M_3^{(a)})^{-1}\|_1$ bounds its worst-case error on $\mathcal{P}_1$. Poincaré scaling $E^{(a)}_{\alpha\beta}(R,t_c) = a^{-12}E^{(1)}_{\alpha\beta}(R/a, t_c/a)$ gives $M_3^{(a)} = a^{-12}M_3^{(1)}$ and hence $(M_3^{(a)})^{-1} = a^{12}(M_3^{(1)})^{-1}$. For the design (6.31), 50-digit interval arithmetic (`verify_04`) encloses the inverse rows and their ℓ_1-norms (6.37), and certifies that for all four inequivalent vertices $s \in \{(1,1,1), (1,1,-1), (1,-1,1), (-1,1,1)\}$, the lifted matrix $Z(s) = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}$ with $(z_1,z_2,z_3)^\top = (M_3^{(1)})^{-1}s$ satisfies $\det Z(s) = z_1z_2 - z_3^2 \in [-577.95, -393.68] < 0$. Hence $Z(s) \in \mathcal{V}_{\le 2}$ at all eight cube vertices, so Theorem 5.1(iv) (just as in Theorem 5.3) implies $L_{\mathrm{pure}}(Q) = L_{\mathrm{SDP}}(Q) = L_{\mathrm{lin}}(Q)$ for every $Q \in \mathrm{Sym}^2(\mathbb{R}^2)$. Finally, taking the ℓ_1-norm of row 3 of (6.35) gives the first equality in (6.38); because $E^{(a)}_{ee}(R_3,t_c)$ and $E^{(a)}_{oo}(R_3,t_c)$ are smooth even functions of t_c while $2E^{(a)}_{eo}(R_3,t_c) = \dot{\mathcal{E}}_{B(R_3,0)}[h^{(+)}_{\mathrm{loc}}]\,t_c + O(t_c^3)$, expanding as $t_c \to 0$ yields (6.38), and when $R_3 \in \{R_1, R_2\}$ the identity $r^{(a)}(R_3,0)^\top (M^{(a)}_{2\times 2})^{-1} = e_j^\top$ together with (6.21) yields (6.39). □

Remark 6.6 (Alternative better-conditioned shifted ball and role of fixed width). Choosing instead $(R_1,R_2;R_3,t_c) = (\frac{1}{2}a, a;\, a, \frac{1}{2}a)$ (matching the $t_0 = 0.5$ ball of (6.24), where $|2E^{(1)}_{eo}(1,\frac{1}{2})| \approx 0.149256$ is three times larger) reduces the mixed-entry amplification slightly from $22.97\,a^{12}$ to $18.27\,a^{12}$ (still with $\det((M_3^{(1)})^{-1}s) < 0$ at all eight vertices), at the cost of a larger rational denominator in $2E^{(1)}_{eo}(1,\frac{1}{2})$; we feature (6.31) for its compact exact arithmetic ($2E^{(1)}_{eo}(\frac{1}{2},\frac{1}{2}) = -\frac{27\pi}{1760}$). Note also that Theorem 6.5 assumes the packet width $a > 0$ is fixed and known. At the zero state $(u,w) = (0,0)$, the perturbation vanishes identically and the width parameter a is unidentifiable. Simultaneously identifying an unknown width $a > 0$ alongside a nonzero state $(u,w) \neq (0,0)$ from centered boundary-ball readouts—as well as arbitrary finite and infinite-dimensional superpositions—is a separate nonlinear inverse problem resolved completely in Section 7.

7 Infinite-Dimensional Second-Order Reconstruction in the Axisymmetric Axial Sector

Theorem 6.5 recovers the two real amplitudes (u,w) of a single localized axial packet $uh^{(a)}_{\mathrm{even}} + wh^{(a)}_{\mathrm{odd}}$ up to global sign when the packet width $a > 0$ is fixed and known. We now carry this programme to its full nonlinear and infinite-dimensional resolution inside $\ker M_1$: unknown-width single-packet recovery, arbitrary finite superpositions of unknown cardinality and widths, and infinite-dimensional compactly supported width-measure superpositions. Along the way we establish four exact structural pillars:

(a) an exact, manifestly positive real-space one-dimensional canonical-energy identity for *every* centered boundary ball $B(R,0,t)$ on the entire radial axial sector—settling classical positivity unconditionally without CFT hypotheses and yielding both an exact $R \to 0^+$ boundary-trace limit and uniform domination by the global energy (Theorem 7.2);

(b) the exact closed-form two-width holomorphic master kernel $\mathcal{K}_R(\alpha,\beta)$ governing both complex-bilinear and Hermitian pairings (verified independently by direct bulk symplectic

pairing, polarized charge–area with surface term, and 64-node quadrature) together with exact elementary frequency-space kernels $\widehat{\mathcal{K}}_R^+(\omega_1,\omega_2)$ and $\widehat{\mathcal{H}}_R(\omega_1,\omega_2)$ (Theorem 7.4);

(c) a sharp dichotomy between *physical rank-one states* and *unrestricted lifted states* at fixed versus multiple radii: at *any single fixed centered radius* $R_0 > 0$ (with zero restriction on R_0 relative to the width support), exact canonical-energy data $t \mapsto \mathcal{E}_{R_0,t}(h[\mu])$ on any nonempty open time interval $I \subset \mathbb{R}$ already determine *every* perturbation $h[\mu]$ in the compact positive width-measure class $\mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$ $(0 < a_{\min} \le a_{\max} < \infty)$ unconditionally up to one overall sign via large-t boundary dominance and a constructive strictly triangular exterior Laurent moment recursion (Theorem 7.7)—removing all finite-N interlacing caveats alongside direct complex slit-endpoint recovery of single unknown-width packets (Theorem 7.5) and finite-superposition widths and squared amplitudes (Theorem 7.6)—whereas the *unrestricted lifted* map collapses analytically for every finite set of K radii once $N \ge 14K$ via indefinite boundary-vs-bulk cancellation (Theorem 7.6(ii)), and requires multi-radius data on an open rectangle $(R_1, R_2) \times I$ to achieve complete infinite-dimensional injectivity on lifted symmetric measures $\Gamma \in \mathcal{M}_{\mathrm{sym}}(K \times K)$ (Theorem 7.9);

(d) sharp quantitative stability bounds distinguishing noiseless uniqueness from noisy finite-window recovery: unconditional local two-point and compact-global Lipschitz stability on the sign quotient $K_{\mathrm{par}}/\langle S_{-1}\rangle$ for well-separated finite-N superpositions even at a single fixed radius $R_0 > 0$ on a continuous window $[-T, T]$ (kept distinct from finite discrete-sample designs, whose rank and global injectivity hypotheses are stated separately; with exact Schur-complement $O(\eta/\epsilon^2)$ small-amplitude width blow-up accounting for amplitude absorption, exact nearby-width global-energy Gram conditioning $\frac{16a^2}{13\delta^2}$, and proved $\Theta(\delta^2)$ lifted readout singular-value collapse on $L^2([-T,T])/L^\infty([-T,T])$) contrasted with an unconditional exponential/logarithmic ill-conditioning barrier governed by the exact Bernstein parameter $\rho_B = (\sqrt{a_{\max}} + \sqrt{a_{\min}})/(\sqrt{a_{\max}} - \sqrt{a_{\min}})$ for continuous width-measure reconstruction, yielding a sharp minimax L^2 error floor of 1 on the unit ball and an $\Omega((\log(1/\eta))^{-2s})$ lower bound on H^s (Theorem 7.11).

Every formula, algebraic identity, and high-precision certificate in this section is reproduced independently by the companion verification modules `reproducibility/verify_05_infinite_dim_reconstruction.py` and `reproducibility/verify_06_single_radius_reconstruction.py`.

7.1 Inverse Problem Formulation, Admissible Measure Class, and Measurement Families

Fix the localized packet convention of Section 6: for $a > 0$,

$$\Phi_a(t,z,x,y) := \frac{1}{\big(-(t-ia)^2 + r^2 + z^2\big)^3}, \qquad g_a := h[\Phi_a] = h_{\mathrm{odd}}^{(a)} + i\, h_{\mathrm{even}}^{(a)}, \tag{7.1}$$

where $r^2 = x^2 + y^2$ and $\varrho := \sqrt{r^2 + z^2}$. Writing a complex amplitude $c = w - iu \in \mathbb{C}$ recovers every real single-packet combination compactly as

$$u\, h_{\mathrm{even}}^{(a)} + w\, h_{\mathrm{odd}}^{(a)} = \mathrm{Re}(c\, g_a) = \frac{1}{2}\big(c\, g_a + \bar{c}\, \overline{g_a}\big). \tag{7.2}$$

To formulate an infinite-dimensional admissible class containing all finite superpositions as atomic measures, fix any compact width interval $[a_{\min}, a_{\max}] \subset (0, \infty)$ bounded away from zero, and let $\mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$ denote the Banach space of finite complex Borel measures $\mu = \mu_w - i\mu_u$ (where μ_u, μ_w are finite real signed Borel measures on $[a_{\min}, a_{\max}]$) equipped with total variation

norm $\|\mu\|_{\mathrm{TV}} < \infty$. To each $\mu \in \mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$ associate the real axisymmetric axial potential and linearized metric perturbation

$$\Phi[\mu](t,\varrho) := \operatorname{Re}\int_{a_{\min}}^{a_{\max}} \Phi_a(t,\varrho)\,\mathrm{d}\mu(a), \qquad h[\mu] := h\big[\Phi[\mu]\big] = \operatorname{Re}\int_{a_{\min}}^{a_{\max}} g_a\,\mathrm{d}\mu(a). \tag{7.3}$$

Finite superpositions $h = \sum_{j=1}^{N}(u_j h_{\mathrm{even}}^{(a_j)} + w_j h_{\mathrm{odd}}^{(a_j)})$ $(0 < a_1 < \cdots < a_N,\ c_j = w_j - iu_j \neq 0)$ correspond to discrete atomic measures $\mu = \sum_{j=1}^{N} c_j\,\delta_{a_j}$, after deleting zero coefficient pairs and merging repeated widths.

Proposition 7.1 (Regularity, Finite Energy, $\ker M_1$ Persistence, and Zero Gauge Redundancy)**.** *For every compact interval $[a_{\min}, a_{\max}] \subset (0,\infty)$ and every $\mu \in \mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$:*

(i) ***Smoothness and*** $\ker M_1$ ***membership:*** *$\Phi[\mu] \in C^\infty(\mathbb{R}\times[0,\infty)\times\mathbb{R}^2)$ is real-analytic in (t,z,x,y) (extending holomorphically to the strip $|\operatorname{Im} t| < a_{\min}$) and solves (6.1). Differentiation commutes with integration against μ, so $h[\mu] = h[\Phi[\mu]]$ is a globally smooth normalizable Fefferman–Graham TT vacuum solution satisfying $DE_{B(R,\mathbf{x}_0,t_0)}[h[\mu]] = 0$ and $DA_{B(R,\mathbf{x}_0,t_0)}[h[\mu]] = 0$ for* every *rest-frame spherical ball $B(R,\mathbf{x}_0,t_0)$ (Theorem 6.1).*

(ii) ***Exchange of polarization and integration, and finite global energy:*** *Denoting the $\mathbb{C}$-bilinear extension of the centered wedge canonical energy $\mathcal{E}_{R,t}(\cdot) := \mathcal{E}_{B(R,0,t)}(\cdot)$ by $\mathcal{E}_{R,t}(\cdot,\cdot)$, Fubini's theorem holds identically for every $R > 0$ and $t \in \mathbb{R}$:*

$$\mathcal{E}_{R,t}\big(h[\mu]\big) = \frac{1}{2}\operatorname{Re}\iint_{[a_{\min},a_{\max}]^2}\Big[\mathcal{E}_{R,t}(g_a,g_b)\,\mathrm{d}\mu(a)\,\mathrm{d}\mu(b) + \mathcal{E}_{R,t}(g_a,\overline{g_b})\,\mathrm{d}\mu(a)\,\mathrm{d}\overline{\mu(b)}\Big], \tag{7.4}$$

and the global Poincaré canonical energy is finite and given by the strictly positive-definite Bergman–Hankel form

$$\mathcal{E}_{\partial_t}\big(h[\mu]\big) = \frac{1}{2}\iint_{[a_{\min},a_{\max}]^2}\frac{10800\,\pi}{(a+b)^{13}}\,\mathrm{d}\mu(a)\,\mathrm{d}\overline{\mu(b)} = \frac{5400\,\pi}{12!}\int_0^\infty \omega^{12}\,\big|\mathcal{L}[\mu](\omega)\big|^2\,\mathrm{d}\omega \;<\; \infty, \tag{7.5}$$

where $\mathcal{L}[\mu](\omega) := \int_{a_{\min}}^{a_{\max}} e^{-a\omega}\,\mathrm{d}\mu(a)$ $(\omega > 0)$ is the Laplace transform of μ.

(iii) ***Zero gauge redundancy and physical equivalence:*** *The linear maps $\mu \mapsto h[\mu] \mapsto T^{(1)}_{\mu\nu}[h[\mu]]$ are injective on $\mathcal{M}_{\mathbb{C}}([a_{\min},a_{\max}])$. In particular, $h[\mu]$ is pure gauge (or has $\mathcal{E}_{\partial_t}(h[\mu]) = 0$) if and only if $\mu = 0$, and physical equivalence up to overall sign reversal satisfies $h[\mu_1] = \pm h[\mu_2] \iff \mu_1 = \pm\mu_2$.*

Proof. (i) For all real (t,z,x,y) and $a \in [a_{\min},a_{\max}]$, we have $|\sigma_a| = |(a+it)^2+\varrho^2| = |(a+it)+i\varrho|\,|(a+it)-i\varrho| \geq a^2 \geq a_{\min}^2 > 0$. More generally, on the closed strip $|\operatorname{Im} t| \leq \frac{1}{2}a_{\min}$ we have $\operatorname{Re}(a+it) = a - \operatorname{Im} t \geq \frac{1}{2}a_{\min} > 0$, whence $|\sigma_a| = |(a+it)+i\varrho|\,|(a+it)-i\varrho| \geq (\operatorname{Re}(a+it))^2 \geq \frac{1}{4}a_{\min}^2 > 0$ uniformly on $\{|\operatorname{Im} t| \leq \frac{1}{2}a_{\min}\}\times[0,\infty)\times\mathbb{R}^2$. All spacetime derivatives of Φ_a are therefore uniformly bounded on compact subsets of that domain, with $O(\varrho^{-6})$ spatial decay as $\varrho\to\infty$ uniformly on compact t-intervals, justifying differentiation under the integral sign against the finite measure μ to all orders and placing $\Phi[\mu]$ inside Theorem 6.1.

(ii) On every compact half-ball $\Sigma_{B(R,0,t)}$, the bilinear symplectic density $\omega^0(g_a,\mathcal{L}_{\xi_B}g_b)$ (and $\omega^0(g_a,\mathcal{L}_{\xi_B}\overline{g_b})$) is uniformly bounded in $(a,b)\in[a_{\min},a_{\max}]^2$ and $z^{-2}\mathrm{d}z\,\mathrm{d}^2x$-integrable (since $\omega^0 = O(z^2)$ near $z=0$), so Fubini's theorem immediately yields (7.4). Taking $R\to\infty$ in $\mathcal{E}_{R,t}(h[\mu])/(\pi R)$ via the exact two-width global limits $\mathcal{E}_{\partial_t}(g_a,g_b) = 0$ and $\mathcal{E}_{\partial_t}(g_a,\overline{g_b}) = 10800\pi/(a+b)^{13}$ proved in Theorem 7.4 below (with dominated convergence against $|\mu|\otimes|\mu|$ justified by Cauchy–Schwarz on the positive-semidefinite form $\mathcal{E}_{R,t}$ and the uniform Hardy bound of Theorem 7.2(iv)), and substituting $(a+b)^{-13} = \frac{1}{12!}\int_0^\infty \omega^{12}e^{-(a+b)\omega}\,\mathrm{d}\omega$, yields (7.5) (which is bounded by $\frac{5400\pi}{(2a_{\min})^{13}}\|\mu\|_{\mathrm{TV}}^2 < \infty$).

(iii) If $T^{(1)}_{\mu\nu}[h[\mu]] \equiv 0$, then in particular $H^{(0)}_{xx}(t,x,y) = 2xy\,\partial_t p[\mu](t,r) \equiv 0$ (by (7.9) below), forcing $\partial_t p[\mu](t,r) \equiv 0$ for all $t \in \mathbb{R}$ and $r > 0$. Since $p[\mu](t,r) \to 0$ as $|t| \to \infty$ by (7.7), this forces $p[\mu](t,r) \equiv 0$ and hence $p[\mu](t,0) \equiv 0$ for all $t \in \mathbb{R}$; by the Fourier–Laplace identity (7.44) below (or by Step 1 of Theorem 7.7), $\mu = 0$. Independently, by (7.5), $\mathcal{E}_{\partial_t}(h[\mu]) = 0$ implies $\mathcal{L}[\mu](\omega) = 0$ for all $\omega > 0$, hence $\mu = 0$ by uniqueness of the finite Borel Laplace transform (Lerch's theorem). Linearity immediately gives $h[\mu_1] = \pm h[\mu_2] \iff \mu_1 = \pm\mu_2$. □

We define two explicit centered measurement families prior to proving injectivity:

- **Single-radius centered time trace $\mathcal{M}_{R,I}$:** Fix a single physical radius $R > 0$ (independent of unknown widths) and a nonempty open time interval $I \subset \mathbb{R}$, and observe $t \mapsto \mathcal{E}_{R,t}(h)$ for $t \in I$ (and analogously $\mathcal{M}_{\{R_1,\dots,R_K\},I}$ for finitely many fixed radii $R_1, \dots, R_K > 0$).
- **Multi-radius centered trace $\mathcal{M}_{(R_1,R_2),I}$:** Observe $(R,t) \mapsto \mathcal{E}_{R,t}(h)$ for (R,t) ranging over any nonempty open rectangle $(R_1,R_2) \times I \subset (0,\infty) \times \mathbb{R}$ (or more generally any set $S_R \times I$ where $S_R \subset (0,\infty)$ has an accumulation point in $[0,\infty)$).

Notice that because $\mathcal{E}_{R,t}(h[\mu])$ is jointly real-analytic in $(R,t) \in (0,\infty) \times \mathbb{R}$ (Theorem 7.4), exact data on an open interval I determine the real-analytic trace on all of $\mathbb{R}$ (and exact data on $(R_1,R_2) \times I$ determine it on all of $(0,\infty) \times \mathbb{R}$); Section 7.7 carefully separates what survives under finite sampling and additive measurement noise.

7.2 Exact Real-Space Positivity Identity for All Centered Balls

Before computing the explicit two-width kernel for Φ_a, we establish a remarkable one-dimensional real-space reduction valid for *every* real spherically symmetric (in $\varrho = \sqrt{r^2+z^2}$) axial potential $\Phi(t,\varrho)$ solving (6.1). Define the single real radial master wave

$$p(t,\varrho) := \left(\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\right)^2 \Phi(t,\varrho). \tag{7.6}$$

For $\Phi = \Phi[\mu]$ in (7.3), direct differentiation yields

$$p[\mu](t,\varrho) = \mathrm{Re}\int_{a_{\min}}^{a_{\max}} \frac{48}{\left(-(t-ia)^2+\varrho^2\right)^5}\,\mathrm{d}\mu(a), \qquad p[\mu](t,0) = \mathrm{Re}\int_{a_{\min}}^{a_{\max}} \frac{48}{(a+it)^{10}}\,\mathrm{d}\mu(a), \tag{7.7}$$

since $-(t-ia)^2 = (a+it)^2$. Because $(\Delta_{\mathbf{x}} + \partial_z^2 + \frac{4}{z}\partial_z)\Phi(t,\varrho) = (\partial_\varrho^2 + \frac{6}{\varrho}\partial_\varrho)\Phi$ and the Darboux intertwining identity $(\frac{1}{\varrho}\partial_\varrho)(\partial_\varrho^2 + \frac{m}{\varrho}\partial_\varrho) = (\partial_\varrho^2 + \frac{m+2}{\varrho}\partial_\varrho)(\frac{1}{\varrho}\partial_\varrho)$ applies twice, $p(t,\varrho)$ satisfies the exact eleven-dimensional flat radial wave equation

$$p_{tt} = p_{\varrho\varrho} + \frac{10}{\varrho}\,p_\varrho. \tag{7.8}$$

Furthermore, substituting $\Phi(t,\varrho)$ into the axial ansatz (6.2) shows that the *entire* 4D metric perturbation $h[\Phi]$ depends on Φ solely through p and its first derivatives: setting $G_p := 4p + \frac{r^2}{\varrho}p_\varrho$,

$$h_{tx} = zyG_p, \quad h_{ty} = -zxG_p, \quad h_{xx} = -h_{yy} = 2zxy\,p_t, \quad h_{xy} = z(y^2-x^2)\,p_t. \tag{7.9}$$

Theorem 7.2 (Exact Real-Space Classical Positivity Identity for All Centered Balls)**.** *Let $\Phi(t,\varrho)$ be a smooth real radial solution of (6.1) even in ϱ at $\varrho = 0$ (more generally, any even-in-ϱ solution such that $p(t,\varrho) := (\frac{1}{\varrho}\partial_\varrho)^2\Phi(t,\varrho) \in C^3(\mathbb{R} \times [0,\infty))$ solves (7.8); in particular, every $\Phi[\mu]$ in Proposition 7.1 satisfies this automatically). Then:*

(i) ***Universal centered-hemisphere extremality and exact surface term:*** *For every axisymmetric $\Phi(t,z,r)$ in (6.2), transforming to polar coordinates (r,φ) around $\mathbf{x}_0 = 0$ gives $h_{tr} = h_{rr} = h_{\varphi\varphi} \equiv 0$ and $\partial_\varphi h_{t\varphi} = \partial_\varphi h_{r\varphi} \equiv 0$ identically (`verify_05`). Hence $w_* \equiv 0$ and $\tau_* \equiv 0$ on every centered hemisphere $\widetilde{B}(R,0,t)$ at all $R > 0$ and all $t \in \mathbb{R}$, and the polarized/quadratic extremal-surface contribution is purely algebraic in $p_t(t,R)$:*

$$-\frac{1}{4}A''_{\mathrm{ext},B(R,0,t)}\big[h[\Phi]\big] \;=\; \frac{4\pi}{693}\,R^{10}\,p_t(t,R)^2. \tag{7.10}$$

(ii) ***Exact one-line manifestly positive canonical-energy identity:*** *For* every *centered radius $R > 0$ and* every *time $t \in \mathbb{R}$, the four-dimensional Hollands–Wald canonical energy $\mathcal{E}_{B(R,0,t)}(h[\Phi]) = -\frac{1}{4}A''_{\mathrm{ext}} - I$ decouples identically into time-even and time-odd Cauchy-data blocks, and integration by parts cancels the time-odd boundary term against (7.10) to exact zero, yielding the manifestly nonnegative sum-of-squares representation (`verify_05`):*

$$\begin{aligned}\mathcal{E}_{B(R,0,t)}\big(h[\Phi]\big) &= \frac{8\pi}{105}\,R^8\,p(t,R)^2 + \frac{\pi}{R}\int_0^R\bigg[\frac{\varrho^{10}(R^2-\varrho^2)}{693}\big(p_{\varrho\varrho}^2 + p_{t\varrho}^2\big)\\ &\qquad + \frac{2\varrho^8(25R^2+33\varrho^2)}{3465}\,p_\varrho^2 + \frac{2\varrho^{10}}{105}\,p_t^2\bigg]\,\mathrm{d}\varrho\\ &= \frac{8\pi}{105}\,R^8\,p(t,R)^2 + \frac{\pi}{R}\int_0^R\bigg[\frac{\varrho^{10}(R^2-\varrho^2)}{693}\big(p_{tt}^2 + p_{t\varrho}^2\big)\\ &\qquad + \frac{16\varrho^{10}}{3465}\,p_\varrho^2 + \frac{2\varrho^{10}}{105}\,p_t^2\bigg]\,\mathrm{d}\varrho \;\geq\; 0.\end{aligned} \tag{7.11}$$

In particular, $\mathcal{E}_{B(R,0,t)}(h[\Phi]) = 0$ if and only if $p(t,\varrho) \equiv 0$ and $p_t(t,\varrho) \equiv 0$ for all $\varrho \in [0,R]$ (equivalently, $h[\Phi] \equiv 0$ throughout the causal wedge $D(B(R,0,t))$).

(iii) ***Exact global energy and small-R boundary-trace limit:*** *Provided $\varrho^4 p, \varrho^5 p_\varrho, \varrho^5 p_t \to 0$ as $\varrho \to \infty$, dividing (7.11) by πR and letting $R \to \infty$ gives the exact global Poincaré energy*

$$\mathcal{E}_{\partial_t}\big(h[\Phi]\big) = \frac{1}{693}\int_0^\infty \varrho^{10}\big(p_{tt}^2 + p_{t\varrho}^2\big)\,\mathrm{d}\varrho = \frac{1}{693}\int_0^\infty \big[\varrho^{10}\big(p_{\varrho\varrho}^2 + p_{t\varrho}^2\big) + 10\,\varrho^8\,p_\varrho^2\big]\,\mathrm{d}\varrho, \tag{7.12}$$

while letting $R \to 0^+$ at any fixed $t \in \mathbb{R}$ isolates the exact boundary-origin trace

$$\lim_{R\to 0^+} R^{-8}\,\mathcal{E}_{B(R,0,t)}\big(h[\Phi]\big) \;=\; \frac{8\pi}{105}\,p(t,0)^2. \tag{7.13}$$

(iv) ***Uniform global-energy domination:*** *Under the same asymptotic decay $\varrho^4 p, \varrho^5 p_\varrho, \varrho^5 p_t \to 0$ as $\varrho \to \infty$ (with $\mathcal{E}_{\partial_t}(h[\Phi]) < \infty$), for* every *$R > 0$ and every $t \in \mathbb{R}$,*

$$0 \;\leq\; \mathcal{E}_{B(R,0,t)}\big(h[\Phi]\big) \;\leq\; \frac{363}{175}\,\pi R\,\mathcal{E}_{\partial_t}\big(h[\Phi]\big). \tag{7.14}$$

Proof. (i) Direct symbolic transformation of (6.2) into polar coordinates (r,φ) around $\mathbf{x}_0 = 0$ for an arbitrary $SO(2)$-invariant $\Phi(t,z,r)$ gives $h_{tr} = h_{rr} = h_{\varphi\varphi} = 0$ and φ-independent $h_{t\varphi}, h_{r\varphi}$ (`verify_05`). On any centered hemisphere $\widetilde{B}(R,0,t) = \{\varrho = R\}$, the induced trace $\mathrm{Tr}_{\widetilde{B}}(h)$ and $\mathrm{Tr}_{\widetilde{B}}(\partial_t h)$, radial-normal projections $h_{\varrho\varrho}, h_{t\varrho}$, and azimuthal divergence $\partial_\varphi h_{t\varphi}$ therefore all vanish identically, forcing $w_* = \tau_* = 0$. On $\widetilde{B}(R,0,t)$ at $y = 0$ ($z = Ru$, $x = R\sqrt{1-u^2}$), (7.9) gives $h_{xy} = -R^3 u(1-u^2)p_t(t,R)$, so $-\frac{1}{4}A''_{\mathrm{ext}} = \frac{1}{2}\pi R^4\int_0^1 u^4 h_{xy}^2\,\mathrm{d}u = \frac{1}{2}\pi R^{10}p_t(t,R)^2\int_0^1 u^6(1-u^2)^2\,\mathrm{d}u = \frac{4\pi}{693}R^{10}p_t(t,R)^2$.

(ii) Substituting (7.9) into the 4D second-order Einstein tensor $D^2\mathcal{F}_{tt}[h,h]|_{y=0}$, eliminating all $p_{tt}, p_{tt\varrho}, p_{ttt}$ via (7.8), and integrating over $u = z/\varrho \in [0,1]$ yields $-I = \frac{\pi}{R}\int_0^R (R^2-\varrho^2)(E_{\mathrm{ev}} +$

$E_{\mathrm{od}})\,\mathrm{d}\varrho$ with zero cross-terms between $(p, p_\varrho, p_{\varrho\varrho})$ and $(p_t, p_{t\varrho}, p_{t\varrho\varrho})$ (all $p_{\varrho\varrho\varrho}$ coefficients cancel identically; `verify_05`):

$$E_{\mathrm{ev}} = \frac{\varrho^6}{3465}\big(8316\,p^2 + 4488\varrho\,pp_\varrho + 264\varrho^2 pp_{\varrho\varrho} + 836\varrho^2 p_\varrho^2 + 116\varrho^3 p_\varrho p_{\varrho\varrho} + 5\varrho^4 p_{\varrho\varrho}^2\big),$$
$$E_{\mathrm{od}} = -\frac{\varrho^8}{3465}\big(693\,p_t^2 + 354\varrho\,p_t p_{t\varrho} + 20\varrho^2 p_t p_{t\varrho\varrho} + 15\varrho^2 p_{t\varrho}^2\big).$$

In the time-odd block, integrating $(R^2-\varrho^2)(-\frac{4}{693}\varrho^{10})p_t p_{t\varrho\varrho}$ by parts (boundary term vanishes at $\varrho = 0, R$) shifts the $p_{t\varrho}^2$ coefficient from $-\frac{1}{231}$ to $+\frac{1}{693}\varrho^{10}(R^2-\varrho^2)$ and leaves a $p_t p_{t\varrho}$ weight $w_1(\varrho) = -\frac{1}{3465}\varrho^9(234R^2 - 194\varrho^2)$, which satisfies $\frac{1}{2}w_1(R)\frac{\pi}{R} = -\frac{4\pi}{693}R^{10}$—canceling the surface term (7.10) *identically*—and transforms the bulk p_t^2 coefficient to $+\frac{2}{105}\varrho^{10}$. In the time-even block, integrating $(R^2-\varrho^2)(\frac{8}{105}\varrho^8 pp_{\varrho\varrho} + \frac{116}{3465}\varrho^9 p_\varrho p_{\varrho\varrho})$ by parts (both boundary terms vanish at $\varrho = 0, R$) produces the positive p_ϱ^2 weight $\frac{2}{3465}\varrho^8(25R^2 + 33\varrho^2)$ and a remaining pp_ϱ weight whose integration by parts gives the boundary term $+\frac{8\pi}{105}R^8 p(t,R)^2$ and cancels the bulk p^2 coefficient to exact zero! This establishes the first line of (7.11). For the second line, substituting $p_{tt}^2 = (p_{\varrho\varrho} + \frac{10}{\varrho}p_\varrho)^2$ and integrating $20\varrho^9(R^2-\varrho^2)p_\varrho p_{\varrho\varrho} = 10\varrho^9(R^2-\varrho^2)(p_\varrho^2)_\varrho$ by parts converts $\frac{1}{693}\varrho^{10}(R^2-\varrho^2)p_{tt}^2 + \frac{16}{3465}\varrho^{10}p_\varrho^2$ back to $\frac{1}{693}\varrho^{10}(R^2-\varrho^2)p_{\varrho\varrho}^2 + \frac{2}{3465}\varrho^8(25R^2+33\varrho^2)p_\varrho^2$ identically. If $\mathcal{E}_{B(R,0,t)}(h[\Phi]) = 0$, every nonnegative term vanishes, forcing $p_\varrho = p_t = 0$ on $[0,R]$ and $p(t,R) = 0$, hence $p(t,\cdot) = p_t(t,\cdot) \equiv 0$ on $[0,R]$.

(iii) Letting $R\to\infty$ in $\mathcal{E}_{B(R,0,t)}/(\pi R)$ drops the $O(R^{-1})$ terms and yields (7.12); letting $R\to 0^+$ makes the integral $O(R^{10})$ and leaves $\frac{8\pi}{105}R^8 p(t,0)^2 + O(R^{10})$, proving (7.13).

(iv) Split (7.11) and (7.12) into time-even and time-odd parts ($\mathcal{E}_{\partial_t} = \mathcal{E}_{\partial_t}^{\mathrm{ev}} + \mathcal{E}_{\partial_t}^{\mathrm{od}}$). For the bulk p_t^2 term, integrating $\frac{\mathrm{d}}{\mathrm{d}\varrho}(\varrho^9 p_t^2) = 9\varrho^8 p_t^2 + 2\varrho^9 p_t p_{t\varrho}$ on $(0,\infty)$ and applying Cauchy–Schwarz gives the weighted Hardy inequality $\int_0^\infty \varrho^8 p_t^2\,\mathrm{d}\varrho \le \frac{4}{81}\int_0^\infty \varrho^{10}p_{t\varrho}^2\,\mathrm{d}\varrho$, whence

$$\frac{\pi}{R}\int_0^R \frac{2\varrho^{10}}{105}\,p_t^2\,\mathrm{d}\varrho \le \pi R\cdot\frac{2}{105}\int_0^\infty \varrho^8 p_t^2\,\mathrm{d}\varrho \le \pi R\cdot\frac{2}{105}\cdot\frac{4}{81}\cdot 693\,\mathcal{E}_{\partial_t}^{\mathrm{od}} = \frac{88}{135}\,\pi R\,\mathcal{E}_{\partial_t}^{\mathrm{od}},$$

giving $\mathcal{E}_{B(R,0,t)}^{\mathrm{od}} \le (1+\frac{88}{135})\pi R\,\mathcal{E}_{\partial_t}^{\mathrm{od}} = \frac{223}{135}\pi R\,\mathcal{E}_{\partial_t}^{\mathrm{od}}$. For the boundary term in the time-even block, $p(t,R) = -\int_R^\infty p_\varrho(t,\varrho)\,\mathrm{d}\varrho$ and Cauchy–Schwarz give $p(t,R)^2 \le (\int_R^\infty \varrho^{-8}\,\mathrm{d}\varrho)(\int_R^\infty \varrho^8 p_\varrho^2\,\mathrm{d}\varrho) = \frac{1}{7R^7}\int_0^\infty \varrho^8 p_\varrho^2\,\mathrm{d}\varrho$; since $\int_0^\infty \varrho^8 p_\varrho^2\,\mathrm{d}\varrho \le \frac{693}{10}\mathcal{E}_{\partial_t}^{\mathrm{ev}}$ from (7.12),

$$\frac{8\pi}{105}R^8 p(t,R)^2 + \frac{\pi}{R}\int_0^R \frac{16\varrho^{10}}{3465}p_\varrho^2\,\mathrm{d}\varrho \le \pi R\left(\frac{8}{735}+\frac{16}{3465}\right)\frac{693}{10}\mathcal{E}_{\partial_t}^{\mathrm{ev}} = \frac{188}{175}\,\pi R\,\mathcal{E}_{\partial_t}^{\mathrm{ev}},$$

giving $\mathcal{E}_{B(R,0,t)}^{\mathrm{ev}} \le (1+\frac{188}{175})\pi R\,\mathcal{E}_{\partial_t}^{\mathrm{ev}} = \frac{363}{175}\pi R\,\mathcal{E}_{\partial_t}^{\mathrm{ev}}$. Combining both blocks yields (7.14). □

Remark 7.3 (Classical positivity on the full centered localized sector). Theorem 7.2(ii) supplies a completely classical, self-contained proof of strict positivity $\mathcal{E}_{B(R,0,t)}(h) > 0$ for *every* centered boundary ball $B(R,0,t)$ and every nonzero perturbation $h = h[\mu]$ in the axisymmetric localized sector (confirming in particular that every centered measurement matrix $A(R,t)$ arising in Theorem 6.5 and below is strictly positive definite without invoking any CFT relative-entropy hypothesis).

7.3 Exact Polarized Two-Width Master Kernel and Frequency-Space Kernels

To evaluate (7.4) explicitly, observe that shifting time by $t\in\mathbb{R}$ in (6.5) sends $a\mapsto a+it$, while complex conjugation obeys the exact identity

$$\overline{\Phi_b(\tilde t + t, \varrho)} = \frac{1}{\big(-(\tilde t + t + ib)^2 + \varrho^2\big)^3} = \Phi_{-b+it}(\tilde t, \varrho). \tag{7.15}$$

Consequently, *both* the complex-bilinear pairing $K_R(a,b;t) := \mathcal{E}_{R,t}(g_a, g_b)$ and the Hermitian pairing $H_R(a,b;t) := \mathcal{E}_{R,t}(g_a, \overline{g_b})$ are specializations of a *single* holomorphic two-variable master kernel $\mathcal{K}_R(\alpha,\beta) := \mathcal{E}_{B(R,0,0)}(h[\Phi_\alpha], h[\Phi_\beta])$ defined on $\Omega_R \times \Omega_R$, where $\Omega_R := \mathbb{C} \setminus i[-R,R]$:

$$K_R(a,b;t) = \mathcal{K}_R(a+it,\ b+it), \qquad H_R(a,b;t) = \mathcal{K}_R(a+it,\ -b+it). \tag{7.16}$$

Equivalently, introducing the conjugate-symmetric measure ν on the symmetric compact set $K := [-a_{\max}, -a_{\min}] \cup [a_{\min}, a_{\max}]$ by $\nu|_{[a_{\min},a_{\max}]} := \frac{1}{2}\mu$ and $\nu(-E) := \overline{\nu(E)}$, (7.4) becomes the single integral

$$\mathcal{E}_{R,t}\big(h[\mu]\big) = \iint_{K\times K} \mathcal{K}_R(x+it,\ y+it)\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y). \tag{7.17}$$

Theorem 7.4 (Exact Two-Width Master Kernel $\mathcal{K}_R(\alpha,\beta)$ and Elementary Frequency Kernels)**.** *Let $R > 0$ and $\Omega_R := \mathbb{C} \setminus i[-R,R]$ (using the principal branch $\arctan(R/\alpha) = \frac{1}{2i}\log\frac{\alpha+iR}{\alpha-iR}$, holomorphic on Ω_R and odd in α).*

(i) ***Exact closed-form two-width master kernel:*** *On $\Omega_R \times \Omega_R$, the polarized canonical energy $\mathcal{K}_R(\alpha,\beta) = \mathcal{E}_{B(R,0,0)}(h[\Phi_\alpha], h[\Phi_\beta])$ equals*

$$\begin{aligned}\frac{\mathcal{K}_R(\alpha,\beta)}{\pi} &= \frac{1080\left(10R^2+\alpha^2+8\alpha\beta+\beta^2\right)}{R\,(\alpha-\beta)^{13}}\left[\arctan\frac{R}{\alpha} - \arctan\frac{R}{\beta}\right] + \frac{10800}{(\alpha-\beta)^{12}} \\ &\quad + \sum_{m=1}^{5}\left[\frac{P_m(\alpha,\beta)}{(R^2+\alpha^2)^m} + \frac{P_m(\beta,\alpha)}{(R^2+\beta^2)^m}\right],\end{aligned} \tag{7.18}$$

where $P_6 \equiv 0$ identically, each $P_m(\alpha,\beta)$ is an explicit rational function homogeneous of degree $2m-12$ whose only denominator factors are powers of $(\alpha-\beta)$ and $(\alpha+\beta)$—in particular, the highest-order endpoint pole coefficient is

$$P_5(\alpha,\beta) = -\frac{3072\,\alpha^8\left(17\alpha^2+11\alpha\beta+22\beta^2\right)}{385\,(\alpha-\beta)^7(\alpha+\beta)^5}, \tag{7.19}$$

and all five rational functions $P_1,\dots,P_5$ are derived identically by both *Route 1 (direct Hollands–Wald bulk symplectic pairing W_{Σ_B}) and Route 2 (polarized 4D Einstein charge–area identity $-\frac{1}{4}A''_{\mathrm{ext}} - I$ with polarized surface term $-\frac{102400\pi R^{10}\alpha\beta}{77(\alpha^2+R^2)^6(\beta^2+R^2)^6}$), with exact symbolic zero difference across every coefficient and independent 64-node 2D Gauss–Legendre corroboration at unequal complex widths to $< 2\times10^{-14}$ (`verify_05`).*

(ii) ***Symmetries, diagonal reductions, and global limits:*** *The singularities along $\alpha = \pm\beta$ in (7.18) are removable on $\Omega_R \times \Omega_R$, and $\mathcal{K}_R$ satisfies exact interchange symmetry $\mathcal{K}_R(\alpha,\beta) = \mathcal{K}_R(\beta,\alpha)$, parity $\mathcal{K}_R(-\alpha,-\beta) = \mathcal{K}_R(\alpha,\beta)$, conjugation $\overline{\mathcal{K}_R(\alpha,\beta)} = \mathcal{K}_R(\bar\alpha,\bar\beta)$, and scaling $\mathcal{K}_{\lambda R}(\lambda\alpha,\lambda\beta) = \lambda^{-12}\mathcal{K}_R(\alpha,\beta)$. On the diagonal $\beta\to\alpha$, $\mathcal{K}_R(\alpha,\alpha) = Q_R(\alpha)$ recovers the rational function (6.25); on the anti-diagonal $\beta\to-\alpha$ ($\alpha = a > 0$), $\mathcal{K}_R(a,-a) = E^{(a)}_{oo}(R,0) + E^{(a)}_{ee}(R,0)$ recovers Theorem 6.3. As $R\to\infty$ for $a,b>0$,*

$$\lim_{R\to\infty}\frac{\mathcal{K}_R(a+it,b+it)}{\pi R} = 0, \qquad \lim_{R\to\infty}\frac{\mathcal{K}_R(a+it,-b+it)}{\pi R} = \frac{10800\,\pi}{(a+b)^{13}}. \tag{7.20}$$

(iii) ***Exact Laplace–Fourier representation and elementary frequency-space kernels:*** *For all $a>0$, $t\in\mathbb{R}$, and $\varrho>0$,*

$$\Phi_a(t,\varrho) = \int_0^\infty e^{-a\omega}e^{-i\omega t}\,\psi(\omega,\varrho)\,\mathrm{d}\omega, \qquad \left(\frac{1}{\varrho}\partial_\varrho\right)^2\psi(\omega,\varrho) = \frac{\omega^9}{8}\,\frac{j_4(\omega\varrho)}{(\omega\varrho)^4}, \tag{7.21}$$

where $\psi(\omega,\varrho)=[(3-\omega^2\varrho^2)\sin(\omega\varrho)-3\omega\varrho\cos(\omega\varrho)]/(8\varrho^5)$ *is odd in* ω *and* j_4 *is the order-4 spherical Bessel function. Writing* $F(\omega):=\mathcal{L}[\mu](\omega)$*, the canonical energy (7.4) takes the exact frequency-space form*

$$\begin{aligned}\mathcal{E}_{R,t}\big(h[\mu]\big)&=\frac{1}{2}\,\mathrm{Re}\iint_{(0,\infty)^2}F(\omega_1)F(\omega_2)\,e^{-i(\omega_1+\omega_2)t}\,\widehat{\mathcal{K}}_R^{+}(\omega_1,\omega_2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\\&\quad+\frac{1}{2}\iint_{(0,\infty)^2}F(\omega_1)\overline{F(\omega_2)}\,e^{-i(\omega_1-\omega_2)t}\,\widehat{\mathcal{H}}_R(\omega_1,\omega_2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2,\end{aligned}\tag{7.22}$$

where $\widehat{\mathcal{H}}_R(\omega_1,\omega_2)=-\widehat{\mathcal{K}}_R^{+}(\omega_1,-\omega_2)$*, all* Si/Ci *($\varrho^{-1}$) antiderivative terms cancel identically, and* $\widehat{\mathcal{K}}_R^{+}$ *is purely elementary in* $\cos((\omega_1\pm\omega_2)R)$ *and* $\sin((\omega_1\pm\omega_2)R)$ *(verified against (7.18) via 35-digit double Laplace integration to* $<10^{-14}$ *in* `verify_05`*). On the frequency diagonal* $\omega_1=\omega_2=\omega$*, setting* $u:=R\omega$*,*

$$\begin{aligned}\frac{R^{10}\,\widehat{\mathcal{H}}_R(\omega,\omega)}{\pi}&=\frac{10u^{12}-252u^{10}+1296u^8+4860u^6+39285u^4+340200u^2+1984500}{665280}\\&\quad-\frac{14u^8-2590u^6+75765u^4-403200u^2+220500}{73920}\cos(2u)\\&\quad+\frac{u\big(26u^6-1713u^4+21840u^2-44100\big)}{7392}\sin(2u),\end{aligned}\tag{7.23}$$

whose exact Taylor series $\sum_{n=9}^{\infty}d_nu^{2n}$ *vanishes for* $0\le n\le 8$ *and has strictly nonzero alternating coefficients for* every *integer* $n\ge 9$*:*

$$\begin{aligned}d_n&=-(-1)^n\,\frac{2^{2n}}{591360\,(2n)!}\,P(n),\\P(n)&:=(n-8)(n-7)(2n-7)(2n-5)(2n-3)(2n-1)\big(7n^2-67n+300\big)>0.\end{aligned}\tag{7.24}$$

Proof. (i)–(ii) In Route 1, evaluating $\omega^0(h[\Phi_\alpha],\mathcal{L}_{\xi_B}h[\Phi_\beta])$ along $y=0$, integrating over $u=z/\varrho\in[0,1]$, setting $s=\varrho^2$, and expanding the rational integrand in s around the two poles $s=-\alpha^2$ and $s=-\beta^2$ reduces the ϱ-integral to the elementary recurrence $J_m(\gamma;R):=\int_0^R(\varrho^2+\gamma^2)^{-m}\,\mathrm{d}\varrho$ ($J_1(\gamma;R)=\frac{1}{\gamma}\arctan(R/\gamma)$, $J_{m+1}=\frac{R}{2m\gamma^2(R^2+\gamma^2)^m}+\frac{2m-1}{2m\gamma^2}J_m$). Collecting the coefficients of $\arctan(R/\alpha)$, $\arctan(R/\beta)$, the R-independent constant, and $(R^2+\alpha^2)^{-m}$ ($m=1,\dots,6$) yields (7.18)–(7.19) with $P_6\equiv 0$. In Route 2, polarizing the 4D Einstein residual $D^2\mathcal{F}_{tt}[h[\Phi_\alpha],h[\Phi_\beta]]$ and adding the polarized surface term $-\frac{1}{4}A''_{\mathrm{ext}}=-\frac{102400\pi R^{10}\alpha\beta}{77(\alpha^2+R^2)^6(\beta^2+R^2)^6}$ (whose partial fraction in R^2 cancels the bulk $(R^2+\alpha^2)^{-6}$ pole identically!) reproduces every rational coefficient of (7.18) with exact symbolic zero difference (`verify_05`). Because $\mathcal{K}_R(\alpha,\beta)$ is defined by an integral whose integrand is holomorphic on $\Omega_R\times\Omega_R$, the apparent poles at $\alpha=\pm\beta$ are removable; symmetries follow directly from $\Phi_{-\gamma}(\tilde t,\varrho)=\Phi_\gamma(-\tilde t,\varrho)$ and scaling covariance; taking $R\to\infty$ uses $\arctan(R/a)\to+\frac{\pi}{2}$ and $\arctan(R/(-b))\to-\frac{\pi}{2}$ for $a,b>0$, so the arctan bracket cancels for $(a+it,b+it)$ and doubles to π for $(a+it,-b+it)$, yielding (7.20).

(iii) Since $\Phi_a(t,\varrho)=\frac{1}{8\varrho^5}\int_0^\infty e^{-a\omega-i\omega t}[(3-\omega^2\varrho^2)\sin(\omega\varrho)-3\omega\varrho\cos(\omega\varrho)]\,\mathrm{d}\omega$ (verified symbolically via the radial ODE $(\partial_\varrho^2+\frac{6}{\varrho}\partial_\varrho+\omega^2)\psi=0$ and numerically to 10^{-16}), applying $(\frac{1}{\varrho}\partial_\varrho)^2$ yields the $\ell=4$ spherical Bessel function (7.21). Polarizing the exact 1D energy identity (7.11) on two monochromatic modes $p_k(t,\varrho)=e^{-i\omega_kt}(\frac{1}{\varrho}\partial_\varrho)^2\psi(\omega_k,\varrho)$ ($k=1,2$) and integrating term-by-term in exponential form $e^{i(\pm\omega_1\pm\omega_2)\varrho}$ over $\varrho\in[0,R]$ yields exact cancellation of every ϱ^{-1} residue (Si/Ci coefficient identically zero) and the elementary closed-form $\widehat{\mathcal{K}}_R^{+}(\omega_1,\omega_2)$ (`verify_05`). Because $\psi(-\omega,\varrho)=-\psi(\omega,\varrho)$, complex conjugation satisfies $\overline{e^{-i\omega t}\psi(\omega,\varrho)}=-e^{+i\omega t}\psi(-\omega,\varrho)$, giving $\widehat{\mathcal{H}}_R(\omega_1,\omega_2)=-\widehat{\mathcal{K}}_R^{+}(\omega_1,-\omega_2)$ and diagonal limit (7.23). For $2n\ge 14$ ($n\ge 7$), the degree-12 polynomial part of (7.23) does not contribute to $[u^{2n}]$, while expanding $\cos(2u)$ and $\sin(2u)$ yields (7.24) directly; since $7n^2-67n+300$ has discriminant $-3911<0$ (and $P(k+9)$ has strictly positive integer coefficients for all $k\ge 0$; `verify_05`), $P(n)>0$ and $d_n\neq 0$ for every integer $n\ge 9$. □

7.4 Single-Packet Unknown-Width Recovery at Any Fixed Radius

Theorem 7.4 immediately settles the single-packet unknown-width inverse problem posed in Remark 6.6: *one* fixed centered radius $R > 0$ already suffices, with zero exceptional radii and no obstruction at pure even or pure odd states.

Theorem 7.5 (Unconditional Single-Packet Unknown-Width Recovery at Any Fixed Radius)**.** *Fix any centered radius $R > 0$ and any nonempty open time interval $I \subset \mathbb{R}$. Let $h = u h^{(a)}_{\mathrm{even}} + w h^{(a)}_{\mathrm{odd}} = \mathrm{Re}(c g_a)$ ($a > 0$, $c = w - iu \neq 0$) be a nonzero localized wavepacket of* unknown *width a and* unknown *coefficients (u, w). Then the measured real-time canonical-energy trace $t \mapsto \mathcal{E}_{R,t}(h)$ on I uniquely determines (a, u, w) up to simultaneous sign reversal $(u, w) \mapsto (-u, -w)$, with* zero *exceptional radii and covering both pure even ($w = 0$) and pure odd ($u = 0$) packets identically.*

Proof. By (7.4) and (7.16), the actual measured real trace is

$$\mathcal{E}_{R,t}(h) = \frac{1}{4} c^2 \, Q_R(a + it) + \frac{1}{4} \bar{c}^2 \, Q_R(-a + it) + \frac{1}{2} |c|^2 \, \mathcal{K}_R(a + it, \, -a + it). \tag{7.25}$$

Because $Q_R(\alpha)$ is meromorphic on $\mathbb{C}$ with poles only at $\alpha = \pm iR$, and $\mathcal{K}_R(\alpha, \beta)$ is holomorphic on $\Omega_R \times \Omega_R$, the function $t \mapsto \mathcal{E}_{R,t}(h)$ is real-analytic on $\mathbb{R}$—hence uniquely determined on $\mathbb{R}$ by its restriction to I—and continues analytically to the cut plane $\mathbb{C} \setminus \big(([-R, R] + ia) \cup ([-R, R] - ia)\big)$. Examine all three terms of (7.25) near the upper-right endpoint $t_* := R + ia$ by setting $t = R + ia + \zeta$ ($\zeta \to 0$), so that $\alpha := a + it = iR + i\zeta$ and $\beta := -a + it = -2a + iR + i\zeta$:

(1) *First complex-bilinear term $\frac{1}{4} c^2 Q_R(a + it)$:* Using (6.25), $\alpha^2 + R^2 = i\zeta(2iR + i\zeta) = -2R\zeta(1 + O(\zeta))$, while the numerator polynomial evaluates at $\alpha = iR$ to $68R^4 - 39R^2(-R^2) + 143R^4 = 250R^4 \neq 0$. Hence this meromorphic term contributes an exact order-12 pole at $t_* = R + ia$ with *R-independent* nonzero leading coefficient (`verify_05`):

$$\lim_{\zeta \to 0} \zeta^{12} \, \frac{1}{4} c^2 Q_R(iR + i\zeta) = \frac{1}{4} c^2 \cdot \frac{6144\pi \, R^8 \cdot 250 R^4}{5005 \, (-2R)^{12}} = \frac{75\,\pi}{4004} \, c^2 \;\neq\; 0. \tag{7.26}$$

(2) *Second complex-bilinear term $\frac{1}{4} \bar{c}^2 Q_R(-a + it)$:* Here $\beta^2 + R^2 = (-2a + iR)^2 + R^2 = 4a(a - iR) \neq 0$ for every $a > 0$ and $R > 0$, so this term is strictly holomorphic at t_*.

(3) *Hermitian term $\frac{1}{2} |c|^2 \mathcal{K}_R(a + it, -a + it)$:* Because $\alpha - \beta = 2a \neq 0$ and $\beta^2 + R^2 = 4a(a - iR) \neq 0$, the only singularities of (7.18) at $\zeta = 0$ are the logarithmic branch point from $\arctan(R/\alpha) = -\frac{1}{2i} \log \zeta + O(1)$ (taken on the cut plane $\mathbb{C} \setminus ([-R, R] + ia)$) and poles $(R^2 + \alpha^2)^{-m} = O(\zeta^{-m})$ for $1 \leq m \leq 5$ (since $P_6 \equiv 0$ and $\alpha + \beta = -2a + 2iR \neq 0$). Thus the Hermitian term is $O(|\zeta|^{-5})$ (including its $\log \zeta$ branch contribution) as $\zeta \to 0$ outside the slit, strictly subleading to ζ^{-12} by seven orders!

Because the Hermitian term also carries a logarithmic branch point at t_*, the full measured trace is not meromorphic at t_*; rather, $t_* = R + ia$ is the unique upper-half-plane **leading singularity of order** 12 **at a slit endpoint** on $\{\mathrm{Re}\, t = R, \mathrm{Im}\, t > 0\}$, defined by the rescaled slit-complement limit

$$c^2 = (w - iu)^2 = \frac{4004}{75\,\pi} \lim_{\substack{\zeta \to 0 \\ R + ia + \zeta \in \mathbb{C} \setminus ([-R, R] + ia)}} \zeta^{12} \, \mathcal{E}_{R, R + ia + \zeta}(h). \tag{7.27}$$

Extracting the unique location $t_* = R + ia$ where this rescaled limit is nonzero recovers $a = \mathrm{Im}\, t_* > 0$ uniquely, and (7.27) determines c^2, hence $c = w - iu$ and (u, w), up to overall sign ± 1—including pure even ($c^2 = -u^2 < 0$) and pure odd ($c^2 = w^2 > 0$) packets, with zero exceptional radii $R > 0$. □

7.5 Finite Superpositions: Single-Radius Lifted Collapse vs. Rank-One Sign Recovery

Now consider a general finite superposition of *unknown* cardinality $N \geq 1$, distinct widths, and nonzero coefficients:

$$h = \sum_{j=1}^{N} \big(u_j h_{\text{even}}^{(a_j)} + w_j h_{\text{odd}}^{(a_j)}\big) = \operatorname{Re} \sum_{j=1}^{N} c_j \, g_{a_j}, \qquad 0 < a_1 < \cdots < a_N, \quad c_j = w_j - i u_j \neq 0. \tag{7.28}$$

For any fixed width tuple $(a_1, \ldots, a_N)$, write $v = (u_1, \ldots, u_N, w_1, \ldots, w_N)^\top \in \mathbb{R}^{2N}$ so that $\mathcal{E}_{R,t}(h) = v^\top A(R,t) v = \operatorname{Tr}(A(R,t)\, vv^\top)$, where the lifted linear map $\mathcal{A}_R : \operatorname{Sym}^2(\mathbb{R}^{2N}) \to C^\omega(\mathbb{R})$ acts by $(\mathcal{A}_R(X))(t) := \operatorname{Tr}(A(R,t)X)$. The next theorem resolves the single-radius (and finite-K-radius) problem sharply, revealing a striking gap between unrestricted lifted injectivity and physical rank-one identifiability.

Theorem 7.6 (Finite Superpositions at One or Finitely Many Fixed Radii)**.** *Let h be a finite superposition (7.28) of unknown $N \geq 1$, widths $0 < a_1 < \cdots < a_N$, and nonzero complex amplitudes $c_j = w_j - i u_j$.*

(i) ***Unconditional recovery of N, all widths, and all c_j^2 at ONE radius $R > 0$:*** *For every $R > 0$ and every nonempty open interval $I \subset \mathbb{R}$, the analytic continuation of $t \mapsto \mathcal{E}_{R,t}(h)$ to $\mathbb{C} \setminus \bigcup_{j=1}^{N}([-R,R] \pm i a_j)$ has leading order-12 slit-endpoint singularities on $\{\operatorname{Re} t = R, \operatorname{Im} t > 0\}$ at precisely $\{R + i a_1, \ldots, R + i a_N\}$, with rescaled limits $\lim_{\zeta \to 0} \zeta^{12} \mathcal{E}_{R, R + i a_j + \zeta}(h) = \frac{75\pi}{4004}\, c_j^2 \neq 0$ (taken outside the slits, strictly dominating the accompanying $O(|\zeta|^{-5})$ poles and $\log \zeta$ branch terms of the Hermitian and mixed cross pairings). Hence $\mathcal{M}_{R,I}$ unconditionally recovers N, $\{a_1, \ldots, a_N\}$, and every amplitude c_j up to an individual sign $\sigma_j \in \{\pm 1\}$.*

(ii) ***Unconditional collapse of the unrestricted lifted map for any finite number K of radii:*** *Fix any $K \geq 1$ radii $R_1, \ldots, R_K > 0$ and any N distinct widths $0 < a_1 < \cdots < a_N$. For each radius R_ℓ and each width a_j, all singularities of $\operatorname{Tr}(A(R_\ell, t)X)$ across the upper slit $[-R_\ell, R_\ell] + i a_j$ are spanned by at most 28 real linear functionals of X (2 from the order-12 diagonal singularity Q_{R_ℓ}, 6 from the quadratic polynomial jump across $(-R_\ell, R_\ell) + i a_j$, and 20 from the order-1 … 5 complex endpoint poles at $\pm R_\ell + i a_j$). Since $\operatorname{Tr}(A(R_\ell, t)X) = O(t^{-20}) \to 0$ as $|t| \to \infty$, Liouville's theorem yields the universal analytic rank bound*

$$\operatorname{rank}\big(\mathcal{A}_{\{R_1,\ldots,R_K\}}\big|_{\operatorname{Sym}^2(\mathbb{R}^{2N})}\big) \;\leq\; 28\,K\,N, \qquad \text{while} \quad \dim \operatorname{Sym}^2(\mathbb{R}^{2N}) = 2N^2 + N. \tag{7.29}$$

Consequently, $\ker \mathcal{A}_{\{R_1,\ldots,R_K\}} \neq \{0\}$ analytically for every $N \geq 14K$ (and on the $t \mapsto -t$-even cross-width subspace $\{(X_{e_j e_k}, X_{o_j o_k})_{j<k}\}$ of dimension $N(N-1)$, each width contributes at most $13K$ real constraints, guaranteeing nontrivial exact lifted null vectors with nullity at least $N(N-1) - 13KN$ for all $N \geq 13K+2$, and indeed earlier via global Cauchy residue relations at infinity). At $K = 1$, $R_1 = 1$, $N = 15$ ($a_j = 1 + \frac{j-1}{10}$), the analytic bound guarantees an exact nullspace of dimension ≥ 15 inside the 210-dimensional t-even cross block; as numerical evidence illustrating this analytic obstruction, `verify_05` constructs an explicit high-precision (100-digit) numerical near-null witness $\Delta \in \operatorname{Sym}^2(\mathbb{R}^{30})$ *normalized to $\|\Delta\|_F = 1$ from the exact symbolic rational 195×210 constraint matrix (with exact two-prime modular lower bound $\operatorname{rank}_{\mathbb{Q}} \geq 184$, 100-digit numerical rank $184/195$ and nullity 26): it satisfies $|\operatorname{Tr}(A(1,t)\Delta)| \leq 1.32 \times 10^{-88}$ at $t \in \{0, 0.5, 1.3\}$ while taking strictly nonzero off-radius absolute trace values $|\operatorname{Tr}(A(R,t)\Delta)| \in [9.6 \times 10^{-20},\ 3.4 \times 10^{-18}]$ at $R \in \{2, \frac{1}{2}\}$ (a relative separation ratio $|\operatorname{Tr}(A(R,t)\Delta)| / |\operatorname{Tr}(A(1,t)\Delta)| > 7 \times 10^{68}$), and its 50-digit numerical spectrum exhibits inertia $(14+, 16-)$ (numerical rank 30, with 4 positive and 4 negative eigenvalues satisfying $|\lambda| > 10^{-3}$), showing it is far from any physical rank-≤ 2 difference $v_1 v_1^\top - v_2 v_2^\top$.*

(iii) ***Physical rank-one relative sign recovery at ONE radius*** $R>0$*: Suppose two finite superpositions* h_1,h_2 *satisfy* $\mathcal{E}_{R,t}(h_1)=\mathcal{E}_{R,t}(h_2)$ *on* I *at a single fixed radius* $R>0$*. By (i),* h_1 *and* h_2 *share the same* $(N,a_1,\dots,a_N)$ *and satisfy* $c_j^{(2)}=\sigma_j c_j^{(1)}$ *with* $\sigma_j\in\{\pm1\}$*. Partition* $\{1,\dots,N\}=S_+\sqcup S_-$ *into* $\sigma_j=\pm1$ *with* $|S_+|=p\geq q=|S_-|$*, and write* $h_\pm:=\operatorname{Re}\sum_{j\in S_\pm}c_j^{(1)}g_{a_j}$*. If* $q\geq1$*, then* $\mathcal{E}_{R,t}(h_+,h_-)\equiv0$ *on* $\mathbb{R}$*, and vanishing of the discontinuity across every open slit* $(-R,R)+ia$ *(*$a\in S_+$*) forces*

$$g_-(a)=g_-'(a)=g_-''(a)=0\qquad\text{and}\qquad h_-(a)=h_-'(a)=h_-''(a)=0\qquad(\forall\,a\in S_+),\tag{7.30}$$

where

$$g_-(x):=\sum_{b\in S_-}w_b\left[\frac{1}{(x-b)^{11}}+\frac{1}{(x+b)^{11}}\right],\qquad h_-(x):=\sum_{b\in S_-}u_b\left[\frac{1}{(x-b)^{11}}-\frac{1}{(x+b)^{11}}\right].\tag{7.31}$$

From the open-slit jump constraints (7.30)–(7.31) alone, physical recovery up to a single global sign $h_2=\pm h_1$ *holds at* any single fixed radius $R>0$*:*

- ***unconditionally for all*** $N\leq5$ *(every partition has* $q\leq2$*) and all widths* $0<a_1<\cdots<a_N$*;*
- ***unconditionally for all*** $N\geq1$ *whenever the partition is separated (*$\max S_-<\min S_+$ *or* $\max S_+<\min S_-$*);*
- ***generically for all*** $N\geq1$*, i.e. for all width tuples* $(a_1,\dots,a_N)$ *outside a proper real-algebraic hypersurface of* $(0,\infty)^N$*.*

Furthermore, Theorem 7.7 below removes the remaining interlaced $N\geq6$ *and non-generic hypersurface caveats completely at the* exact same single fixed radius $R>0$*, establishing unconditional global-sign recovery for* every $N\geq1$ *and* every *width tuple* $0<a_1<\cdots<a_N$ *(as well as arbitrary compactly supported Borel width measures) without any radius variation.*

Proof. (i) Every diagonal term $\frac{1}{4}c_j^2Q_R(a_j+it)$ contributes a meromorphic order-12 pole at $t=R+ia_j$ with coefficient $\frac{75\pi}{4004}c_j^2\neq0$ by (7.26), while every diagonal Hermitian term $\mathcal{K}_R(a_j+it,-a_j+it)$ and every off-diagonal mixed term $\mathcal{K}_R(\pm a_j+it,\pm a_k+it)$ $(j\neq k)$ contributes at most order-5 poles plus $\log\zeta$ branch points along $[-R,R]\pm ia_j$ (since $a_j\neq a_k$ keeps $(\alpha-\beta)$ and $(\alpha+\beta)$ bounded away from 0 at $t=R+ia_j$). Taking the rescaled slit-complement limit $\lim_{\zeta\to0}\zeta^{12}\mathcal{E}_{R,R+ia+\zeta}(h)$ along $\{\operatorname{Re}t=R,\operatorname{Im}t>0\}$ therefore isolates the leading order-12 slit-endpoint singularities cleanly from all lower-order mixed and logarithmic terms, recovering N, $\{a_j\}$, and $\{c_j^2\}$ uniquely.

(ii) For fixed R_ℓ and $X\in\operatorname{Sym}^2(\mathbb{R}^{2N})$, the function $f_\ell(t):=\operatorname{Tr}(A(R_\ell,t)X)$ is real on $\mathbb{R}$ (hence Schwarz-symmetric, $f_\ell(\bar t)=\overline{f_\ell(t)}$), holomorphic on $\mathbb{C}\setminus\bigcup_{j=1}^N([-R_\ell,R_\ell]\pm ia_j)$, and decays as $O(t^{-20})$ as $|t|\to\infty$ (as every term in (7.18) is $O(t^{-20})$ when $t\to\infty$ with $\alpha-\beta$ fixed). By Schwarz reflection, f_ℓ has no singularities in $\mathbb{C}$ iff its singularities on every upper slit $[-R_\ell,R_\ell]+ia_j$ vanish identically, in which case $f_\ell\equiv0$ by Liouville's theorem. Along the open slit $t=s+ia_j$ $(s\in(-R_\ell,R_\ell))$, setting $\alpha_j=is$ and $\beta=\gamma+is$ $(\gamma=\pm a_k-a_j\neq0)$ gives $10R_\ell^2+\alpha_j^2+8\alpha_j\beta+\beta^2=(10R_\ell^2+\gamma^2)+10i\gamma s-10s^2$, so the jump across $(-R_\ell,R_\ell)+ia_j$ is a complex quadratic polynomial in s (3 complex = 6 real functionals of X); at each endpoint $t=\pm R_\ell+ia_j$, the cross-terms contribute poles of orders $1,\dots,5$ (10 complex = 20 real functionals), while the diagonal term $Q_{R_\ell}(a_j+it)$ contributes 1 complex = 2 real functionals (fixed by $X_{e_je_j}-X_{o_jo_j}$ and $X_{e_jo_j}$). Summing over $j=1,\dots,N$ and $\ell=1,\dots,K$ proves (7.29). On the t-even cross-width subspace, $f_\ell(-t)=f_\ell(t)$ forces the singularities at $-R_\ell+ia_j$ to be conjugates of those at $+R_\ell+ia_j$ and makes the jump even in s, leaving at most 13 real functionals per width per radius, hence an exact nullspace of dimension at least $N(N-1)-13KN$ whenever $N\geq13K+2$. For

$K = 1$, $N = 15$, $R_1 = 1$ ($a_j = 1 + \frac{j-1}{10}$), as high-precision numerical evidence illustrating this analytic nullspace, `verify_05` derives the exact symbolic rational right-endpoint pole functions directly from $P_1, \dots, P_5$ via truncated polynomial inversion, checks an exact two-prime modular lower bound $\operatorname{rank}_{\mathbb{Q}}(M_{\mathrm{even}}) \geq 184$ on the 195×210 rational constraint matrix, and performs 100-digit row-scaled Gaussian elimination (numerical rank 184/195, reflecting 11 global Cauchy relations at infinity, numerical nullity 26, with a 38-order pivot gap 2.9×10^{-32} vs. 1.6×10^{-70}) to produce a unit-Frobenius ($\|\Delta\|_F = 1$) high-precision numerical near-null witness (algebraic residual 1.25×10^{-88}, row-scaled residual 2.04×10^{-102}) satisfying $|\operatorname{Tr}(A(1,t)\Delta)| \leq 1.32 \times 10^{-88}$ alongside nonzero off-radius absolute traces $|\operatorname{Tr}(A(R,t)\Delta)| \in [9.6 \times 10^{-20},\ 3.4 \times 10^{-18}]$ at $R \in \{2, \frac{1}{2}\}$ (relative separation ratio $> 7 \times 10^{68}$) and a 50-digit numerical spectrum with inertia $(14+, 16-)$ (numerical rank 30, including 4 positive and 4 negative eigenvalues with $|\lambda| > 10^{-3}$).

(iii) Since $h_1 = h_+ + h_-$ and $h_2 = h_+ - h_-$, we have $\mathcal{E}_{R,t}(h_1) - \mathcal{E}_{R,t}(h_2) = 4\mathcal{E}_{R,t}(h_+, h_-) \equiv 0$. Along the open slit $t = s + ia$ ($a \in S_+$, $s \in (-R, R)$), the only terms of $\mathcal{E}_{R,t}(h_+, h_-)$ with a jump discontinuity are $\frac{1}{2} c_a \sum_{b \in S_-} [c_b \mathcal{K}_R(a + it, b + it) + \bar{c}_b \mathcal{K}_R(a + it, -b + it)]$. Since $c_a \neq 0$, vanishing of the coefficients of s^2, s^1, s^0 in the $\arctan(R/\alpha)$ jump is equivalent to $\sum_{b \in S_-} [c_b (a-b)^{-m} + \bar{c}_b (a+b)^{-m}] = 0$ for $m = 11, 12, 13$. Taking real and imaginary parts (with $c_b = w_b - i u_b$) yields (7.30)–(7.31) identically.

Now analyze (7.30) when $q = |S_-| \geq 1$ (so at least one of $(w_b)_{b \in S_-}$ or $(u_b)_{b \in S_-}$ is nonzero):

- *Case* $q = 1$ ($S_- = \{b_1\}$): For every $x \in (0, \infty) \setminus \{b_1\}$, $|x - b_1| < x + b_1$, so $(x - b_1)^{-11} \pm (x + b_1)^{-11} \neq 0$; thus $g_-(a) = h_-(a) = 0$ forces $w_{b_1} = u_{b_1} = 0$, contradiction.

- *Case* $q = 2$ ($S_- = \{b_1, b_2\}$, $0 < b_1 < b_2$): Write $\phi_k(x) := (x - b_k)^{-11} + (x + b_k)^{-11}$ and $\psi_k(x) := (x - b_k)^{-11} - (x + b_k)^{-11}$ ($k = 1, 2$). A nontrivial solution of $g_-(a) = g_-'(a) = 0$ (or $h_-(a) = h_-'(a) = 0$) at any $a \in S_+$ requires the 2×2 Wronskian $W(\phi_1, \phi_2)(a) = 0$ (or $W(\psi_1, \psi_2)(a) = 0$). We show both Wronskians are strictly nonzero on all three connected components of $(0, \infty) \setminus \{b_1, b_2\}$:

  (1) On (b_1, b_2): $\phi_1(x) > 0$, $\phi_2(x) < 0$, while $\phi_k'(x) = -11[(x-b_k)^{-12} + (x+b_k)^{-12}] < 0$ for both $k = 1, 2$, giving $W(\phi_1, \phi_2)(x) = \phi_1 \phi_2' - \phi_1' \phi_2 < 0$ strictly; identically $\psi_1(x) > 0$, $\psi_2(x) < 0$, $\psi_k'(x) = -11[(x - b_k)^{-12} - (x + b_k)^{-12}] < 0$, giving $W(\psi_1, \psi_2)(x) < 0$.

  (2) On (b_2, ∞): we write

$$\phi_k(x) = \frac{2}{10!} \int_0^\infty s^{10} e^{-sx} \cosh(sb_k)\, \mathrm{d}s, \qquad \psi_k(x) = \frac{2}{10!} \int_0^\infty s^{10} e^{-sx} \sinh(sb_k)\, \mathrm{d}s.$$

  By Andréief's identity,

$$W(\phi_1, \phi_2)(x) = -\frac{4}{(10!)^2} \iint_{0 < s_1 < s_2} (s_1 s_2)^{10} e^{-(s_1+s_2)x} (s_2 - s_1) \times \det\big(\cosh(s_i b_j)\big)_{i,j=1}^2 \, \mathrm{d}s_1\, \mathrm{d}s_2 < 0$$

  because $b \mapsto b \tanh(sb)$ is strictly increasing on $(0, \infty)$ (so $s \mapsto \cosh(sb_2)/\cosh(sb_1)$ is strictly increasing, making the 2×2 determinant strictly positive). Similarly, $b \mapsto b \coth(sb)$ is strictly increasing on $(0, \infty)$, so $s \mapsto \sinh(sb_2)/\sinh(sb_1)$ is strictly increasing and $W(\psi_1, \psi_2)(x) < 0$.

  (3) On $(0, b_1)$: we write

$$\phi_k(x) = -\frac{2}{10!} \int_0^\infty s^{10} e^{-sb_k} \sinh(sx)\, \mathrm{d}s, \qquad \psi_k(x) = -\frac{2}{10!} \int_0^\infty s^{10} e^{-sb_k} \cosh(sx)\, \mathrm{d}s.$$

  Andréief's identity gives $W(\phi_1, \phi_2)(x) < 0$ and $W(\psi_1, \psi_2)(x) < 0$ because $\det(e^{-s_i b_j}) = e^{-s_1 b_1 - s_2 b_2} - e^{-s_1 b_2 - s_2 b_1} < 0$ for $0 < s_1 < s_2$ and $0 < b_1 < b_2$, while both $s \mapsto s \coth(sx)$ and $s \mapsto s \tanh(sx)$ are strictly increasing on $(0, \infty)$.

Since every partition of $N \le 5$ has $q = \min(|S_+|, |S_-|) \le 2$, no sign collision can exist for any $N \le 5$.

- *Separated partitions for arbitrary $N \ge 1$:* If $\max S_- < \min S_+$, then the $q \times q$ evaluation matrix $M_\pm(a,b) := \big((a_i - b_j)^{-11} \pm (a_i + b_j)^{-11}\big)_{i,j=1}^{q}$ (taking the first q points $a_1 < \cdots < a_q$ of S_+) has integral representation $\frac{2}{10!}\int_0^\infty s^{10} e^{-sa_i} \{\substack{\cosh \\ \sinh}\}(sb_j)\, \mathrm{d}s$. On $(0,\infty)$, any nontrivial sum $f(s) = \sum_{j=1}^q c_j \cosh(sb_j) = \frac{1}{2}\sum_{j=1}^q c_j (e^{b_j s} + e^{-b_j s})$ is an even combination of $2q$ distinct real exponentials, which by Descartes' rule of signs has at most $2q-1$ real zeros and hence at most $q-1$ positive zeros (as positive zeros pair with negative zeros); likewise, $g(s) = \sum_{j=1}^q c_j \sinh(sb_j)$ is odd with a zero at $s = 0$, hence also has at most $q-1$ positive zeros. Thus $\det(\cosh(s_i b_j))$ and $\det(\sinh(s_i b_j))$ never vanish on the connected simplex $0 < s_1 < \cdots < s_q$ and are strictly positive (checked as $s_q \gg \cdots \gg s_1$). Since $\det(e^{-s_i a_j})$ likewise has constant nonzero sign $(-1)^{\binom{q}{2}}$ on $0 < s_1 < \cdots < s_q$, the Cauchy–Binet (Andréief) formula yields $(-1)^{\binom{q}{2}} \det M_\pm(a,b) > 0$, hence $\det M_\pm(a,b) \neq 0$! The symmetric case $\max S_+ < \min S_-$ is identical with (a,b) swapped in $e^{-sb}\{\substack{\sinh \\ \cosh}\}(sa)$.
- *Generic width tuples for arbitrary $N \ge 1$:* For any fixed partition $S_+ \sqcup S_-$ ($p \ge q \ge 1$), choose any q elements $a_1, \dots, a_q \in S_+$. The $q \times q$ matrices $M_\pm(a,b) := \big((a_i - b_j)^{-11} \pm (a_i + b_j)^{-11}\big)_{i,j=1}^{q}$ have rational determinants in $(a_1, \dots, a_N)$ that do not vanish identically: letting $a_i = b_i + \varepsilon$ ($i = 1, \dots, q$) as $\varepsilon \to 0^+$ yields $\det M_\pm(b+\varepsilon, b) = \varepsilon^{-11q}(1 + O(\varepsilon^{11})) \neq 0$. Hence the union over all $2^{N-1} - 1$ nontrivial partitions of $\{\det M_+ = 0\} \cup \{\det M_- = 0\}$ is a proper real-algebraic hypersurface in $(0,\infty)^N$, outside which (7.30) forces $w_b = u_b = 0$ for all $b \in S_-$. □

7.6 Infinite-Dimensional Reconstruction: Single-Radius Physical Uniqueness vs. Multi-Radius Lifted Injectivity

Theorem 7.6(ii) shows that any *finite* set of radii collapses the unrestricted lifted map $\mathcal{A}_{\{R_1,\dots,R_K\}}$ once $N \ge 14K$. We now prove that the physical rank-one constraint transforms the inverse problem dramatically: *one* single centered ball of fixed arbitrary radius $R_0 > 0$, observed over any nonempty open time interval $I \subset \mathbb{R}$, already determines *every* perturbation $h[\mu]$ in the entire infinite-dimensional compactly supported Borel measure class $\mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$ unconditionally up to one overall sign (Theorem 7.7)—resolving every interlaced finite superposition ($N \ge 6$) and every overlapping or singular continuous measure in a single stroke. Varying the radius over an open interval (R_1, R_2) is needed *only* to achieve injectivity on the unrestricted lifted symmetric measure space $\mathcal{M}_{\mathrm{sym}}(K \times K)$ (Theorem 7.9(ii)), while also yielding an independent small-R Fourier–Laplace proof of physical recovery (Theorem 7.9(i)).

Theorem 7.7 (Unconditional Single-Radius Infinite-Dimensional Physical Reconstruction)**.** *Fix* any single *centered radius $R_0 > 0$ (independently of the unknown field, with no restriction on R_0 relative to $[a_{\min}, a_{\max}]$), any compact width interval $[a_{\min}, a_{\max}] \subset (0,\infty)$, and any nonempty open time interval $I \subset \mathbb{R}$. Then:*

(i) ***Unconditional physical rank-one injectivity on the full measure class:*** *For every pair of finite complex Borel measures $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$,*

$$\mathcal{E}_{B(R_0,0,t)}\big(h[\mu_1]\big) = \mathcal{E}_{B(R_0,0,t)}\big(h[\mu_2]\big) \quad \forall\, t \in I \qquad \Longrightarrow \qquad \mu_1 = \pm\mu_2 \tag{7.32}$$

(equivalently $h[\mu_1] = \pm h[\mu_2]$, or in polarized form: $\mathcal{E}_{R_0,t}(h[\nu], h[\lambda]) = 0$ for all $t \in \mathbb{R}$ forces $\nu = 0$ or $\lambda = 0$). In particular, every *finite superposition* (7.28) *of arbitrary unknown cardinality $N \ge 1$ and arbitrary distinct widths $0 < a_1 < \cdots < a_N$ (including every interlaced $N \ge 6$ configuration) is unconditionally uniquely determined up to one overall sign from the single-radius trace $\mathcal{M}_{R_0,I}$, with zero exceptional widths and zero exceptional radii.*

(ii) ***Constructive strictly triangular Laurent moment reconstruction at fixed*** $R_0 > 0$***:*** *For any nonzero* $\mu = \mu_w - i\mu_u \in \mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$*, the analytically continued single-radius trace* $\mathcal{E}_{R_0,t}(h[\mu])$ *has an absolutely convergent exterior Laurent expansion* $\mathcal{E}_{R_0,t}(h[\mu]) = \sum_{m=0}^{\infty} E_m\, t^{-(20+m)}$ *on* $\{t \in \mathbb{C} : |t| > R_0 + a_{\max}\}$ *(uniquely fixed by exact data on* I*). Letting*

$$\begin{aligned} c_n[\mu] &:= -48\binom{n+9}{9}\,\mathrm{Re}\Big(i^n \int_{a_{\min}}^{a_{\max}} a^n \,\mathrm{d}\mu(a)\Big) \\ &= \begin{cases} -48(-1)^r\binom{2r+9}{9}\int_{a_{\min}}^{a_{\max}} a^{2r}\,\mathrm{d}\mu_w(a), & n = 2r, \\ -48(-1)^r\binom{2r+10}{9}\int_{a_{\min}}^{a_{\max}} a^{2r+1}\,\mathrm{d}\mu_u(a), & n = 2r+1, \end{cases} \end{aligned} \tag{7.33}$$

and denoting by $k \geq 0$ *the unique smallest index with* $c_k[\mu] \neq 0$*, one has* $E_m = 0$ *for* $0 \leq m < 2k$*,*

$$E_{2k} = \frac{8\pi}{105}\, R_0^8\, c_k[\mu]^2 > 0, \tag{7.34}$$

which fixes k *and* $c_k[\mu]$ *up to one overall sign* ± 1*, and for every integer* $j \geq 1$*,*

$$E_{2k+j} = \frac{16\pi}{105}\, R_0^8\, c_k[\mu]\, c_{k+j}[\mu] + \mathcal{P}_j\big(R_0;\ c_k[\mu], c_{k+1}[\mu], \ldots, c_{k+j-1}[\mu]\big), \tag{7.35}$$

where $\mathcal{P}_j$ *is an explicit universal quadratic polynomial in the preceding coefficients (with all Darboux radial-expansion and bulk canonical-energy integral terms involving only indices* $\leq k+j-2$*). Solving (7.35) sequentially recovers every coefficient* $c_n[\mu]$ *(*$n \geq 0$*), hence every even moment of* μ_w *and odd moment of* μ_u*, determining* μ *uniquely via Stone–Weierstrass / Hausdorff moment uniqueness (*`verify_06`*).*

Proof. (i) Suppose $\mathcal{E}_{R_0,t}(h[\mu_1]) = \mathcal{E}_{R_0,t}(h[\mu_2])$ for all $t \in I$. By Proposition 7.1 and Theorem 7.4(i), each trace $t \mapsto \mathcal{E}_{R_0,t}(h[\mu_j])$ is real-analytic on $\mathbb{R}$ (holomorphic on $\mathbb{C} \setminus ([-R_0, R_0] \pm i[a_{\min}, a_{\max}])$), so equality extends to all $t \in \mathbb{R}$. Setting $\nu := \mu_1 + \mu_2$ and $\lambda := \mu_1 - \mu_2$ in $\mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$, bilinearity of the polarized centered-ball canonical energy (including its surface term) yields

$$\mathcal{E}_{R_0,t}\big(h[\nu], h[\lambda]\big) = \mathcal{E}_{R_0,t}(h[\mu_1]) - \mathcal{E}_{R_0,t}(h[\mu_2]) = 0 \qquad \forall\, t \in \mathbb{R}. \tag{7.36}$$

Assume for contradiction that *both* $\nu \neq 0$ and $\lambda \neq 0$.

Step 1 (Exterior analyticity and moment nonvanishing of the boundary-origin trace). For any $\nu = \nu_w - i\nu_u \in \mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$, rewrite (7.7) at $\varrho = 0$ for real $t \in \mathbb{R}$ as

$$p[\nu](t, 0) = \frac{1}{2}\int_{a_{\min}}^{a_{\max}} \frac{48}{(a+it)^{10}}\,\mathrm{d}\nu(a) + \frac{1}{2}\int_{a_{\min}}^{a_{\max}} \frac{48}{(a-it)^{10}}\,\mathrm{d}\overline{\nu}(a).$$

Because $|a \pm iz| \geq |z| - a_{\max} > 0$ uniformly for $a \in [a_{\min}, a_{\max}]$ whenever $|z| > a_{\max}$, the right-hand side defines a holomorphic function $P_\nu(z)$ on $\{z \in \mathbb{C} : |z| > a_{\max}\}$ agreeing with $p[\nu](t, 0)$ for all real $|t| > a_{\max}$. Setting $u := 1/z$ ($|u| < a_{\max}^{-1}$) and using $(\pm i)^{10} = -1$, we have the uniformly convergent binomial expansions $(a \pm iz)^{-10} = -u^{10}(1 \mp iau)^{-10} = -u^{10}\sum_{n=0}^{\infty}\binom{n+9}{9}(\pm ia)^n u^n$ on $[a_{\min}, a_{\max}]$. Integrating term-by-term against ν and $\overline{\nu}$ gives the convergent exterior Laurent series

$$P_\nu(z) = \sum_{n=0}^{\infty} c_n[\nu]\, z^{-(10+n)} \qquad (|z| > a_{\max}), \tag{7.37}$$

with real coefficients $c_n[\nu]$ given verbatim by (7.33) (verified symbolically in `verify_06`). We claim that $c_n[\nu]$ cannot vanish for all $n \geq 0$: if $c_{2r}[\nu] = 0$ for all $r \geq 0$, then $\int_{a_{\min}}^{a_{\max}} (a^2)^r\,\mathrm{d}\nu_w(a) = 0$

for all $r \geq 0$; since $a \mapsto a^2$ is strictly increasing (hence point-separating) on the compact interval $[a_{\min}, a_{\max}] \subset (0, \infty)$, the subalgebra $\mathbb{R}[a^2]$ is uniformly dense in $C([a_{\min}, a_{\max}], \mathbb{R})$ by the Stone–Weierstrass theorem, forcing $\nu_w = 0$ by the Riesz representation theorem. Likewise, $c_{2r+1}[\nu] = 0$ for all $r \geq 0$ implies $\int_{a_{\min}}^{a_{\max}} (a^2)^r (a \, \mathrm{d}\nu_u(a)) = 0$ for all $r \geq 0$, forcing $a \, \mathrm{d}\nu_u = 0$ and hence $\nu_u = 0$ since $a \geq a_{\min} > 0$. Thus $\nu \neq 0$ implies there exists a unique smallest integer $k \geq 0$ with $c_k[\nu] \neq 0$, and identically $\lambda \neq 0$ implies there exists a unique smallest integer $m \geq 0$ with $c_m[\lambda] \neq 0$.

Step 2 (Uniform large-t Darboux expansion on $[0, R_0]$ and Cauchy derivative bounds). Set $T_* := 4(R_0 + a_{\max})$. For every real $t \geq T_*$, every $\varrho \in [0, R_0]$, and every $a \in [a_{\min}, a_{\max}]$, we have $|a \pm it| = \sqrt{a^2 + t^2} \geq t > 4R_0 \geq 4\varrho$, so $\varrho^2/|a \pm it|^2 \leq 1/16 < 1$. Expanding $((a \pm it)^2 + \varrho^2)^{-5} = \sum_{r=0}^{\infty} (-1)^r \binom{r+4}{4} \varrho^{2r} (a \pm it)^{-(10+2r)}$ and applying the exact Darboux coefficient identity $c_r (10)_{2r} = \binom{r+4}{4}$ with $c_r := [4^r r! (11/2)_r]^{-1}$ together with $\partial_t^{2r}[(a \pm it)^{-10}] = (-1)^r (10)_{2r} (a \pm it)^{-(10+2r)}$ gives, for every $t \geq T_*$ and $\varrho \in [0, R_0]$,

$$p[\nu](t, \varrho) = \sum_{r=0}^{\infty} c_r \, \varrho^{2r} \, \partial_t^{2r} P_\nu(t), \tag{7.38}$$

where absolute and uniform convergence on $[a_{\min}, a_{\max}] \times [0, R_0] \times [T_*, \infty)$ (dominated by geometric ratio $\leq 1/16$) justifies both Fubini integration against $\nu, \overline{\nu}$ and termwise differentiation in (t, ϱ) through second order.

Now write $P_\nu(z) = z^{-(10+k)} g_\nu(1/z)$, where $g_\nu(u) := \sum_{j=0}^{\infty} c_{k+j}[\nu] \, u^j$ is holomorphic on $\{|u| < a_{\max}^{-1}\}$, and set $M_\nu := \sup_{|u| \leq (2a_{\max})^{-1}} |g_\nu(u)| < \infty$. For any real $t \geq T_* = 4(R_0 + a_{\max})$, every z on the circle $|z - t| = t/2$ satisfies $|z| \geq t/2 \geq 2(R_0 + a_{\max}) > 2a_{\max}$, hence $|P_\nu(z)| \leq M_\nu (t/2)^{-(10+k)} = 2^{10+k} M_\nu \, t^{-(10+k)}$. By Cauchy's integral formula for the n-th derivative at t,

$$\left|\partial_t^n P_\nu(t)\right| \;\leq\; \frac{n!}{(t/2)^n} \sup_{|z-t|=t/2} |P_\nu(z)| \;\leq\; 2^{10+k} M_\nu \, n! \, 2^n \, t^{-(10+k+n)} \qquad (\forall \, n \geq 0, \;\; t \geq T_*). \tag{7.39}$$

Observe the exact factorial ratio

$$c_r \, (2r)! = \frac{4^r \, r! \, (1/2)_r}{4^r \, r! \, (11/2)_r} = \frac{(1/2)_r}{(11/2)_r} \;\leq\; 1 \qquad (\forall \, r \geq 0), \tag{7.40}$$

and note that $2R_0/t \leq \frac{1}{2}$ for $t \geq T_*$. Substituting (7.39)–(7.40) into (7.38) and its termwise derivatives yields uniformly for $\varrho \in [0, R_0]$ as $t \to +\infty$:

$$\begin{aligned}
\left|p[\nu](t, \varrho) - P_\nu(t)\right| &\leq 2^{10+k} M_\nu \, t^{-(10+k)} \sum_{r=1}^{\infty} \left(\frac{2R_0}{t}\right)^{2r} = O\big(t^{-(12+k)}\big), \\
\left|\partial_t p[\nu](t, \varrho)\right| &\leq 2^{11+k} M_\nu \, t^{-(11+k)} \sum_{r=0}^{\infty} (2r+1) \left(\frac{2R_0}{t}\right)^{2r} = O\big(t^{-(11+k)}\big), \\
\left|\partial_\varrho p[\nu](t, \varrho)\right| &\leq \frac{2^{10+k} M_\nu}{R_0} \, t^{-(10+k)} \sum_{r=1}^{\infty} 2r \left(\frac{2R_0}{t}\right)^{2r} = O\big(t^{-(12+k)}\big), \\
\left|\partial_{tt} p[\nu](t, \varrho)\right| &\leq 2^{12+k} M_\nu \, t^{-(12+k)} \sum_{r=0}^{\infty} (2r+2)(2r+1) \left(\frac{2R_0}{t}\right)^{2r} = O\big(t^{-(12+k)}\big), \\
\left|\partial_{t\varrho} p[\nu](t, \varrho)\right| &\leq \frac{2^{11+k} M_\nu}{R_0} \, t^{-(11+k)} \sum_{r=1}^{\infty} 2r(2r+1) \left(\frac{2R_0}{t}\right)^{2r} = O\big(t^{-(13+k)}\big).
\end{aligned}$$

In particular, at $\varrho = R_0$ we have $p[\nu](t, R_0) = c_k[\nu] \, t^{-(10+k)} + O\big(t^{-(11+k)}\big)$, and the exact same estimates hold for λ with $(k, c_k[\nu])$ replaced by $(m, c_m[\lambda])$.

Step 3 (Boundary dominance in polarized Theorem 7.2(ii)). Substitute these uniform estimates on $\varrho \in [0, R_0]$ into the polarized real-space canonical-energy identity (7.11) of Theorem 7.2(ii) at the fixed radius $R_0 > 0$:

- the boundary term gives $\frac{8\pi}{105}R_0^8\,p[\nu](t,R_0)\,p[\lambda](t,R_0) = \frac{8\pi}{105}R_0^8\,c_k[\nu]\,c_m[\lambda]\,t^{-(20+k+m)} + O\big(t^{-(21+k+m)}\big)$;
- the four bulk integrand products scale uniformly on $[0,R_0]$ as $\partial_t p[\nu]\,\partial_t p[\lambda] = O(t^{-(22+k+m)})$, $\partial_\varrho p[\nu]\,\partial_\varrho p[\lambda] = O(t^{-(24+k+m)})$, $\partial_{tt} p[\nu]\,\partial_{tt} p[\lambda] = O(t^{-(24+k+m)})$, and $\partial_{t\varrho} p[\nu]\,\partial_{t\varrho} p[\lambda] = O(t^{-(26+k+m)})$, so the entire bulk integral over $[0,R_0]$ is $O\big(t^{-(22+k+m)}\big)$.

Hence

$$\lim_{t\to+\infty} t^{20+k+m}\,\mathcal{E}_{R_0,t}\big(h[\nu],h[\lambda]\big) = \frac{8\pi}{105}\,R_0^8\,c_k[\nu]\,c_m[\lambda] \neq 0, \tag{7.41}$$

contradicting (7.36)! Therefore $\nu = 0$ or $\lambda = 0$, i.e. $\mu_1 = \pm\mu_2$ (and $h[\mu_1] = \pm h[\mu_2]$ by Proposition 7.1(iii)).

(ii) On $\{t\in\mathbb{C} : |t| > R_0 + a_{\max}\}$, the reverse triangle inequality gives $|a \pm it| \geq |t| - a_{\max} > R_0 \geq \varrho$ uniformly for $(a,\varrho)\in[a_{\min},a_{\max}]\times[0,R_0]$, so both the binomial expansion (7.37) and the Darboux series (7.38) converge absolutely and uniformly on compact subsets of $\{|t| > R_0 + a_{\max}\}$ (and termwise through second derivatives), yielding a convergent exterior Laurent expansion $\mathcal{E}_{R_0,t}(h[\mu]) = \sum_{m=0}^{\infty} E_m\,t^{-(20+m)}$. Substituting $P_\mu(t) = \sum_{n=k}^{\infty} c_n[\mu]\,t^{-(10+n)}$ into (7.38) and (7.11), each monomial pair $(t^{-(10+a)}, t^{-(10+b)})$ produces a series $\sum_{s=0}^{\infty} W_s(a,b;R_0)\,t^{-(20+a+b+2s)}$ whose $s=0$ boundary term is $W_0(a,b;R_0) = \frac{8\pi}{105}R_0^8$ and whose $s\geq 1$ Darboux and bulk terms carry R_0^{8+2s} (for example, $W_1(a,b;R_0) = \frac{2\pi R_0^{10}}{1155}(2A^2+AB+2A+2B^2+2B)$ with $A = 10+a, B = 10+b$; `verify_06`). Collecting powers of t^{-1} gives $E_m = 0$ for $m < 2k$, $E_{2k} = \frac{8\pi}{105}R_0^8 c_k[\mu]^2 > 0$, and at order $m = 2k+j$ $(j\geq 1)$:

$$\begin{aligned} E_{2k+j} = {}& \frac{16\pi}{105}R_0^8\,c_k[\mu]\,c_{k+j}[\mu] + \frac{8\pi}{105}R_0^8\sum_{a=k+1}^{k+j-1} c_a[\mu]\,c_{2k+j-a}[\mu] \\ & + \sum_{s=1}^{\lfloor j/2\rfloor}\sum_{a=k}^{k+j-2s} W_s(a,2k+j-2s-a;R_0)\,c_a[\mu]\,c_{2k+j-2s-a}[\mu], \end{aligned}$$

which is (7.35). Since $c_k[\mu]\neq 0$, solving sequentially for $j = 1,2,\ldots$ determines $\{c_n[\mu]\}_{n\geq 0}$ uniquely up to the single overall sign of $c_k[\mu]$, which determines $\mu = \mu_w - i\mu_u$ uniquely by Step 1. (In `verify_06`, this exact triangular recursion recovers 12 consecutive moments $c_2,\ldots,c_{13}$ of a mixed atomic-plus-continuous complex measure with $c_0 = c_1 = 0$ at a single large fixed radius $R_0 = 5 > a_{\max} = 2$ to relative error $< 2\times 10^{-33}$, and confirms (7.41) on both maximally interlaced $N = 6$ atomic partitions across $R_0\in\{0.2,1,5\}$ and overlapping continuous densities at $R_0 = 10$ with high vanishing orders $k = 5, m = 4$.) □

Remark 7.8 (Mechanism of the single-radius physical-vs-lifted dichotomy, and noiseless uniqueness vs. noisy stability). Two structural lessons follow immediately from Theorem 7.7 juxtaposed with Theorems 7.6(ii) and 7.11:

(1) **Why unrestricted lifted states evade Theorem 7.7:** For any physical rank-one cross-pair (ν,λ), the boundary term $\frac{8\pi}{105}R_0^8 c_k[\nu]c_m[\lambda]\,t^{-(20+k+m)}$ is strictly two orders in t^{-1} larger than the bulk integral $O(t^{-(22+k+m)})$ and cannot cancel. By contrast, an indefinite high-rank lifted matrix $\Delta\in\ker\mathcal{A}_{R_0}$ can cancel its leading boundary moments across opposite-sign eigenmodes and then cancel every subsequent boundary order identically against the bulk integral! As high-precision numerical evidence illustrating this mechanism, evaluating the $N = 15, R_0 = 1$ numerical near-null witness $\Delta\in\mathrm{Sym}^2(\mathbb{R}^{30})$ of Theorem 7.6(ii) at large time (`verify_06`) confirms directly that at $t = 200$ its boundary and bulk traces are individually nonzero and opposite ($\pm 1.68\times 10^{-67}$, scaling as t^{-24}), canceling to relative accuracy $< 7\times 10^{-62}$ ($|\mathrm{bnd}+\mathrm{bulk}|\approx 1.16\times 10^{-128}$). Varying R over (R_1,R_2) breaks this fixed-R_0 boundary-vs-bulk cancellation by resolving the distinct R^{8+2s} scalings, restoring full lifted injectivity in Theorem 7.9(ii) below.

(2) **Noiseless uniqueness vs. finite-window noisy stability:** Theorem 7.7 resolves exact *noiseless* single-radius uniqueness on the entire infinite-dimensional class $\mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])/\{\pm 1\}$ for every $R_0 > 0$. Under additive noise on a finite continuous observation window $[-T, T]$, however, the distinction between finite-N and continuum recovery remains sharp: for fixed cardinality $N \geq 1$ on any compact separated parameter set K_{par}, combining Theorem 7.7(i) with the unconditional $3N$-parameter Jacobian injectivity of Theorem 7.11(iv)(a) yields unconditional *compact-global Lipschitz stability on the sign quotient* $K_{\mathrm{par}}/\langle S_{-1}\rangle$ *even at one fixed radius* $R_0 > 0$ for all $N \geq 1$ (Theorem 7.11(iv)(b), whereas finite discrete sample grids require separate rank and global injectivity hypotheses, part (iv)(c)); for continuous densities $\rho \in L^2([a_{\min}, a_{\max}])$, extracting the large-$t$ Laurent moments (7.35) from noisy data on $[-T, T]$ compounds analytic continuation to $t \to \infty$ (across $O(t^{-20})$ signal decay) with Hausdorff moment inversion, and therefore remains subject to the unconditional Bernstein exponential/logarithmic ill-conditioning barrier of Theorem 7.11(iii).

Theorem 7.9 (Multi-Radius Centered-Ball Reconstruction: Small-R Physical Proof and Unrestricted Lifted Injectivity)**.** *Fix any compact width interval* $[a_{\min}, a_{\max}] \subset (0, \infty)$*, any nonempty open time interval* $I \subset \mathbb{R}$*, and any nonempty open radius interval* $(R_1, R_2) \subset (0, \infty)$ *(or any sequence* $\{R_n\}_{n\geq 1} \subset (0, \infty)$ *having an accumulation point in* $[0, \infty)$*).*

(i) ***Independent small-R physical rank-one proof:*** *For every* $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$*,*

$$\mathcal{E}_{B(R,0,t)}\big(h[\mu_1]\big) = \mathcal{E}_{B(R,0,t)}\big(h[\mu_2]\big) \ \ \forall\, (R,t) \in (R_1, R_2) \times I \implies \mu_1 = \pm\mu_2 \tag{7.42}$$

follows directly from the $R \to 0^+$ *trace limit* (7.13) *and Fourier–Laplace uniqueness (providing an independent fixed-time counterpart to the fixed-radius large-t proof of Theorem 7.7).*

(ii) ***Unrestricted lifted infinite-dimensional injectivity:*** *Let* $K := [-a_{\max}, -a_{\min}] \cup [a_{\min}, a_{\max}]$ *and let* $\Gamma \in \mathcal{M}_{\mathrm{sym}}(K \times K)$ *be any finite complex Borel measure symmetric under* $(x, y) \mapsto (y, x)$*. Then*

$$\iint_{K\times K} \mathcal{K}_R(x+it,\ y+it)\, \mathrm{d}\Gamma(x,y) = 0 \quad \forall\, (R,t) \in (R_1, R_2)\times I \qquad \Longrightarrow \qquad \Gamma \equiv 0. \tag{7.43}$$

Proof. First observe that for any $\Gamma \in \mathcal{M}_{\mathrm{sym}}(K \times K)$, the function

$$\mathcal{F}_\Gamma(R,t) := \iint_{K\times K} \mathcal{K}_R(x+it,\ y+it)\, \mathrm{d}\Gamma(x,y)$$

is jointly real-analytic in $(R, t) \in (0, \infty) \times \mathbb{R}$ and extends holomorphically in R to the disk $\{|R| < a_{\min}\}$ at every fixed $t \in \mathbb{R}$ (indeed, $(x + it)^2 + R^2$ vanishes only when $R = \pm i(x + it)$, so $|R| < |x| \leq |x + it|$ avoids every pole and branch point uniformly for $x \in K$ and $t \in \mathbb{R}$!). By the identity theorem for analytic functions in two variables, $\mathcal{F}_\Gamma(R, t) = 0$ on $(R_1, R_2) \times I$ (or on $\{R_n\} \times I$) forces $\mathcal{F}_\Gamma(R, t) \equiv 0$ for all $(R, t) \in (0, \infty) \times \mathbb{R}$ and for all $|R| < a_{\min}, t \in \mathbb{R}$.

(i) *Direct physical rank-one proof (self-contained and independent of part (ii)):* Notice that this physical reconstruction proof relies solely on the small-R boundary-trace limit of Theorem 7.2(iii) and Fourier–Laplace uniqueness, completely independently of the lifted small-R expansion in part (ii). Suppose $\mathcal{E}_{B(R,0,t)}(h[\mu_1]) = \mathcal{E}_{B(R,0,t)}(h[\mu_2])$ on $(R_1, R_2) \times I$. By joint real-analyticity as above, the equality extends to all $(R, t) \in (0, \infty) \times \mathbb{R}$. Applying the exact small-R limit (7.13) of Theorem 7.2(iii) gives

$$p[\mu_1](t, 0)^2 = p[\mu_2](t, 0)^2 \qquad \forall\, t \in \mathbb{R}.$$

Thus $\big(p[\mu_1](t, 0) - p[\mu_2](t, 0)\big)\big(p[\mu_1](t, 0) + p[\mu_2](t, 0)\big) \equiv 0$ on $\mathbb{R}$. Because $p[\mu_j](t, 0) = 48\,\mathrm{Re}\int_{a_{\min}}^{a_{\max}} (a + it)^{-10}\, \mathrm{d}\mu_j(a)$ is real-analytic on $\mathbb{R}$ (Proposition 7.1(i)), the ring of real-analytic

functions on $\mathbb{R}$ is an integral domain, so there exists a *single* global sign $\sigma \in \{\pm 1\}$ such that $p[\mu_1](t,0) = \sigma\, p[\mu_2](t,0)$ for all $t \in \mathbb{R}$. Finally, compute the Fourier transform of $p[\mu](t,0)$ for any $\omega > 0$: since $(a+it)^{-10} = -(t-ia)^{-10}$ has a single pole at $t = +ia$ in the upper half-plane while $(a-it)^{-10}$ has its pole at $t = -ia$ in the lower half-plane, closing the contour in the upper half-plane (and applying Fubini's theorem on the $L^1(\mathbb{R})$ integrand $O(|t|^{-10})$) gives the exact Fourier–Laplace inversion identity (verified numerically to $< 6 \times 10^{-16}$ in `verify_05`):

$$\begin{aligned}\int_{-\infty}^{\infty} e^{+i\omega t}\, p[\mu](t,0)\, \mathrm{d}t &= \frac{1}{2}\int_{a_{\min}}^{a_{\max}} \left[\int_{-\infty}^{\infty} e^{+i\omega t}\, \frac{48}{(a+it)^{10}}\, \mathrm{d}t\right] \mathrm{d}\mu(a) \\ &= +\frac{\pi}{7560}\, \omega^9\, \mathcal{L}[\mu](\omega) \qquad (\omega > 0). \end{aligned} \tag{7.44}$$

Hence $\mathcal{L}[\mu_1](\omega) = \sigma\, \mathcal{L}[\mu_2](\omega)$ for all $\omega > 0$, forcing $\mu_1 = \sigma\, \mu_2$ by uniqueness of the finite Borel Laplace transform.

(ii) *Unrestricted lifted Stone–Weierstrass proof (direct real-space small-R chain for all signed widths):* We derive the uniform small-R expansion of $\mathcal{K}_R(x+it, y+it)$ and the all-orders diagonal coefficient $\tilde{\gamma}_{k,k} > 0$ directly from the polarized real-space identity of Theorem 7.2(ii), avoiding any need to continue a double-Laplace integral to negative or mixed signed widths.

First, for *any* complex parameter $\alpha = x + it$ with $x \in K = [-a_{\max}, -a_{\min}] \cup [a_{\min}, a_{\max}]$ (whether $x > 0$ or $x < 0$) and $t \in \mathbb{R}$, we have $|\alpha| \geq |x| \geq a_{\min} > 0$. Writing $p_\alpha(\tilde{t}, \varrho) := 48\big(-(\tilde{t} - i\alpha)^2 + \varrho^2\big)^{-5}$, its binomial series in ϱ^2/α^2 converges absolutely and uniformly (together with all $\tilde{t}$- and ϱ-derivatives) on $\varrho \in [0, R]$ whenever $|R| < a_{\min}$, uniformly over $(x,t) \in K \times \mathbb{R}$. Since $\partial_{\tilde{t}}^{2m}[48(\alpha + i\tilde{t})^{-10}]|_{\tilde{t}=0} = 48(-1)^m (10)_{2m} \alpha^{-(10+2m)}$ and

$$c_m := \frac{1}{4^m\, m!\, (11/2)_m} \quad \text{satisfies} \quad c_m\, (10)_{2m} = \frac{(2m+9)!}{9!\, 4^m\, m!\, (11/2)_m} = \binom{m+4}{4} \qquad (\forall\, m \geq 0), \tag{7.45}$$

we have the exact uniformly convergent Darboux–Taylor representation (matching the radial wave recursion $(\partial_\varrho^2 + \frac{10}{\varrho}\partial_\varrho)\varrho^{2m} = 2m(2m+9)\varrho^{2m-2}$):

$$p_\alpha(\tilde{t}, \varrho) = \sum_{m=0}^{\infty} c_m\, \varrho^{2m}\, \partial_{\tilde{t}}^{2m} p_\alpha(\tilde{t}, 0) \qquad (\varrho \in [0, R],\ |R| < a_{\min},\ x \in K,\ \tilde{t} \text{ near } 0). \tag{7.46}$$

Substituting (7.46) for (p_α, p_β) ($\alpha = x + it, \beta = y + it$; all sign combinations of $x, y \in K$ permitted!) into the polarized real-space identity (7.11) of Theorem 7.2(ii) and integrating term-by-term over $\varrho \in [0, R]$ yields an absolutely and uniformly convergent even power series on $|R| < a_{\min}$:

$$\mathcal{K}_R(x + it,\ y + it) = \sum_{k=0}^{\infty} R^{8+2k}\, S_k(x + it,\ y + it), \tag{7.47}$$

where $S_k(x + it, y + it) = \mathcal{D}_k(f, g)(t)$ is a universal symmetric bilinear differential operator of total order $2k$ acting on $f(t) := 48(x + it)^{-10}$ and $g(t) := 48(y + it)^{-10}$ (inspecting (7.11): $R^8 p(R)^2$, $R^{-1}\int_0^R \varrho^{10} p_\varrho^2$, $R^{-1}\int_0^R \varrho^{10} p_t^2$, $R^{-1}\int_0^R \varrho^{10}(R^2 - \varrho^2) p_{t\varrho}^2$, and $R^{-1}\int_0^R \varrho^{10}(R^2 - \varrho^2) p_{tt}^2$ each couple $f^{(a)} g^{(b)}$ with $a + b = 2k$ and $a \equiv b \pmod 2$ at order R^{8+2k}). Its bivariate symbol $q_k(z_1, z_2) = \sum_{a+b=2k} \kappa_{a,b}^{(k)} z_1^a z_2^b$ is a homogeneous symmetric polynomial of even degree $2k$, hence expands uniquely in the elementary symmetric basis $\{(z_1 + z_2)^{2(k-j)}(z_1 z_2)^j\}_{j=0}^k$ as $q_k(z_1, z_2) = \sum_{j=0}^k \alpha_{k,j}(z_1 + z_2)^{2(k-j)}(z_1 z_2)^j$. Translating $(z_1 + z_2) \leftrightarrow \partial_t$ via Leibniz's rule gives the universal real-space operator identity

$$\mathcal{D}_k(f, g)(t) = \sum_{j=0}^{k} \alpha_{k,j}\, \partial_t^{2(k-j)}\big[f^{(j)}(t)\, g^{(j)}(t)\big]. \tag{7.48}$$

To determine the top diagonal coefficient $\alpha_{k,k}$ for *every* $k \geq 0$, evaluate the exact same universal real-space expansion $\mathcal{E}_{R,t}(p_1, p_2) = \sum_{k=0}^{\infty} R^{8+2k} \mathcal{D}_k(p_1(\cdot,0), p_2(\cdot,0))(t)$ on the monochromatic pair $p_1(t,\varrho) := e^{-i\omega t} \sum_{m=0}^{\infty} (-1)^m c_m (\omega\varrho)^{2m} = \frac{7560}{\omega^9} e^{-i\omega t} (\frac{1}{\varrho}\partial_\varrho)^2 \psi(\omega,\varrho)$ and $p_2(t,\varrho) := \overline{p_1(t,\varrho)}$ ($\omega > 0$, verified symbolically in `verify_05`). On the right-hand side of (7.48), $p_1^{(j)}(t,0)\, p_2^{(j)}(t,0) = (-i\omega)^j (+i\omega)^j = \omega^{2j}$ is independent of t, so every term with $j < k$ is annihilated by $\partial_t^{2(k-j)}$, leaving $\mathcal{E}_{R,t}(p_1,p_2) = \sum_{k=0}^{\infty} \alpha_{k,k} R^{8+2k} \omega^{2k}$. On the left-hand side, by definition of the diagonal Hermitian frequency kernel (7.23)–(7.24), $\mathcal{E}_{R,t}(p_1,p_2) = (7560/\omega^9)^2 \widehat{\mathcal{H}}_R(\omega,\omega) = 7560^2 \pi \sum_{k=0}^{\infty} d_{k+9} R^{8+2k} \omega^{2k}$. Comparing coefficients of $R^{8+2k}\omega^{2k}$ yields $\alpha_{k,k} = 7560^2 \pi d_{k+9}$ for all $k \geq 0$.

Now substitute $f(t) = 48(x+it)^{-10}$ and $g(t) = 48(y+it)^{-10}$ back into (7.48). Since $9!/48 = 7560$ gives $f^{(j)}(t) = 48(-i)^j (10)_j (x+it)^{-(10+j)} = \frac{(-i)^j (9+j)!}{7560} (x+it)^{-(10+j)}$, we have $f^{(j)}(t) g^{(j)}(t) = \frac{(-1)^j (9+j)!^2}{7560^2} (x+it)^{-(10+j)} (y+it)^{-(10+j)}$, and setting $\tilde{\gamma}_{k,j} := (-1)^k \frac{(9+j)!^2}{7560^2} \alpha_{k,j}$ yields the exact convergent expansion on $|R| < a_{\min}$ for *all* signed widths $x, y \in K$:

$$S_k(x+it, y+it) = \sum_{j=0}^{k} \tilde{\gamma}_{k,j}\, (-1)^{k-j}\, \partial_t^{2(k-j)} \Big[(x+it)^{-(10+j)} (y+it)^{-(10+j)} \Big], \tag{7.49}$$

where at $j = k$ the factor 7560^2 cancels identically, yielding for *every* integer $k \geq 0$:

$$\tilde{\gamma}_{k,k} = (-1)^k\, (9+k)!^2\, \pi\, d_{k+9} = \frac{\pi\, 2^{2k+18}\, (9+k)!^2}{591360\, (2k+18)!}\, P(k+9) \;>\; 0 \tag{7.50}$$

by (7.24)! (As an independent consistency check, on the positive quadrant $\operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0$ where the double-Laplace formula converges directly, (7.49) agrees term-by-term with the double-Laplace transform of the R^{8+2k} Taylor coefficient of $\widehat{\mathcal{K}}_R^+(\omega_1,\omega_2)$, and because each $S_k(\alpha,\beta)$ is a polynomial in α^{-1}, β^{-1}, that algebraic formula automatically continues to all $\alpha, \beta \in \mathbb{C}\setminus\{0\}$; `verify_05` verifies (7.49)–(7.50) symbolically for $k = 0,1,2,3$—with $\tilde{\gamma}_{0,0} = \frac{6144\pi}{35}$, $\tilde{\gamma}_{1,1} = \frac{92160\pi}{77}$, $\tilde{\gamma}_{2,2} = \frac{5038080\pi}{1001}$, $\tilde{\gamma}_{3,3} = \frac{2359296\pi}{143}$—and numerically across positive, negative, and mixed signed widths.)

Because (7.47) converges uniformly on $K \times K$ for every $|R| < a_{\min}$ and fixed $t \in \mathbb{R}$, Fubini's theorem integrates term-by-term against the finite measure $\Gamma \in \mathcal{M}_{\mathrm{sym}}(K \times K)$, so $\mathcal{F}_\Gamma(R,t) \equiv 0$ for $|R| < a_{\min}$ forces $\iint_{K\times K} S_k(x+it, y+it)\, \mathrm{d}\Gamma(x,y) \equiv 0$ for every $k \geq 0$ and every $t \in \mathbb{R}$. Define $M_j(t) := \iint_{K\times K} (x+it)^{-(10+j)} (y+it)^{-(10+j)}\, \mathrm{d}\Gamma(x,y)$ (differentiation in t commutes with integration against Γ since $|x+it| \geq a_{\min} > 0$). By strong induction on $k \geq 0$: for $k = 0$, $\tilde{\gamma}_{0,0} M_0(t) \equiv 0 \implies M_0(t) \equiv 0$; assuming $M_0 \equiv \cdots \equiv M_{k-1} \equiv 0$ on $\mathbb{R}$, all lower terms $\partial_t^{2(k-j)} M_j(t)$ ($0 \leq j \leq k-1$) vanish identically, leaving $\tilde{\gamma}_{k,k} M_k(t) \equiv 0$, whence $M_k(t) \equiv 0$ for all $t \in \mathbb{R}$ since $\tilde{\gamma}_{k,k} > 0$!

Finally, differentiate $M_k(t) \equiv 0$ at $t = 0$: let $u := x^{-1}$ and $v := y^{-1}$ on the compact set $K^{-1} := [-a_{\min}^{-1}, -a_{\max}^{-1}] \cup [a_{\max}^{-1}, a_{\min}^{-1}]$. Because $\frac{\mathrm{d}}{\mathrm{d}(it)}\Big|_{t=0} (x+it)^{-n} = -n x^{-(n+1)}$, differentiating $M_k(t) \equiv 0$ with respect to it exactly m times at $t = 0$ yields

$$\iint_{K\times K} u^{10} v^{10}\, (uv)^k \sum_{r=0}^{m} \binom{m}{r} (10+k)_r (10+k)_{m-r}\, u^r v^{m-r}\, \mathrm{d}\Gamma(x,y) = 0 \qquad \forall\, k, m \geq 0.$$

Observe that for fixed k and each degree $m \geq 0$, the homogeneous symmetric polynomial $q_{k,m}(u,v) := \sum_{r=0}^{m} \binom{m}{r} (10+k)_r (10+k)_{m-r} u^r v^{m-r}$ has strictly positive leading coefficient $(10+k)_m$ on $u^m + v^m$, and its highest power of $(u+v)$ is $(10+k)_m (u+v)^m$ modulo lower powers of $(u+v)$ multiplied by $(uv)^s$ ($s \geq 1$)! Hence by induction on m (for all $k \geq 0$ simultaneously), the linear span of $\{(uv)^k q_{k,m}(u,v) : k, m \geq 0\}$ equals the *entire* algebra of symmetric polynomials $\mathbb{C}[u+v, uv]$ in (u,v)! Because $u^{10} v^{10} \geq a_{\max}^{-20} > 0$ everywhere on $K \times K$, the symmetric complex

measure $u^{10}v^{10}\,\mathrm{d}\Gamma$ annihilates $\mathbb{C}[u+v, uv]$—which separates points of the compact quotient $(K \times K)/\mathbb{Z}_2$ and is closed under conjugation—so the Stone–Weierstrass theorem and Riesz representation theorem force $u^{10}v^{10}\,\mathrm{d}\Gamma \equiv 0$, hence $\Gamma \equiv 0$. □

Remark 7.10 (Spectral R-band separation between bilinear and Hermitian lifted components). In addition to the small-R algebraic proof of Theorem 7.9(ii), the frequency representation (7.22) reveals a striking physical separation mechanism across radii: at any fixed temporal Fourier frequency $\Omega > 0$, the complex-bilinear density $F(\omega)F(\Omega-\omega)$ $(0 < \omega < \Omega)$ couples through $\widehat{\mathcal{K}}^{+}_{R}(\omega, \Omega-\omega)$ and oscillates in R at spatial wavenumbers $k_R = |2\omega - \Omega| \in [0, \Omega)$, whereas the Hermitian density $F(\omega+\Omega)\overline{F(\omega)}$ $(\omega > 0)$ couples through $\widehat{\mathcal{H}}_R(\omega+\Omega, \omega)$ and oscillates in R at wavenumbers $k_R = 2\omega + \Omega \in (\Omega, \infty)$. Thus varying R places the bilinear and Hermitian lifted correlations in completely disjoint R-frequency bands!

7.7 Quantitative Stability and Analytic-Continuation Ill-Conditioning

Having settled exact uniqueness completely, we now quantify the stability of recovery on stated domains under additive readout noise $|\delta\mathcal{E}_{R,t}| \le \eta$, distinguishing the four regimes highlighted by the inverse problem: small coefficients, nearby widths, continuous width-measure densities, and finite space-time observation windows.

Theorem 7.11 (Quantitative Stability and Exponential Ill-Conditioning Barriers)**.** *Under additive measurement noise $\delta\mathcal{E}_{R,t}$ on the centered-ball canonical energy (with stated domain and observation norms specified in each regime):*

(i) ***Small-amplitude scaling (exact Hölder-*$1/2$ *state error and Schur-complement* $O(\eta/\epsilon^2)$ *width blow-up):*** *Fix any centered radius $R > 0$ and compact time window $J = [-T, T]$ $(T > 0)$, equipped with observation norm $\|\cdot\|_{\mathrm{obs}} \in \{\|\cdot\|_{L^2(J)}, \|\cdot\|_{L^\infty(J)}\}$ under $\|\delta\mathcal{E}_{R,\cdot}\|_{\mathrm{obs}} \le \eta$. For a single packet $h(a,u,w) = uh^{(a)}_{\mathrm{even}} + wh^{(a)}_{\mathrm{odd}} = \mathrm{Re}(cg_a)$ with amplitude $\epsilon := |c| = \sqrt{u^2+w^2}$ and unit-amplitude state $\hat{h}_{a,\theta} := h/\epsilon$ $(c = \epsilon e^{i\theta})$, homogeneity $\mathcal{E}_{R,t}(h) = \epsilon^2\mathcal{E}_{R,t}(\hat{h}_{a,\theta})$ implies that near $h = 0$ the width a is unidentifiable and worst-case amplitude recovery has unavoidable Hölder-$1/2$ error $\Omega(\sqrt{\eta})$ (and $\Theta(\sqrt{\eta})$ at fixed width). At any small nonzero amplitude $\epsilon > 0$, the three-parameter Fréchet derivative of $\Phi_R(a,u,w) := \mathcal{E}_{R,\cdot}(h(a,u,w))$ is*

$$D\Phi_R(a,u,w)[\delta a, \delta u, \delta w] = \epsilon^2(\delta a)\,\psi_a + \epsilon(\delta u)\,\psi_u + \epsilon(\delta w)\,\psi_w,$$

where $\psi_a := \partial_a\mathcal{E}_{R,\cdot}(\hat{h}_{a,\theta})$, $\psi_u := 2\mathcal{E}_{R,\cdot}(\hat{h}_{a,\theta}, h^{(a)}_{\mathrm{even}})$, and $\psi_w := 2\mathcal{E}_{R,\cdot}(\hat{h}_{a,\theta}, h^{(a)}_{\mathrm{odd}})$. Because unknown amplitude shifts $(\delta u, \delta w)$ can absorb part of ψ_a, the linearized width sensitivity is governed by the Schur-complement distance to the amplitude tangent plane,

$$d_a\big(a,\theta; R, \|\cdot\|_{\mathrm{obs}}\big) := \inf_{(\alpha,\beta)\in\mathbb{R}^2} \|\psi_a + \alpha\psi_u + \beta\psi_w\|_{\mathrm{obs}} \;>\; 0, \tag{7.51}$$

which is strictly positive unconditionally for every $a > 0$, $R > 0$, $T > 0$, and $\theta \in [0, 2\pi)$ (since $\{\psi_a, \psi_u, \psi_w\}$ are linearly independent on J). Consequently, for $\eta \ll \epsilon^2 a^{-12}$ on the scaled window $J_a = [-T_0 a, T_0 a]$ at $R = \hat{R}a$ (fixed dimensionless $\hat{R}, T_0 > 0$, in $L^\infty(J_a)$ norm), local amplitude errors scale as $O(\eta a^{12}/\epsilon)$ while the linearized width-estimation error obeys

$$\sup_{\|\delta\mathcal{E}_{R,\cdot}\|_{L^\infty(J_a)}\le\eta} |\delta a|_{\mathrm{lin}} = \frac{\eta}{\epsilon^2\, d_a(a,\theta; R, L^\infty(J_a))} = \Theta\!\left(\frac{\eta\, a^{13}}{\epsilon^2}\right) \qquad (\epsilon \to 0^+), \tag{7.52}$$

uniformly in $\theta \in [0, 2\pi)$ (and similarly in $L^2(J_a)$ with $a^{12.5}$; numerically at $a = 1, R = 1, J = [-2, 2]$ in `verify_05`*, amplitude shifts absorb up to $\approx 98.9\%$ of $\|\psi_a\|_{L^2}$, yet $\min_\theta d_a \approx 0.478 > 0$).*

(ii) ***Nearby-width dipole conditioning*** *($a_2 = a_1 + \delta \to a_1$): Fix $a > 0$, $R > 0$, and $T > 0$, and consider two packets at widths $a_1 = a$ and $a_2 = a + \delta$ ($\delta \to 0^+$):*

(a) Exact global-energy Bergman Gram asymptotic: *The 2×2 global-energy Bergman Gram matrix $G_{\mathrm{glob}}(a, a+\delta)_{jk} = 10800\pi/(a_j + a_k)^{13}$ obeys (`verify_05`)*

$$\frac{\det G_{\mathrm{glob}}(a, a+\delta)}{G_{\mathrm{glob}}(a,a)^2} = \frac{13}{4a^2}\,\delta^2 + O(\delta^3), \qquad \kappa_2\big(G_{\mathrm{glob}}(a, a+\delta)\big) = \frac{16\,a^2}{13\,\delta^2} + O(\delta^{-1}). \tag{7.53}$$

(b) Lifted readout smallest singular value on stated norms: *Equip the lifted domain $\mathrm{Sym}^2(\mathbb{R}^4)$ with the Frobenius norm $\|X\|_F = \sqrt{\mathrm{Tr}(X^2)}$ and the observation space $\mathcal{Y}$ with either $\|\cdot\|_{L^2([-T,T])}$ or $\|\cdot\|_{L^\infty([-T,T])}$, and define $\sigma_{\min}(\mathcal{A}_{R,\delta}) := \inf_{\|X\|_F=1} \|\mathcal{A}_{R,\delta}(X)\|_{\mathcal{Y}}$. Testing against the unit-Frobenius width-difference tensor $X_\delta := \frac{1}{2} v_- v_-^\top$ ($v_- = (1,-1,0,0)^\top$, $\|X_\delta\|_F = 1$) via Hardy domination (7.14) yields the explicit upper bound*

$$0 \le \big(\mathcal{A}_{R,\delta}(X_\delta)\big)(t) = \frac{1}{2}\mathcal{E}_{R,t}\big(h^{(a)}_{\mathrm{even}} - h^{(a+\delta)}_{\mathrm{even}}\big) \le \frac{363 \cdot 61425\,\pi^2 R}{350 \cdot 2048\,a^{15}}\,\delta^2 + O(\delta^3) \quad (\forall\, t \in \mathbb{R}), \tag{7.54}$$

proving $\sigma_{\min}(\mathcal{A}_{R,\delta}) = O(\delta^2)$ in every observation norm dominated by $L^\infty(\mathbb{R})$. Conversely, factoring $\mathcal{A}_{R,\delta}(X) = \widetilde{\mathcal{A}}_{R,\delta}(D_\delta(QXQ^\top))$ through an orthogonal change of basis $Q \in O(4)$ and proving analytically that the limiting confluent jet operator $\widetilde{\mathcal{A}}_{R,0} : \mathrm{Sym}^2(\mathbb{R}^4) \to \mathcal{Y}$ on $\{h^{(a)}_{\mathrm{even}}, h^{(a)}_{\mathrm{odd}}, \partial_a h^{(a)}_{\mathrm{even}}, \partial_a h^{(a)}_{\mathrm{odd}}\}$ is injective (full rank 10) for every *$a > 0$, $R > 0$, and $T > 0$ establishes the matching lower bound $\sigma_{\min}(\mathcal{A}_{R,\delta}) = \Omega(\delta^2)$, hence the proved asymptotic*

$$\sigma_{\min}\big(\mathcal{A}_{R,\delta} : (\mathrm{Sym}^2(\mathbb{R}^4), \|\cdot\|_F) \to \mathcal{Y}\big) = \Theta(\delta^2) \qquad \big(\delta \to 0^+,\ \ \mathcal{Y} \in \{L^2([-T,T]),\, L^\infty([-T,T])\}\big). \tag{7.55}$$

On the discrete 16-point grid $t_k = -2 + 4k/15$ ($k = 0, \dots, 15$) at $a = 1, R = 1$ with Frobenius-to-$\ell_2(\mathbb{R}^{16})$ norms, 100-digit numerical evaluation in `verify_05` corroborates $\sigma_{\min}(\mathcal{A}_{1,\delta})/\delta^2 \to 2.395 \times 10^{-2}$ as $\delta \to 0^+$ (numerical rate matching the discrete confluent Schur limit).

(iii) ***Unconditional exponential/logarithmic ill-conditioning of continuous width-measure recovery:*** *Fix any compact width interval $[a_{\min}, a_{\max}] \subset (0, \infty)$ of length $L := a_{\max} - a_{\min}$, set $a_0 := \frac{1}{2}(a_{\min} + a_{\max})$, and define the Bernstein ellipse parameter*

$$\rho_B := \frac{\sqrt{a_{\max}} + \sqrt{a_{\min}}}{\sqrt{a_{\max}} - \sqrt{a_{\min}}} \ > \ 1. \tag{7.56}$$

Identify absolutely continuous width measures $\mathrm{d}\mu(a) = \rho(a)\,\mathrm{d}a$ with densities $\rho = \rho_w - i\rho_u \in L^2([a_{\min}, a_{\max}], \mathbb{C})$ (containing real densities $\rho \in L^2([a_{\min}, a_{\max}], \mathbb{R})$ as the pure-odd subspace $\rho_u = 0$). Because $h[\rho] = \mathrm{Re} \int g_a \rho(a)\,\mathrm{d}a$ is $\mathbb{R}$-linear, physical equivalence is governed by the real *overall sign $\sigma \in \{\pm 1\}$ (Proposition 7.1(iii)), with sign-quotient L^2 error $\min_{\sigma \in \{\pm 1\}} \|\widehat{\rho} - \sigma\rho\|_{L^2([a_{\min}, a_{\max}])}$. For every integer $n \ge 0$, let $\tilde{\rho}_n(a) := \sqrt{\frac{2n+1}{L}}\, P_n\Big(\frac{2(a-a_0)}{L}\Big) \in L^2([a_{\min}, a_{\max}], \mathbb{R})$ be the L^2-orthonormal Legendre density on $[a_{\min}, a_{\max}]$ ($\|\tilde{\rho}_n\|_{L^2([a_{\min}, a_{\max}])} = 1$). Under the additive observation model $y(R,t) = \mathcal{E}_{B(R,0,t)}(h[\rho]) + \delta\mathcal{E}_{R,t}$ on $(0, R_{\max}] \times \mathbb{R}$ with uniform noise bound $\|\delta\mathcal{E}\|_{L^\infty((0,R_{\max}]\times\mathbb{R})} := \sup_{0<R\le R_{\max},\, t\in\mathbb{R}} |\delta\mathcal{E}_{R,t}| \le \eta$ (which in particular dominates any finite-radius, finite-window L^∞ or normalized L^2 observation norm on $[R_1, R_2] \times [-T, T]$ or $\{R_0\} \times [-T, T]$),*

the global energy and every *centered-ball canonical energy of* $h[\tilde{\rho}_n]$ *across all* $R \in (0, R_{\max}]$ *and all* $t \in \mathbb{R}$ *obey the exact non-asymptotic exponential decay bound*

$$\sup_{0<R\le R_{\max},\, t\in\mathbb{R}} \mathcal{E}_{B(R,0,t)}\big(h[\tilde{\rho}_n]\big) \le \frac{363}{175}\,\pi R_{\max}\,\mathcal{E}_{\partial_t}\big(h[\tilde{\rho}_n]\big)$$
$$\le C_0\,R_{\max}\,(2n+1)\,\frac{(2n+12)!}{(2n)!}\,\frac{\binom{2n}{n}}{4^n}\,\rho_B^{-2n}, \tag{7.57}$$

where $C_0 := \frac{363}{175}\cdot\frac{5400\,\pi^2 L}{12!\,(2a_{\min})^{13}}$ *(verified directly against* 64*-node quadrature on* $[1,2]$ *for* $n = 0,\dots,9$ *in* `verify_05`*, where* $\rho_B = 3+2\sqrt{2} \approx 5.8284$ *and the Bergman operator eigenvalues decay by* ≈ 0.02–0.035 *per mode, reaching* $\lambda_9 \approx 1.03\times10^{-15}$*). Consequently, even given* all *centered radii* $R \in (0, R_{\max}]$ *and* all *times* $t \in \mathbb{R}$ *(for either the real unit ball* $B_{L^2(\mathbb{R})}(0,1)$ *or the complex unit ball* $B_{L^2(\mathbb{C})}(0,1)$*):*

- *on the unit ball* $\|\rho\|_{L^2([a_{\min},a_{\max}])} \le 1$, no *estimator (linear or nonlinear) admits a uniform modulus of continuity: the worst-case sign-quotient* $L^2([a_{\min},a_{\max}])$ *error* $\sup_{\|\rho\|_2\le1,\,\|\delta\mathcal{E}\|_\infty\le\eta}\min_{\sigma\in\{\pm1\}}\|\widehat{\rho}(\mathcal{E}(h[\rho])+\delta\mathcal{E})-\sigma\rho\|_{L^2([a_{\min},a_{\max}])}$ *is* ≥ 1 *(and hence has minimax value identically equal to* 1*) for every noise level* $\eta > 0$*;*
- *under a Sobolev smoothness prior* $\|\rho\|_{H^s([a_{\min},a_{\max}])} \le M$ *(*$s > 0$*), the minimax sign-quotient* $L^2([a_{\min},a_{\max}])$ *error of* any *estimator satisfies the logarithmic lower bound* $\Omega\big((\log(1/\eta))^{-2s}\big)$ *as* $\eta \to 0^+$*.*

(iv) ***Finite space-time coverage, analytic continuation, and finite-*** N ***local vs. compact-global Lipschitz stability:*** *Although Theorems 7.5, 7.7, and 7.9 require only an open interval* $I \subset \mathbb{R}$ *(at a single fixed* $R_0 > 0$ *for physical states, or over* $(R_1,R_2) \subset (0,\infty)$ *for lifted states) in the* exact *noiseless problem, reconstructing general infinite-dimensional states from noisy data on a finite window* $[-T,T]$ *(or* $[R_1,R_2]\times[-T,T]$*) compounds unstable analytic continuations—continuing* t *off* $[-T,T]$ *to* $t\to\infty$ *across* $O(t^{-20})$ *decay (or into the complex plane to* $\operatorname{Im} t = a_j$*, amplifying high-frequency noise by* $e^{a_j\omega}$*) and, in the lifted multi-radius problem, extrapolating* $R \in [R_1,R_2] \to 0^+$ *across an* $O(R^8)$ *vanishing readout—on top of the real Laplace / Hausdorff moment inversion barrier (7.57). By contrast, for fixed cardinality* $N \ge 1$*, write* $\theta = (a,u,w) \in U_N \subset (0,\infty)^N\times\mathbb{R}^N\times\mathbb{R}^N$ *with* $a = (a_1,\dots,a_N)$*,* $u = (u_1,\dots,u_N)$*,* $w = (w_1,\dots,w_N)$ *on the open parameter domain* $U_N := \{(a,u,w) : 0 < a_1 < \cdots < a_N,\ c_j = w_j - iu_j \ne 0\ \forall\, j\}$*, define the overall sign action on* U_N *by*

$$S_\sigma(a,u,w) := (a,\ \sigma u,\ \sigma w), \qquad \sigma \in \{\pm1\} \tag{7.58}$$

(which leaves the positive widths a *unchanged, acts isometrically on* $\mathbb{R}^{3N}$*, preserves* U_N *and every separated compact set* K_{par} *below, and satisfies* $h[S_\sigma\theta] = \sigma h[\theta]$ *and* $\Phi_R(S_\sigma\theta) = \Phi_R(\theta)$*), and consider the smooth continuous-window parameter map* $\Phi_R(\theta) := \mathcal{E}_{R,\cdot}(h[\theta])$ *into* $\mathcal{Y} \in \{L^2([-T,T]), L^\infty([-T,T])\}$ *(*$T > 0$*):*

(a) Continuous-window local two-point Lipschitz stability: *For* every *single fixed radius* $R > 0$*, every* $T > 0$*, every* $N \ge 1$*, and every* $\theta_* \in U_N$*, the* $3N$*-parameter Fréchet derivative* $D\Phi_R(\theta_*) : \mathbb{R}^{3N} \to \mathcal{Y}$ *is* unconditionally injective *(*$\sigma_{\min}(D\Phi_R(\theta_*)) := \inf_{\|v\|_2=1}\|D\Phi_R(\theta_*)v\|_{\mathcal{Y}} > 0$*), and* $\|D^2\Phi_R(\theta)\|_{\mathrm{op}}$ *is uniformly bounded on every closed ball* $\overline{B_\rho(\theta_*)} \subset U_N$ *(and in particular on every compact separated parameter set*

$$K_{\mathrm{par}} := \big\{\theta = (a,u,w) \in U_N : a_{\min} \le a_1 < \cdots < a_N \le a_{\max},$$
$$a_{j+1} - a_j \ge \delta_{\min},\ \epsilon_{\min} \le |c_j| \le \epsilon_{\max}\big\}$$

with $a_{\min}, \delta_{\min}, \epsilon_{\min} > 0$*). Consequently, there exists* $r_*(\theta_*) > 0$ *with* $\overline{B_{r_*}(\theta_*)} \subset U_N$ *such that* every *nearby pair* $\theta_1, \theta_2 \in \overline{B_{r_*}(\theta_*)}$ *satisfies the two-point local lower bound*

$$\|\Phi_R(\theta_1) - \Phi_R(\theta_2)\|_{\mathcal{Y}} \;\geq\; \frac{1}{2}\,\sigma_{\min}\big(D\Phi_R(\theta_*)\big)\,\|\theta_1 - \theta_2\|_2. \tag{7.59}$$

(b) Continuous-window compact-global Lipschitz stability on the sign quotient: *Because Theorem 7.7(i) establishes global injectivity on the sign quotient* $K_{\mathrm{par}}/\langle S_{-1}\rangle$ unconditionally for every single fixed radius $R > 0$ and every $N \geq 1$ *(and likewise on any compact rectangle* $[R_1, R_2] \times [-T, T]$ *by Theorem 7.9(i)), compactness of* K_{par} *combined with the two-point local lower bound* (7.59) *yields an unconditional* compact-global *Lipschitz stability constant* $C(K_{\mathrm{par}}, R, T) < \infty$ *at* any *single fixed radius* $R > 0$ *and any* $T > 0$*:*

$$\min_{\sigma\in\{\pm1\}} \|\theta_1 - S_\sigma\theta_2\|_2 \;\leq\; C(K_{\mathrm{par}}, R, T)\,\|\Phi_R(\theta_1) - \Phi_R(\theta_2)\|_{\mathcal{Y}} \qquad \forall\,\theta_1, \theta_2 \in K_{\mathrm{par}}. \tag{7.60}$$

(c) Distinct finite discrete-sample designs (conditional statement): *By contrast with the unconditional continuous-window bounds in (a)–(b), for a finite discrete sample grid* $\{(R_\ell, t_\ell)\}_{\ell=1}^M \subset (0,\infty)\times\mathbb{R}$ *with readout map* $\Phi_{\mathrm{disc}}(\theta) := (\mathcal{E}_{R_\ell,t_\ell}(h[\theta]))_{\ell=1}^M \in \mathbb{R}^M$ *(in any norm* $\|\cdot\|_{\mathrm{disc}}$ *on* $\mathbb{R}^M$*), neither full Jacobian rank nor global sign-quotient injectivity follows automatically from Theorem 7.7(i). Instead, local two-point Lipschitz stability around* $\theta_* \in U_N$ *holds whenever the* $M\times 3N$ *sampled Jacobian* $D\Phi_{\mathrm{disc}}(\theta_*)$ *has full column rank* $3N$*; and compact-global Lipschitz stability* $\min_{\sigma\in\{\pm1\}} \|\theta_1 - S_\sigma\theta_2\|_2 \leq C_{\mathrm{disc}}\|\Phi_{\mathrm{disc}}(\theta_1) - \Phi_{\mathrm{disc}}(\theta_2)\|_{\mathrm{disc}}$ *holds on* K_{par} *under the two explicit hypotheses that (1)* $\operatorname{rank} D\Phi_{\mathrm{disc}}(\theta) = 3N$ *for every* $\theta \in K_{\mathrm{par}}$ *and (2)* Φ_{disc} *is globally injective on the sign quotient* $K_{\mathrm{par}}/\langle S_{-1}\rangle$ *(i.e.* $\Phi_{\mathrm{disc}}(\theta_1) = \Phi_{\mathrm{disc}}(\theta_2)$ *with* $\theta_1, \theta_2 \in K_{\mathrm{par}}$ *implies* $\theta_1 = S_\sigma\theta_2$ *for some* $\sigma \in \{\pm1\}$*).*

Proof. (i) Homogeneity gives the $h = 0$ Hölder-1/2 lower bound immediately: comparing $h = 0$ against $h = \epsilon\hat{h}_{a,\theta}$ with $\epsilon^2\|\mathcal{E}_{R,\cdot}(\hat{h}_{a,\theta})\|_{\mathrm{obs}} = \eta$ yields indistinguishable states separated in amplitude by $\sqrt{\eta/\|\mathcal{E}_{R,\cdot}(\hat{h}_{a,\theta})\|_{\mathrm{obs}}}$. To prove $d_a(a,\theta;R,\|\cdot\|_{\mathrm{obs}}) > 0$ for every $a, R, T > 0$ and θ, suppose $\gamma_a\psi_a(t) + \gamma_u\psi_u(t) + \gamma_w\psi_w(t) = 0$ on $J = [-T, T]$ for some $(\gamma_a, \gamma_u, \gamma_w) \in \mathbb{R}^3$. By real-analyticity, this linear combination vanishes identically on $\mathbb{C}\setminus([-R, R] \pm ia)$. Examine the slit endpoint $t_* = R + ia + \zeta$ ($\zeta \to 0$ outside the slit) using (7.25) and (7.26): differentiating the diagonal term $\frac{1}{4}\hat{c}^2 Q_R(a + it)$ with respect to a ($\partial_a = -i\partial_\zeta$) yields a leading singularity $+12i\,\frac{75\pi}{4004}\hat{c}^2\,\zeta^{-13} + O(\zeta^{-12})$, whereas ψ_u and ψ_w have leading singularities $\frac{150\pi}{4004}\hat{c}(-i)\zeta^{-12}$ and $\frac{150\pi}{4004}\hat{c}\,\zeta^{-12}$, and all derivatives of the holomorphic term $Q_R(-a + it)$ and the Hermitian term $\mathcal{K}_R(a+it, -a+it)$ are at most $O(|\zeta|^{-6})$. Hence $\lim_{\zeta\to0}\zeta^{13}(\gamma_a\psi_a+\gamma_u\psi_u+\gamma_w\psi_w) = 12i\frac{75\pi}{4004}\hat{c}^2\gamma_a = 0 \implies \gamma_a = 0$, and then $\lim_{\zeta\to0}\zeta^{12}(\gamma_u\psi_u + \gamma_w\psi_w) = \frac{150\pi}{4004}\hat{c}(\gamma_w - i\gamma_u) = 0 \implies \gamma_u = \gamma_w = 0$. Thus $\{\psi_a, \psi_u, \psi_w\}$ are linearly independent on J, so $d_a > 0$ continuously on the compact circle $\theta \in [0, 2\pi)$ (hence bounded away from zero uniformly in θ). By exact AdS_4 scaling $\mathcal{E}_{\lambda R,\lambda t}(\hat{h}_{\lambda a,\theta}) = \lambda^{-12}\mathcal{E}_{R,t}(\hat{h}_{a,\theta})$, we have $\psi_a \propto a^{-13}$ and $\psi_u, \psi_w \propto a^{-12}$ in $L^\infty([-T_0a, T_0a])$ at $R = \hat{R}a$, yielding $d_a(a,\theta;\hat{R}a, L^\infty(J_a)) = a^{-13}d_a(1,\theta;\hat{R}, L^\infty([-T_0, T_0]))$ and (7.52).

(ii) For (a), expanding $\det G_{\mathrm{glob}}(a, a+\delta) = (10800\pi)^2[(4a(a+\delta))^{-13} - (2a+\delta)^{-26}] = (10800\pi)^2(2a+\delta)^{-26}[(1 - \frac{\delta^2}{(2a+\delta)^2})^{-13} - 1]$ yields (7.53) directly (`verify_05`). For (b), take the unit-Frobenius width-difference tensor $X_\delta = \frac{1}{2}v_-v_-^\top$ ($v_- = (1, -1, 0, 0)^\top$). By Theorem 7.2(iv) and (7.5), for every $t \in \mathbb{R}$,

$$0 \leq \big(\mathcal{A}_{R,\delta}(X_\delta)\big)(t) = \frac{1}{2}\mathcal{E}_{R,t}\big(h^{(a)}_{\mathrm{even}} - h^{(a+\delta)}_{\mathrm{even}}\big) \leq \frac{363}{350}\pi R\cdot 5400\pi\left[\frac{1}{(2a)^{13}} - \frac{2}{(2a+\delta)^{13}} + \frac{1}{(2a+2\delta)^{13}}\right],$$

whose Taylor expansion at $\delta = 0$ is $\frac{363}{350}\pi R\cdot\frac{61425\pi}{2048a^{15}}\delta^2 + O(\delta^3)$ (`verify_05`), proving (7.54) and $\sigma_{\min}(\mathcal{A}_{R,\delta}) = O(\delta^2)$.

For the matching lower bound on $\mathcal{Y} \in \{L^2([-T,T]), L^\infty([-T,T])\}$, apply the orthogonal transformation $Q \in O(4)$ mapping (u_1,u_2,w_1,w_2) to $(u_+,w_+,u_-,w_-) := \frac{1}{\sqrt{2}}(u_1+u_2, w_1+w_2, u_2-u_1, w_2-w_1)$ (which preserves $\|QXQ^\top\|_F = \|X\|_F$ identically), and define the block diagonal scaling $D_\delta\left(\begin{smallmatrix} Y_{++} & Y_{+-} \\ Y_{+-}^\top & Y_{--}\end{smallmatrix}\right) := \left(\begin{smallmatrix} Y_{++} & \delta Y_{+-} \\ \delta Y_{+-}^\top & \delta^2 Y_{--}\end{smallmatrix}\right)$, which satisfies $\|D_\delta(Y)\|_F \geq \delta^2\|Y\|_F$ for $0<\delta\leq 1$. Then $\mathcal{A}_{R,\delta}(X) = \widetilde{\mathcal{A}}_{R,\delta}(D_\delta(QXQ^\top))$, where $\widetilde{\mathcal{A}}_{R,\delta}$ is the lifted canonical-energy operator on the smoothly varying basis $\{\frac{h_{\mathrm{ev}}^{(a)}+h_{\mathrm{ev}}^{(a+\delta)}}{\sqrt{2}}, \frac{h_{\mathrm{od}}^{(a)}+h_{\mathrm{od}}^{(a+\delta)}}{\sqrt{2}}, \frac{h_{\mathrm{ev}}^{(a+\delta)}-h_{\mathrm{ev}}^{(a)}}{\sqrt{2}\delta}, \frac{h_{\mathrm{od}}^{(a+\delta)}-h_{\mathrm{od}}^{(a)}}{\sqrt{2}\delta}\}$, converging in operator norm at rate $O(\delta)$ as $\delta\to 0^+$ to the confluent jet operator $\widetilde{\mathcal{A}}_{R,0}$ on $\{\sqrt{2}h_{\mathrm{ev}}^{(a)}, \sqrt{2}h_{\mathrm{od}}^{(a)}, \frac{1}{\sqrt{2}}\partial_a h_{\mathrm{ev}}^{(a)}, \frac{1}{\sqrt{2}}\partial_a h_{\mathrm{od}}^{(a)}\}$. We prove analytically that $\widetilde{\mathcal{A}}_{R,0}$ is injective (rank 10) on $[-T,T]$ for every $a,R,T>0$: suppose $F_Y(t) := \widetilde{\mathcal{A}}_{R,0}(Y)(t) \equiv 0$ on $[-T,T]$ (hence on $\mathbb{C}\setminus([-R,R]\pm ia)$). Expanding Y in the 6 real complex-bilinear parameters ($c_{00},c_{01},c_{11}\in\mathbb{C}$) multiplying $K_{00}=Q_R(a+it)$, $K_{10}=\frac{1}{2}Q_R'(a+it)$, and $K_{11}=\partial_a\partial_b\mathcal{K}_R(a+it,b+it)|_{b=a}$, plus the 4 real Hermitian parameters multiplying $H_{00}=\mathcal{K}_R(a+it,-a+it)$, $H_{\mathrm{sym}}=(\partial_a+\partial_b)\mathcal{K}_R(a+it,-b+it)|_{b=a}$, $H_{\mathrm{anti}}=i(\partial_a-\partial_b)\mathcal{K}_R(a+it,-b+it)|_{b=a}$, and $H_{11}=\partial_a\partial_b\mathcal{K}_R(a+it,-b+it)|_{b=a}$:

(1) Direct evaluation of polarized Theorem 7.2(ii) on $(\partial_\alpha p_\alpha, \partial_\alpha p_\alpha)$ (`verify_05`) shows that $K_{11}(\alpha,\alpha)$ is purely rational, equal at $R=1$ to $\frac{30720\pi(572\alpha^6-429\alpha^4+450\alpha^2-5)}{1001(\alpha^2+1)^{14}}$ (and by scaling $K_{11}(\alpha,\alpha;R)=R^{-14}K_{11}(\alpha/R,\alpha/R;1)$ in general), with R-independent nonzero order-14 leading singularity $-\frac{30\pi}{11}\zeta^{-14}$ at $t_*=R+ia+\zeta$. Since $K_{10}=\frac{1}{2}(-i\partial_\zeta)Q_R(iR+i\zeta)\sim\frac{450i\pi}{1001}\zeta^{-13}$, $K_{00}\sim\frac{75\pi}{1001}\zeta^{-12}$ (both also R-independent), and all four Hermitian functions have singularities of order at most 6 at t_*, taking $\lim_{\zeta\to0}\zeta^{14}F_Y$, $\lim_{\zeta\to0}\zeta^{13}F_Y$, and $\lim_{\zeta\to0}\zeta^{12}F_Y$ successively forces $c_{11}=c_{01}=c_{00}=0$.

(2) Among the remaining four Hermitian functions, H_{anti} is the unique odd-in-t function and has nonzero slit jump across $(-R,R)+ia$, so its coefficient vanishes. For the three even-in-t functions $(H_{00},H_{\mathrm{sym}},H_{11})$, extracting the s^0 and s^2 open-slit jump coefficients from (7.18) produces a rank-2 system whose unique 1-dimensional kernel is spanned by $\frac{91}{3a^2}H_{00}+\frac{35}{6a}H_{\mathrm{sym}}+H_{11}$ (`verify_05`). Evaluating the order-6 endpoint pole at $t_*=R+ia$ (arising solely from $\partial_\alpha[P_5/(R^2+\alpha^2)^5]$) on this kernel combination yields $-\frac{5iR^9(35iR^3-91R^2a-99iRa^2+22a^3)}{77a^8(a-iR)^6}$ times $(R^2+\alpha^2)^{-6}$, whose numerator modulus squared is

$$25R^{18}\big(1225R^6+1351R^4a^2+5797R^2a^4+484a^6\big) \;>\; 0 \qquad \forall\, a>0,\ R>0.$$

Hence all three even Hermitian coefficients vanish as well, proving $\ker\widetilde{\mathcal{A}}_{R,0}=\{0\}$.

Since $\mathrm{Sym}^2(\mathbb{R}^4)$ is finite-dimensional, $\sigma_{\min}(\widetilde{\mathcal{A}}_{R,0})>0$, whence $\sigma_{\min}(\mathcal{A}_{R,\delta})\geq(\sigma_{\min}(\widetilde{\mathcal{A}}_{R,0})-O(\delta))\delta^2=\Omega(\delta^2)$, completing the proof of (7.55).

(iii) By Rodrigues' formula $P_n(s)=\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}(s^2-1)^n$ and n integrations by parts on $[-1,1]$, the Laplace transform of $\tilde\rho_n$ satisfies

$$\big|\mathcal{L}[\tilde\rho_n](\omega)\big| = \sqrt{\frac{2n+1}{L}}\,\frac{L}{2}\,\frac{(L\omega/4)^n}{n!}\int_{-1}^{1}(1-s^2)^n\,e^{-(a_0+\frac{L}{2}s)\omega}\,\mathrm{d}s.$$

Substituting this into (7.5), applying Cauchy–Schwarz in $s\in[-1,1]$ (which introduces a factor $\int_{-1}^1\mathrm{d}s=2$), and evaluating the ω-integral first via $\int_0^\infty\omega^{2n+12}e^{-(2a_0+Ls)\omega}\,\mathrm{d}\omega=(2n+12)!/(2a_0+Ls)^{2n+13}$ yields

$$\mathcal{E}_{\partial_t}\big(h[\tilde\rho_n]\big)\leq\frac{5400\,\pi\,(2n+1)\,L}{2\cdot 12!}\,\frac{(2n+12)!}{(n!)^2}\left(\frac{L}{4}\right)^{2n}\int_{-1}^{1}\frac{(1-s^2)^{2n}}{(2a_0+Ls)^{2n+13}}\,\mathrm{d}s.$$

On $[-1,1]$, we have $2a_0+Ls\geq 2a_{\min}>0$, while the positive function $\phi(s):=(1-s^2)/(2a_0+Ls)$ achieves its unique maximum on $[-1,1]$ at $s_*=(-2a_0+\sqrt{4a_0^2-L^2})/L=-\rho_B^{-1}\in(-1,0)$,

where $\phi'(s_*) = 0$ implies $\phi(s_*) = -2s_*/L = 2/(L\rho_B)$, so $\frac{L}{4}\phi(s_*) = \frac{1}{2\rho_B}$. Pulling out $\int_{-1}^{1} \mathrm{d}s = 2$ and $\big(\frac{L}{4}\phi(s_*)\big)^{2n} = 4^{-n}\rho_B^{-2n}$, and combining with the uniform Hardy domination (7.14) of Theorem 7.2(iv), yields (7.57) verbatim!

For the unit-ball minimax lower bound of 1, fix any noise level $\eta > 0$. Since $(2n+1)\frac{(2n+12)!}{(2n)!}\frac{\binom{2n}{n}}{4^n} = O(n^{12.5})$ while $\rho_B > 1$, the right-hand side of (7.57) tends to 0 as $n \to \infty$. Hence there exists $N(\eta) \in \mathbb{N}_0$ such that

$$\sup_{0<R\le R_{\max},\, t\in\mathbb{R}} \mathcal{E}_{B(R,0,t)}\big(h[\tilde\rho_n]\big) \;\le\; \eta \qquad (\forall\, n \ge N(\eta)).$$

Consequently, *every* member of the entire orthonormal tail $\{\tilde\rho_n : n \ge N(\eta)\}$ lies in the unit ball $\{\|\rho\|_{L^2([a_{\min},a_{\max}])} \le 1\}$ (both real and complex) and is compatible with the *same* identically zero observation $y \equiv 0$ under the admissible additive noise realization $\delta\mathcal{E}_{R,t} := -\mathcal{E}_{B(R,0,t)}(h[\tilde\rho_n])$, which satisfies $\|\delta\mathcal{E}\|_{L^\infty((0,R_{\max}]\times\mathbb{R})} \le \eta$. Let $\widehat\rho$ be any estimator (linear or nonlinear) and let $f := \widehat\rho(0) \in L^2([a_{\min},a_{\max}],\mathbb{C})$ (or $\mathbb{R}$) denote its fixed estimate on the zero observation $y \equiv 0$. For every $n \ge N(\eta)$, the squared L^2 estimation error modulo the physical real sign $\sigma \in \{\pm1\}$ is

$$\min_{\sigma\in\{\pm1\}} \|f - \sigma\tilde\rho_n\|_{L^2}^2 = \|f\|_{L^2}^2 + \|\tilde\rho_n\|_{L^2}^2 - 2\,|\mathrm{Re}\langle f,\tilde\rho_n\rangle_{L^2}| = 1 + \|f\|_{L^2}^2 - 2\,|\mathrm{Re}\langle f,\tilde\rho_n\rangle_{L^2}|\,.$$

Because $\{\tilde\rho_n\}_{n=0}^{\infty}$ is an orthonormal sequence in $L^2([a_{\min},a_{\max}])$, Bessel's inequality gives $\sum_{n=0}^{\infty}|\langle f,\tilde\rho_n\rangle_{L^2}|^2 \le \|f\|_{L^2}^2 < \infty$, forcing $\langle f,\tilde\rho_n\rangle_{L^2} \to 0$ as $n\to\infty$, whence

$$\lim_{n\to\infty}\min_{\sigma\in\{\pm1\}} \|f-\sigma\tilde\rho_n\|_{L^2}^2 \;=\; 1 + \|f\|_{L^2}^2 \;\ge\; 1.$$

Taking the supremum over $n \ge N(\eta)$ proves that for every $\eta > 0$, the worst-case sign-quotient L^2 error of $\widehat\rho$ over the unit ball is at least $\sqrt{1+\|f\|_{L^2}^2} \ge 1$ (and equality 1 is achieved by the zero estimator $\widehat\rho \equiv 0$).

Separately, for the Sobolev logarithmic lower bound under $\|\rho\|_{H^s([a_{\min},a_{\max}])} \le M$ ($s > 0$), we retain the two-state comparison between $\rho_0 := 0$ and $\rho_n := \alpha_n\tilde\rho_n$: the L^2 Markov inequality (extended to real $s > 0$ by Sobolev interpolation) gives $\|\tilde\rho_n\|_{H^s} \le C_s n^{2s}$, so setting $\alpha_n := (M/C_s)n^{-2s}$ places both 0 and ρ_n inside the Sobolev ball $\{\|\rho\|_{H^s} \le M\}$. By quadratic homogeneity,

$$\sup_{0<R\le R_{\max},\, t\in\mathbb{R}} \mathcal{E}_{B(R,0,t)}\big(h[\rho_n]\big) = \alpha_n^2 \sup_{0<R\le R_{\max},\, t\in\mathbb{R}} \mathcal{E}_{B(R,0,t)}\big(h[\tilde\rho_n]\big) \;\le\; C'\, n^{12.5-4s}\, \rho_B^{-2n} \;\le\; \eta$$

once $n = \lceil \frac{\log(1/\eta)}{2\log\rho_B} + O(\log\log(1/\eta))\rceil$, making both 0 and ρ_n compatible with the zero observation $y \equiv 0$ under $\|\delta\mathcal{E}\|_{L^\infty((0,R_{\max}]\times\mathbb{R})} \le \eta$. For any estimator with output $f = \widehat\rho(0)$ and any sign $\sigma \in \{\pm1\}$, the triangle inequality gives $\|f-0\|_{L^2} + \|f-\sigma\rho_n\|_{L^2} \ge \|\sigma\rho_n\|_{L^2} = \alpha_n$, whence $\max\big\{\|f\|_{L^2},\ \min_{\sigma\in\{\pm1\}}\|f-\sigma\rho_n\|_{L^2}\big\} \ge \frac12\alpha_n = \Omega\big((\log(1/\eta))^{-2s}\big)$ as $\eta \to 0^+$.

(iv) For (a), fix any $R > 0$, $T > 0$, $N \ge 1$, and $\theta_* = (a_*, u_*, w_*) \in U_N$, and suppose $F(t) := D\Phi_R(\theta_*)[\delta\theta](t) = \sum_{j=1}^{N}(\gamma_{a,j}\partial_{a_j} + \gamma_{u,j}\partial_{u_j} + \gamma_{w,j}\partial_{w_j})\mathcal{E}_{R,t}(h[\theta_*]) = 0$ for all $t \in [-T,T]$. By real-analyticity, $F \equiv 0$ on $\mathbb{C}\setminus\bigcup_{j=1}^{N}([-R,R] \pm ia_{*,j})$. For each $j \in \{1,\dots,N\}$, examine $t = R + ia_{*,j} + \zeta$ ($\zeta \to 0$ outside the slits): because $a_{*,1} < \cdots < a_{*,N}$ are distinct, every diagonal term $Q_R(a_{*,k} + it)$ ($k \ne j$) and every $Q_R(-a_{*,k} + it)$ is holomorphic at $R + ia_{*,j}$, while every Hermitian and mixed cross-term $\mathcal{K}_R(\pm a_{*,k} + it, \pm a_{*,\ell} + it)$ has singularity order at most 5 (plus $\log\zeta$), so its first partial derivatives in (a_j, u_j, w_j) are at most $O(|\zeta|^{-6})$. Only the j-th diagonal term $\frac14 c_{*,j}^2 Q_R(a_{*,j} + it)$ contributes above order 6, giving

$$\lim_{\zeta\to0}\zeta^{13}F(R + ia_{*,j} + \zeta) = 12i\,\frac{75\pi}{4004}\,c_{*,j}^2\,\gamma_{a,j} = 0 \implies \gamma_{a,j} = 0 \qquad (\forall\, j = 1,\dots,N),$$

since $c_{*,j} \neq 0$, and subsequently

$$\lim_{\zeta\to 0} \zeta^{12} F(R + ia_{*,j} + \zeta) = \frac{150\pi}{4004}\, c_{*,j}\,(\gamma_{w,j} - i\gamma_{u,j}) = 0 \implies \gamma_{u,j} = \gamma_{w,j} = 0 \qquad (\forall\, j = 1, \dots, N).$$

Thus $D\Phi_R(\theta_*) : \mathbb{R}^{3N} \to \mathcal{Y}$ is unconditionally injective, so $\sigma_* := \sigma_{\min}(D\Phi_R(\theta_*)) := \inf_{\|v\|_2=1} \|D\Phi_R(\theta_*)v\|_{\mathcal{Y}} > 0$ on the finite-dimensional domain $\mathbb{R}^{3N}$. Since U_N is open, choose $\rho > 0$ small enough that the closed convex Euclidean ball $\overline{B_\rho(\theta_*)}$ is contained in U_N (so all widths on $\overline{B_\rho(\theta_*)}$ stay inside a compact subinterval $[a'_{\min}, a'_{\max}] \subset (0, \infty)$). Because $\mathcal{K}_R(x + it, y + it)$ is C^∞ on $([-a'_{\max}, -a'_{\min}] \cup [a'_{\min}, a'_{\max}])^2 \times [-T, T]$, the map $\Phi_R : U_N \to C([-T, T]) \hookrightarrow \mathcal{Y}$ is C^∞, and its second derivative is uniformly bounded on $\overline{B_\rho(\theta_*)}$: $M_2(\theta_*, \rho) := \sup_{\theta\in\overline{B_\rho(\theta_*)}} \|D^2\Phi_R(\theta)\|_{\mathrm{op}} < \infty$. For any $r \in (0, \rho]$ and any two nearby points $\theta_1, \theta_2 \in \overline{B_r(\theta_*)}$, the connecting segment $\gamma(s) := \theta_2 + s(\theta_1 - \theta_2)$ ($s \in [0, 1]$) lies entirely inside the convex ball $\overline{B_r(\theta_*)} \subset U_N$ with $\|\gamma(s) - \theta_*\|_2 \le r$. Subtracting $D\Phi_R(\theta_*)(\theta_1 - \theta_2)$ from the fundamental-theorem integral $\Phi_R(\theta_1) - \Phi_R(\theta_2) = \int_0^1 D\Phi_R(\gamma(s))(\theta_1 - \theta_2)\,\mathrm{d}s$ yields

$$\Phi_R(\theta_1) - \Phi_R(\theta_2) - D\Phi_R(\theta_*)(\theta_1 - \theta_2) = \int_0^1 \big(D\Phi_R(\gamma(s)) - D\Phi_R(\theta_*)\big)(\theta_1 - \theta_2)\,\mathrm{d}s.$$

Applying the mean-value bound along the radial segment from θ_* to $\gamma(s)$ inside $\overline{B_r(\theta_*)}$ gives $\|D\Phi_R(\gamma(s)) - D\Phi_R(\theta_*)\|_{\mathbb{R}^{3N}\to\mathcal{Y}} \le M_2(\theta_*, \rho)\,\|\gamma(s) - \theta_*\|_2 \le M_2(\theta_*, \rho)\, r$ for all $s \in [0, 1]$, whence the reverse triangle inequality yields

$$\|\Phi_R(\theta_1) - \Phi_R(\theta_2)\|_{\mathcal{Y}} \;\ge\; \big(\sigma_* - M_2(\theta_*, \rho)\, r\big)\,\|\theta_1 - \theta_2\|_2 \qquad (\forall\, \theta_1, \theta_2 \in \overline{B_r(\theta_*)}).$$

Setting $r_*(\theta_*) := \min\big\{\rho,\ \frac{\sigma_*}{2(M_2(\theta_*,\rho)+1)}\big\} > 0$ establishes the two-point local lower bound (7.59) for all $\theta_1, \theta_2 \in \overline{B_{r_*}(\theta_*)}$.

For (b), suppose for contradiction that (7.60) fails on K_{par}. Then there exist sequences $\theta_n^{(1)}, \theta_n^{(2)} \in K_{\mathrm{par}}$ with $\min_{\sigma\in\{\pm1\}} \|\theta_n^{(1)} - S_\sigma\theta_n^{(2)}\|_2 > 0$ such that

$$\frac{\|\Phi_R(\theta_n^{(1)}) - \Phi_R(\theta_n^{(2)})\|_{\mathcal{Y}}}{\min_{\sigma\in\{\pm1\}} \|\theta_n^{(1)} - S_\sigma\theta_n^{(2)}\|_2} \;\longrightarrow\; 0 \qquad (n \to \infty).$$

Because $K_{\mathrm{par}} \subset U_N$ is compact, its diameter is finite, so $\|\Phi_R(\theta_n^{(1)}) - \Phi_R(\theta_n^{(2)})\|_{\mathcal{Y}} \to 0$; passing to a convergent subsequence yields $\theta_n^{(1)} \to \theta^{(1)} \in K_{\mathrm{par}}$ and $\theta_n^{(2)} \to \theta^{(2)} \in K_{\mathrm{par}}$, with $\Phi_R(\theta^{(1)}) = \Phi_R(\theta^{(2)})$ by continuity of Φ_R. Thus $\mathcal{E}_{R,t}(h[\theta^{(1)}]) = \mathcal{E}_{R,t}(h[\theta^{(2)}])$ for all $t \in (-T, T)$. By Theorem 7.7(i) (unconditional single-radius physical injectivity on $(-T, T)$), the associated atomic measures satisfy $\mu[\theta^{(1)}] = \sigma_*\,\mu[\theta^{(2)}]$ for a unique $\sigma_* \in \{\pm1\}$; since the widths in U_N are strictly ordered ($a_1 < \cdots < a_N$) and all $c_j \neq 0$, this forces $a^{(1)} = a^{(2)}$ and $(u^{(1)}, w^{(1)}) = \sigma_*(u^{(2)}, w^{(2)})$, i.e. $\theta^{(1)} = S_{\sigma_*}\theta^{(2)}$. Set $\theta_* := \theta^{(1)} \in K_{\mathrm{par}} \subset U_N$ and define $\tilde\theta_n^{(2)} := S_{\sigma_*}\theta_n^{(2)} \in K_{\mathrm{par}}$ (using $S_{\sigma_*}(K_{\mathrm{par}}) = K_{\mathrm{par}}$). Then $\Phi_R(\tilde\theta_n^{(2)}) = \Phi_R(\theta_n^{(2)})$, both $\theta_n^{(1)} \to \theta_*$ and $\tilde\theta_n^{(2)} \to S_{\sigma_*}\theta^{(2)} = \theta_*$, and $0 < \min_{\sigma\in\{\pm1\}} \|\theta_n^{(1)} - S_\sigma\theta_n^{(2)}\|_2 = \min_{\sigma\in\{\pm1\}} \|\theta_n^{(1)} - S_\sigma\tilde\theta_n^{(2)}\|_2 \le \|\theta_n^{(1)} - \tilde\theta_n^{(2)}\|_2$ (with equality for all large n, since $\|\theta_* - S_{-1}\theta_*\|_2 = 2\|(u_*, w_*)\|_2 \ge 2\sqrt{N}\epsilon_{\min} > 0$). For all sufficiently large n so that $\theta_n^{(1)}, \tilde\theta_n^{(2)} \in \overline{B_{r_*}(\theta_*)}$, the two-point local lower bound (7.59) gives

$$\frac{\|\Phi_R(\theta_n^{(1)}) - \Phi_R(\theta_n^{(2)})\|_{\mathcal{Y}}}{\min_{\sigma\in\{\pm1\}} \|\theta_n^{(1)} - S_\sigma\theta_n^{(2)}\|_2} \;\ge\; \frac{\|\Phi_R(\theta_n^{(1)}) - \Phi_R(\tilde\theta_n^{(2)})\|_{\mathcal{Y}}}{\|\theta_n^{(1)} - \tilde\theta_n^{(2)}\|_2} \;\ge\; \frac{1}{2}\,\sigma_{\min}\big(D\Phi_R(\theta_*)\big) \;>\; 0,$$

contradicting convergence of the ratio to 0 and proving (7.60). (The multi-radius version on $[R_1, R_2] \times [-T, T]$ follows identically using Theorem 7.9(i).)

For (c), the finite-sample map $\Phi_{\mathrm{disc}} : U_N \to \mathbb{R}^M$ is likewise C^∞ and invariant under S_σ. Whenever $\operatorname{rank} D\Phi_{\mathrm{disc}}(\theta_*) = 3N$, one has $\sigma_{\min}(D\Phi_{\mathrm{disc}}(\theta_*)) := \inf_{\|v\|_2=1} \|D\Phi_{\mathrm{disc}}(\theta_*)v\|_{\mathrm{disc}} > 0$ and the exact same segment-integral argument yields the two-point local lower bound on a ball $\overline{B_{r_*}(\theta_*)} \subset U_N$; assuming in addition full column rank $3N$ at every $\theta \in K_{\mathrm{par}}$ together with global injectivity of Φ_{disc} on $K_{\mathrm{par}}/\langle S_{-1}\rangle$, the exact same compactness proof on the sign quotient yields compact-global Lipschitz stability on K_{par}. □

8 Discussion, Literature Comparison, and Open Problems

We conclude by situating our results within the existing mathematical-physics literature on holographic bulk reconstruction, canonical energy, extremal-surface shape variations, and quadratic inverse problems, and by delineating the exact mathematical boundaries and open problems left by this work.

8.1 Comparison with Prior Literature

Covariant phase space and holographic quantum Fisher information. Our starting point is the Hollands–Wald canonical energy functional $\mathcal{E}_B(h) = W_{\Sigma_B}(\gamma, \mathcal{L}_{\xi_B}\gamma)$ [3, 4], whose identification with boundary quantum Fisher information (second variation of relative entropy) for spherical regions in vacuum-perturbed holographic CFTs was established by Lashkari and Van Raamsdonk [7] (building on the first-order linearized Einstein derivation of [5] and the CHM modular map [2]). While [7] proved general positivity of $\mathcal{E}_B$ via energy-density inequalities and noted that Hollands–Wald gauge fixes the extremal surface, explicit closed-form evaluations of the Lorentzian timelike HRT surface variation $A''_{\mathrm{ext}}[h_{\mathrm{odd}}]$ and its exact spectral competition against the bulk kinetic term $-I[h_{\mathrm{odd}}]$ had not been carried out. Our exact Green's function (Lemma 3.1), one-line time-even identity (3.17), sharp spectral theorem $\|T\|_{\mathrm{op}} = 1/10$ (Theorem 3.2), and explicit $O(\varepsilon_z)$ conformal-boundary corner audit (Section 3.4) give a complete analytical accounting of both bulk and boundary contributions in AdS_4.

Direct extension of Rodgers (2026). Rodgers [6, 8] formulated the boundary-ball readout map (M_1, M_2), characterized the fixed-frame first-order kernel $\ker M_1$, derived the charge–area identity eliminating k in Poincaré AdS_4, and evaluated $\mathcal{E}_B$ explicitly for the time-even polynomial mode $E_{\mathrm{Rodgers}} = \widetilde{E}_1 + \frac{1}{3}E_0$ and oscillatory time-even modes h_c. The present paper resolves the open problems posed in [8, Section 6] and carries the programme to an infinite-dimensional reconstruction theorem inside $\ker M_1$: we derive the $+\frac{1}{8\pi}$ off-shell charge–area prefactor directly with consistent form signs and boundary orientations, extend the exact computation to all polynomial degrees and the full analytically diagonalized Lorentzian time-odd sector, construct explicit closed-form second-order vacuum completions k on the Poincaré patch, prove exact all-radii non-identifiability and all-times polarization-chirality blindness, compute the exact lifting under time shifts, $m = 2/m = 4$ multipoles, and time-translated Lorentz boosts (including an explicit six-ball constructive recovery design with sharp $v, \tau \to 0$ conditioning rates), determine the exact minimax stability and SDP phase transition of M_2, construct smooth finite-energy localized wavepackets strictly inside $\ker M_1$ with a minimal constructive three-ball reconstruction theorem, prove an exact manifestly positive one-dimensional real-space canonical-energy identity for every centered ball (Theorem 7.2), and establish our headline infinite-dimensional reconstruction theorem (Theorem 7.7): at any fixed centered radius $R_0 > 0$, exact canonical-energy data on any nonempty open time interval determine every perturbation in the compact positive width-measure class $\mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$ $(0 < a_{\min} \le a_{\max} < \infty)$ up to one overall sign, sharply separated from the fixed-radius lifted rank obstruction (Theorem 7.6(ii)), multi-radius lifted injectivity (Theorem 7.9(ii)), and finite-window noisy stability barriers (Theorem 7.11).

Linear holographic bulk reconstruction and integral geometry. Classical bulk reconstruction programmes—HKLL smearing [16], bulk-cone singularities [17], hole-ography and kinematic space [15, 18], entanglement-based linearized metric reconstruction [19, 20], and extremal-area inversion [21]—operate primarily at first order in the metric perturbation (or invert the X-ray / Radon transform of δA_B over all boundary regions, including boosted and deformed regions). Our results clarify precisely what is lost and what is recovered when measurements are restricted to *spherical rest-frame balls*: on a fixed time slice, the entire homogeneous TT space V_{hom} and the infinite-dimensional localized axial sector $h[\Phi]$ lie in $\ker M_1$ for all rest-frame balls, so first-order spherical area readouts are completely blind to them. Second-order canonical energy detects every nonzero mode in V_{hom} and our localized family $h[\mu]$, yet still exhibits exact rest-frame degeneracies at $t = 0$ (Theorem 4.1) that require time shifts, $m = 2/m = 4$ shape deformations, or time-translated Lorentz boosts to resolve (Theorems 4.4–4.5). Once time evolution on any nonempty open interval $I \subset \mathbb{R}$ is observed, Theorem 7.7 shows that even a *single* fixed centered radius $R_0 > 0$ suffices for exact noiseless physical reconstruction across the entire compact positive width-measure class $\mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])/\{\pm 1\}$—without varying the ball radius at all. Under measurement noise, however, both Theorem 5.8 (exponential bulk-depth extrapolation instability $(R_*/R_{\max})^{4n+2}$ in V_n^{even}) and Theorem 7.11(iii) (Bernstein-rate exponential/logarithmic ill-conditioning with unit-ball minimax error floor 1 and H^s rate $\Omega((\log(1/\eta))^{-2s})$) quantify severe stability barriers mirroring the analytic-continuation instability well known in Euclidean and HKLL bulk reconstruction.

Shape dependence of holographic entanglement entropy. Deformed spherical entangling surfaces in pure AdS_4 were analyzed in [9, 10, 13, 22], establishing the exact $\mathbb{H}^3$ minimal-surface harmonics $\psi_m(y) = (1 + my)\left(\frac{1-y}{1+y}\right)^{m/2}$ and computing vacuum shape Hessians and first-order stress-tensor couplings. Theorem 4.3 pushes this programme to *second order in a Lorentzian time-odd bulk metric perturbation*: because time-odd modes vanish on the $t = 0$ slice, they remain invisible to first-order area variations DA_{B_ϵ} and local energy-density smears of *every* static shape ($m = 2$ ellipses included), and their polarization orientation is first detected linearly in the shape parameter ϵ at $m = 4$ (quadrupole) via resonant coupling to the $m = 2$ timelike HRT extremum τ_*, yielding $-\frac{\pi R^8}{1200}(b_+^2 - b_\times^2)$ in $\partial_\epsilon A''_{\mathrm{ext},B_\epsilon^{(4)}}|_{\epsilon=0}$ and $-\frac{\pi R^8}{4800}(b_+^2 - b_\times^2)$ in $\partial_\epsilon \delta^{(2)} S_{B_\epsilon^{(4)}}|_{\epsilon=0}$ for every $\pi/4$-rotation-invariant second-order completion k.

Phase retrieval, PhaseLift, and optimal recovery. Our stability theory in Section 5 connects directly to the phase-retrieval and optimal-recovery literatures [11, 12, 14, 23–25]. Unlike standard PhaseLift [23], where measurement matrices $A_k = \phi_k \phi_k^\top$ are rank-one and randomized measurements ensure SDP exactness in high dimensions, gravitational boundary-ball canonical energies are *full-rank positive-definite* matrices $A_n(R)$ whose radial dependence is constrained by AdS_4 scaling to the $(2n-1)$-dimensional Hankel subspace $\{R^{6+2s}\}$. As a result, SDP relaxation is universally exact at $n = 2$ (Theorem 5.3), yet collapses completely ($L_{\mathrm{SDP}} = \infty$) for every $n \geq 3$ even with continuum radial measurements, while pure-state recovery remains globally injective and Lipschitz stable from $2n - 1$ radii (Theorems 5.5–5.6).

8.2 Limitations and Open Mathematical Problems

We emphasize five clear scope boundaries and open mathematical problems:

(1) Classical canonical energy vs. CFT state admissibility. Throughout this manuscript, $\mathcal{E}_B(h)$ is defined and analyzed purely as the classical Hollands–Wald gravitational canonical energy of an on-shell linearized vacuum solution h (Remark 2.2). While $\mathcal{E}_B(h)$ agrees formally with the holographic quantum Fisher information $\delta^{(2)} S(\rho_B \| \sigma_B)$ whenever a one-parameter family of holographic CFT states dual to the second-order

bulk completion $g(\lambda) = \bar{g} + \lambda h + \frac{1}{2}\lambda^2 k + O(\lambda^3)$ exists [7], we do *not* prove the existence of dual CFT states for the non-normalizable homogeneous modes in V_{hom} (which carry infinite global energy on the Poincaré patch). For the localized finite-energy wavepackets $h^{(a)}_{\mathrm{odd/even}}$ and infinite-dimensional superpositions $h[\mu]$ of Sections 6–7, the boundary stress tensor decays as $O(r^{-9})$, the global energy is finite, and Theorem 7.2 establishes strict classical positivity of $\mathcal{E}_{B(R,0,t)}$ for every centered ball without invoking CFT relative entropy, though constructing explicit dual CFT coherent states remains outside our purely gravitational analysis.

(2) Second-order vacuum completions for localized wavepackets. Proposition 3.4 constructs explicit closed-form Fefferman–Graham second-order vacuum completions k solving $D\mathcal{F}[k] + D^2\mathcal{F}[h,h] = 0$ for every polynomial mode in V_{hom}. Constructing an explicit globally regular second-order vacuum completion k_{loc} (with finite global ADM/Poincaré mass matching $\mathcal{E}_{\partial_t} = \frac{675\pi}{1024\, a^{13}}$ and smooth Fefferman–Graham asymptotics) for the localized wavepackets $h^{(a)}_{\mathrm{odd/even}}$ of Section 6 is left open. Notice, however, that our calculations of $\mathcal{E}_{B(R,0,t)}$ via both the charge–area identity and direct Fefferman–Graham bulk symplectic pairing $W_{\Sigma_B}(h, \mathcal{L}_{\xi_B} h)$ (Theorems 6.3, 7.2, and 7.4) are rigorously independent of k_{loc}.

(3) Extension beyond the axisymmetric radial axial sector. Theorems 7.7 and 7.9 completely settle the axisymmetric radial axial inverse problem on compactly supported width measures $\mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$—proving that a *single* fixed centered ball radius $R_0 > 0$ already determines every physical rank-one state unconditionally up to overall sign (resolving arbitrary interlaced finite superpositions and continuous or singular measures simultaneously), while varying R over (R_1, R_2) restores unrestricted lifted injectivity on $\mathcal{M}_{\mathrm{sym}}(K \times K)$ where every finite set of K radii collapses at $N \geq 14K$. Extending these infinite-dimensional reconstruction theorems beyond the axisymmetric radial axial sector $\Phi(t, \varrho)$ to general non-axisymmetric potentials $\Phi(t, z, x, y)$ using off-center boundary balls $B(R, \mathbf{x}_0, t)$—where w_* and τ_* no longer vanish identically—remains an open geometric inverse problem.

(4) Pure-vs-SDP stability gap for $n \geq 4$. Theorem 5.3 proves $L_{\mathrm{pure}} \equiv L_{\mathrm{SDP}}$ at $n = 2$, and Theorem 5.6 proves a sharp zero-or-infinite dichotomy at $n = 3$ (so no strict finite gap $L_{\mathrm{pure}}(Q) < L_{\mathrm{SDP}}(Q) < \infty$ exists for $n \leq 3$). At $n \geq 4$, $\dim\ker\mathcal{A} = \binom{n-1}{2}$ still equals the real codimension of the symmetric rank-≤ 2 variety, so it remains open whether every Type II target ($Q \in (\ker\mathcal{A})^{\perp}$) continues to have zero gap for $n \geq 4$, or whether a strict finite pure-vs-SDP stability gap first appears at $n = 4$.

(5) Higher dimensions $d \neq 3$ and global AdS_4. Our explicit Green's function, sharp spectral constant $\|T\|_{\mathrm{op}} = 1/10$, Axial Potential Theorem, and real-space positivity identity (7.11) are proved in Poincaré AdS_4 ($d = 3$). Extending the timelike HRT spectral operator T_d and infinite-dimensional axial reconstruction to general AdS_{d+1} ($d \geq 4$, where the radial wave equation has $\frac{d+1}{z}\partial_z$ and $\widetilde{B} \cong \mathbb{H}^{d-1}$) and testing whether $\|T_d\|_{\mathrm{op}}$ remains strictly below 1 across all d—as well as formulating the analogous inverse problem on global AdS_4 where boundary spheres are compact and normalizable modes are discrete—are natural next steps.

8.3 Conclusion

Taken together, the results of this paper establish a sharp mathematical picture of second-order gravitational boundary-ball reconstruction inside the first-order kernel $\ker M_1$ of Poincaré AdS_4. At the headline level (Theorem 7.7), at *any fixed centered radius* $R_0 > 0$, exact canonical-energy data $t \mapsto \mathcal{E}_{B(R_0,0,t)}(h[\mu])$ on any nonempty open time interval $I \subset \mathbb{R}$ determine *every*

perturbation $h[\mu]$ in the compact positive width-measure class $\mathcal{M}_{\mathbb{C}}([a_{\min}, a_{\max}])$ ($0 < a_{\min} \le a_{\max} < \infty$) up to one overall sign—both conceptually via large-t boundary dominance in the exact manifestly positive real-space identity (Theorem 7.2) and constructively via a strictly triangular exterior Laurent moment recursion, without requiring any variation of the ball radius. Simultaneously, the surrounding theorems delineate every structural boundary of this recovery: the physical sign quotient must be kept strictly separate from unrestricted lifted recovery (which collapses analytically at any finite set of K radii once $N \ge 14K$ and requires multi-radius data on $(R_1, R_2) \times I$ for infinite-dimensional lifted injectivity, Theorems 7.6(ii) and 7.9(ii)), and exact continuous-time noiseless uniqueness must be kept strictly separate from finite-window noisy stability (where separated finite-N families enjoy unconditional compact-global Lipschitz stability on the sign quotient $K_{\mathrm{par}}/\langle S_{-1}\rangle$ over continuous windows $[-T, T]$, whereas continuous width densities face an unconditional Bernstein-rate barrier with unit-ball minimax error floor 1 and logarithmic H^s lower bound $\Omega((\log(1/\eta))^{-2s})$, Theorem 7.11).

Remark 8.1 (Methodological independence). Although the initial questions investigated here were motivated by an audit of the source–readout gravity programme [6, 8], every theorem, proof, and computational certificate in this paper stands entirely on standard four-dimensional Lorentzian general relativity, the Hollands–Wald covariant phase-space formalism, extremal-surface variational calculus, classical harmonic and complex analysis, and finite-dimensional convex and real-algebraic optimization, without invoking any external axioms or specialized frameworks.

A Exact Four-Dimensional $D^2\mathcal{F}_{ab}$ Components and Hollands–Wald Flux

For a spatially homogeneous TT perturbation $h_{xy}(t, z) = zH(t, z)$ (all other components zero) on Poincaré AdS_4, $\bar{g} = z^{-2}(-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}x^2 + \mathrm{d}y^2)$, expanding the vacuum Einstein tensor $\mathcal{F}_{ab}[g] = G_{ab}[g] - 3g_{ab}$ to second order $g = \bar{g} + \lambda h$ gives $D\mathcal{F}_{xy}[h] = \frac{z^2}{2}\big(zH_{tt} - zH_{zz} - 4H_z\big)$ (all other $D\mathcal{F}_{ab} = 0$) and the exact nonvanishing second-order components (`verify_01`):

$$D^2\mathcal{F}_{tt}[h,h] = \frac{z^4}{2}\Big(4z^2HH_{zz} - z^2H_t^2 + 3z^2H_z^2 + 34zHH_z + 27H^2\Big), \tag{A.1}$$

$$D^2\mathcal{F}_{tz}[h,h] = z^5\Big(2zHH_{tz} + zH_tH_z + 9HH_t\Big), \tag{A.2}$$

$$D^2\mathcal{F}_{zz}[h,h] = \frac{z^4}{2}\Big(4z^2HH_{tt} + 3z^2H_t^2 - z^2H_z^2 + 2zHH_z + 15H^2\Big), \tag{A.3}$$

$$D^2\mathcal{F}_{xx}[h,h] = D^2\mathcal{F}_{yy}[h,h] = \frac{z^4}{2}\Big(4z^2HH_{tt} - 4z^2HH_{zz} + z^2H_t^2 - z^2H_z^2 - 22zHH_z - 9H^2\Big). \tag{A.4}$$

Whenever H satisfies the linearized wave equation $H_{tt} = H_{zz} + \frac{4}{z}H_z$, these components satisfy the exact background Bianchi identity $\bar{\nabla}^a D^2\mathcal{F}_{ab}[h,h] \equiv 0$ identically (`verify_01`), guaranteeing local integrability of the second-order vacuum system $D\mathcal{F}_{ab}[k] + D^2\mathcal{F}_{ab}[h,h] = 0$.

For the constant-velocity mode O_0 ($h_{xy} = zt$) with Hollands–Wald gauge vector $V^a = \big(\tau_*,\, t\partial_z\tau_*,\, t\partial_x\tau_*,\, t\partial_y\tau_*\big)$ of (3.25) ($\tau_* = -\frac{1}{20}xyz^3$, evaluating on $t = 0$ to $V|_{t=0} = (-\frac{1}{20}xyz^3, 0, 0, 0)$ with $\partial_t V^i|_{t=0} = \partial_i\tau_*$), the exact densitized flux components $\rho^{0i}_{\mathrm{flux}} := \sqrt{-\bar{g}}\big(\chi^{0i}(\gamma; [\xi_B, V]) - \chi^{0i}(\mathcal{L}_{\xi_B}h; V)\big)$ entering eq. (2.10) on the $t = 0$ slice (in the normalization

(2.7)–(2.8), which already includes $\frac{1}{16\pi}$) are (`verify_01`):

$$\rho^{0z}_{\text{flux}} = -\frac{3z}{3200R}\Big(6x^4y^2 + 6x^2y^4 + x^4z^2 + y^4z^2 + x^2z^4 + y^2z^4 - 6R^2x^2y^2 - R^2x^2z^2 - R^2y^2z^2\Big), \tag{A.5}$$

$$\rho^{0x}_{\text{flux}} = -\frac{xz^2}{3200R}\Big(9x^2y^2 + 9y^4 - 19x^2z^2 + 22y^2z^2 - 19z^4 - 9R^2y^2 + 19R^2z^2\Big), \tag{A.6}$$

$$\rho^{0y}_{\text{flux}} = -\frac{yz^2}{3200R}\Big(9x^4 + 9x^2y^2 + 22x^2z^2 - 19y^2z^2 - 19z^4 - 9R^2x^2 + 19R^2z^2\Big), \tag{A.7}$$

which satisfy $\partial_z\rho^{0z}_{\text{flux}} + \partial_x\rho^{0x}_{\text{flux}} + \partial_y\rho^{0y}_{\text{flux}} \equiv \sqrt{-\bar{g}}\big(\omega^t(\gamma, \mathcal{L}_{\xi_B}\gamma) - \omega^t(h, \mathcal{L}_{\xi_B}h)\big)$ pointwise throughout Σ_B, integrate over $z = \varepsilon_z$ to $-\frac{3\pi\varepsilon_z(R^2-\varepsilon_z^2)^3(R^2+7\varepsilon_z^2)}{51200R} = O(\varepsilon_z) \to 0$, and integrate over the hemisphere $\widetilde{B}(R)$ to $-\frac{\pi R^8}{12600}$ identically.

B Exact Symbolic Sum-of-Squares Certificate for Theorem 5.3

On $V_2^{\text{even}} = \text{span}\{E_0, \widetilde{E}_1\}$ with $x_k = R_k^2 > 0$ $(k = 1, 2, 3)$, solving $\text{Tr}(A_2(R_k)\Delta) = 1$ $(k = 1, 2, 3)$ for $\Delta \in \text{Sym}^2(\mathbb{R}^2)$ gives $\det\Delta(1,1,1) = N_{111}(x_1, x_2, x_3)/(29\,\pi^2 x_1^6 x_2^6 x_3^6)$ (`verify_03`). Subtracting the three manifestly negative symmetric sum-of-squares blocks

$$S_{\text{SOS}}(x_1, x_2, x_3) := -\sum_{1\le i<j\le 3} x_i^4 x_j^4\Big[5{,}115{,}600\,(x_i - x_j)^2 + 7{,}526{,}400\,x_i x_j\Big] < 0 \tag{B.1}$$

from the exact numerator $N_{111}(x_1, x_2, x_3)$ leaves a symmetric homogeneous degree-10 remainder polynomial $R_{\text{rem}}(x_1, x_2, x_3) := N_{111} - S_{\text{SOS}}$ consisting of exactly 27 monomials $c_{abc}\,x_1^a x_2^b x_3^c$ with $a, b, c \ge 1$ and $a + b + c = 10$, every single coefficient of which is strictly negative, lying in $[-27{,}400{,}800,\ -4{,}821{,}600]$ (`verify_03`). Thus $N_{111}(x_1, x_2, x_3) = S_{\text{SOS}}(x_1, x_2, x_3) + R_{\text{rem}}(x_1, x_2, x_3) < 0$ identically on the entire positive orthant $(0, \infty)^3$, certifying $\det\Delta(1,1,1) < 0$ for all triples $0 < R_1 < R_2 < R_3$.

C Companion Reproducibility Suite

Every symbolic identity, closed-form integral, exact rational matrix, and numerical certificate in this paper is reproduced by six self-contained Python scripts (using `SymPy`, `NumPy`, `SciPy`, and `mpmath`) executed in sequence by `reproducibility/run_all.py`, with full captured logs and JSON receipts stored in `reproducibility/results/`:

1. `verify_01_timelike_hrt_and_hw.py`: 4D Einstein tensors $D\mathcal{F}$, $D^2\mathcal{F}_{ab}$ and $\bar{\nabla}^a D^2\mathcal{F}_{ab} = 0$; off-shell $+\frac{1}{8\pi}$ charge–area derivation directly verifying $\mathrm{d}\chi = -\delta C_{\xi_B}$ and oriented Stokes boundary fluxes on both the on-shell planar mass mode and off-shell $v_{xx} = v_{yy} = z^3$; closed-form $\mathbb{H}^2$ Green's function; all-k polynomial HRT identity and matrices $\alpha^{(4)}$, $\gamma^{(3)}$, α_{mon}, γ_{mon}; one-line time-even integral and oscillatory table cross-check; analytic invariant-flag diagonalization of T $(k = 0, \ldots, 4$ plus exact $T = 2G/(yy'))$ check) and 240-node Nyström sanity check; direct Hollands–Wald eqs. (2.9)–(2.10) and $O(\varepsilon_z)$ corner audit; closed-form Fefferman–Graham completions k with c_M-cancellation; smooth triangular Green's quadrature for h_s.

2. `verify_02_blind_spots_and_lifting.py`: single-polarization counterpair $(h^{(1)}, h^{(2)})$ and exact cubic time-shift expansion $(R_* = \sqrt{45/11},\ \frac{\mathrm{d}^2}{\mathrm{d}t_0^2}\Delta\mathcal{E}|_{t_0=0} = \frac{2\pi R^6}{35})$; $\mathbb{H}^3$ minimal surfaces $\psi_m(y)$ for $m = 1, 2, 3, 4$; $m = 2$ elliptical DA_{B_ϵ}, $m = 4$ quadrupolar $\partial_\epsilon A''_{\text{ext},B_\epsilon^{(4)}}[O_0]$ $(m = 2$ null control, 64-node GL check, and $SO(2)$-isotropy of $D^2\mathcal{F}[b_+O_0^+ + b_\times O_0^\times])$;

exact first-order boosted readouts $M_1^{(v)}$ with independent $(\mathbf{x}_0, t_c)$ regression sweep, 3-mode bulk $\frac{1}{4}DA_{B^{(v)}}$ cross-check, and single-ball counterpair blind-locus verification $(2(t_c^2+\sigma) = b_{0*}t_c$, threshold $\sigma \le 5/12)$; constructive six-ball recovery Designs A and B (determinants, inverses, ℓ_1 conditioning bounds, chirality separation); exact $O(v^2)$ and all-v symbolic $t_c = 0$ boosted canonical energies $\mathcal{E}_{B^{(v)}(R,\mathbf{x}_0,0)}$ showing exact $O(v^2)$ cancellation and $+\frac{4\pi R^8 v^4}{315(1-v^2)^2}$ quartic lifting.

3. `verify_03_higher_dim_st2_and_stability.py`: $n = 2$ symbolic SOS/27-monomial certificate; $n = 3$ kernel K_3, all-radii mixed-vs-pure counterpair, symbolically verified monic cubic coefficients and exact rational-Vandermonde full-cube $[-1,1]^5$ upper bound $11{,}717{,}038.408\ldots < 1.1718 \times 10^7$ for $L_{\text{pure}}(e_1 e_1^\top)$, exact Type II IVT pure-state witnesses, and $n = 4$ rank check; universal Lagrange scaling table $(n = 1, \ldots, 6)$, continuum linear/SDP Chebyshev lower bounds, and unconditional pure-state lower bounds $(R_*/R_{\max})^{4n+2}$.

4. `verify_04_wavepackets_and_completions.py`: general C^5 Axial Potential Theorem (all 10 Einstein equations + exact total ∂_φ trace identity on arbitrary $B(R, \mathbf{x}_0, t_0)$) plus explicit counterexample tests ($\Phi_{\text{bad}} = txyz^{-3}$ and vanishing-image potentials); complexified AdS_4 inversion wavepackets $h^{(a)}_{\text{odd}}, h^{(a)}_{\text{even}}$; exact closed-form radial densities, wedge canonical energies, rational difference (6.17), and finite global energy $\frac{675\pi}{1024}$; exact Bianchi-flux + surface time-shift derivative $\dot{\mathcal{E}}_{B(R,0)}[h^{(\pm)}_{\text{loc}}]$ and exact holomorphic finite-time separation identity (6.23) corroborated by direct Hollands–Wald bulk symplectic pairing W_{Σ_B} at $t_0 = 0, 0.1, 0.5$ (GL64 agreement $< 10^{-15}$); complete exact algebraic and 50-digit `mpmath.iv` interval certification of the minimal three-ball localized design $(R_1, t_1) = (\frac{1}{2}a, 0)$, $(R_2, t_2) = (a, 0)$, $(R_3, t_3) = (\frac{1}{2}a, \frac{1}{2}a)$ (exact $2E^{(1)}_{eo}(\frac{1}{2}, \frac{1}{2}) = -\frac{27\pi}{1760}$, GL64 corroboration $< 1.4 \times 10^{-16}$, three-negative-term $\det M_{2\times 2}$, PSD $\det A_3 > 0$, inverse row ℓ_1-norms, $1/|t_c|$ rate, and all-vertex rank-one zero-gap certificate).

5. `verify_05_infinite_dim_reconstruction.py`: exact axisymmetric polar simplification ($w_* = \tau_* = 0$ on all centered hemispheres) and exact real-space 1D classical positivity identity (7.11) (verifying exact surface cancellation, bulk p^2 cancellation, and Hardy uniform domination constant $\frac{363}{175}$); independent two-route symbolic derivation of the polarized two-width master kernel $\mathcal{K}_R(\alpha, \beta)$ (Route 1 direct bulk symplectic W_{Σ_B} vs. Route 2 polarized 4D Einstein charge–area with exact surface term $-\frac{102400\pi R^{10}\alpha\beta}{77(\alpha^2+R^2)^6(\beta^2+R^2)^6}$, exact symbolic zero difference across all coefficients $A_\alpha, C_0, P_1, \ldots, P_6 = 0$, exact global Bergman limit $\frac{10800\pi}{(a+b)^{13}}$, and 4 unequal-width 64-node GL direct symplectic challenges $< 10^{-13}$); exact elementary frequency-space kernels $\widehat{\mathcal{K}}^+_R, \widehat{\mathcal{H}}_R$ (zero Si/Ci terms, 35-digit double Laplace check $< 10^{-14}$), exact boundary-trace Fourier–Laplace identity (7.44) ($< 10^{-15}$), direct real-space Darboux/Bessel chain verifying (7.50) across positive, negative, and mixed signed widths ($P(n) > 0\ \forall n \ge 9$, exact S_k for $k = 0, \ldots, 3$); single-packet leading order-12 slit-endpoint singularity $\frac{75\pi}{4004}c^2$, $q = 2$ Wronskian signs on all three intervals, exact symbolic right-endpoint pole functions from $P_1, \ldots, P_5$, exact two-prime modular lower bound $\operatorname{rank}_{\mathbb{Q}}(M_{\text{even}}) \ge 184$, and a 100-digit numerical near-null witness $\Delta \in \mathrm{Sym}^2(\mathbb{R}^{30})$ ($\|\Delta\|_F = 1$, 100-digit numerical rank 184/195, nullity 26, $|\operatorname{Tr}(A(1,t)\Delta)| \le 1.32 \times 10^{-88}$, off-radius absolute traces $|\operatorname{Tr}(A(R,t)\Delta)| \in [9.6\times 10^{-20}, 3.4\times 10^{-18}]$ at $R \in \{2, \frac{1}{2}\}$ for relative separation ratio $> 7 \times 10^{68}$, and 50-digit spectrum ruling out rank ≤ 2); single-packet Schur-complement width sensitivity d_a across all phases, exact nearby-width global-energy Gram conditioning ($\frac{16a^2}{13\delta^2}$), exact width-difference global energy ($\frac{61425\pi}{2048a^{15}}\delta^2$), exact symbolic rank-10 proof of confluent jet injectivity (verifying R-independence of the leading K_{11}, K_{10} singularities and proving $\Theta(\delta^2)$ on $L^2/L^\infty([-T,T])$) plus 100-digit Frobenius-to-ℓ_2 16-point discrete corroboration ($\sigma_{\min}/\delta^2 \to 2.395 \times 10^{-2}$), exact Legendre–Bergman

exponential decay bound (7.57) on $[1,2]$ ($n = 0,\dots,9$), orthonormal Legendre-tail verification of the unit-ball minimax error floor 1, sign-action S_σ invariance and two-point segment-integral local lower-bound verification for $N = 2$ separated packets, and top-10 Nyström Bergman eigenvalues.

6. `verify_06_single_radius_reconstruction.py`: exact symbolic proof of the Darboux factorial ratio $c_r(2r)! = (1/2)_r/(11/2)_r \le 1$, exact exterior moment identity (7.33), and exact symbolic large-t Laurent coefficients $W_s(a,b;R_0)$ ($s = 0,\dots,3$) proving $O(t^{-2})$ bulk-to-boundary suppression for arbitrary fixed $R_0 > 0$; 100-digit verification of the polarized limit (7.41) on maximally interlaced $N = 6$ atomic partitions ($S_+ = \{1.0, 1.4, 1.8\}$, $S_- = \{1.2, 1.6, 2.0\}$ with vanishing orders $k = 2, m = 3$, order t^{-25}) across $R_0 \in \{0.2, 1.0, 5.0\}$ and on overlapping continuous densities on $[1,2]$ at large radius $R_0 = 10 > a_{\max}$ with high vanishing orders $k = 5, m = 4$ (order t^{-29}, confirming exact $O(t^{-2})$ bulk-to-boundary decay); 80-digit constructive strictly triangular Laurent moment recovery (7.35) at one fixed radius $R_0 = 5.0$ on a mixed atomic-plus-continuous complex measure with $c_0 = c_1 = 0$ (recovering 12 consecutive moments $c_2,\dots,c_{13}$ to $< 1.23\times10^{-33}$ relative error); and 100-digit large-t boundary-vs-bulk evaluation on the $N = 15, R_0 = 1$ lifted near-null witness Δ_{30} at $t \in \{10, 50, 200\}$, demonstrating exact opposite-sign boundary-vs-bulk cancellation ($\pm 1.68 \times 10^{-67}$ canceling to relative accuracy 6.9×10^{-62} at $t = 200$).

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